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Théorie de Ramsey sans principe des tiroirs et applications à la preuve de dichotomies d'espaces de Banach

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Titre : Théorie de Ramsey sans principe des tiroirs et applications à la preuve de dichotomies d'espaces de Banach

Résumé : Dans les années 90, Gowers démontre un théorème de type Ramsey pour les bloc-suites dans les espaces de Banach, afin de prouver deux dichotomies d'espaces de Banach. Ce théorème, contrairement à la plupart des résultats de type Ramsey en dimension infinie, ne repose pas sur un principe des tiroirs, et en conséquence, sa formulation doit faire appel à des jeux. Dans une première partie de cette thèse, nous développons un formalisme abstrait pour la théorie de Ramsey en dimension infinie avec et sans principe des tiroirs, et nous démontrons dans celui-ci une version abstraite du théorème de Gowers, duquel on peut déduire à la fois le théorème de Mathias-Silver et celui de Gowers. On en donne à la fois une version exacte dans les espaces dénombrables, et une version approximative dans les espaces métriques séparables. On démontre également le principe de Ramsey adverse, un résultat généralisant à la fois le théorème de Gowers abstrait et la détermination borélienne des jeux dénombrables. On étudie aussi les limitations de ces résultats et leurs généralisations possibles sous des hypothèses supplémentaires de théorie des ensembles.

Dans une seconde partie, nous appliquons les résultats précédents à la preuve de deux dichotomies d'espaces de Banach. Ces dichotomies ont une forme similaire à celles de Gowers, mais sont Hilbert-évitantes : elles assurent que le sous-espace obtenu n'est pas isomorphe à un espace de Hilbert. Ces dichotomies sont une nouvelle étape vers la résolution d'une question de Ferenczi et Rosendal, demandant si un espace de Banach séparable non-isomorphe à un espace de Hilbert possède nécessairement un grand nombre de sous-espaces, à isomorphisme près.

Mots clefs : Logique, Théorie des ensembles, Théorie de Ramsey, Détermination, Analyse fonctionnelle, Géométrie des espaces de Banach

Title: Ramsey theory without pigeonhole principle and applications to the proof of Banach-space dichotomies

Abstract: In the 90's, Gowers proves a Ramsey-type theorem for block-sequences in Banach spaces, in order to show two Banach-space dichotomies. Unlike most infinite-dimensional Ramsey-type results, this theorem does not rely on a pigeonhole principle, and therefore it has to have a partially game-theoretical formulation. In a first part of this thesis, we develop an abstract formalism for Ramsey theory with and without pigeonhole principle, and we prove in it an abstract version of Gowers' theorem, from which both Mathias-Silver's theorem and Gowers' theorem can be deduced. We give both an exact version of this theorem in countable spaces, and an approximate version of it in separable metric spaces. We also prove the adversarial Ramsey principle, a result generalising both the abstract Gowers' theorem and Borel determinacy of countable games. We also study the limitations of these results and their possible generalisations under additional set-theoretical hypotheses.

In a second part, we apply the latter results to the proof of two Banach-space dichotomies. These dichotomies are similar to Gowers' ones, but are Hilbert-avoiding, that is, they ensure that the subspace they give is not isomorphic to a Hilbert space. These dichotomies are a new step towards the solution of a question asked by Ferenczi and Rosendal, asking whether a separable Banach space non-isomorphic to a Hilbert space necessarily contains a large number of subspaces, up to isomorphism.

Keywords: Logic, Set theory, Ramsey theory, Determinacy, Functional analysis, Geometry of Banach spaces

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[1]ou à la secte, comme l'aurait dit mon père

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Courte introduction en français

Pour des raisons pratiques et afin de la rendre accessible à un public plus large, cette thèse a entièrement été rédigée en anglais. Seule cette courte introduction sera rédigée en français ; elle présente dans les grandes lignes les principales questions et travaux qui ont dirigé mes recherches, ainsi que les résultats démontrés. Elle sera suivi d'une introduction plus longue, en anglais, présentant de façon détaillée les notions nécessaires à la compréhension de cette thèse et les travaux antérieurs sur lesquels elle s'appuie.

Les résultats présentés dans ce manuscrit prennent leurs racines dans les travaux de Gowers. Dans les années 90, ce dernier a montré une dichotomie d'espaces de Banach [24] qui, combinée avec un résultat antérieur dû à Komorowski et Tomczak-Jaegermann [34], a répondu par la positive à une célèbre question de Banach, le problème de l'espace homogène. Ce problème était le suivant : ℓ_2 est-il le seul espace de Banach, a isomorphisme près, qui est isomorphe à tous ses sous-espaces ?

Ce résultat a ouvert plusieurs nouvelles directions de recherche, qui seront étudiées dans cette thèse. La première est de nature combinatoire et ensembliste. En effet, la preuve de la dichotomie de Gowers utilise des méthodes mêlant théorie de Ramsey et théorie des jeux ; plus précisément, cette dichotomie est déduite d'un théorème de type Ramsey dans les espaces de Banach avec base, fortement inspiré de résultats de théorie de Ramsey en dimension infinie plus classiques, dont le résultat fondateur est le théorème de Mathias–Silver [43, 58]. Néanmoins, le théorème de type Ramsey de Gowers diffère significativement de ces résultats classiques en cela qu'il ne repose pas sur un principe des tiroirs, contrairement à eux. La conséquence est qu'il est plus faible et a une formulation faisant intervenir des jeux. D'autre part, le fait qu'il soit énoncé dans un espace non-dénombrable nécessite une approximation métrique.

Une partie de cette thèse a pour but d'étudier de façon plus systématique la théorie de Ramsey en dimension infinie sans principe des tiroirs, énoncée à l'aide de jeux, et de la comparer avec la théorie de Ramsey avec principe des tiroirs. De même qu'un formalisme abstrait pour la théorie de Ramsey avec principe des tiroirs a été introduit par Todorčević [61], permettant de déduire le théorème de Mathias–Silver ainsi que d'autres résultats similaires dans différents contextes, nous introduirons ici un formalisme abstrait unifiant théorie de Ramsey avec et sans principe des tiroirs, celui des *espaces de Gowers*. Ce formalisme est inspiré de la version exacte du théorème de Gowers donnée par Rosendal dans les espaces vectoriels dénombrables [56], et peut s'appliquer à divers types de structures dénombrables. En particulier, une version abstraite du théorème de Rosendal sera démontrée (théorème II.14). Nous introduirons aussi une version approximative des espaces de Gowers, permettant de travailler dans des espaces non-dénombrables avec approximation métrique, et destinée a permettre de prouver facilement des dichotomies d'espaces de Banach dans la même veine que celle de Gowers. En particulier, un théorème abstrait généralisant à la fois le théorème de Mathias–Silver et celui de Gowers sera démontré (théorème III.17).

Un étude plus approfondie des espaces de Gowers et de leurs propriétés combinatoires sera effectuée. En particulier, on démontrera le principe de Ramsey adverse (théorème II.4), un résultat conjecturé par Rosendal généralisant à la fois sa version du théorème de Gowers et la détermination Borélienne des jeux dénombrables. On étudiera aussi les limitations et les possibles extensions des résultats présentés. La plupart d'entre eux sont démontrés pour les ensembles boréliens ou analytiques ; on verra sous quelles conditions ces résultats sont optimaux dans ZFC, dans quels cas ils peuvent être étendus à de plus grandes classes d'ensembles sous des hypothèses supplémentaires de théorie des ensembles. On étudiera aussi la force métamathématique de ces résultats. Cela fera apparaître une grande différence de comportement entre les espaces satisfaisant le principe des tiroirs et les espaces ne le satisfaisant pas. On peut en particulier citer le résultat suivant : le principe de Ramsey adverse, lorsqu'énoncé dans un espace sans principe des tiroirs, a la force de la détermination Borélienne, alors qu'il peut être démontré dans ZC pour les espaces avec principe des tiroirs.

La seconde direction de recherche ouverte par la preuve de la dichotomie de Gowers est connue sous le nom de "programme de Gowers". L'idée est de donner une classification "faible" mais la plus précise possible des espaces de Banach séparables "a sous-espace près". Plus précisément, on veut construire une liste de classes d'espaces de Banach séparables (généralement appelée *liste de Gowers*), aussi grande que possible, satisfaisant les critères suivants :

1. Les classes sont, dans un certain sens, héréditaires (closes par prise de sous-espaces, ou au moins de bloc-sous-espaces, pour les classes définies par les propriétés des bases) ;

2. Les classes sont deux à deux disjointes ;

3. Chaque espace possède au moins un sous-espace dans une des classes ;

4. Les classes sont naturelles, dans le sens ou savoir qu'un espace est dans une classe donne de nombreuses informations sur sa structure.

La dichotomie de Gowers fournit une telle classification en deux classes, la première étant la classe des espaces avec base inconditionnelle, et la seconde celle des espaces héréditairement indécomposables, c'est-à dire des espaces ne contenant aucune somme directe topologique de deux sous-espaces fermés de dimension infinie. Gowers a lui même démontré, dans le même article [24], une seconde dichotomie, allongeant cette liste à trois classes, et d'autres dichotomies ont par la suite été démontrées par d'autres auteurs, en particulier Ferenczi et Rosendal [19]. La tendance générale de ces dichotomies est de tracer une frontière entre, d'un côté, les espaces "simples", ayant un comportement proche des ℓ_p et de c_0, et d'un autre côté les espaces "pathologiques".

La troisième direction de recherche, ouverte plus particulièrement par la solution du problème de l'espace homogène, est celle de ses possibles extensions. On sait qu'un espace de Banach séparable non-isomorphe à ℓ_2 doit avoir au moins deux sous-espaces, à isomorphisme près, mais combien peut-il en avoir ? Cette question a été initialement posée par Gilles Godefroy. Elle s'exprime bien dans le langage de la classification des relations d'équivalence analytiques sur un espace Polonais : en étudiant la complexité de la relation d'isomorphisme entre les sous-espaces d'un espace donné (qu'on peut voir comme une relation d'équivalence analytique sur un espace Polonais), on obtiendra strictement plus d'informations qu'en étudiant uniquement le nombre de classes. Dans cet esprit, Ferenczi et Rosendal ont conjecturé que pour un espace de Banach séparable X non-isomorphe à ℓ_2, la relation d'équivalence $\mathbf{E_0}$ devait être réductible à l'isomorphisme entre les sous-espaces de X (un espace satisfaisant cette dernière propriété sera appelé un espace *ergodique*). En particulier, le nombre de classes d'isomorphisme devrait avoir la puissance du continu. Une conjecture plus faible, émise par Johnson, est la suivante : il n'existe pas d'espace de Banach séparable possédant exactement deux sous-espaces, à isomorphisme près. Ces deux problèmes sont encore largement ouverts à l'heure actuelle.

Ces deux dernières directions de recherche s'avèrent être liées, et seront étudiées dans le dernier chapitre de ce manuscrit. On étudiera la conjecture de Ferenczi et Rosendal, ainsi que celle de Johnson, et en particulier la question de savoir si on peut, pour démontrer ces conjectures, se ramener au cas d'espaces ayant une base inconditionnelle. Plus précisément, on s'intéressera aux conjectures suivantes :

(1) Tout espace de Banach séparable non-ergodique, non isomorphe à ℓ_2, possède un sous-espace non-isomorphe à ℓ_2 ayant une base inconditionnelle.

(2) Tout espace de Banach séparable possédant exactement deux sous-espaces à isomorphisme près, doit posséder une base inconditionnelle.

On ne démontrera pas ces conjectures, mais on parviendra à les réduire à des problèmes semblant plus abordables. Leur énoncé fait appel à une nouvelle classe d'espaces introduite dans ce manuscrit, les espaces héréditairement Hilbert-primaires (HHP), qu'on peut voir comme une généralisation des espaces héréditairement indécomposables ou bien comme une variante des espaces primaires. Un espace X sera dit HHP s'il ne contient aucune somme directe topologique de deux sous-espaces fermés, de dimension infinie, et non-isomorphes à ℓ_2. Les résultats suivants seront démontrés :

- Pour montrer la conjecture **(1)**, il suffit de montrer que tout espace HHP non-isomorphe à ℓ_2 possède un sous-espace non-isomorphe à ℓ_2 dans lequel il ne peut pas se plonger ;

- Pour montrer la conjecture **(2)**, il suffit de montrer qu'un espace HHP non-isomorphe à ℓ_2 possède au moins trois sous-espaces deux-à-deux non-isomorphes.

Ces résultats semblent plausibles, car ils sont proches du résultat dû a Gowers et Maurey affirmant qu'un espace héréditairement indécomposable n'est isomorphe à aucun de ses sous-espaces propres. On donnera d'ailleurs à la fin de ce manuscrit une nouvelle preuve du théorème de Gowers et Maurey, basée uniquement sur la théorie de Fredholm, et qui pourrait être un point de départ pour montrer qu'un espace HHP non-isomorphe à ℓ_2 possède suffisamment de sous-espaces deux-à-deux non-isomorphes.

Les deux résultats précédents sont conséquences de deux dichotomies d'espaces de Banach qui seront démontrées dans le chapitre IV de cette thèse. Ces dichotomies sont dans l'esprit du programme de Gowers, mis à part qu'elle sont Hilbert-évitantes, c'est-à-dire qu'on assure que le sous-espace qu'elles produisent sera non-isomorphe à ℓ_2. Avoir de telles dichotomies est très utile lorsqu'on s'attaque à la question du nombre de sous-espaces, car lorsqu'on utilise les dichotomies traditionnelles, rien n'assure que le sous-espace produit ne sera pas isomorphe à ℓ_2, même si l'espace de départ est très complexe. La première dichotomie (théorème IV.12) est une variante ℓ_2-évitante de la première dichotomie de Gowers et la seconde est une variante ℓ_2-évitante d'une dichotomie dûe à Ferenczi et Rosendal [19]. On peut les voir comme les premières pierres d'une liste de Gowers pour les espaces non-isomorphes à ℓ_2. Ces dichotomies sont prouvées en utilisant les résultats de type Ramsey abstraits démontrés dans les chapitres II et III de cette thèse, en particulier le théorème de Gowers abstrait (théorème III.17) ainsi que le principe de Ramsey adverse (théorème II.4).

Contents

Notations and conventions

Following the tradition in set theory, the set of nonnegative integers will be denoted by ω. An integer $n \in \omega$ will usually be viewed as the set of its predecessors, $n = \{0, 1, \ldots, n-1\}$. Given two nonempty subsets $A, B \subseteq \omega$, we will say that $A < B$ if $\forall i \in A\ \forall j \in B\ i < j$. Given $n \in \omega$ and a nonempty $A \subseteq \omega$, we say that $n < A$ if $\forall i \in A\ n < i$.

If X and Y are two sets, X^Y will denote the set of mappings from Y to X. In particular, X^ω is the set of infinite sequences of elements of X, and for $n \in \omega$, X^n is the set of n-uples of elements of X. We will denote by $X^{<\omega} = \bigcup_{n\in\omega} X^n$ the set of finite sequences of elements of X, and $X^{\leqslant\omega} = X^{<\omega} \cup X^\omega$. We denote by $\mathrm{Seq}(X) = X^{<\omega}\backslash\{\varnothing\}$ the set of finite sequences of elements of X having at least one term. Given $s, t \in X^{\leqslant\omega}$, we let $s \subseteq t$ if s is an initial segment of t; this is, actually, the usual set-theoretical inclusion. If $s \in X^{\leqslant\omega}$, we denote by $|s|$ the length of s, i.e. the unique ordinal α (ω or an integer) such that $s \in X^\alpha$. For $s \in X^{<\omega}$ and $t \in X^{\leqslant\omega}$, we denote by $s \frown t$ the concatenation of s and t; for instance, if $s = (s_0, \ldots, s_{m-1})$ and $t = (t_0, \ldots, t_{n-1})$, then $s \frown t = (s_0, \ldots, s_{m-1}, t_0, \ldots, t_{n-1})$. If $f \in X^Y$ and $Z \subseteq Y$, we will denote by $f_{\restriction Z} \in X^Z$ the restriction of f to Z; in particular, if $s \in X^{\leqslant\omega}$ and $n \leqslant |s|$, $s_{\restriction n}$ will denote the sequence of the n first terms of s (unless otherwise specified, because for convenience of notation, we will sometimes derogate to this rule).

