Řešené příklady na maxima a minima

Definice. Nechť $Y_1,\dots,Y_n,\dots,Y$ jsou náhodné veličiny. Pak $Y_n \xrightarrow{P} Y$, pokud $\forall \varepsilon>0: \mathbb{P}[|Y_n-Y|>\varepsilon] \xrightarrow{n\to\infty} 0$.

Tvrzení 1. $Y_n \xrightarrow{P} Y \Rightarrow Y_n \xrightarrow{d} Y$; $Y_n \xrightarrow{d} c \Rightarrow Y_n \xrightarrow{P} c$

Tvrzení 2. Nechť $X_1,\dots,X_n$ je náhodný výběr z rozdělení s CDF $F_X(x), x\in\mathbb{R}$. Pak CDF n. veličiny $Y_n=\max_{1\le i\le n} X_i$ je $F_{Y_n}(y)=F_{X_1}^n(y)$ a CDF n. veličiny $Z_n=\min_{1\le i\le n} X_i$ je $F_{Z_n}(z)=1-(1-F_X(z))^n$.

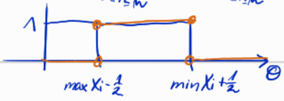
• Všechny systémy hustot jsou neregulární (nosič závisí na parametru.)

Příklad 55 $X_1,\dots,X_n \sim Unif(\theta-\frac{1}{2},\theta+\frac{1}{2})$ s hustotou $f(x,\theta)=\mathbb{1}[\theta-\frac{1}{2}\le x\le\theta+\frac{1}{2}]$. MLE + slabá konzistence.

Řešení $L_n(\theta)=\prod_{i=1}^n \mathbb{1}[\theta-\frac{1}{2}\le X_i\le\theta+\frac{1}{2}]=\mathbb{1}[\theta-\frac{1}{2}\le\min_{1\le i\le n}X_i\le\max_{1\le i\le n}X_i\le\theta+\frac{1}{2}]$

jako funkce θ

$=\mathbb{1}[\max_{1\le i\le n}X_i-\frac{1}{2}\le\theta\le\min_{1\le i\le n}X_i+\frac{1}{2}]$ $\quad \theta\le\frac{1}{2}+\min_{1\le i\le n}X_i \quad \max_{1\le i\le n}X_i-\frac{1}{2}\le\theta$



$\hat{\theta}_{MLE}$ je libovolná hodnota z intervalu $[\max X_i-\frac{1}{2},\min X_i+\frac{1}{2}]$

• interval je vž neprázdný, protože $0\le\max_{1\le i\le n}X_i-\min_{1\le i\le n}X_i\le 1$

chceme ukázat $\underset{=m_n}{\min X_i}\xrightarrow{P}\theta-\frac{1}{2}$, $\underset{=M_n}{\max X_i}\xrightarrow{P}\theta+\frac{1}{2}$, pak lib. hodnota z $[\max X_i-\frac{1}{2},\min X_i+\frac{1}{2}]$ konv. v P k θ, protože krajní hodnoty $\xrightarrow{P}\theta$.

$$F_{X_1}(x)=\begin{cases}1 & x\ge\theta+\frac{1}{2}\\ x-\theta+\frac{1}{2} & x\in(\theta-\frac{1}{2},\theta+\frac{1}{2})\\ 0 & x<\theta-\frac{1}{2}\end{cases} \rightsquigarrow \text{Tvrzení 2} \rightsquigarrow F_{m_n}(x)=\begin{cases}1 & x\ge\theta+\frac{1}{2}\\ 1-(\theta+\frac{1}{2}-x)^n & x\in(\theta-\frac{1}{2},\theta+\frac{1}{2})\\ 0 & x\le\theta-\frac{1}{2}\end{cases},\ F_{M_n}(x)=\begin{cases}1 & x\ge\theta+\frac{1}{2}\\ (x-\theta+\frac{1}{2})^n & x\in(\theta-\frac{1}{2},\theta+\frac{1}{2})\\ 0 & x\le\theta-\frac{1}{2}\end{cases}$$

ukážeme pak $\lim_{n\to\infty}F_{m_n}(x)=\begin{cases}1 & x>\theta-\frac{1}{2}\\ 0 & x\le\theta-\frac{1}{2}\end{cases}$, což je po upravení bodu nespojitosti CDF degenerované n.v. $\theta-\frac{1}{2}$

