Assume that x is a convex combination of affinely dependent points $x_1, \ldots, x_k$. Then the preceding argument shows that one of these points is redundant and x is a convex combination of at most $k-1$ of these points. Repeating this procedure, we arrive at a representation of x as a convex combination of an affinely independent subset of $\{x_1, \ldots, x_k\}$. In particular, any point of the convex hull of A is the convex combination of at most $n+1$ points of A. Clearly, the choice of these points from A will depend on the point x.

For the results of this section, linear analogues can be stated and proved which are concerned with vectors, linear hulls, convex cones, positive hulls instead of points, affine hulls, convex sets, and convex hulls. Often results can either be proved in a similar way or deduced from the corresponding counterpart.

Exercises and Supplements for Sect. 1.2

Dűsledék Hellyhez vég

1. (a) Show by an example that Theorem 1.8 is wrong in general if the sets in $\mathcal{A}$ are only assumed to be closed (and not necessarily compact).
   (b) Show by an example that the result is also wrong in general if the sets are bounded but not closed.
   (c) Construct an example of four sets in the plane, three of which are compact and convex (one can even choose rectangles), such that any three of the sets have a nonempty intersection, but such that the intersection of all sets is the empty set.
   (d) Show that Theorem 1.8 remains true if all sets in $\mathcal{A}$ are closed and convex and one of the sets is compact and convex.
2. (a) Let $\mathcal{R}$ be a finite set of paraxial rectangles. For any two rectangles $R, R' \in \mathcal{R}$ let $R \cap R' \neq \emptyset$. Show that all rectangles in $\mathcal{R}$ have a common point.
   (b) Let $\mathcal{S}$ be a finite family of arcs in $\mathbb{S}^1$, each of which is contained in an open semi-circular arc of the circle. Any three arcs of $\mathcal{S}$ have a point in common. Show that all arcs have a point in common.
   Is it sufficient to assume that any two arcs of $\mathcal{S}$ have a common point?
3. In an old German fairy tale, a brave little tailor claimed the fame to have 'killed seven at one blow'. A closer examination showed that the victims were in fact flies which had landed on a toast covered with jam. The tailor had used a fly-catcher of convex shape for his sensational victory. As the remains of the flies on the toast showed, it was possible to kill any three of them with one stroke of the (suitably) shifted fly-catcher without even turning the direction of the handle.
   Is it possible that the tailor told the truth (if it is assumed that the flies are points)?
4. Let $k \in \mathbb{N}$ and $k \geq n+1$. Let $A, A_1, \ldots, A_k \subset \mathbb{R}^n$ be nonempty and convex. Assume that for any set $I \subset \{1, \ldots, k\}$ with $|I| = n+1$ there is a vector $t_I \in \mathbb{R}^n$ such that

$$A_i \subset A + t_I \text{ for } i \in I.$$

Show that there is a vector $t \in \mathbb{R}^n$ such that $t \in A_i + (-A)$ for $i \in \{1, \dots, k\}$. If the sets $A_1, \dots, A_k$ are singletons, then $A_i \subset A + t$ for $i \in \{1, \dots, k\}$.