A *tree* on a set X is a set $T \subseteq X^{<\omega}$ such that for every $s, t \in X^\omega$, if $s \subseteq t$ and $t \in T$, then $s \in T$. An element of a tree is usually called a *node*, and a *terminal node* of T is an $s \in T$ that is maximal in T for the inclusion. A *pruned tree* is a tree without terminal nodes. An *infinite branch* of the tree T is an $x \in X^\omega$ such that for every $n \in \omega$, we have $x_{\restriction n} \in T$; the set of infinite branches of T is denoted by $[T]$.

We will denote by $\mathfrak{c}$ the cardinality of the continuum, $2^{\aleph_0}$.

If X is a topological space we define by induction, for $n \in \omega$, the sets $\mathbf{\Sigma}^1_n(X)$ and $\mathbf{\Pi}^1_n(X)$ of subsets of X in the following way:

- $\mathbf{\Sigma}^1_0(X)$ is the set of open subsets of X;
- $\mathbf{\Pi}^1_n(X)$ is the set of $A \subseteq X$ such that $A^c \in \mathbf{\Sigma}^1_n(X)$;
- $\mathbf{\Sigma}^1_{n+1}(X)$ is the set of $A \subseteq X$ that are the first projection of a set $B \in \mathbf{\Pi}^1_n(X \times \omega^\omega)$.

We also let $\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$. In particular, if X is Polish, then $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$ and $\mathbf{\Delta}^1_n(X)$ are respectively the set of analytic, coanalytic, and Borel subsets of X. As in Polish spaces, we call $\bigcup_{n\in\omega} \mathbf{\Sigma}^1_n(X)$ the class of *projective subsets* of X.

We say that a class Γ of subsets of Polish spaces is *suitable* if it contains the class of Borel sets and is stable under finite unions, finite intersections and Borel inverse images. For such a class, let $\exists\Gamma$ be the class of projections of Γ-sets; in other words, for A a subset of a Polish space X, we say that $A \in \exists\Gamma$ if and only if there exist $B \in X \times 2^\omega$ such that $B \in \Gamma$ and A is the first projection of B (we could have taken any uncountable Polish space instead of 2^ω in this definition, since Γ is closed under Borel inverse images). The class $\exists\Gamma$ is itself suitable.

In this thesis, we will call *Banach space* an *infinite-dimensional* complete normed vector space. Unless otherwise specified, all Banach spaces will be over $\mathbb{R}$; however, most of the time, the results we present apply as well to complex spaces. Unless otherwise specified, we will call a *subspace* of a Banach space E an *infinite-dimensional, closed* vector subspace of E. The unit sphere of E will be denoted by S_E. Usually, the norm on a Banach space will be denoted by $\|\cdot\|$. If E and F are Banach spaces, we will equip, unless otherwise specified, the space $\mathcal{L}(E,F)$ of bounded operators from E to F, and the space E^* of continuous linear forms on E, with the operator norm, that will usually be denoted by $|||\cdot|||$. When we refer to topological notions about Banach spaces without further explanation, these notion are always considered in respect to the norm topology.

An *isomorphism* between two Banach spaces E and F is a bijective bounded operator $T : E \longrightarrow F$ whose inverse is bounded. Such an isomrphism is said to be a C-isomorphism, where $C \geqslant 1$, if $|||T||| \cdot |||T^{-1}||| \leqslant C$. If $|||T||| = |||T^{-1}||| = 1$, we say that T is an *isometry*. An *embedding* (resp. a C-embedding) of E into F is an isomorphism (resp. a C-isomorphism) between E and a subspace of F. If there exists an embedding (resp. a C-embedding) of E into F, we say that E embeds (resp. C-embeds) into F, and this is denoted by $E \sqsubseteq F$ (resp. $E \sqsubseteq_C F$). Given two finite-dimensional vector spaces E and F with the same dimension, we denote by $d_{BM}(E,F)$ the *Banach-Mazur distance* between E and F, i.e. the infimum of the nuumbers $\log(|||T||| \cdot |||T^{-1}|||)$, where $T : E \longrightarrow F$ is an isomorphism. Sometimes, we will also use this notation for infinite-dimensional spaces, and in the case where E and F are not isomorphic, we will say that $d_{BM}(E,F) = \infty$. Two spaces are isometric if and only if the Banach-Mazur distance between them is 0.

If X is a compact Hausdorff space, $\mathcal{C}(X)$ will denote the space of continuous functions $X \longrightarrow \mathbb{R}$ with the sup norm $\|\cdot\|_\infty$.

Chapter I

Introduction and history

The results presented in this thesis have their roots in the work of Gowers. In the 90's, he proved a Banach-space dichotomy [24] which, combined with a result by Komorowski and Tomczak-Jaegermann [34] gave a positive answer to a celebrated question by Banach. This question, known as the homogeneous space problem, asked whether ℓ_2 was the only Banach space, up to isomorphism, that was isomorphic to all of its subspaces.

His proof opened several new research directions. The first one is combinatorial and set-theoretical. The methods used in the proof of Gowers' dichotomy are much more combinatorial than analytical. The proof indeed relies on a Ramsey-type result in Banach spaces, inspired by Mathias' [43] and Silver's [58] infinite-dimensional version of Ramsey's theorem. However, this Ramsey-type result is slightly different from most infinite-dimensional Ramsey results, since it has a partially game-theoretic formulation. This led several authors, for instance Bagaria and López-Abad [7, 8] or Rosendal [56, 57] to study this result in more details, its possible extensions, and its links with the determinacy of games.

The second research direction opened by Gower's work is known as Gowers' program. The idea is to give a "loose" classification of Banach spaces "up to subspaces", i.e. to give a list of natural classes of Banach spaces that are pairwise disjoints and such that every space has a subspace in one of the classes. This could be done by proving other Banach-spaces dichotomies by the same Ramsey-theoretic methods. This work has been initiated by Gowers in [24], and continued by several authors and in particular by Ferenczi and Rosendal in [19].

The third research direction comes from a question by Godefroy. He asked how many subspaces could have, up to isomorphism, a Banach space non-isomorphic to ℓ_2. This question, that can be asked more precisely in the formalism of the classification of analytic equivalence relations on standard Borel spaces, led to several partial results, for example [14, 17, 4, 11].

In this manuscript, we will mostly investigate the first and the third direction, that turn out to be widely linked. We start by introducing more precisely these results and their history, before presenting the organisation of this manuscript.

I.1 Determinacy

Determinacy is not the central subject of this thesis, however, since statements based on determinacy are often taken as axioms in set theory and have consequences on several results that will be presented in this manuscript, it is worth to start by introducing it. Determinacy is the study of the existence of winning strategies in two-player games with perfect information. Here, we will restrict our attention to games with length ω. Such a game will be represented by a set X (the set of *possible moves*), by a nonempty tree $T \subseteq X^{<\omega}$ without terminal nodes (the *rule*), and by a set $\mathcal{X} \subseteq [T]$ (the *target set*). Two players, denoted by **I** and **II**, choose alternately an element $x_i \in X$:

I	x_0		x_2		$\dots$	
II		x_1		x_3		$\dots$

and they have to preserve the following property: for every $i \in \omega$, $(x_0, \dots x_i) \in T$. Player **I** wins if $(x_i)_{i\in\omega} \in \mathcal{X}$, and otherwise, player **II** wins. This game will be denoted by $\mathcal{G}(T, \mathcal{X})$.

A *winning strategy* for a player is a strategy that enables him or her to win whatever the other player plays. Formally, a *strategy* for player **I** is a function τ that associate to every $s \in T$ with even length an $x \in X$ such that $s^\frown x \in T$. Saying that ***I** plays according to the strategy* τ means that, if the current state of the game is the following:

I	x_0		$\dots$	x_{2i-2}	
II		x_1	$\dots$		x_{2i-1} ,

then **I** plays $x_{2i} = \tau(x_0, x_1, \dots, x_{2i-2}, x_{2i-1})$. We say that this strategy is *winning* when for every sequence $(x_i)_{i\in\omega} \in [T]$, if for every $i \in \omega$ we have $x_{2i} = \tau(x_0, x_1, \dots, x_{2i-2}, x_{2i-1})$, then $(x_i)_{i\in\omega} \in \mathcal{X}$. We define in the same way the notion of a *strategy*, and of a *winning strategy*, for player **II**.

It will also often be convenient to define games without specifying a target set. Such games are defined with an *outcome*, which is a function of the sequence of moves of the players during the game (most of the time, it will be a subsequence of the sequence of moves). Formally, an outcome is a mapping F from $[T]$ to some set Y; if the sequence of moves during the game is $(x_i)_{i\in\omega} \in [T]$, then the outcome of the game will be $F((x_i)_{i\in\omega})$. The game whose rule is a tree T and whose outcome is a function F will be denoted by $\mathcal{G}(T, F)$, or simply by $\mathcal{G}(T)$ if F is the identity. For games that are defined with an outcome rather than a target set, we will not speak about winning strategies but rather about strategies to reach some sets. For example, if $\mathcal{Y} \subseteq Y$, we will say that player **I** has a strategy in the game $\mathcal{G}(T, F)$ to reach $\mathcal{Y}$ if he has a strategy to ensures that the outcome of the game will be in the set $\mathcal{Y}$; formally, such a strategy will be a winning strategy in the game $\mathcal{G}(T, F^{-1}(\mathcal{Y}))$.

We say that the game $\mathcal{G}(T, \mathcal{X})$ is determined if one of the players has a winning strategy in this game. When there is no ambiguity on the tree T, we will also say that the set $\mathcal{X} \subseteq [T]$ is determined. In many cases we will study, the tree T will be the whole $X^{<\omega}$; when we say, without further explanation, that a set $\mathcal{X} \subseteq X^\omega$ is determined, it will always be understood that the underlying tree is $X^{<\omega}$. A game whose rule tree is $X^{<\omega}$ will be called a *game on* X.

Not all games are determined: we can easily build counterexamples using the axiom of choice (see [47], exercise 6A.6 for a construction of a subset of 2^ω that is not determined; this easily implies that such sets also exist in X^ω for every set X with cardinality greater that 2). It is then natural to look for positive results under topological restrictions. Here, we will endow X with the discrete topology, and $[T]$ with the topology induced by the product topology on X^ω.

Gale and Stewart [21] proved that every *closed game* (i.e., a game with a closed target set) is determined. It was then extended by Wolfe [62] to $\mathbf{\Sigma}^0_2$ games, by Davis [12] to $\mathbf{\Sigma}^0_3$ games, and finally, Martin proved in 1975 that every Borel game is determined [39] (a proof can also be found in [32], theorem 20.5). Martin's result is optimal in ZFC: in $ZFC + V = L$, it is possible to build $\mathbf{\Sigma}^1_1$ subsets of ω^ω that are not determined (see [47], exercise 6A.12).

In this manuscript, for Γ a class of subsets of Polish spaces, we will denote by $\mathrm{Det}_\omega(\Gamma)$ the assumption "every Γ-subset of ω^ω is determined". This implies that every Γ-game whose rule is an at most countable tree is determined, as soon as Γ is suitable. We will also denote by $\mathrm{Det}_{\mathbb{R}}(\Gamma)$ the statement "*when* $\mathbb{R}$ *is endowed with its usual Polish topology, and* $\mathbb{R}^\omega$ *with the product topology*, every Γ-subset of $\mathbb{R}^\omega$ is determined". Here, we consider the Polish topology on $\mathbb{R}$ and not the discrete one, since it will be enough to prove the results we want.

Determinacy has strong links with set theory. The first remark is that, while Gale and Stewart's, Wolfe's and Davis' results can be proved in second-order arithmetic (so in particular, in the theory ZC), Martin's proof of Borel determinacy uses a much larger fragment of ZFC. In fact, Friedman proved [20] that any proof of Borel determinacy should make use of the replacement scheme and of the powerset axiom. Many determinacy statements have also been shown equiconsistent with large cardinal hypotheses. Martin [38] proved in 1970 that, if there exists a measurable cardinal κ, then every analytic game on a set X with cardinality strictly lower that κ was determined. Harrigton [26] showed then that $\mathrm{Det}_\omega(\mathbf{\Sigma}^1_1)$ was equivalent to a slightly weaker hypothesis than the existence of a measurable cardinal, the existence of $x^\#$ for every real x. Then, the works of Martin and Steel [41, 42] and of Woodin [63] (see also [42] and [50] for proofs of unpublished results by Woodin) led to proofs of the statements $\mathrm{Det}_\omega(\mathbf{\Sigma^1_n})$ (for $n \geqslant 2$) and "every subset of ω^ω in $L(\mathbb{R})$ is determined" assuming large cardinal axioms, based on the notion of *Woodin cardinals*. Later works by Woodin (see [48, 33]) showed the last statements to be equiconsistent with statements involving large cardinals. Martin, Steel and Woodin's work show, in particular, that assuming the consistency of some large cardinal axioms, the following theories are consistant:

- $ZFC + PD$, where PD is the *axiom of projective determinacy*, i.e. the statement $\forall n \in \omega\ \mathrm{Det}_\omega(\mathbf{\Sigma}^1_n)$;
- $ZF + DC + AD$, where AD is the *axiom of determinacy*, i.e. the statement "every subset of ω^ω is determined".

The axioms PD and AD have many interesting consequences on the structure of sets of reals, and hence they are widely studied. Of course AD is incompatible with ZFC. Some stronger theories are often also considered, for example:

- $ZFC + PD_{\mathbb{R}}$, where $PD_{\mathbb{R}}$ is the statement $\forall n \in \omega \ \mathrm{Det}_{\mathbb{R}}(\mathbf{\Sigma}^1_n)$;
- $ZF + DC + AD_{\mathbb{R}}$, where $AD_{\mathbb{R}}$ is the statement "every subset of $\mathbb{R}^\omega$ is determined".

I.2 Infinite-dimensional Ramsey theory

The fundamental result in Ramsey theory is Ramsey's theorem [54]:

Theorem I.1. *Let $d \in \omega$. For every colouring of $[\omega]^d$ with a finite number of colors, there exists an infinite $M \subseteq \omega$ such that $[M]^d$ is monochromatic.*

The set M is usually said to be *homogeneous* for this colouring, and the integer d is called the *dimension* of the Ramsey result. Ramsey initially proved his theorem as a lemma in a logic article, however it later found applications in many other fields. Many generalizations, or variants of this theorem in other contexts, were also proved, forming a field that we now call *Ramsey theory*. An interesting way to generalize this result is to look at what happens when the number d is infinite (this number d is called the *dimension* of the Ramsey result). We call this direction of research *infinite-dimensional Ramsey theory*. We restrict our attention to colourings with two colors, since it is more convenient and since extensions to an arbitrary finite number of colors are easily deduced by induction. A colouring with two colors, blue and red, can be viewed as a set $\mathcal{X}$, the set of blue sets for this colouring. It is thus natural to give the following definition.

Definition I.2. A set $\mathcal{X} \subseteq [\omega]^\omega$ is *Ramsey* if for every infinite $M \subseteq \omega$, there exists an infinite $N \subseteq M$ such that either $[N]^\omega \subseteq \mathcal{X}$, or $[N]^\omega \subseteq \mathcal{X}^c$.