$(m_n\xrightarrow{d}\theta-\frac{1}{2}\overset{T1}{\Rightarrow}m_n\xrightarrow{P}\theta-\frac{1}{2})$

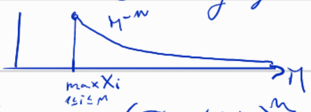
vnitřek závorky <1

analogicky $\lim_{n\to\infty}F_{M_n}(x)=\begin{cases}1 & x\ge\theta+\frac{1}{2}\\ \lim_{n\to\infty}(x-\theta+\frac{1}{2})^n, & x\in(\theta-\frac{1}{2},\theta+\frac{1}{2})\\ 0 & x\le\theta-\frac{1}{2}\end{cases}=\begin{cases}1 & x\ge\theta+\frac{1}{2}\\ 0 & x<\theta+\frac{1}{2}\end{cases}$ CDF deg. n.v. $\theta+1/2$ $(M_n\xrightarrow{d}\theta+\frac{1}{2}\overset{T1}{\Rightarrow}M_n\xrightarrow{P}\theta+\frac{1}{2})$

$\Big\}\ \hat{\theta}_{MLE}\xrightarrow{P}\theta$

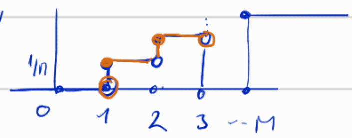
Příklad 56. $X_1,\dots,X_n\sim Unif\{1,\dots,M\}$ $f(x,M)=\frac{1}{M}$, $x\in\{1,\dots,M\}$ (nereg. sys.) MLE + konzistence M

Řešení. $L_n(M)=M^{-n}\cdot\mathbb{1}[\max_{1\le i\le n}X_i\le M]$



$k\in[0,M]$, jinak 0 nebo 1

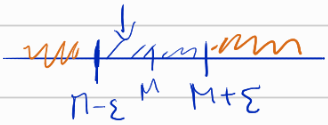
$\rightarrow\hat{M}_{MLE}=\max_{1\le i\le n}X_i\rightsquigarrow$ rozdělení $F_{\hat{M}}(k)=(F_{X_1}(k))^n=\left(\frac{\lfloor k\rfloor}{M}\right)^n$



• slabá konzistence: zvolme $\varepsilon>0$ (stačí $\varepsilon\in(0,M)$)

$\mathbb{P}[|M-\hat{M}|>\varepsilon]=\mathbb{P}[\hat{M}<M-\varepsilon]\le\mathbb{P}[\hat{M}\le M-\varepsilon]$

$=\left(\frac{M-\varepsilon}{M}\right)^n=\left(1-\frac{\varepsilon}{M}\right)^n\to 0\quad\forall\varepsilon>0$



$F(x;\lambda,\vartheta)=\begin{cases}0 & x<\vartheta\\ 1-e^{-\lambda(x-\vartheta)} & x\ge\vartheta\end{cases}$

Příklad 67 $X_1,\dots,X_n$ iid $f(x;\lambda,\vartheta)=\lambda e^{-\lambda(x-\vartheta)}\mathbb{1}[x\in(\vartheta,\infty)]$, MLE, slabá konzistence, $\mathbb{P}[n(\hat{\vartheta}-\vartheta)\le x]$

Řešení MLE $L_n(\lambda,\vartheta)=\lambda^n\exp\{-\lambda(\sum X_i-\vartheta)\}\,\mathbb{1}[\vartheta<\min_{1\le i\le n}X_i]$

