by (1.4), a contradiction. Therefore, $K = \widehat{K}$. Since $y_x \in \exp K$, for $x \in \mathbb{R}^n$, we obtain

$$K = \widehat{K} \subset \text{cl conv exp } K \subset K,$$

hence $K = \text{cl conv exp } K$. □

**Corollary 1.6 (Straszewicz's Theorem)** *Let $K \subset \mathbb{R}^n$ be compact and convex. Then*

$$\text{ext } K \subset \text{cl} \exp K.$$

***Proof*** By Theorems 1.23 and 1.12, we have

$$K = \text{cl conv exp } K \subset \text{cl conv cl exp } K = \text{conv cl exp } K \subset K,$$

hence

$$K = \text{conv cl exp } K.$$

By Theorem 1.21, this implies that $\text{ext } K \subset \text{cl} \exp K$. □

## *Exercises and Supplements for Sect. 1.5*

1. Let $A \subset \mathbb{R}^n$ be closed and convex. Show that $\text{ext } A \neq \emptyset$ if and only if $A$ does not contain any line.
2. Let $K \subset \mathbb{R}^n$ be compact and convex.
   (a) If $n = 2$, show that $\text{ext } K$ is closed.
   (b) If $n \geq 3$, show by an example that $\text{ext } K$ need not be closed.
3.* Let $A \subset \mathbb{R}^n$ be closed and convex. A subset $M \subset A$ is called an *extreme set* in $A$ (or a *face* of $A$) if $M$ is convex and if $x, y \in A$, $(x, y) \cap M \neq \emptyset$ implies that $[x, y] \subset M$. The set $A$ and $\emptyset$ are faces of $A$, all other faces are called proper. Prove the following assertions.
   (a) Extreme sets $M$ are closed.
   (b) Each support set of $A$ is extreme.
   (c) If $M, N \subset A$ are extreme, then $M \cap N$ is extreme. (This extends to arbitrary families of extreme sets.)
   (d) If $M$ is extreme in $A$ and $N \subset M$ is extreme in $M$, then $N$ is extreme in $A$.
   (e) If $M, N \subset A$ are extreme and $M \neq N$, then $\text{relint } M \cap \text{relint } N = \emptyset$.
   (f) Let $B$ be a nonempty and relatively open subset of $A$. Then there is a unique face $F$ of $A$ such that $B \subset \text{relint}(F)$.

(g) Let $\mathcal{E}(A) := \{M \subset A : M \text{ extreme}\}$. Then $A = \bigcup_{M \in \mathcal{E}(A)} \operatorname{relint} M$ is a disjoint union.
(h) Extreme sets of $A$ of dimension $\dim(A) - 1$ are always support sets.

4. A real $(n, n)$-matrix $A = ((\alpha_{ij}))$ is called *doubly stochastic* if $\alpha_{ij} \geq 0$ and

$$\sum_{k=1}^{n} \alpha_{kj} = \sum_{k=1}^{n} \alpha_{ik} = 1$$

for $i, j \in \{1, \ldots, n\}$. A doubly stochastic matrix with components in $\{0, 1\}$ is called a *permutation matrix*.
Prove the following statements.

(a) The set $K \subset \mathbb{R}^{n^2}$ of doubly stochastic matrices is compact and convex.
(b) The extreme points of $K$ are precisely the permutation matrices.

5. Let $P \subset \mathbb{R}^n$ be a polyhedral set, but not an affine subspace (a flat). Let

$$P = \operatorname{aff}(P) \cap \bigcap_{i=1}^{m} H^-(u_i, \alpha_i)$$

with $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and $u_1, \ldots, u_m \in \mathbb{R}^n \setminus \{0\}$ be a representation of $P$ in which none of the halfspaces $H^-(u_i, \alpha_i)$ can be omitted. Put $F_i := P \cap H(u_i, \alpha_i)$. Then

$H^-(u,\alpha) := \{x : \langle x, u \rangle \leq \alpha\}$

(a) $\operatorname{relint}(P) = \bigcap_{i=1}^{m} \{x \in P : \langle x, u_i \rangle < \alpha_i\}$.
(b) $\operatorname{relbd}(P) = \bigcup_{i=1}^{m} F_i$.
(c) $F_1, \ldots, F_m$ are precisely the facets of $P$.
(d) Each proper face $F$ of $P$ is equal to the intersection of all facets of $P$ which contain $F$.
(e) The number of faces of $P$ is finite. Each face of $P$ is a support set and polyhedral (or empty).

6. Let $P \subset \mathbb{R}^n$ be a polyhedral set, but not an affine subspace (a flat). Prove the following statements.