Basing ourselve on Ramsey's theorem, it would be natural to conjecture that every subset of $[\omega]^\omega$ is Ramsey. However, it is easy to build a counterexample, using the axiom of choice (see for example [32], II.19.C., for a construction by a diagonal argument). It is then natural to look for positive results when we put topological restrictions on the set $\mathcal{X}$. Here, we equip $[\omega]^\omega$ with the topology inherited from the Cantor space $\mathcal{P}(\omega) = 2^\omega$ with the product topology. Alternately, we can see this topology as inherited from the product topology on ω^ω, when we see $[\omega]^\omega$ as a subset of ω^ω by identifying an infinite set of integer with its increasing enumeration. Basing themselve on previous works by Nash-Williams [49] and Galvin and Prikry [22], Mathias [43] and Silver [58] finally proved the following result:

Theorem I.3 (Mathias–Silver)**.** *Every analytic subset of $[\omega]^\omega$ is Ramsey.*

Other results about Ramsey sets were proved next. Ellentuck [13] gave a topological characterisation of subsets of $[\omega]^\omega$ that are, in some sense, Ramsey "at every scale"; this characterisation is based on a toplogy on $[\omega]^\omega$ that is finer than the usual topology. If $V = L$, it is not hard to build a $\mathbf{\Sigma}^1_2$-subset of $[\omega]^\omega$ that is not Ramsey (this is folklore; a more general result will be proved in section II.4 of this manuscript). In particular, Mathias–Silver's result is optimal in ZFC. Results were also proved under stronger set-theoretical assumptions. Mathias [43] proved, assuming the consistency of large cardinal hypotheses, the consistency of the theory $ZF + DC+$ "every subset of $[\omega]^\omega$ is Ramsey". Silver [58] proved that if there exists a measurable cardinal, then every $\mathbf{\Sigma}^1_2$-subset of $[\omega]^\omega$ is Ramsey. Harrington and Kechris [27], and independently Woodin [64] proved that under PD, every projective subset of $[\omega]^\omega$ is Ramsey. Martin and Steel [40] proved a result implying that if AD holds in $L(\mathbb{R})$, then in $L(\mathbb{R})$, every subset of $[\omega]^\omega$ is Ramsey. In particular, this implies that under a strong enough large cardinal assumption, every subset of $[\omega]^\omega$ that is in $L(R)$ is Ramsey. These last results are not directly obtained by the determinacy of some game, but are proved using heavy set-theoretical machinery.

Kastanas [31] defined for the first time in 1983 a game that is directly related to the Ramsey property. Tanaka [60] gave then an unfolded version of Kastanas' game and used it to give a new proof of Mathias–Silver's theorem, based on the determinacy of $\mathbf{\Sigma}^0_2$-sets. Since Kastanas' game will play a central role in the proof of a result of this thesis, we will here recall its definition. Given an infinite set of integers M, Kastanas' game below M, denoted by K_M, is defined as follows:

I	M_0		M_1		$\dots$	
II		n_0, N_0		n_1, N_1		$\dots$

where the M_i's and the N_i's are elements of $[\omega]^\omega$, and the n_i's are elements of ω. The rules are the following:

- for **I**: $M_0 \subseteq M$, and for all $i \in \omega$, $M_{i+1} \subseteq N_i$;
- for **II**: for all $i \in \omega$, $n_i \in M_i$, $N_i \subseteq M_i$, and $n_i < N_i$.

The outcome of the game is the set $\{n_0, n_1, \dots\} \in [\omega]^\omega$.

Remark that this game is a game on reals: the players play elements of $[\omega]^\omega$ (or of $\omega \times [\omega]^\omega$ for player **II**) that can be viewed as real numbers. The result proved by Kastanas is the following:

Theorem I.4 (Kastanas)**.** *Let $\mathcal{X} \subseteq [\omega]^\omega$ and $M \subseteq \omega$ be infinite.*

1. *If player **I** has a strategy in K_M to reach $\mathcal{X}^c$, then there exists an infinite $N \subseteq M$ such that $[N]^\omega \subseteq \mathcal{X}^c$.*
2. *If player **II** has a strategy in K_M to reach $\mathcal{X}$, then there exists an infinite $N \subseteq M$ such that $[N]^\omega \subseteq \mathcal{X}$.*

In particular, this theorem, combined with Borel determinacy of games on reals, show that every Borel subset of $[\omega]^\omega$ is Ramsey. Tanaka actually proved that it was possible to deduce the Ramsey property for analytic subsets of $[\omega]^\omega$ from the determinacy of an unfolded version of Kastanas' game with a $\mathbf{\Sigma}^0_2$ target set, and the Ramsey property for $\mathbf{\Sigma}^1_2$-subsets of $[\omega]^\omega$ from the determinacy of the same game with an analytic target set. In particular, Tanaka's method enables to recover Mathias-Silver's theorem, and Silver's result that under the existence of a measurable cardinal, every $\mathbf{\Sigma}^1_2$-subset of $[\omega]^\omega$ is Ramsey. However, it does not enable to recover results about the Ramsey property under PD or under AD in $L(\mathbb{R})$, since Kastanas' game is not a game on integers. And it is still not known today whether in $ZF + DC + AD$, every subset of $[\omega]^\omega$ is Ramsey.

An important remark is that in Tanaka's proof of Mathias–Silver's theorem, the sets M and N are often seen as *subspaces*, i.e. elements of a poset (in the game K_M, players play subsets of ω that are smaller and smaller), while an element of $[N]^\omega$ is rather seen as an *infinite sequence*, the increasing sequence of its elements (this sequence being a subsequence of the sequence of moves of the players). This distinction between sets seen as subspaces and sets seen as sequences of points also appear in more classical proofs of Mathias–Silver's theorem and is actually central in infinite-dimensional Ramsey theory. In the decades that followed the proof of Mathias–Silver's theorem, several similar results arose in different contexts (words, trees, etc.), constituting what we call now *infinite-dimensional Ramsey theory.* All of these have the same form: we color infinite sequences of points satisfying some structural condition (being increasing, being block-sequences, etc.) and the theorem ensures that we can find a monochromatic subspace. To illustrate this, we give here another example due to Milliken. Let K be a field, and E be a countably-infinite dimensional vector space over K with a basis $(e_i)_{i\in\omega}$. If $x = \sum_{i\in\omega} x^i e_i \in E$, we define the *support* of x as the set $\mathrm{supp}(x) = \{i \in \omega \mid x^i \neq 0\}$. A *block-sequence* is an infinite sequence $(x_i)_{i\in\omega}$ of nonzero vectors of E such that $\mathrm{supp}(x_0) < \mathrm{supp}(x_1) < \ldots$. A *block-subspace* of E is a vector subspace of E spanned by a block-sequence. Remark that, since every vector of E has finite support, every infinite-dimensional subspace of E contains a block subspace. We can endow E with the discrete topology and E^ω with the product topology. Miliken's result is the following:

Theorem I.5 (Milliken)**.** *Suppose that E is a countably-infinite dimensional vector space over $K = \mathbb{F}_2$, with a basis. Let $\mathcal{X}$ be an analytic set of block-sequences of E. Then for every block-subspace $X \subseteq E$, there exists a block-subspace $Y \subseteq X$ such that:*

- *either every block-sequence in Y belongs to $\mathcal{X}$;*
- *or every block-sequence in Y belongs to $\mathcal{X}^c$.*

This theorem is usually formulated in terms of finite subsets of ω rather that vector spaces over $\mathbb{F}_2$, however we chose here this formulation because a link with other results presented in this manuscript will appear more clearly. For a proof, see [61], corollary 5.23.

It turns out that the proofs of all results in infinite-dimensional Ramsey theory use in an essential way what we call a *pigeonhole principle*. A pigeonhole principle is, in general, a one-dimensional Ramsey result, ensuring that for a colouring of points of some space with a finite number of colors (or equivalently, two colors), there exists a monochromatic subspace. The pigeonhole principle associated to an infinite-dimensional Ramsey theorem is the result we get if we restrict this theorem to colorations of sequences that only depend on the first term of the sequence. For instance, the pigeonhole principle associated to Mathias–Silver's theorem is the trivial fact that for every infinite $M \subseteq \omega$ and every $A \subseteq \omega$, there exists an infinite $N \subseteq M$ such that either $N \subseteq A$, or $N \subseteq A^c$. The pigeonhole principle associated to Milliken's theorem, however, is not trivial at all; it is the following result by Hindman (see [61], theorem 2.25):

Theorem I.6 (Hindman)**.** *Suppose that E is a countably-infinite dimensional vector space over $K = \mathbb{F}_2$, with a basis. Then for every colouring of the nonzero vectors of E with a finite number of colors, and for every block-subspace $X \subseteq E$, there exists a block-subspace $Y \subseteq X$ such that $Y \backslash \{0\}$ is monochromatic.*

Many examples of infinite-dimensional Ramsey theorems, and of their associated pigeonhole principles, can be found in Todorčević book [61], where a general framework to deduce an infinite dimensional Ramsey result from its associated pigeonhole principle is also developped.

I.3 Gowers' Ramsey-type theorem in Banach spaces and adversarial Gowers' games

The first infinite-dimensional Ramsey-type result that was not relying on a pigeonhole principle was proved by Gowers, in the 90's. The aim of Gowers was to solve a celebrated problem asked by Banach, the homogeneous space problem, asking whether ℓ_2 was the only infinite-dimensional Banach space, up to isomorphism, that was isomorphic to all of its closed, infinite-dimensional subspaces. Gowers proved a dichotomy [24] that, combined with a result by Komorowski and Tomczak-Jaegermann [34], provided a positive answer to Banach's question. The proof of this dichotomy relies on a Ramsey-type theorem in separable Banach spaces, that we will state now. We start by recalling some basic notions about bases in Banach spaces; these notions will be central in all of this manuscript. For proofs and more details, see [2].

Let E be a Banach space. A *Schauder basis* of E is a sequence $(e_i)_{i\in\omega} \in E^\omega$ such that every $x \in E$ can be written in a unique way as an infinite sum $\sum_{i=0}^{\infty} x^i e_i$, where $x^i \in \mathbb{R}$. In this manuscript, we will only consider *normalized* Schauder bases: we will add to the definition that vectors of a Schauder basis must have norm 1 (this restriction is not usual, but here, it will make things simpler). A Schauder basis is not a basis in the algebraic sense, however algebraic bases (that are often called *Hamel bases*) do not have much interest in the study of Banach spaces, so when speaking about a Banach space, in this manuscript, Schauder bases will often be simply called *bases*. Given a basis (e_i) of a Banach space E, we can define , for every n, a projection $P_n : E \longrightarrow \mathrm{span}\{e_i \mid i < n\}$ by $P_n\left(\sum_{i=0}^{\infty} x^i e_i\right) = \sum_{i<n} x^i e_i$. It can be shown that all these projections are bounded and that $C := \sup_{n\in\omega} \|P_n\| < \infty$. This constant C is called the *basis constant* of (e_i).

A normalized sequence $(x_i)_{i\in\omega} \in E^\omega$ that is a basis of the closed subspace of E it spans is called a *basic sequence*. It can be shown that a normalized sequence $(x_i)_{i\in\omega} \in E^\omega$ is a basic sequence if and only if there exists a constant C such that for every integers $m \leqslant n$ and for every $(a_i)_{i<n} \in \mathbb{R}^n$, we have $\|\sum_{i<m} a_i x_i\| \leqslant \|\sum_{i<n} a_i x_i\|$; in this case, the basis constant of (x_i) is the least such C. A classical result asserts that every Banach space contains a basic sequence, and that moreover, the constant of this basic sequence can be choosen as close as 1 as we want.

If E is a Banach space with a basis (e_i), we define the *support* of a vector $x = \sum_{i=0}^{\infty} x^i e_i$, denoted by $\mathrm{supp}(x)$, as the set $\{i \in \omega \mid x^i \neq 0\}$. A *block-sequence* of (e_i) is an infinite normalized sequence $(x_n)_{n\in\omega}$ of vectors of E with $\mathrm{supp}(x_0) < \mathrm{supp}(x_1) < \ldots$. A consequence of the previous characterisation of basic sequences is that a block-sequence of (e_i) is a basic sequence with constant not greather than the constant of (e_i). A closed subspace of E generated by a block-sequence is called a *block-subspace*.

For X a block-subspace of E, we denote by $[X]$ the set of block-sequences all of whose terms are in X (if (x_n) is a block-sequence generating X, then these sequences are exactly the block-sequences of (x_n)). We can equip $[E]$ with a natural topology by seeing it as a subspace of $(S_E)^\omega$ with the product topology (where S_E is endowed with the norm topology), which makes it a Polish space. For $\mathcal{X} \subseteq [E]$ and $\Delta = (\Delta_n)_{n\in\omega}$ a sequence of positive real numbers, we let $(\mathcal{X})_\Delta = \{(x_n)_{n\in\omega} \in [E] \mid \exists (y_n)_{n\in\omega} \in \mathcal{X}\ \forall n \in \omega\ \|x_n - y_n\| \leqslant \Delta_n\}$, a set called the Δ-*expansion* of $\mathcal{X}$. In order to state Gowers' theorem, we need a last definition.

Definition I.7. Let X be a block-subspace of E. *Gowers' game* below X, denoted by G_X, is the following infinite two-players game (whose players will be denoted by **I** and **II**):

I	Y_0		Y_1		$\ldots$	
II		y_0		y_1		$\ldots$

where the Y_i's are block-subspaces of X, and the y_i's are normalized elements of E with finite support, with the constraints for **II** that for all $i \in \omega$, $y_i \in Y_i$ and $\mathrm{supp}(y_i) < \mathrm{supp}(y_{i+1})$. The outcome of the game is the sequence $(y_i)_{i\in\omega} \in [E]$.

Remark that saying that player **II** has a strategy in G_X to reach $\mathcal{X}$ means, in a certain way, that "a lot" of block sequences of X belong to $\mathcal{X}$. We can now state Gowers' theorem:

Theorem I.8 (Gowers' Ramsey-type theorem)**.** *Let $\mathcal{X} \subseteq [E]$ be an analytic set, $X \subseteq E$ a block-subspace, and Δ be an infinite sequence of positive real numbers. Then there exists a block-subspace Y of X such that either $[Y] \subseteq \mathcal{X}^c$, or player **II** has a strategy in G_Y to reach $(\mathcal{X})_\Delta$.*

While one of the possible conclusions of this theorem, $[X] \subseteq \mathcal{X}^c$, is very similar to "For every infinite $S \subseteq M$, we have $S \in \mathcal{X}^c$" in Mathias–Silver's theorem, the other one is much weaker, for two reasons: the use of metrical approximation and the use of a game. As we will see later, the necessity of the approximation is due to a lack of finiteness, while the necessity for one of the possible conclusions to involve a game matters much more and is due to the lack of a pigeonhole principle in this context. In some Banach spaces, a pigeonhole principle holds, and in these spaces, Gowers gave a strengthening of his theorem, involving no game, that we will introduce now. We start by stating the general form of the pigeonhole principle that we will use in Banach spaces; since an exact pigeonhole principle is never satisfied in this context, and would anyways be useless since approximation is needed for other reasons, we will only state an approximate pigeonhole principle. For a Banach space E, a set $A \subseteq S_E$, and $\delta > 0$, we let $(A)_\delta = \{x \in S_E \mid \exists y \in A \ \|x - y\| \leqslant \delta\}$.