$\rightsquigarrow$ pro pevné $\lambda\rightsquigarrow\hat{\vartheta}_{MLE}=\min X_i$ a pro dané ϑ je $\hat{\lambda}_{MLE}$ řešení $\frac{n}{\lambda}-\sum x_i+n\vartheta=0$ $\rightsquigarrow\hat{\lambda}=\frac{1}{\bar{X}_n-\hat{\vartheta}}$

$\rightsquigarrow(\hat{\vartheta},\hat{\lambda})=(\min X_i,\frac{1}{\bar{X}_n-\hat{\vartheta}})\xrightarrow{P}(\vartheta,\lambda)$

$F_{\hat{\vartheta}}(y)\overset{T2}{=}\begin{cases}1-(e^{-\lambda(x-\vartheta)})^n\\ 0\end{cases}=\begin{cases}1-e^{-n\lambda(x-\vartheta)}, & x\ge\vartheta\\ 0, & x<\vartheta\end{cases}\Rightarrow\hat{\vartheta}-\vartheta\sim Exp(n\lambda)$ a $n(\hat{\vartheta}-\vartheta)\sim Exp(\lambda)$

$$\mathbb{P}[n(\hat{\vartheta}-\vartheta) \le x] = \mathbb{P}[\hat{\vartheta}-\vartheta \le \tfrac{x}{n}] \stackrel{\uparrow}{=} 1 - e^{-\lambda n (\frac{x}{n})} = 1 - e^{-\lambda x}$$

nejen asymptoticky

Příklad 68 $X_1, \dots, X_n \sim Unif(a,b)$ MLE, sl. konzistence, $\mathbb{P}[n(\hat{b}-b) \le x]$

Řešení $L_n(a,b) = \left(\frac{1}{b-a}\right)^n \mathbb{1}[a < \min X_i < \max X_i < b]$

$(\hat{a}, \hat{b}) = (\min X_i, \max X_i)$ $\rightarrow$ slabá konzistence

$F_X(x) = \begin{cases} 1 & x \ge b \\ \frac{x-a}{b-a} & x \in (a,b) \\ 0 & x < a \end{cases} \rightsquigarrow F_{\hat{b}}(x) = \begin{cases} 1 & x \ge b \\ \left(\frac{x-a}{b-a}\right)^n & x \in (a,b) \\ 0 & x \le a \end{cases}$ $\lim\limits_{n\to\infty} F_{\hat{b}}(x) = \begin{cases} 1 & x \ge b \\ 0 & x < b \end{cases} \rightarrow \hat{b} \xrightarrow{d} b \Rightarrow \hat{b} \xrightarrow{P} b$

konvergence i sdruž. v P $\begin{pmatrix}\hat{a}\\ \hat{b}\end{pmatrix} \xrightarrow{P} \begin{pmatrix}a\\ b\end{pmatrix}$

$F_{\hat{a}}(x) = \begin{cases} 1 & x \ge b \\ 1-\left(\frac{b-x}{b-a}\right)^n & x \in (a,b) \\ 0 & x \le a \end{cases}$, $\lim\limits_{n\to\infty} F_{\hat{a}}(x) = \begin{cases} 1 & x > a \\ 0 & x \le a \end{cases}$ + mod. v bodě nespojitosti $\hat{a} \xrightarrow{d} a \Rightarrow \hat{a} \xrightarrow{P} a$

$$P[n(\hat{b}-b) \le x] = P[\hat{b} \le \tfrac{x}{n} + b] = \left(\frac{\frac{x}{n}+b-a}{b-a}\right)^n = \left(1 + \frac{x}{n(b-a)}\right)^n \xrightarrow[n\to\infty]{} e^{\frac{x}{b-a}} \text{ pro } x < 0$$

$\rightarrow n(b-\hat{b}) \xrightarrow{d} Exp\left(\frac{1}{b-a}\right)$