It is easy to check that A_i is a closed convex cone, and clearly $h_K = \langle x_i, \cdot \rangle$ is linear on A_i. Thus h_K is piecewise linear.

Conversely, suppose that h_K is linear on the convex cones $A_1, \dots, A_k$ which cover $\mathbb{R}^n$. Then we may assume that A_i is closed, since we can replace A_i by $\mathrm{cl}\, A_i$, which is still a convex cone, and use that h_K remains linear on $\mathrm{cl}\, A_i$. Moreover, we can assume that all closed convex cones $A_1, \dots, A_k$ have interior points, since lower-dimensional closed sets A_i can be omitted (if a point $x_0 \in \mathbb{R}^n$ lies only in lower-dimensional closed convex cones, then there is a point in a neighbourhood of x_0 which is not covered by the union of all cones, a contradiction). Then, for $i = 1, \dots, k$ there is some (uniquely determined) $x_i \in \mathbb{R}^n$ such that $\langle x_i, \cdot \rangle = h_K$ on A_i. This already implies that

$$\max\{\langle x_i, \cdot \rangle : i = 1, \dots, k\} \geq h_K.$$

For the reverse inequality, let $x \in \mathbb{R}^n$. For $i \in \{1, \dots, k\}$, we can choose $z \in \mathrm{int}(A_i) \setminus \{x\}$. Then there are $y \in A_i$ and $\lambda \in (0, 1)$ such that $z = \lambda x + (1 - \lambda) y$. Hence,

$$\begin{aligned} \langle x_i, z \rangle = h_K(z) &= h_K(\lambda x + (1 - \lambda) y) \\ &\leq \lambda h_K(x) + (1 - \lambda) h_K(y) = \lambda h_K(x) + (1 - \lambda) \langle x_i, y \rangle, \end{aligned}$$

and thus $\langle x_i, x \rangle \leq h_K(x)$. Since this holds for $i = 1, \dots, k$, we get

$$\max\{\langle x_i, x \rangle : i = 1, \dots, k\} \leq h_K(x)$$

for $x \in \mathbb{R}^n$. This finally yields that $h_K = \max\{\langle x_i, \cdot \rangle : i = 1, \dots, k\}$. □

Exercises and Supplements for Sect. 2.3

1.* Let $f : \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous and twice continuously partially differentiable on $\mathbb{R}^n \setminus \{0\}$. Show that there are $K, L \in \mathcal{K}^n$ such that

$$f = h_K - h_L.$$

Hint: Use Exercise 2.2.4 (b).

2. Let $K \in \mathcal{K}^n$ with $0 \in \mathrm{int}\, K$, and let K° be the polar of K (see Exercise 1.1.14). Show that

   (a) K° is compact and convex with $0 \in \mathrm{int}\, K^\circ$,
   (b) $K^{\circ\circ} := (K^\circ)^\circ = K$,
   (c) K is a polytope if and only if K° is a polytope,
   (d) $h_K = d_{K^\circ}$.

$d_A(x) := \inf\{\alpha \geq 0 : x \in \alpha A\},\ x \in \mathbb{R}^{n}$.

3. Let $K \in \mathcal{K}^n$ with $0 \in \operatorname{int} K$. Then the *radial function* of K is defined by

$$\rho(K, x) := \rho_K(x) := \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Show that $\rho(K, x) = h(K^\circ, x)^{-1}$ for $x \in \mathbb{R}^n \setminus \{0\}$. See Exercise 1.1.14 for the definition of the polar body K°.
4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be sublinear. Then there is a uniquely determined $K \in \mathcal{K}^n$ satisfying $f = h(K, \cdot)$. Prove the existence of K by using Helly's theorem.
5.* Let $K \subset \mathbb{R}^n$ be a compact set, and for $k \in \mathbb{N}$ let $f_k : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Assume that for $x \in \mathbb{R}^n$ the limit $f(x) := \lim_{k\to\infty} f_k(x)$ exists. Prove the following assertions.

   (a) $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.
   (b) There is a constant $M \in \mathbb{R}$ such that $|f_k(x)| \leq M$ for $x \in K$ and $k \in \mathbb{N}$.
   (c) The sequence $(f_k)_{k\in\mathbb{N}}$ converges uniformly on K to f.

6. Let $K, L \in \mathcal{K}^n$ and let $C > 0$ be a constant such that $K, L \subset B^n(0, C)$. Show that

   (a) h_K is Lipschitz continuous with

   $$|h_K(x) - h_K(y)| \leq C\|x - y\|, \quad x, y \in \mathbb{R}^n.$$

   (b)

   $$|h_K(u) - h_L(v)| \leq C\|u - v\| + \|h_K - h_L\|, \quad u, v \in \mathbb{S}^{n-1},$$

   where $\|f\| := \sup\{|f(u)| : u \in \mathbb{S}^{n-1}\}$ for a function $f : \mathbb{S}^{n-1} \to \mathbb{R}$.

7. Let $0 < r \leq R < \infty$, $K, L \in \mathcal{K}^n$ with $B^n(0, r) \subset K, L \subset B^n(0, R)$ and $u, v \in \mathbb{S}^{n-1}$. Prove that

$$|\rho(K, u) - \rho(L, v)| \leq \frac{R}{r}\|h_K - h_L\| + \frac{R^2}{r}\|u - v\|.$$

8. Let $K \in \mathcal{K}^n$ with $\operatorname{int} K \neq \emptyset$ and

$$l(K, u) := \sup\{\|x - y\| : x, y \in K, x - y \in \mathbb{R}u\}, \quad u \in \mathbb{S}^{n-1}.$$

Prove the following assertions.

   (a) The supremum in the definition of $l(K, u)$ is a maximum. (This is the maximal length of a segment in K having direction u.)
   (b) $l(K, u) = \rho(K - K, u)$ for $u \in \mathbb{S}^{n-1}$.
   (c) $\min\{l(K, u) : u \in \mathbb{S}^{n-1}\} = \min\{h(K, u) + h(K, -u) : u \in \mathbb{S}^{n-1}\}$.

9. Determine the support functions of the following convex bodies (cube and cross-polytope).

   (a) $K_1 := \{(x_1, \dots, x_n)^\top \in \mathbb{R}^n : |x_i| \le r \text{ for } i = 1, \dots, n\}, r \ge 0.$
   (b) $K_2 := \{(x_1, \dots, x_n)^\top \in \mathbb{R}^n : |x_1| + \dots + |x_n| \le r\}, r \ge 0.$

10. Let $K_1, K_2 \subset \mathbb{R}^n$ be convex bodies. Determine the support function of the convex body

$$\operatorname{conv}(K_1 \cup K_2).$$

Suppose in addition that $K_1 \cap K_2 \neq \emptyset$. Prove that the function

$$g(u) := \inf\{h(K_1, u_1) + h(K_2, u_2) : u_1 + u_2 = u\}, \quad u \in \mathbb{R}^n,$$

is sublinear. In fact, g is the support function of $K_1 \cap K_2$.

11. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be convex functions, and let $x \in \mathbb{R}^n$. Show that

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

12. Hints to the literature: Analytic aspects of convexity and convex functions are the subject of [37, 45, 55, 56, 65, 76, 87]. There exists a vast literature on optimization related to convexity. In this respect, we mention only the classical text [89].