Definition I.9. Say that a Banach space E with a Schauder basis satisfies the *approximate pigeonhole principle* if for every $A \subseteq S_E$, for every block-subspace $X \subseteq E$, and for every $\delta > 0$, there exists a block-subspace $Y \subseteq X$ such that either $S_Y \subseteq A^c$, or $S_Y \subseteq (A)_\delta$.

Recall that an infinite-dimensional Banach space E is said to be c_0*-saturated* if c_0 can be embedded in all of its infinite-dimensional, closed subspaces. A combination of results of Gowers [23], Odell and Schlumprecht [51], and Milman [46] shows the following:

Theorem I.10. *A space E with a Schauder basis satisfies the approximate pigeonhole principle if an only if it is c_0-saturated.*

Thus, in c_0-saturated spaces, we have a strengthening of Gowers' theorem:

Theorem I.11 (Gowers' Ramsey-type theorem for c_0)**.** *Suppose that E is c_0-saturated. Let $\mathcal{X} \subseteq [E]$ be an analytic set, $X \subseteq E$ be a block-subspace, and Δ be an infinite sequence of positive real numbers. Then there exists a block-subspace Y of X such that either $[Y] \subseteq \mathcal{X}^c$, or $[Y] \subseteq (\mathcal{X})_\Delta$.*

For a complete survey of Gowers' Ramsey-type theory in Banach spaces, see [6], part B, chapter IV.

In 2010, in [56], Rosendal proves an exact version (without approximation) of Gowers' theorem, in countable vector spaces, which easily implies Gowers' theorem in Banach spaces. In this theorem, to be able to remove the approximation, we have to weaken the non-game-theoretical conclusion by introducing a new game, the *asymptotic game*. We present here Rosendal's theorem in more details. Let E be a countably infinite-dimensional vector space over an at most countable field K and $(e_i)_{i\in\omega}$ be a basis (in the algebraic sense) of E. To a block-subspace $X \subseteq E$, we associate two games defined as follows:

Definition I.12.

1. *Gowers' game* below X, denoted by G_X, is defined in the following way:

$$\begin{array}{llllll} \textbf{I} & Y_0 & & Y_1 & & \dots \\ \textbf{II} & & y_0 & & y_1 & & \dots \end{array}$$

   where the Y_i's are block-subspaces of X, and the y_i's are nonzero elements of E, with the constraint for **II** that for all $i \in \omega$, $y_i \in Y_i$. The outcome of the game is the sequence $(y_i)_{i\in\omega} \in E^\omega$.

2. The *asymptotic game* below X, denoted by F_X, is defined in the same way as G_X, except that this time, the Y_i's are moreover required to have finite codimension in X.

We endow E with the discrete topology and E^ω with the product topology; since E is countabe, E^ω is a Polish space. Rosendal's theorem is then the following:

Theorem I.13 (Rosendal)**.** *Let $\mathcal{X}$ be an analytic subset of E^ω. Then for every block-subspace $X \subseteq E$, there exists a block-subspace $Y \subseteq X$ such that either **I** has a strategy in F_Y to reach $\mathcal{X}^c$, or **II** has a strategy in G_Y to reach $\mathcal{X}$.*

We say that a set $\mathcal{X} \subseteq E^\omega$ is *strategically Ramsey* if it satisfies the conclusion of this theorem. Remark that if $K = \mathbb{F}_2$, then this theorem is implied by Milliken's theorem I.5. However, the immediate generalization of Milliken's theorem is false for fields with more than two elements, in particular (but not only) because the associated pigeonhole principle (i.e. the immediate generalization of Hindman's theorem I.6) is not true for these fields. In general, for vector spaces over an at most countable field, we cannot have a better result than theorem I.13. However, here, the use of an asymptotic game in one side of the alternative is not much weaker than a non-game-theoretical conclusion as in Milliken's theorem. This will be discussed in more details in section II.2 of this manuscript.

In the same paper as the last theorem, Rosendal, inspired by the work of Pelczar [52], and by a common work with Ferenczi [19], introduced a new Ramsey principle which is, unlike theorem I.13, symmetrical. His result was then refined in [57]. It involves two games, known as the *adversarial Gowers' games*, obtained by mixing the games G_X and F_X.

Definition I.14.

1. For a block-subspace $X \subseteq E$, the game A_X is defined in the following way:

$$\begin{array}{llllllll} \textbf{I} & & x_0, Y_0 & & x_1, Y_1 & & \dots \\ \textbf{II} & X_0 & & y_0, X_1 & & y_1, X_2 & & \dots \end{array}$$

   where the x_i's and the y_i's are nonzero vectors of X, the X_i's are block-subspaces of X, and the Y_i's are block-subspaces of X with finite codimension. The rules are the following:

- for **I**: for all $i \in \omega$, $x_i \in X_i$;
- for **II**: for all $i \in \omega$, $y_i \in Y_i$;

and the outcome of the game is the sequence $(x_0, y_0, x_1, y_1, \ldots) \in E^\omega$.

2. The game B_X is defined in the same way as A_X, except that this time the X_i's are required to have finite codimension in X, whereas the Y_i's can be arbitrary block-subspaces of X.

The result Rosendal proves in [57] is the following:

Theorem I.15 (Rosendal). *Let $\mathcal{X} \subseteq E^\omega$ be $\mathbf{\Sigma}^0_3$ or $\mathbf{\Pi}^0_3$. Then for every block-subspace $X \subseteq E$, there exists a block-subspace $Y \subseteq X$ such that either **I** has a strategy in A_Y to reach $\mathcal{X}$, or **II** has a strategy in B_Y to reach $\mathcal{X}^c$.*

Let us say that a set $\mathcal{X} \subseteq E^\omega$ is *adversarially Ramsey* if it satisfies the conclusion of this theorem. Then, a natural question to ask is for which complexity of the set $\mathcal{X}$ one can ensure that it is adversarially Ramsey.

There are two things to remark. Firstly, let $\mathcal{X} \subseteq E^\omega$ and define $\mathcal{X}' = \{(x_i)_{i\in\omega} \in E^\omega \mid (x_{2i})_{i\in\omega} \in \mathcal{X}\}$. Then by forgetting the contribution of player **II** to the outcome of the adversarial Gowers' games and switching the roles of players **I** and **II**, we see that $\mathcal{X}$ is strategically Ramsey if and only if $\mathcal{X}'$ is adversarially Ramsey. So, for a class Γ of subsets of Polish spaces, closed under continuous inverse image, saying that all Γ-subsets of E^ω are adversarially Ramsey is stronger than saying that all Γ-subsets of E^ω are strategically Ramsey. The second remark is that, if the field K is infinite, then the adversarial Ramsey property for Γ-subsets of E^ω also implies that all Γ-subsets of ω^ω are determined. To see this, remark that when playing vectors in A_X or B_X, no matter the constraint imposed by the other player, players **I** and **II** have total liberty for choosing the first non-zero coordinate of the vectors they play. Therefore, by making $\mathcal{X}$ only depend on the first nonzero coordinate of each vector played, we recover a classical Gale-Stewart game in $(K^*)^\omega$. For this reason, there is no hope, in ZFC, to prove the adversarial Ramsey property for a class larger than Borel sets. Then, Rosendal asks the following questions in [57]:

Question I.16 (Rosendal). *Is every Borel set adversarially Ramsey?*

Question I.17 (Rosendal). *In the presence of large cardinals, is every analytic set adversarially Ramsey?*

A part of chapter II in this thesis will be devoted to answer these questions.

I.4 Banach-spaces dichotomies and complexity of the isomorphism

As we already said at the beginning of this introduction, Gowers introduced his Ramsey-type theorem I.8 in order to prove a Banach-space dichotomy that was instrumental in the solution of a celebrated question asked by Banach in his book *Théorie des Opérations Linéaires* [9]. Say that a Banach space is *homogeneous* if it is isomorphic to all of its subspaces. Obviously, ℓ_2 is homogeneous, and a homogeneous space has to be separable. Banach's question is the following:

Question I.18 (Banach's homogeneous space problem). *Is every homogeneous Banach space isomorphic to ℓ_2?*

This problem was solved by the positive in the 90's by a combination of results by Gowers and Maurey [25], Komorowski and Tomczak-Jaegeramnn [34] , and Gowers [24]. We will briefly expose the main steps of this solution, and then present the new research directions that this problem, and its solution, have raised. For this, we need to recall some notions in Banach-space geometry.

Let $(e_i)_{i\in\omega}$ be a basis of a Banach space E. Remark that, if $A \subseteq \omega$ is infinite and coinfinite, then a projection on the closed subspace generated by the e_i's, for $i \in A$, does not necessarily exist: actually, if $x = \sum_{i=0}^{\infty} x^i e_i$ converges, the sums $\sum_{i\in A} x^i e_i$ do not need to converge unless A is finite or cofinite. We say that the basis (e_i) is an *unconditional basis* if for every $x = \sum_{i=0}^{\infty} x^i e_i \in E$ and for every $A \subseteq \omega$, the sum $\sum_{i\in A} x^i e_i$ converges. It can be shown that, in this case, for every $a = (a_i)_{i\in\omega} \in \ell_\infty$ and for every $x = \sum_{i=0}^{\infty} x^i e_i \in E$, the sum $D_a(x) = \sum_{i=0}^{\infty} a_i x^i e_i$ converges, and that the operator $D_a : E \longrightarrow E$ it defines is bounded (such an operator is called a *diagonal operator*). Moreover, there exists a constant K such that for every $a \in \ell_\infty$, $|||D_a||| \leqslant K\|a\|_\infty$. In this case, the sequence is said to be K-unconditional, and the least such K is called the *unconditional constant* of the basis (e_i). The unconditional constant is greater, but in general not equal, to the basis constant.

An *unconditional basic sequence* (or simply an *unconditional sequence*) is a normalized sequence $(x_i)_{i\in\omega} \in E^\omega$ that is an unconditional basis of the closed subspace of E it generates. It can be shown that a normalized sequence $(x_i)_{i\in\omega} \in E^\omega$ is K-unconditional if and only if for every $n \in \omega$, every $(a_i)_{i<n} \in \mathbb{R}^n$, and every sequence of signs $(\varepsilon_i)_{i<n} \in \{-1, 1\}^n$, we have $\|\sum_{i<n} \varepsilon_i a_i x_i\| \leqslant K \|\sum_{i<n} a_i x_i\|$ (so in particular, (x_i) is unconditional if and only if there exist a constant K satisfying this property). This shows that a block-sequence of a (K-)unconditional sequence is itself (K-)unconditional.

The canonical bases of the spaces c_0, and ℓ_p for $1 \leqslant p < \infty$, are 1-unconditional. Spaces with an unconditional basis can be seen as quite regular spaces; in particular, many bounded operators are definable on them (all the D_a's for $a \in \ell_\infty$) and they share many of the good properties of the ℓ_p's and of c_0. For more details, proofs of the previous results, and properties of spaces with an unconditional basis, see [2].

In 1995, Komorowski and Tomczak-Jaegermann [34] (with an erratum [35]) showed the following result:

Theorem I.19 (Komorowski–Tomczak-Jaegermann)**.** *Every separable Banach space either has a subspace isomorphic to ℓ_2, or a subspace without unconditional basis.*

An immediate consequence of this theorem is that a homogeneous space that is not isomorphic to ℓ_2 cannot contain an unconditional sequence. The question whether a Banach space should always contain an unconditional sequence was itself a longstanding problem, asked by Banach in the same book [9] and called *the unconditional basic sequence problem.* This problem was solved by the negative by Gowers and Maurey [25] a few years before the proof of Komorowski–Tomczak-Jaegermann's theorem, in 1992. The counterexample built by Gowers and Maurey actually had a slightly stronger property than not containig any unconditional sequence: it was *hereditarily indecomposable.*

Definition I.20.

1. A Banach space E is *indecomposable* if there are no subspaces $X, Y \subseteq E$ such that $E = X \oplus Y$.

2. A Banach space E is *hereditarily indecomposable* (or simply *HI*) if every subspace of E is indecomposable.

(Obviously, in the definition of an indecomposable Banach space, we only quantify on infinite-dimensional closed subspaces, since every finite-dimensional subspace has a closed complement, and since every vector subspace is the complement of another vector subspace.)

A space E with an unconditional basis $(e_i)_{i\in\omega}$ is not indecomposable: indeed, for every $A \subseteq \omega$, we have $E = \overline{\operatorname{span}(\{e_i \mid i \in A\})} \oplus \overline{\operatorname{span}(\{e_i \mid i \in A^c\})}$. In particular, an HI space cannot contain an unconditional sequence. However, the converse is not true: for instance, it can easily be shown that the direct sum of two HI spaces cannot contain an unconditional sequence, however it is not HI. Surprisingly, all the natural counterexamples to the unconditional basic sequence problem that Gowers and Maurey managed to build where HI, as if HI spaces were the basic building blocks of such spaces. To explain this phenomenon, Gowers proved a few years later his celebrated first dichotomy [24]:

Theorem I.21 (Gowers' first dichotomy)**.** *Every Banach space either contains a unconditional basic sequence, or contains a HI subspace.*

This is in order to prove this dichotomy that Gowers proved his Ramsey-type theorem I.8. A consequence of this dichotomy, combined with Komorowski–Tomczak-Jaegermann's theorem, is that if a homogeneous space is not isomorphic to ℓ_2, then it has to be HI. So to solve the homogeneous space problem, it only remains to prove that an HI space cannot be homogeneous. This is actually a consequence of general results by Gowers and Maurey about HI spaces. In the same paper [25] where they built the first HI space, they proved the following theorem:

Theorem I.22 (Gowers–Maurey)**.** *Let X be a complex HI space. Then every bounded operator $X \longrightarrow X$ has the form $\lambda \operatorname{Id}_X + S$, where $\lambda \in \mathbb{C}$ and S is a strictly singular operator (that is, an operator that induces no isomorphism between two subspaces of X).*

This theorem was proved using spectral theory and is only valid for complex spaces. Using Fredholm theory, we can easily deduce from it the following result (valid as well for real spaces as for complex spaces):

Theorem I.23 (Gowers–Maurey)**.** *A (real or complex) HI space cannot be isomorphic to one of its proper subspaces.*

In particular, such a space is very far from being homogeneous, and this last result ends the solution of the homogeneous space problem.

The first research direction opened by the solution of the homogeneous space problem, and in particular by Gowers' first dichotomy, is a project initiated by Gowers at the end of his article [24]. He suggested that, using Ramsey-theoretic methods to prove Banach-space dichotomies in the same vein as his first dichotomy, we could build a "loose" classification of separable Banach spaces, up to subspaces. The idea is to build a list of classes of separable Banach spaces (called a *Gowers list*), as precise as possible, satisfying the following conditions:

(1) The classes should be *hereditary*, i.e. if a space E belongs to one class, then every subspace of E must belong to the same class (or every block-subspace, if the class is defined by a property of bases);

(2) The classes should be disjoint, for obvious reasons;

(3) Every Banach space should have at least one subspace belonging to one of the classes;

(4) Knowing that a space belongs to a class should give much information about the space, and in particular about the operators that can be defined on this space.

Gowers' first dichotomy gives an example of a Gowers list with two classes, the class of spaces with an unconditional basis, and the class of HI spaces. Properties **(1)** and **(2)** are obvious, and property **(3)** is given by the dichotomy. This Gowers list illustrates particularly well property **(4)**, since spaces with an unconditional basis have many operators (in particular, all the diagonal operators), whereas HI spaces have very few of them (in paticular, all diagonal operators on a HI space with a basis are trivial).

The interest of a Gowers list is also to draw a border between "nice", well-behaved spaces (those sharing many good properties of the ℓ_p's and of c_0) and "pathological" one, like HI spaces, that were mostly discovered in order to provide counterexamples. In the same paper [24], Gowers proved a second dichotomy, enabling him to get a Gowers' list with three classes, and then, Ferenczi and Rosendal [19] proved three other dichotomies. We will present one of them here, since it is an inspiration for a part of the work of this thesis.