5. Let $\mathcal{F}$ be a family of parallel closed segments in $\mathbb{R}^2$, $|\mathcal{F}| \geq 3$. Suppose that for any three segments in $\mathcal{F}$ there is a line intersecting all three segments. Show that there is a line in $\mathbb{R}^2$ intersecting all segments in $\mathcal{F}$. (The problem is slightly easier if it is assumed that $\mathcal{F}$ is a finite family of segments.)
6. Prove the following version of Carathéodory's theorem:
   Let $A \subset \mathbb{R}^n$ and $x_0 \in A$ be fixed. Then conv A is the union of all simplices with vertices in A and such that x_0 is one of the vertices.
7.* Establish the following refined form of Carathéodory's theorem (due to Fenchel, Stoelinga, Bunt, see also [7] for a discussion):
   Let $A \subset \mathbb{R}^n$ be a set with at most n connected components. Then conv A is the union of all simplices with vertices in A and dimension at most $n - 1$. In other words, any point of conv A is in the convex hull of at most n points of A.
8. Suggestions for further reading: The combinatorial results of this section have been extended and applied in various directions. For instance there exist colourful, fractional, dimension-free and topological versions and generalizations of the theorems of Radon, Helly and Carathéodory (see [1, 6, 8, 9]). For a colourful version of Carathéodory's theorem, see also Exercise 1.4.7.
9. Applications of combinatorial results to containment problems are discussed in [52, 63]. Here are two examples from these works. The first is considered in [63] by E. Lutwak:
   Let $K, L \subset \mathbb{R}^n$ be compact convex sets. Suppose that for every simplex Δ such that $L \subset \Delta$, there exists a $v \in \mathbb{R}^n$ such that $K + v \subset \Delta$. Then there exists a $v_0 \in \mathbb{R}^n$ such that $K + v_0 \subset L$.
   An inscribed counterpart is discussed in [52]:
   Suppose that $K, L \subset \mathbb{R}^n$ have nonempty interiors. If every simplex contained in K can be translated inside L, then K can be translated inside L.
10. Let $K \subset \mathbb{R}^n$ be an n-dimensional compact convex set. Show that there exists a point $c \in K$ such that whenever $a \in K$, $b \in \text{bd}\, K$ with $c \in [a, b]$, then

$$\|a - c\| \leq \frac{n}{n+1} \|a - b\|. \qquad (*)$$

Hint: Consider the sets

$$K_x := x + \frac{n}{n+1}(K - x), \qquad x \in K.$$

Verify that for any points $x_0, \dots, x_n \in K$ we have

$$\frac{1}{n+1}(x_0 + \cdots + x_n) \in \bigcap K_{x_i}.$$

Now Helly's theorem can be applied.

11.* Let $K \subset \mathbb{R}^n$ be an n-dimensional compact convex set. Show that $c \in K$ has the property $(*)$ stated in Exercise 1.2.10 if and only if $-(K - c) \subset n(K - c)$.

12. In $\mathbb{R}^2$ the points

$$x_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad x_2 = \begin{pmatrix}1\\3\end{pmatrix}, \quad x_3 = \begin{pmatrix}4\\3\end{pmatrix}, \quad x_4 = \begin{pmatrix}4\\0\end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix}7/4\\5/4\end{pmatrix}$$

are given. Confirm that

$$x = \frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{6}x_3 + \frac{1}{12}x_4.$$

Use the method of proof for Carathéodory's theorem to express x as a convex combination of x_1, x_2, x_3.

13. Is the decomposition in Radon's theorem uniquely determined for $m = n + 2$ points in $\mathbb{R}^n$?
Hint: See [31].

14. Let $u_1, \ldots, u_m \in \mathbb{R}^n \setminus \{0\}$. Show that

$$0 \in \operatorname{conv}\{u_1, \ldots, u_m\} \iff \mathbb{R}^n = \bigcup_{i=1}^{m} H^+(u_i, 0).$$

Let $\mathbb{R}^n = \bigcup_{i=1}^{m} H^+(u_i, 0)$. Show that there is a set $I \subset \{1, \ldots, m\}$ with at most $n + 1$ elements such that

$$\mathbb{R}^n = \bigcup_{i \in I} H^+(u_i, 0).$$

In words: If N closed halfspaces containing the origin in their boundaries cover $\mathbb{R}^n$, then at most $n + 1$ of these halfspaces are needed to cover $\mathbb{R}^n$.

1.3 Topological Properties

Although convexity is a purely algebraic property, it has some useful topological consequences. For instance, we shall see that a nonempty convex set always has a nonempty relative interior. In order to prove this seemingly obvious fact, we first need an auxiliary result. We recall the following definitions and basic observations.

- Intersections of affine subspaces are affine subspaces (or the empty set).
- The affine hull, aff A, of a nonempty set $A \subset \mathbb{R}^n$ is the intersection of all affine subspaces containing A.