(a) If $0 \leq j \leq k$, $F^j \in \mathcal{F}_j(P)$, $F^k \in \mathcal{F}_k(P)$ and $F^j \subset F^k$, then there are $F^i \in \mathcal{F}_i(P)$ for $i = j+1, \ldots, k-1$ with $F^j \subset F^{j+1} \subset \cdots \subset F^{k-1} \subset F^k$.
(b) If $0 \leq j \leq k < \dim(P)$ and $F^j \in \mathcal{F}_j(P)$, then $F^j = \bigcap\{F \in \mathcal{F}_k(P) : F^j \subset F\}$.
(c) If $\dim(P) = n$, then each $(n-2)$-dimensional face of $P$ is contained in precisely two facets of $P$.

7. Let $\emptyset \neq P \subset \mathbb{R}^n$ be closed and convex. If the number of different support sets of $P$ is finite, then $P$ is a polyhedral set.

8. Let $A \subset \mathbb{R}^n$ be a closed convex set with $A \neq \text{conv}(\text{relbd}(A))$. Then $A$ is a flat or a semi-flat.
9. Let $\emptyset \neq A \subset \mathbb{R}^n$ be closed and convex. The *recession cone* of $A$ is defined by

$$\text{rec}(A) := \{u \in \mathbb{R}^n : A + u \subset A\}.$$

Show that the recession cone of $A$ is a closed convex cone.
For each $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$, let $G^+(x, u) := \{x + \lambda u : \lambda \geq 0\}$. Then, for an arbitrary $x \in A$,

$$\text{rec}(A) = \{u \in \mathbb{R}^n : G^+(x, u) \subset A\}.$$

Moreover, if $G^+(x, u) \subset A$ and $y \in A$, then also $G^+(y, u) \subset A$.
10. Each closed convex set $A \subset \mathbb{R}^n$ can be written in the form $A = \overline{A} \oplus V$, where $V$ is a linear subspace and $\overline{A}$ is a line-free, closed convex set which is contained in a linear subspace complementary to $V$.
11. Let $A \subset \mathbb{R}^n$ be closed and convex. A ray $G^+(x, u) \subset A$ is called extreme if it is an extreme set in $A$. The union of all extreme rays of $A$ is denoted by $\text{extr}(A)$. By definition, it is clear that if $G^+(x, u)$ is an extreme ray of $A$, then $x \in \text{ext}(A)$. Prove the following representation result.
If $A \subset \mathbb{R}^n$ is line-free, closed and convex, then

$$A = \text{conv}(\text{ext}(A) \cup \text{extr}(A)) = \text{conv}(\text{ext}(A)) + \text{rec}(A).$$

12. A set of the form

$$\text{pos}\{a_1, \ldots, a_m\} = \left\{ \sum_{i=1}^{m} \lambda_i a_i : \lambda_i \geq 0 \text{ for } i = 1, \ldots, m \right\}$$

with $a_1, \ldots, a_m \in \mathbb{R}^n$ is called a *finitely generated convex cone*.
The convex hull of finitely many points is a polytope and hence a polyhedral set (as shown before). Show the following fact, which states that the positive hull of finitely many vectors is a polyhedral set: A finitely generated convex cone in $\mathbb{R}^n$ is a polyhedral set, in particular, it is a closed set.
13. Let $P \subset \mathbb{R}^n$ be a polyhedral set. Then there are points $a_1, \ldots, a_m \in \mathbb{R}^n$, $m \geq 1$, and vectors $b_1, \ldots, b_p \in \mathbb{R}^n$ with $P = \text{conv}\{a_1, \ldots, a_m\} + \text{pos}\{b_1, \ldots, b_p\}$. In particular, a bounded polyhedral set is a polytope.
14. Let points $a_1, \ldots, a_m \in \mathbb{R}^n$ with $m \geq 1$ and vectors $b_1, \ldots, b_p \in \mathbb{R}^n$ be given. Then $P := \text{conv}\{a_1, \ldots, a_m\} + \text{pos}\{b_1, \ldots, b_p\}$ is a polyhedral set.
15. Hints to the literature: Convex polytopes and polyhedral sets are treated in greater detail in [2, 23, 39, 40, 49, 64, 71, 93, 94]. Combinatorial aspects of convexity are in the focus of [5, 17, 29, 44, 60, 68, 71, 75, 90]. The connection between discrete and convex geometry is the subject of [17, 19, 21, 36, 38, 60, 71, 77, 95–97]. For algorithmic aspects and combinatorial geometry, see [16, 28, 29, 36, 49, 54, 74, 75].