Definition I.24.

1. A Banach space E is said to be *minimal* if it can be embedded into all of its subspaces.

2. Let $(e_i)_{i\in\omega}$ be a basis of some Banach space E. A Banach space X is *tight in* the basis (e_i) if there is an infinite sequence of intervals $I_0 < I_1 < \ldots$ of integers such that for every infinite $A \subseteq \omega$, we have $X \not\sqsubseteq \overline{\operatorname{span}\left(\left\{e_i \middle| i \notin \bigcup_{j\in A} I_j\right\}\right)}$.

3. A basis $(e_i)_{i\in\omega}$ is said to be *tight* if every Banach space is tight in it. A Banach space X is *tight* if it has a tight basis.

The class of minimal spaces is another example of a class of "nice" spaces. For example, the ℓ_p's and c_0 are minimal. This is obviously a hereditary class. On the other hand, a tight space cannot be minimal, and it is not hard to see that a block-sequence of a tight basis is itself tight. Tight spaces are more pathological spaces, an example of them is Tsirelson's space, the first example of a space in which none of the ℓ_p's, neither c_0, can be embedded (see [2], section 10.3). The dichotomy is the following:

Theorem I.25 (Ferenczi–Rosendal)**.** *Every Banach space either has a minimal subspace, or has a tight subspace.*

This dichotomy does not precisely satisfy the condition **(4)** in the definition of a Gowers list, since the operators on tight spaces have not been studied much. However, as we will see, knowing that a space is tight gives much information about the isomorphism relation between its subspaces. In particular, such spaces are highly non-homogeneous, and this will be a useful information in the study of the number of non-isomorphic subspaces of a separable Banach space.

This is, indeed, the second research direction raised by the solution of the homogeneous space problem. As soon as a separable Banach space is not isomorphic to ℓ_2, it must have at least two non-isomorphic subspaces; but more precisely, how many subspaces can such a space have, up to isomorphism? This very general question was asked by Godefroy, and many particular cases of it were studied. This turn out to be quite difficult questions, and most of the time, we only have partial results about it. For example, the following question was asked by Johnson:

Question I.26. *Does there exist a separable Banach space having exactly two subspaces, up to isomorphism?*

Even this question is still open, and will be studied in the present manuscript. A separable Banach space with exactly two subspaces, up to isomorphism, will be called a *Johnson space.*

It turns out that the right setting to study Godefroy's question is the theory of the *classification of equivalence relations on Polish spaces.* Let us recall the basic notions of this theory. We will study nonempty Polish spaces equiped with an equivalence relation (that will often be analytic). If X and Y are two nonempty Polish spaces, and if E and F are equivalence relations respectively on X and Y, we say that E *Borel-reduces* to F, denoted by $(X, E) \leqslant_B (Y, F)$ (or simply $E \leqslant_B F$) if there is a Borel mapping $f : X \longrightarrow Y$ (called a *reduction*) such that for every $x, y \in X$, we have $x \mathrel{E} y \Leftrightarrow f(x) \mathrel{F} f(y)$ (if such an f can be choosen continuous, we will say that E *continuously reduces* to F, denoted by $E \leqslant_c F$). We say that E and F are Borel-equivalent, denoted by $E \equiv_B F$, if $E \leqslant_B F$ and $F \leqslant_B E$. Saying that E reduces to F means that E is less complex than F, and that if we know F, then we can, in some sense "compute" E. Remark that a reduction f from (X, E) to (Y, F) induces a one-to-one mapping $X/E \to Y/F$, and in particular, if $E \leqslant_B F$, then E has less classes than F. Thus, studying the complexity of an equivalence relation gives us at least as much information than counting it classes.

If E has at most countably-many classes, then $E \leqslant_B F \Leftrightarrow |X/E| \leqslant |Y/F|$. Thus, for equivalence relations with at most countably many classes, the number of classes completely determines the equivalence type of the relation, and we have the following exhaustive hierarchy:

$$(1, =) \leqslant_B (2, =) \leqslant_B (3, =) \leqslant_B \ldots \leqslant_B (\omega, =).$$

However, for relations with uncountably many classes, the situation is more complex. Restricting our attention to Borel equivalence relations, we will present two dichotomies that give the two next steps of this hierarchy. The first one is valid even for coanalytic relations, and is due to Silver [59] (for a more modern proof, see [44]).

Theorem I.27 (Silver)**.** *Let E be a coanalytic equivalence relation on a Polish space X. Then either E has at most countably many classes (and thus, Borel reduces to the equality on ω), or $(2^\omega, =) \leqslant_c (X, E)$.*

The next equivalence relation is the relation $\mathbf{E_0}$ on the Cantor space 2^ω, defined by $x \, \mathbf{E_0} \, y$ if and only if there exists $n \in \omega$ such that for every $m \geqslant n$, we have $x(m) = y(m)$. Using standard category arguments (see [28]), it can be shown that $\mathbf{E_0}$ is *generically ergodic*, that is, every $\mathbf{E_0}$-invariant Baire-measurable set is either meager or comeager, and thus that it is not Borel-reducible to the equality on the Cantor space. In [28], Harrington, Kechris and Louveau show the following dichotomy (for a more modern proof, see [45]):

Theorem I.28. *Let E be a Borel equivalence relation on a Polish space X. Then either $(X, E) \leqslant_B (2^\omega, =)$, or $(2^\omega, \mathbf{E_0}) \leqslant_c (X, E)$.*

Thus, we have the following exhaustive hierarchy for Borel equivalence relation that reduce to $\mathbf{E_0}$:

$$(1, =) \leqslant_B (2, =) \leqslant_B (3, =) \leqslant_B \ldots \leqslant_B (\omega, =) \leqslant_B (2^\omega, =) \leqslant_B (2^\omega, \mathbf{E_0}).$$

The situation is more complex for analytic equivalence relations, for instance there are such relations E that are not reducible to the equality on the Cantor space, but such that $\mathbf{E_0}$ does not reduce to E.

The main application of the complexity of equivalence relations appears in the study of the classification of mathematical objects: one can, for example, put a convenient Borel structure on a class of mathematical objects and study the isomorphism relation between these objects (that is, in general, analytic). Knowing the complexity of this relation enables to estimate how difficult it will be to classify these objects up to isomorphism. In general, we consider that a class of objects is classifiable if the isomorphism relation on this class is reducible to the equality on the Cantor space; indeed, this means that the structures in this class can be described, up to isomorphism, by a real number, or by a sequence of integers (for instance, the isomorphism between Bernoulli shifts is classified by a real number, its entropy).

This theory can indeed be applied to the isomorphism relation between subspaces of a Banach space. Given E a separable Banach space, denote by $\mathrm{Sub}(E)$ the set of its subspaces. On $\mathrm{Sub}(E)$, we will put the *Effros Borel structure*. Recall that if X is a Polish space and $\mathcal{F}(X)$ the set of its open subsets, the Effros Borel structure on $\mathcal{F}(X)$ is the σ-algebra generated by sets of the form $\{F \in \mathcal{F}(X) \mid F \cap U \neq \varnothing\}$, where U varies over open subsets of X. It can be shown that this gives $\mathcal{F}(X)$ a structure of standard Borel space (see [32], theorem 12.6); actually, if $\hat{X}$ is a compactification of X, then this Borel space can be seen as a subspace of the set compacts subsets of $\hat{X}$ with the Hausdorff distance, so it is quite natural. If E is a separable Banach space, it is not hard to see that $\mathrm{Sub}(E)$ is a Borel subset of $\mathcal{F}(E)$, so $\mathrm{Sub}(E)$ with the Effros Borel structure is itself a standard Borel space. Moreover, the isomorphism relation $\cong$ on $\mathrm{Sub}(E)$ is analytic. For more results about the structure of $\mathrm{Sub}(E)$, see [10].

The complexity of the isomorphism relation of $\mathrm{Sub}(\ell_2)$ is minimal among analytic equivalence relations. On the other hand, Ferenczi, Louveau and Rosendal [16] proved the following:

Theorem I.29 (Ferenczi–Louveau–Rosendal)**.** *The isomorphism relation on* $\mathrm{Sub}(C([0,1]))$ *is analytic-complete, that is, every analytic equivalence relation on a Polish space is Borel-reducible to it.*

As, by Banach–Mazur's theorem (theorem 1.4.3 in [2]), every separable Banach space can be isometrically embedded in $\mathrm{Sub}(C([0,1]))$, this result can be interpreted by saying that the isomorphism relation between separable Banach spaces is analytic-complete and in particular, that these spaces are not classifiable, up to isomorphism. This justifies Gowers' idea of rather trying to build a "loose" classification of Banach spaces.

We have, on one side, a space for whose the complexity of the isomorphism between subspaces is minimal, and on the other side, a space for whose this complexity is maximal among analytic equivalence relations, and we are tempted to ask what lies inbetween. Ferenczi and Rosendal defined a new class of spaces based on their complexity:

Definition I.30. A separable Banach space E is *ergodic* if $(2^\omega, \mathbf{E_0}) \leqslant_B (\mathrm{Sub}(E), \cong)$.

In particular, the subspaces of these spaces are not classifiable by real numbers, so they can be seen as rather complex spaces. In the papers [18, 17, 55], Ferenczi and Rosendal studied the properties of non-ergodic spaces, that appeared to behave quite well. Among others, we can cite the following nice results for spaces with an unconditional basis:

Theorem I.31 (Ferenczi–Rosendal). *Let E be a non-ergodic separable Banach space with an unconditional basis. Then E is isomorphic to its hyperplanes, to its square, and to every direct sum $E \oplus X$, where X is a block-subspace of E generated by a subsequence of the basis.*

All of their results led them to conjecture the following generalization to the homogeneous space problem:

Conjecture I.32 (Ferenczi–Rosendal). *Every separable Banach space non-isomorphic to ℓ_2 is ergodic.*

This conjecture will be refered as the *ergodic conjecture* in the rest of this manuscript. Much progress have been made by now on this conjecture. Rosendal proved [55] that an HI space must be ergodic. In particular, using Gowers' first dichotomy, we can deduce that every non-ergodic Banach space contains a subspace with an unconditional basis. Then, an important result was proved by Ferenczi [14]:

Theorem I.33 (Ferenczi). *Every non-ergodic Banach space contains a minimal subspace.*

The proof of this theorem was the main inspiration for Ferenczi and Rosendal's dichotomy I.25, that was proved a few years later. Actually, this result can be seen as a consequence of the dichotomy: indeed, in [15], Ferenczi and Godefroy give a categorical caracterisation of tightness which, combined with a result of Rosendal ([55], proposition 15), easily proves that a tight space must be ergodic. All of this results show that the question of the number of non-isomorphic subspaces and this of the loose classification of Banach spaces are closely related.

In another direction, progress have been made by Anisca [4], who proved that an asymptotically Hilbertian separable Banach space that is not isomorphic to ℓ_2 has to be ergodic; we will not recall here the definition of an asymptotically Hilbertian space, but this results says in some sense that spaces that are too close to ℓ_2 have to be ergodic. Then, Cuellar-Carrera proved [11] that a non-ergodic separable Banach space must have type p and cotype q for every $p < 2 < q$. Without recalling the definitions, it means that such a space still needs to be rather close to ℓ_2. In particular, a consequence of this result is that the ℓ_p's, for $1 \leqslant p \neq 2 < \infty$, and c_0, are ergodic.

I.5 Organisation of the results

In chapter II of this thesis, we present an abstract setting for Ramsey theory, the setting of *Gowers spaces*. The goal of this abstract setting is to enable to prove as well Ramsey results with a pigeonhole principle like Mathias–Silver's theorem I.3 or Milliken's theorem I.5, and strategical Ramsey results without a pigeonhole principle like Rosendal's theorem I.13. An abstract Ramsey theorem, having a version without pigeonhole principle (theorem II.14), and a version with a pigeonhole principle (corollary

II.21) and implying these results, will be shown. Remark that all possible conclusions of this theorem involve games, since our setting is too weak to allow to get directly "genuine" Ramsey-type conclusions as in Mathias–Silver's or Milliken's theorem. However, the results we get are very close to that, and this drawback will be corrected in chapter III by adding a feature to our setting. In chapter II, we will also give an answer to Rosendal's questions I.16 and I.17 by proving an abstract adversarial Ramsey principle (theorem II.4), unifying our strategical Ramsey theory with the determinacy of games on integers. An emphasis will be put on the strength of the latter result, that is implied by the determinacy of games on reals but seems slightly above this of games on integers. Finally, we will study the differences between Gowers spaces with a pigeonhole principle and Gowers space without a pigeonhole principle, and see that they behave very differently. In particular, in spaces without a pigeonhole principle, the adversarial Ramsey principle is much stronger than is spaces where the pigeonhole principle holds.

Gowers spaces are countable, and as we will see at the beginning of chapter III, the result proved in chapter II are not true in the uncountable case. The goal of section III is to adapt the formalism of Gowers spaces to the case of uncountable metric space, in order to prove approximate results in the vein of Gowers' theorems I.8 and I.11. Results of chapter II will be extended to this setting, and a feature will also be added to our formalism, enabling to deduce non-strategical Ramsey results from strategical ones. In particular, both Mathias–Silver's theorem and Gower's theorems will be direct consequences of our main result, corollary III.17. The interest of the results presented in this chapter is more practical that theoretical: they are powerful tools to prove Banach-space dichotomies.

In chapter IV, we work on Johnson's problem and on Ferenczi and Rosendal's ergodic conjecture, and in particular on the following question: if counterexamples to these conjectures exist, do there necessarily exist counterexamples having an unconditional basis? We are not able to solve this question completely, however we prove two Banach-space dichotomies that could help a lot. The first one, theorem IV.12, is very similar to Gowers' first dichotomy between spaces with an unconditional basis and HI spaces, and the second one, theorem IV.14, is very similar to Ferenczi and Rosendal's dichotomy between minimal and tight spaces; however, the difference is that the dichotomies we prove are *Hilbert-avoiding*, that is, we can ensure that the subspace they provide is not isomorphic to ℓ_2. The proofs of these dichotomies makes an essential use of the Ramsey-type results proved in chapters II and III. These dichotomies enable to reduce the question of the existence of counterexamples to the ergodic conjecture with an unconditional basis to a conjecture having many similarities with Gowers–Maurey's result that HI spaces are not isomorphic to their proper subspaces. We were not able to solve this conjecture, however, at the end of the chapter, we give a new and simpler proof of Gowers–Maurey's theorem, only based on Fredholm theory, that could be a good starting point to solve it.

Chapter II

Ramsey theory with and without pigeonhole principle

In this chapter, we present an abstract setting for Ramsey theory with and without pigeonhole principle: the setting of Gowers spaces. Inspired by the examples given in the introduction, we define a formalism with two notions, a notion of *subspaces* and a notion of *points*. The idea is that we will color infinite sequences of points and try to find subspaces such that many sequences of points in this subspace share the same color. This “many” will be expressed, as in Rosendal’s results and conjectures, in terms of games; in particular, in Gowers spaces, we will be able to define the asymptotic game, Gowers’ game, and the adversarial Gowers’ games. These games will enable us to define strategically Ramsey sets and adversarially Ramsey sets in such spaces.

In section II.1, we define the formalism of Gowers spaces and the notion of strategically Ramsey sets in these spaces; then, we prove that every Borel sets is strategically Ramsey (theorem II.4), thus giving a positive answer to Rosendal’s question I.16. The proof of this theorem is based on the determinacy of a game on real numbers, thus, it will also enable to prove that, assuming enough determinacy for such games, we can get the adversarial Ramsey property for more than Borel sets (see theorems II.8 and II.10).

In section II.2, we define the asymptotic game, Gowers’ game, and the notion of a strategically Ramsey set in a Gowers space. We prove an abstract version of Rosendal’s theorem I.13 from the adversarial Ramsey principle proved in the previous section; this enables as well to get the strategical Ramsey property for more complex sets if we assume more determinacy. Then, we introduce the pigeonhole principle in a Gowers space, and we show that the strategical Ramsey property can be strengthened to a symmetrical result, very close to Mathias–Silver’s and Milliken’s theorem, in spaces that satisfy it (corollary II.21).

The two next sections are devoted to the study of the differences of behavior between spaces satisfying the pigeonhole principle, and spaces that don’t. Such a study is done in what concerns the adversarial Ramsey property in section II.3, where we show that in spaces with a pigeonhole principle, this property is not stronger than the strategical

Ramsey property, whereas in spaces without it, it has the strength of determinacy of games on integers (see proposition II.24 and theorem II.25). In section II.4, we carry out the same kind of study for strategically Ramsey sets, studying in particular the limitations on the complexities for which we can ensure this property in ZFC; this turns out to depend on the truth of the pigeonhole principle.

Since the adversarial Ramsey property is a consequence of the determinacy of games on reals, and implies the determinacy of games on integers, a natural question to ask is where lies the strength of the adversarial Ramsey property between the two others. In section II.5, we discuss some consequences of large cardinal assumptions on strategically Ramsey sets, allowing us to better see what could be the strength of this property.

II.1 Gowers spaces and the aversarial Ramsey property

In this section, we will introduce the notion of a *Gowers space*, which will be our abstract setting for infinite-dimensional Ramsey theory; then, we will prove in this setting the adversarial Ramsey principle, our most general Ramsey result without pigeonhole principle, which will give a positive answer to question I.16.

Definition II.1. A *Gowers space* is a quintuple $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$, where P is a nonempty set (the set of *subspaces*), X is an at most countable nonempty set (the set of *points*), $\leqslant$ and $\leqslant^*$ are two quasiorders on P (i.e. reflexive and transitive binary relations), and $\lhd \subseteq \mathrm{Seq}(X) \times P$ is a binary relation, satisfying the following properties:

1. for every $p, q \in P$, if $p \leqslant q$, then $p \leqslant^* q$;
2. for every $p, q \in P$, if $p \leqslant^* q$, then there exists $r \in P$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;
3. for every $\leqslant$-decreasing sequence $(p_i)_{i \in \omega}$ of elements of P, there exists $p^* \in P$ such that for all $i \in \omega$, we have $p^* \leqslant^* p_i$;
4. for every $p \in P$ and $s \in X^{<\omega}$, there exists $x \in X$ such that $s \frown x \lhd p$;
5. for every $s \in \mathrm{Seq}(X)$ and every $p, q \in P$, if $s \lhd p$ and $p \leqslant q$, then $s \lhd q$.

We say that $p, q \in P$ are *compatible* if there exists $r \in P$ such that $r \leqslant p$ and $r \leqslant q$. To save writing, we will often write $p \lessapprox q$ when $p \leqslant q$ and $q \leqslant^* p$. Remark that by 2., the p^* in 3. can be chosen in such a way that $p^* \leqslant p_0$; this will be useful in many proofs.

In most usual cases, the fact that $s \lhd p$ will only depend on p and on the last term of s; the spaces satisfying this property will be called *forgetful Gowers spaces*. In these spaces, we will allow us to view $\lhd$ as a binary relation on $X \times P$. However, for some applications (see, for example, the proof of theorem III.6), it is sometimes useful to make the fact that $s \lhd p$ also depend on the the length of the sequence s; we do not know if there are any interesting applications where it would be useful to make it depend on all the terms of the sequence, however we would like to present results that are as general as possible.

When thinking about a Gowers space, we should have the two following examples in mind:

- The *Mathias–Silver space* $\mathcal{N} = ([\omega]^\omega, \omega, \subseteq, \subseteq^*, \lhd)$, where $[\omega]^\omega$ is the set of all infinite sets of integers, $M \subseteq^* N$ iff $M \backslash N$ is finite and $(x_0, \ldots, x_n) \lhd M$ iff $x_n \in M$. Here, we have that $M \lessapprox N$ iff M is a cofinite subset of N, and M and N are compatible iff $M \cap N$ in infinite.

- The *Rosendal space* over an at most countable field K, $\mathcal{R}_K = (P, E\backslash\{0\}, \subseteq, \subseteq^*, \lhd)$, where E is a countably infinite-dimensional K-vector space with a basis $(e_i)_{i\in\omega}$, P is the set of all block subspaces of E relative to this basis, $X \subseteq^* Y$ iff Y contains some finite-codimensional block subspace of X, and $(x_0, \ldots, x_n) \lhd X$ iff $x_n \in X$. Here, we have that $X \lessapprox Y$ iff X is a finite-codimensional subspace of Y, and X and Y are compatible iff $X \cap Y$ is infinite-dimensional.

Remark that both of these spaces are forgetful, so we could have defined $\lhd$ as a relation between points and subspaces (and that is what we will do, in such cases, in the rest of this paper); in this way, in both cases, $\lhd$ is the membership relation. It is easy to verify that, for these examples, the axioms 1., 2., 4., and 5. are satisfied; we briefly explain how to prove 3.. For the Mathias–Silver space, if $(M_i)_{i\in\omega}$ is a $\subseteq$-decreasing sequence of infinite subsets of ω, then we can, for each $i \in \omega$, choose $n_i \in M_i$ in such a way that the sequence $(n_i)_{i\in\omega}$ is increasing, and let $M^* = \{n_i \mid i \in \omega\}$. Then the set M^* is as wanted. For the Rosendal space, the idea is the same: given $(F_i)_{i\in\omega}$ a decreasing sequence of block subspaces of E, we can pick, for each i, a nonzero vector $x_i \in F_i$, in such a way that for $i \geqslant 1$, we have $\mathrm{supp}(x_{i-1}) < \mathrm{supp}(x_i)$. In this way, $(x_i)_{i\in\omega}$ is a block sequence, and the block subspace F^* spanned by this sequence is as wanted.

Also remark that in the definition of the Rosendal space, choosing $E\backslash\{0\}$ and not E for the set of points is totally arbitrary, and here, we only made this choice in order to use the same convention as Rosendal in his papers [56, 57]; but the results we will show apply as well when the set of points is E. Also, we could have taken for P the set of all infinite-dimensional subspaces of E (where, here, the relation $\subseteq^*$ is defined by $X \subseteq^* Y$ iff $X \cap Y$ has finite codimension in X) instead of only block subspaces. However, the abstract results we will prove are slightly stronger in the case when we consider only block subspaces; this is due to the fact that, while every infinite-dimensional subspace of E contains a block subspace, there are finite-codimensional subspaces that do not contain any finite-codimensional block subspace.

In the rest of this section, we fix a Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$. For $p \in P$, we define the *adversarial Gowers' Games below* p as follows:

Definition II.2.

1. The game A_p is defined in the following way:

$$\begin{array}{llllllll} \mathbf{I} & & x_0, q_0 & & x_1, q_1 & & \ldots & \\ \mathbf{II} & p_0 & & y_0, p_1 & & y_1, p_2 & & \ldots \end{array}$$

   where the x_i's and the y_i's are elements of X, and the p_i's and the q_i's are elements of P. The rules are the following:

   - for **I**: for all $i \in \omega$, $(x_0, y_0, \ldots, x_{i-1}, y_{i-1}, x_i) \lhd p_i$ and $q_i \lessapprox p$;
   - for **II**: for all $i \in \omega$, $(x_0, y_0 \ldots, x_i, y_i) \lhd q_i$ and $p_i \leqslant p$.

   The outcome of the game is the sequence $(x_0, y_0, x_1, y_1, \ldots) \in X^\omega$.

2. The game B_p is defined in the same way as A_p, except that this time the we require $p_i \lessapprox p$, whereas we only require $q_i \leqslant p$.

As in the particular case of vector spaces, we can define the *adversarial Ramsey property* for subsets of X^ω:

Definition II.3. A set $\mathcal{X} \subseteq X^\omega$ is said to be *adversarially Ramsey* if for every $p \in P$, there exists $q \leqslant p$ such that either player **I** has a strategy to reach $\mathcal{X}$ in A_q, or player **II** has a strategy to reach $\mathcal{X}^c$ in B_q.

Informally, the adversarial Ramsey property for $\mathcal{X}$ means that up to taking a subspace, one of the players has a winning strategy in the game that is the most difficult for him. Remark that the property that **I** has a strategy in A_p to reach some set $\mathcal{X}$ (resp. the property that **II** has a strategy in B_p to reach $\mathcal{X}^c$) is strongly hereditary in the sense that if **I** has a strategy to reach $\mathcal{X}$ in A_p, then he also has one in $A_{p'}$ for every $p' \leqslant^* p$ (and the same holds for **II** in B_p). Indeed, we can simulate a play of $A_{p'}$ with a play of A_p: when, in A_p, player **I**'s strategy tells him to play x_i and q_i, then in $A_{p'}$ he can play the same x_i and a q'_i such that $q'_i \lessapprox p'$ and $q'_i \leqslant q_i$, in such a way that the next y_i played by **II** in $A_{p'}$ will be also playable in A_p (the existence of such a q'_i is guaranteed by condition 2. in the definition of a Gowers space). And when, in $A_{p'}$, player **II** plays y_i and p'_{i+1}, then in A_p, **I** can make her play the same y_i and a p_{i+1} such that $p_{i+1} \leqslant p$ and $p_{i+1} \leqslant p'_{i+1}$, in such a way that the next x_{i+1} played by **I** in A_p according to his strategy will also be playable in $A_{p'}$. In this way, the outcomes of both games are the same, and since **I** reaches $\mathcal{X}$ in A_p, then he also does in $A_{p'}$.

On the other hand, it is clear that if **I** has a strategy to reach some set $\mathcal{X}$ in A_p, then he also has one in B_p, so **II** cannot have a strategy to reach $\mathcal{X}^c$ in B_p. Thus, the fact that $\mathcal{X}$ has the adversarial Ramsey property gives a genuine dichotomy between two disjoint and strongly hereditary classes of subspaces.

We endow the set X with the discrete topology and the set X^ω with the product topology. The main result of this section is the following:

Theorem II.4 (Adversarial Ramsey principle, abstract version)**.** *Every Borel subset of X^ω is adversarially Ramsey.*

In the case of the Rosendal Space, the adversarial Gowers games defined here are exactly the same as those defined in the introduction. Thus, theorem II.4 applied to this space provides a positive answer to question I.16.

Also remark that if $P = \{\mathbb{1}\}$ and if we have $s \lhd \mathbb{1}$ for every $s \in \mathrm{Seq}(X)$, then both $A_{\mathbb{1}}$ and $B_{\mathbb{1}}$ are the classical Gale-Stewart game in X, so the adversarially Ramsey subsets of X^ω are exactly the determined ones. So in this space, theorem II.4 is nothing more than Borel determinacy for games on integers; hence, we get that theorem II.4 has at least the metamathematical strength of Borel determinacy for games on integers. Therefore, by the work of Friedman [20], any proof of theorem II.4 should make use of the powerset axiom and of the replacement scheme. We also get that it is not provable in ZFC that every analytic (or coanalytic) set in every Gowers space is adversarially Ramsey. Actually, it turns out that there is a large class of Gowers spaces for which Borel determinacy can be recovered from the version of theorem II.4 in these spaces; this will be shown in section II.3.

We will deduce theorem II.4 from Borel determinacy for games on real numbers. For this purpose, we follow an approach firstly used by Kastanas in [31]: in this paper Kastanas deduced the Ramsey property for subsets of $[\omega]^\omega$ from the determinacy of a game. In what follows, we adapt Kastanas' game in order to get the adversarial Ramsey property.

Definition II.5. For $p \in P$, *Kastanas' game K_p* below p is defined as follows:

I		x_0, q_0		x_1, q_1		$\ldots$	
II	p_0		y_0, p_1		y_1, p_2		$\ldots$

where the x_i's and the y_i's are elements of X, and the p_i's and the q_i's are elements of P. The rules are the following:

- for **I**: for all $i \in \omega$, $(x_0, y_0, \ldots, x_{i-1}, y_{i-1}, x_i) \lhd p_i$ and $q_i \leqslant p_i$;
- for **II**: $p_0 \leqslant p$, and for all $i \in \omega$, $(x_0, y_0 \ldots, x_i, y_i) \lhd q_i$ and $p_{i+1} \leqslant q_i$.

The outcome of the game is the sequence $(x_0, y_0, x_1, y_1, \ldots) \in X^\omega$.

The exact result we will show is the following:

Proposition II.6. *Let $p \in P$ and $\mathcal{X} \subseteq X^\omega$.*

1. *If **I** has a strategy to reach $\mathcal{X}$ in K_p, then there exists $q \leqslant p$ such that **I** has a strategy to reach $\mathcal{X}$ in A_q;*
2. *If **II** has a strategy to reach $\mathcal{X}^c$ in K_p, then there exists $q \leqslant p$ such that **II** has a strategy to reach $\mathcal{X}^c$ in B_q.*

Once this proposition is proved, theorem II.4 will immediately follow from the Borel determinacy of Kastanas' game.

Since the proof of *1.* and *2.* of proposition II.6 are exactly the same, we only prove *2.*. In order to do this, let us introduce some notation. During the whole proof, we fix a strategy τ for **II** in K_p to reach $\mathcal{X}^c$. A partial play of K_p ending with a move of **II** and during which **II** always plays according to her strategy will be called a *state*. We say that a state $\mathcal{s}$ *realises* a finite sequence $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ if $\mathcal{s}$ has the form $(p_0, x_0, ..., q_{n-1}, y_{n-1}, p_n)$; we say that a state realising a sequence of length $2n$ *has rank* n. We define in the same way the notions of a *total state* (which is a total play of K_p) and of realisation for a total state; the restriction of a total state $\mathcal{s} = (p_0, x_0, q_0, y_0, p_1, ...)$ to a state of rank n, denoted by $\mathcal{s}_{\upharpoonright n}$, is the state $(p_0, x_0, ..., q_{n-1}, y_{n-1}, p_n)$. If an infinite sequence $(x_0, y_0, x_1, y_1, \ldots)$ is realised by a total state, then this sequence belongs to $\mathcal{X}^c$.

We will use the following lemma:

Lemma II.7. *Let $\mathcal{S}$ be an at most countable set of states, and $r \in P$. Then there exists $r^* \leqslant r$ satisfying the following property: for all $\mathcal{s} \in \mathcal{S}$ and $x, y \in X$ if there exists $u, v \in P$ such that:*

1. ***I** can legally continue the play $\mathcal{s}$ by the move (x, u);*
2. *$\tau(\mathcal{s} \frown (x, u)) = (y, v)$;*
3. *v and r^* are compatible;*

then there exists $u', v' \in P$ satisfying 1., 2., and 3. and such that, moreover, we have $r^* \leqslant^* v'$.

Proof. Let $(\mathcal{s}_n, x_n, y_n)_{n\in\omega}$ be a (non-necessarily injective) enumeration of $\mathcal{S} \times X^2$. Define $(r_n)_{n\in\omega}$ a decreasing sequence of elements of P in the following way. Let $r_0 = r$. For $n \in \omega$, suppose r_n defined. If there exists a pair $(u, v) \in P^2$ such that:

- **I** can legally continue the play $\mathcal{s}_n$ by the move (x_n, u);
- $\tau(\mathcal{s}_n \frown (x_n, u)) = (y_n, v)$;
- v and r_n are compatible;

then choose (u_n, v_n) such a pair and let r_{n+1} be a common lower bound to r_n and v_n. Otherwise, let $r_{n+1} = r_n$. This achieves the construction.

By the definition of a Gowers space, there exists $r^* \in P$ such that $r^* \leqslant r$ and for all $n \in \omega$, $r^* \leqslant^* r_n$. We show that r^* is as required. Let $n \in \omega$, and suppose that there exists $(u, v) \in P^2$ satisfying properties *1.*, *2.*, and *3.* as in the statement of the lemma for the triple $(\mathcal{s}_n, x_n, y_n)$. Since $r^* \leqslant^* r_n$ and since v and r^* are compatible, then v and r_n are also compatible. This show that the pair (u_n, v_n) has been defined; by construction, this pair satisfies properties *1.* and *2.* for $(\mathcal{s}_n, x_n, y_n)$, and we have $r_{n+1} \leqslant v_n$, so $r^* \leqslant^* v_n$, which shows that $(u', v') = (u_n, v_n)$ is as required.

□

Proof of proposition II.6. Define $(q_n)_{n\in\omega}$ a decreasing sequence of elements of P and $(\mathcal{S}_n)_{n\in\omega}$ a sequence where, for every $n \in \omega$, $\mathcal{S}_n$ is an at most countable set of states of rank n, in the following way. Let $q_0 = \tau(\varnothing)$ and $\mathcal{S}_0 = \{(\tau(\varnothing))\}$. For $n \in \omega$, suppose q_n and $\mathcal{S}_n$ being defined. Let q_{n+1} be the result of the application of lemma II.7 to q_n and the set of states $\mathcal{S}_n$. For $\mathscr{s} \in \mathcal{S}_n$, let $A_{\mathscr{s}}$ be the set of all pairs (x, y) such that there exists $(u, v) \in P^2$ satisfying:

1. **I** can legally continue the play $\mathscr{s}$ by the move (x, u);
2. $\tau(\mathscr{s} ^\frown (x, u)) = (y, v)$;
3. v and q_{n+1} are compatible.

Then by construction of q_{n+1}, for all $(x, y) \in A_{\mathscr{s}}$, there exists a pair $(u, v) \in P^2$ satisfying 1., 2., and 3., and such that moreover $q_{n+1} \leqslant^* v$. For each $(x, y) \in A_{\mathscr{s}}$, choose $(u_{\mathscr{s},x,y}, v_{\mathscr{s},x,y})$ such a pair. Let $\mathcal{S}_{n+1} = \{\mathscr{s} ^\frown (x, u_{\mathscr{s},x,y}, y, v_{\mathscr{s},x,y}) \mid \mathscr{s} \in \mathcal{S}_n,\ (x, y) \in A_{\mathscr{s}}\}$; this is clearly a countable set of states of rank $n + 1$. This achieves the construction.

Now let $q \in P$ be such that $q \leqslant q_0$ and for all $n \in \omega$, we have $q \leqslant^* q_n$. Remark that since $q_0 \leqslant p$, we have $q \leqslant p$. We show that q is as required, by describing a strategy for **II** in B_q to reach $\mathcal{X}^c$. In order to do this, we simulate the play $\mathscr{s} = (v_0, x_0, u_0, y_0, v_1, ...)$ of B_q that **I** and **II** are playing by a play $\mathscr{s}' = (v'_0, x_0, u'_0, y_0, v'_1, ...)$ of K_p having the same outcome and during which **II** always plays according to her strategy τ. This will ensure that the outcome $(x_0, y_0, x_1, y_1, \ldots)$ of both games lies in $\mathcal{X}^c$ and so that the strategy for **II** in B_q that we described enables her to reach her goal. We do this construction in such a way that at each turn n, the following conditions are kept satisfied:

(a) $\mathscr{s}'_{\restriction n} \in \mathcal{S}_n$;

(b) $v_n \leqslant v'_n$.

The moves of the players at the $(n + 1)^{\text{th}}$ turn in both games that are described in the following proof are represented in the diagrams below. The third diagram, called "Fictive K_p", represents a fictive situation that will be studied for technical reasons in the proof, and in which the moves of both players are the same as in K_p until the n^{th} turn but differ from the $(n + 1)^{\text{th}}$ turn.

B_q	**I**	$\ldots$		$x_n,\, u_n$	
	II		$\ldots,\, v_n$		$y_n,\, v_{n+1}$

K_p	**I**	$\ldots$		$x_n,\, u'_n$	
	II		$\ldots,\, v'_n$		$y_n,\, v'_{n+1}$

Fictive K_p	**I**	$\ldots$		$x_n,\, u''_n$	
	II		$\ldots,\, v'_n$		$y_n,\, v''_{n+1}$

Let us describe the strategy of **II** in B_q. At the first turn, this strategy will consist in playing $v_0 = q$; and, according to her strategy τ, **II** will play $v'_0 = \tau(\varnothing)$ in K_p. Now, suppose that both games have been played until the n^{th} turn, that is, the last moves of player **II** in the games B_q and K_p are respectively v_n and v'_n. Player **I** plays (x_n, u_n) in B_q. By the rules of the game B_q and the induction hypothesis, we have that $u_n \leqslant q \leqslant^* v_n \leqslant v'_n$; so there exists $u''_n \in P$ such that $u''_n \lessapprox u_n$ and $u''_n \leqslant v'_n$. We also have that $(x_0, y_0, \ldots, x_n) \lhd v_n \leqslant v'_n$, so it is legal for **I** to pursue the game K_p by playing (x_n, u''_n); this fictive situation is represented in the third diagram above, called "Fictive K_p". In this fictive situation, the strategy τ of **II** would lead her to answer with a move (y_n, v''_{n+1}) satisfying $(x_0, y_0, \ldots, x_n, y_n) \lhd u''_n$ and $v''_{n+1} \leqslant u''_n$. We have, by construction of q, that $v''_{n+1} \leqslant u''_n \leqslant u_n \leqslant q \leqslant^* q_{n+1}$; so in particular, v''_{n+1} and q_{n+1} are compatible. Recalling that $s\restriction_n \in \mathcal{S}_n$, we see that the pair (u''_n, v''_n) witnesses that $(x_n, y_n) \in A_{s\restriction_n}$.

Now let us leave the fictive situation and come back to the "real" K_p. Since $(x_n, y_n) \in A_{s\restriction_n}$, we know that the pair $(u_{s\restriction_n, x_n, y_n}, v_{s\restriction_n, x_n, y_n})$ has been defined; we denote this pair by (u'_n, v'_{n+1}). In the "real" K_p, we make **I** play (x_n, u'_n). By definition of (u'_n, v'_{n+1}), this move is legal, and **II** will answer, according to her strategy, with (y_n, v'_{n+1}). Remark that the required condition (a) in the induction hypothesis is satisfied by these moves since, by the definition of $\mathcal{S}_{n+1}$, we have $s\restriction_n \frown (x_n, u'_n, y_n, v'_{n+1}) \in \mathcal{S}_{n+1}$. We also have that $q \leqslant^* q_{n+1} \leqslant^* v'_{n+1}$, so there exists $v_{n+1} \in P$ such that $v_{n+1} \leqslant v'_{n+1}$ and $v_{n+1} \lessapprox q$. For this reason, and since we also have (as we already saw) $(x_0, y_0, \ldots, x_n, y_n) \lhd u''_n \leqslant u_n$, we get that (y_n, v_{n+1}) is a legal move for **II** in B_q, that satisfies the condition (b) in the induction hypothesis. So we just have to define her strategy as leading her to play this move, and this achieves the proof.

$\square$

We actually proved a little more than theorem II.4. Say that the Gowers space $\mathcal{G}$ is *analytic* if P is an analytic subset of a Polish space and if the relations $\leqslant$ and $\lhd$ are Borel subsets of P^2 and of $\mathrm{Seq}(X) \times P$ respectively. For most of the spaces we actually use, P can be indentified to an analytic subset of $\mathcal{P}(X)$, the relation $\leqslant$ to the inclusion, and the relation $(x_0, \ldots, x_n) \lhd p$ to the membership relation $x_n \in p$; thus, these spaces are analytic. This is, for instance, the case for the Mathias–Silver space and the Rosendal space introduced at the beginning of this section. Then an easy consequence of proposition II.6 is the following:

Corollary II.8. *Let Γ be a suitable class of subsets of Polish spaces. If every Γ-subset of $\mathbb{R}^\omega$ is determined, then for an analytic Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$, every Γ-subset of X^ω is adversarially Ramsey.*

Proof. Fix $\mathcal{X} \subseteq X^\omega$ a Γ-subset, and $p \in P$. By proposition II.6, it is enough to show that in the game K_p, either player **I** has a strategy to reach $\mathcal{X}$, or player **II** has a strategy to reach $\mathcal{X}^c$. Let $\varphi : \mathbb{R} \longrightarrow P$ be a surjective Borel mapping, and consider the following game K'_p:

I		$x_0, \widetilde{q}_0$		$x_1, \widetilde{q}_1$		$\ldots$	
II	$\widetilde{p}_0$		$y_0, \widetilde{p}_1$		$y_1, \widetilde{p}_2$		$\ldots$

where the x_i's and the y_i's are elements of X and the $\widetilde{p}_i$'s and the $\widetilde{q}_i$'s are real numbers, with the constraint that $\varphi(\widetilde{p}_0) \leqslant p$, for all $i \in \omega$, $\varphi(\widetilde{q}_i) \leqslant \varphi(\widetilde{p}_i)$, $\varphi(\widetilde{p_{i+1}}) \leqslant \varphi(\widetilde{q}_i)$, $(x_0, y_0, \ldots, x_i) \lhd \varphi(\widetilde{p}_i)$, and $(x_0, y_0, \ldots, x_i, y_i) \lhd \varphi(\widetilde{q}_i)$, and whose outcome is the sequence $(x_0, y_0, x_1, y_1, \ldots) \in X^\omega$. This game is clearly equivalent to K_p: **I** has a strategy to reach $\mathcal{X}$ in K_p if and only if he has one in K'_p, and **II** has a strategy to reach $\mathcal{X}^c$ in K_p if and only if she has one in K'_p. Since K'_p is a game on real numbers with Borel rules and since $\mathcal{X}$ is in Γ, we deduce that in this game, either **I** has a strategy to reach $\mathcal{X}$, or **II** has a strategy to reach $\mathcal{X}^c$, what concludes the proof.

□

Corollary II.8 shows in particular that, in an analytic Gowers space, under $PD_{\mathbb{R}}$, every projective set is adversarially Ramsey. Recall that Harrington and Kechris [27], and independently Woodin [64] proved that under PD, every projective subset of $[\omega]^\omega$ is Ramsey. Using ideas from Woodin's proof, Bagaria and López-Abad [8] showed that under PD, every projective set of block sequences of a basis of a Banach space is strategically Ramsey (i.e. satisfies the conclusion of Gowers' theorem I.8). Basing ourselve on these facts, we can formulate the following conjecture:

Conjecture II.9. *Under PD, if the Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ is analytic, then every projective subset of X^ω is adversarially Ramsey.*

Clearly, the method presented in the present paper does not enable to prove this.

Also remark that the proof of proposition II.6 can almost entierly be done in $ZF + DC$; the only use of the full axiom of choice is made to choose $u''_n \in P$ such that $u''_n \lessapprox u_n$ and $u''_n \leqslant v'_n$, and $v_{n+1} \in P$ such that $v_{n+1} \leqslant v'_{n+1}$ and $v_{n+1} \lessapprox q$, so actually to apply axiom 2. in the definition of a Gowers space. For this reason, say that the Gowers space $\mathcal{G}$ is *effective* if in this axiom 2., the subspace r can be chosen in an effective way, that is, if there exist a function $f : P^2 \longrightarrow P$ such that for every $p, q \in P$, if $p \leqslant^* q$, then we have $f(p, q) \lessapprox p$ and $f(p, q) \leqslant q$. For instance:

- The Mathias–Silver space is effective: indeed, if $M \subseteq^* N$, then we can take $f(M, N) = M \cap N$.
- The Rosendal space is effective. Indeed, if X and Y are block subspaces such that $X \subseteq^* Y$, let $(x_n)_{n\in\omega}$ be a block sequence spanning X. Then we can let $f(X, Y)$ be the subspace spanned by the largest final segment of (x_n) all of whose terms are in Y (this subspace does not depend on the choice of (x_n)).

To prove proposition II.6 for an effective Gowers space, we only need dependant choices. Thus, we have the following result:

Corollary II.10 ($ZF + DC + AD_{\mathbb{R}}$)**.** *Let $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ be an effective Gowers space such that P is a subset of a Polish space. Then every subset of X^ω is adversarially Ramsey.*

Proof. Recall that in $ZF + DC + AD$, every subset of a Polish space is either at most countable, or contains a Cantor set, and is thus in bijection with $\mathbb{R}$ (this is a consequence of theorem 21.1 in [32], that can be proved in $ZF + DC$). So if P is countable, then Kastanas' game can be viewed as a game on integers and is thus determined, and if P is uncountable, then Kastanas' game can be viewed as a game on real numbers, that is also determined. The conclusion follows from proposition II.6.

□

As above, we cannot prove in this way that the same result holds under AD instead of $AD_{\mathbb{R}}$, but we conjecture that it does so. As we will see in the next section, if this is true, this would imply that under AD, every subset of $[\omega]^\omega$ is Ramsey, which is still conjectural today.

Since for sufficiently regular Gowers spaces (analytic ones, or effective ones with P being subset of a Polish space, depending on the case), we only need the determinacy of Γ-subsets of $\mathbb{R}^\omega$ to prove the adversarial Ramsey property for Γ-sets, and since from this property in every sufficiently regular space, we can deduce the determinacy of Γ-subsets of ω^ω, another interesting question is the following:

Question II.11. *Where does the adversarial Ramsey property for Γ-sets in sufficiently regular Gowers spaces lie between the determinacy of Γ-subsets of ω^ω and the determinacy of Γ-subsets of $\mathbb{R}^\omega$?*

This question can be asked both in terms of implication and of consistency strength. In particular, we don't know whether there exists an analytic Gowers space $\mathcal{G}$ and a suitable class Γ of subsets of Polish spaces such that ZFC doesn't prove that the determinacy of Γ-subsets of ω^ω implies the adversarial Ramsey property for Γ-sets in $\mathcal{G}$, neither if there exists some such that the consistency strength of $ZFC+$ *"Every Γ-set in $\mathcal{G}$ is adversarially Ramsey"* is strictly above the consistency strength of $ZFC+$ *"Every Γ-subset of ω^ω is determined"*.

II.2 Strategically Ramsey sets and the pigeonhole principle

The aim of this section is to prove a version of Rosendal's theorem I.13 in the general setting of Gowers spaces. We also introduce the notion of the *pigeonhole principle* for a Gowers space and see that the last result can be strengthened in the case where this principle holds. This will enable us to see the fundamental difference between the Mathias–Silver space and the Rosendal space over a field with at least three elements. We start by introducing Gowers' game and the asymptotic game in the setting of Gowers spaces, and the notion of a strategically Ramsey set. In this whole section, we fix a Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$.

Definition II.12. Let $p \in P$.

1. *Gowers' game below* p, denoted by G_p, is defined in the following way:

$$\begin{array}{lcccccc} \mathbf{I} & p_0 & & p_1 & & \dots & \\ \mathbf{II} & & x_0 & & x_1 & & \dots \end{array}$$

   where the x_i's are elements of X, and the p_i's are elements of P. The rules are the following:

   - for **I**: for all $i \in \omega$, $p_i \leqslant p$;
   - for **II**: for all $i \in \omega$, $(x_0, \dots, x_i) \lhd p_i$.

   The outcome of the game is the sequence $(x_i)_{i\in\omega} \in X^\omega$.

2. The *asymptotic game below* p, denoted by F_p, is defined in the same way as G_p, except that this time the we moreover require that $p_i \lessapprox p$.

Definition II.13. A set $\mathcal{X} \subseteq X^\omega$ is said to be *strategically Ramsey* if for every $p \in P$, there exists $q \leqslant p$ such that either player **I** has a strategy to reach $\mathcal{X}^c$ in F_q, or player **II** has a strategy to reach $\mathcal{X}$ in G_q.

The general version of Rosendal's theorem I.13 is then the following:

Theorem II.14 (Abstract Rosendal's theorem)**.** *Every analytic subset of X^ω is strategically Ramsey.*

Remark that theorem I.13 is exactly the result of the application of theorem II.14 to the Rosendal space.

Proof. We firstly prove the result for Borel sets. In order to do this, consider another space $\widetilde{\mathcal{G}} = (P, X, \leqslant, \leqslant^*, \widetilde{\lhd})$, where P, X, $\leqslant$, and $\leqslant^*$ are the same as in $\mathcal{G}$, but we replace $\lhd$ by the relation $\widetilde{\lhd}$ defined by $(x_0, y_0, x_1, y_1, \dots, x_n, y_n) \mathrel{\widetilde{\lhd}} p$ iff $(y_0, y_1, \dots, y_n) \lhd p$, and $(x_0, y_0, x_1, y_1, \dots, x_n) \mathrel{\widetilde{\lhd}} p$ iff $(x_0, x_1, \dots, x_n) \lhd p$. Now, to each set $\mathcal{X} \subseteq X^\omega$, associate a set $\widetilde{\mathcal{X}} \subseteq X^\omega$ defined by $(x_0, y_0, x_1, y_1, \dots) \in \widetilde{\mathcal{X}} \Leftrightarrow (y_0, y_1, \dots) \in \mathcal{X}$. Then, when players try to reach $\widetilde{\mathcal{X}}$ or $\widetilde{\mathcal{X}}^c$ in the games A_p and B_p of $\widetilde{\mathcal{G}}$, the p_i's played by **II** and the x_i's played by **I** don't matter at all; so a strategy for **I** in the game A_p of $\widetilde{\mathcal{G}}$ to reach $\widetilde{\mathcal{X}}^c$ becomes a strategy for **I** in the game F_p of $\mathcal{G}$ to reach $\mathcal{X}^c$, and a strategy for **II** in the game B_p of $\widetilde{\mathcal{G}}$ to reach $\widetilde{\mathcal{X}}$ becomes a strategy for **II** in the game G_p of $\mathcal{G}$ to reach $\mathcal{X}$. Thus, the strategical Ramsey property for $\mathcal{X}$ in $\mathcal{G}$ is equivalent to the adversarial Ramsey property for $\widetilde{\mathcal{X}}^c$ in $\widetilde{\mathcal{G}}$, so the strategical Ramsey property for Borel sets in $\mathcal{G}$ follows from theorem II.4.

From the result for Borel sets, we now deduce the result for arbitrary analytic sets using an unfolding argument. Let $\mathcal{X} \subseteq X^\omega$ be analytic, and $p \in P$. Let $X' = X \times \{0,1\}$, whose elements will be denoted by the letters (x, ε). Define the binary relation $\lhd' \subseteq \mathrm{Seq}(X') \times P$ by $(x_0, \varepsilon_0, \ldots, x_n, \varepsilon_n) \lhd' p$ if $(x_0, \ldots, x_n) \lhd p$, and consider the Gowers space $\mathcal{G}' = (P, X', \leqslant, \leqslant^*, \lhd')$. In this proof, we will use the notations F_q and G_q to denote respectively the asymptotic game and Gowers' game in the space $\mathcal{G}$, whereas the notations F'_q and G'_q will be used for these games in the space $\mathcal{G}'$. We denote by π the projection $X'^\omega \longrightarrow X^\omega$. Let $\mathcal{X}' \subseteq X'^\omega$ be a G_δ set such that $\mathcal{X} = \pi(\mathcal{X}')$. Since $\mathcal{X}'$ is $\mathcal{G}_\delta$, it is strategically Ramsey; let $q \leqslant p$ witnessing so. If player **II** has a strategy in G'_q to reach $\mathcal{X}'$, then a run of the game G_q where **II** uses this strategy but omits to display the ε_i's produces an outcome lying in $\mathcal{X}$; hence, **II** has a strategy to reach $\mathcal{X}$ in G_q. Then, our result will follow from the following fact:

Fact II.15. *If **I** has a strategy to reach $\mathcal{X}'^c$ in F'_q, then he has a strategy to reach $\mathcal{X}^c$ in F_q.*

Proof. Let τ' be a strategy enabling **I** to reach $\mathcal{X}'^c$ in F'_q. In order to save notation, in this proof, we consider that in the games F'_q and F_q, player **II** is allowed not to respect the rules (i.e. to play x_i's such that $(x_0, \ldots, x_i) \ntriangleleft p_i$), but loses the game if she does. Then, the strategy τ' can be viewed as a mapping $X'^{<\omega} \longrightarrow P$ such that for every $(x_0, \varepsilon_0, \ldots, x_{n-1}, \varepsilon_{n-1}) \in X'^{<\omega}$, we have $\tau'(x_0, \varepsilon_0, \ldots, x_{n-1}, \varepsilon_{n-1}) \lessapprox q$. Remark that if $(p_j)_{j \in J}$ is a finite family of elements of P such that $\forall j \in J,\ p_j \lessapprox q$, then by applying iteratively the property 2. in the definition of a Gowers space, we can get $p^* \in P$ such that $p^* \lessapprox q$ and $\forall j \in J\ p^* \leqslant p_j$. Thus, for every $(x_0, \ldots, x_{n-1}) \in X^{<\omega}$, we can choose $\tau(x_0, \ldots, x_{n-1}) \in P$ such that $\tau(x_0, \ldots, x_{n-1}) \lessapprox q$ and such that for every $(\varepsilon_0, \ldots, \varepsilon_{n-1}) \in \{0,1\}^n$, we have $\tau(x_0, \ldots, x_{n-1}) \leqslant \tau'(x_0, \varepsilon_0 \ldots, x_{n-1}, \varepsilon_{n-1})$. We have hence defined a mapping $\tau : X^{<\omega} \longrightarrow P$; we show that this is a strategy for **I** in F_q enabling him to reach $\mathcal{X}^c$.

Consider a run of the game F_q during which **II** respects the rules and **I** plays according to his strategy τ:

I	p_0		p_1		$\ldots$	
II		x_0		x_1		$\ldots$

We have to show that $(x_i)_{i\in\omega} \notin \mathcal{X}$, that is, for every $(\varepsilon_i)_{i\in\omega} \in \{0,1\}^\omega$, $(x_i, \varepsilon_i)_{i\in\omega} \notin \mathcal{X}'$. Let $(\varepsilon_i)_{i\in\omega} \in \{0,1\}^\omega$; it is enough to show that $(x_i, \varepsilon_i)_{i\in\omega}$ is the outcome of a run of the game F'_q during which **I** always follows his strategy τ' and **II** always respects the rules.

Letting $p'_i = \tau'(x_0, \varepsilon_0, \ldots, x_{n-1}, \varepsilon_{n-1})$, this means that during the following run of the game F'_q, player **II** always respects the rules:

I	p'_0		p'_1		$\ldots$	
II		x_0, ε_0		x_1, ε_1		$\ldots$

But for every $i \in \omega$, we have that $p_i = \tau(x_0, \ldots, x_{n-1})$ and $p'_i = \tau'(x_0, \varepsilon_0, \ldots, x_{n-1}, \varepsilon_{n-1})$, so by definition of τ, we have $p_i \leqslant p'_i$. Since player **II** respects the rules in F_q, we have that $(x_0, \ldots, x_i) \lhd p_i$, so $(x_0, \varepsilon_0, \ldots, x_i, \varepsilon_i) \lhd p'_i$, and **II** also respects the rules in F'_q. This concludes the proof.

□

□

Remark that in the proof of theorem II.14, we only need theorem II.4 for G_δ sets, and hence determinacy for G_δ games. Hence, unlike theorem II.4 in its generality, the last result is provable in ZC. Actually, as previously, for effective Gowers spaces, it is even provable in $Z + DC$.

Again, we actually proved a little more. Indeed, the proof of theorem II.14, combined with corollaries II.8 and II.10, actually shows the following:

Corollary II.16.

1. *Let Γ be a suitable class of subsets of Polish spaces. If every Γ-subset of $\mathbb{R}^\omega$ is determined, then for an analytic Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$, every $\exists\Gamma$-subset of X^ω is strategically Ramsey.*

2. *($ZF + DC + AD_{\mathbb{R}}$) Let $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ be an effective Gowers space such that P is a subset of a Polish space. Then every subset of X^ω is strategically Ramsey.*

The rest of this section aims at explaining how we can, in certain cases, get symmetrical Ramsey results like Mathias–Silver's theorem from theorem II.14, which is asymmetrical. By *asymmetrical*, we mean here that unlike Mathias–Silver's theorem, in theorem II.14, both possible conclusion don't have the same form. Actually, one of these conclusions is stronger than the other (and, as it will turn out later, *strictly* stronger in general), as it is shown by the following lemma.

Lemma II.17. *Let $\mathcal{X} \subseteq X^\omega$ and $p \in P$. Suppose that **I** has a strategy in F_p to reach $\mathcal{X}$. Then **II** has a strategy in G_p to reach $\mathcal{X}$.*

Proof. Fix τ a strategy enabling **I** to reach $\mathcal{X}$ in F_p. We describe a strategy for **II** in G_p by simulating a play $(q_0, x_0, q_1, x_1, \ldots)$ of G_p by a play $(p_0, x_0, p_1, x_1, \ldots)$ of F_p having the same outcome and during which **I** always plays according to τ; this will ensure that $(x_0, x_1, \ldots) \in \mathcal{X}$ and that this play of G_p will be winning for **II**.

Suppose that the first n turns of both games have been played, which means that the p_i's, the q_i's and the x_i's have been choosen for every $i < n$. For the next turn, in G_p, player **I** plays $q_n \leqslant p$, and in F_p, the strategy τ tells **I** to play $p_n \lessapprox p$. Then $q_n \leqslant^* p_n$, so by axiom 2. in the definition of a Gowers space, there exists $r_n \in P$ such that $r_n \leqslant p_n$ and $r_n \leqslant q_n$. Let $x_n \in X$ such that $(x_0, \ldots, x_n) \lhd r_n$ (existing by axiom 4.). Then x_n can be legally played by **II** in both F_p and G_p, what concludes the proof.

□

Actually, the fact that **I** has a strategy in F_p to reach some set $\mathcal{X}$ is in general much stronger than the fact, for **II**, to have a strategy in G_p to reach the same set, and the first statement is in fact very close to a "genuine" Ramsey statement. By a "genuine" Ramsey statement, we mean a non-game-theoretical statement of the form "every sequence $(x_n)_{n\in\omega}$ such that $\forall n \in \omega\ (x_0, \ldots, x_n) \lhd p$, and moreover satisfiying some structural condition, belongs to $\mathcal{X}$"; this is, for example, the form of both possible conclusions of Mathias–Silver's theorem (that have the form "every infinite subset of N belongs to $\mathcal{X}$"; here, we identify infinite sets of integers with strictly increasing sequences of integers, the fact of being "strictly increasing" being in this case the structural condition mentionned above). In the case of the Mathias–Silver space, the link between the existence of a strategy for **I** in the asymptotic game and a genuine Ramsey statement is given by the following lemma:

Lemma II.18. *Work in the Mathias–Silver space, and let $\mathcal{X} \subseteq \omega^\omega$. Suppose that, for some $M \in [\omega]^\omega$, player **I** has a strategy in F_M to reach $\mathcal{X}$. Then there exists an infinite $N \subseteq M$ such that every infinite $S \subseteq N$ belongs to $\mathcal{X}$ (here, we identify infinite subsets of ω with increasing sequences of integers).*

Obviously, a weak converse of this lemma holds: if every infinite $S \subseteq M$ belongs to $\mathcal{X}$, then **I** has a strategy in F_M to reach $\mathcal{X}$. Indeed, he can always ensure that the outcome of this game is an increasing sequence.

Proof of lemma II.18. Without loss of generality, assume $M = \omega$. As in the proof of fact II.15, consider that in F_ω, player **II** is allowed to play against the rules, but loses if she does. Let τ be a strategy for player **I** in F_ω, enabling him to reach $\mathcal{X}$; in this context, this strategy can be viewed as a mapping associating to each finite sequence of integers a cofinite subset of ω. Without loss of generality, we can assume that these cofinite subsets are final segments of ω; for $s \in \omega^{<\omega}$, let $\tau_0(s) = \min \tau(s)$. Now define, by induction, a strictly increasing sequence $(n_i)_{i\in\omega}$ of integers in the following way: let $n_0 = \tau_0(\varnothing)$, and for $i \in \omega$, let n_{i+1} be the maximum of $n_i + 1$ and of the $\tau_0(n_{i_0}, \ldots, n_{i_{k-1}})$'s for $k \in \omega$ and $0 \leqslant i_0 < \ldots < i_{k-1} = i$. Let $N = \{n_i \mid i \in \omega\}$; then N is as required. Indeed, an infinite subset of N has the form $\{n_{i_k} \mid k \in \omega\}$ for a strictly increasing sequence of integers $(i_k)_{k\in\omega}$. To prove that $(n_{i_k})_{k\in\omega} \in \mathcal{X}$, it is enough to prove this sequence is the outcome of some legal run of the game F_ω during which player **I** always plays according to the strategy τ. In other words, letting, for all $k \in \omega$, $P_k = \tau(n_{i_0}, \ldots, n_{i_{k-1}})$, we have to show that during the following run of the game F_ω, player **II** always respects the rules:

I	P_0		P_1		$\ldots$
II		n_{i_0}		n_{i_1}	$\ldots$

But by construction, we have that $n_{i_0} \geqslant n_0 = \tau_0(\varnothing) = \min P_0$, and for $k \geqslant 1$, $n_{i_k} \geqslant n_{i_{k-1}+1} \geqslant \tau_0(n_{i_0}, \ldots, n_{i_{k-1}}) = \min P_k$, which concludes the proof.

□

The setting of Gowers spaces does not give enough structure to get such a result in general. A general version of this result will be given in section III.3, in the setting of *approximate asymptotic spaces* with some additional structure; and, in a very different way, the setting of *Ramsey spaces* presented in [61] is also convenient to get non game-theoretical infinite-dimensional Ramsey results.

In the setting of Gowers spaces, however, the best kinds of conclusions we can get in general are those involving strategies for **I** in the asymptotic game. As, in the case of the Mathias–Silver space, we are able to get an alternative both side of whose are "genuine" Ramsey statements, it would be tempting to wonder whether, for Gowers spaces satisfying some additional property, it would be possible to get an alternative involving a strategy for player **I** in the asymptotic game in both sides. It turns out that such a property exists, called the *pigeonhole principle*.

In the rest of this paper, we denote by $q \subseteq_s A$, for $q \in P$, $s \in X^{<\omega}$ and $A \subseteq X$, the fact that for every $x \in X$ such that $s^\frown x \lhd q$, we have $x \in A$. This notation could sound strange, however, in spaces where $P \subseteq \mathcal{P}(X)$ and $(x_0, \ldots, x_n) \lhd q \Leftrightarrow x_n \in q$, we have that $q \subseteq_s A$ iff $q \subseteq A$. Let us introduce the pigeonhole principle.

Definition II.19. The Gowers space $\mathcal{G}$ is said to satisfy the *pigeonhole principle* if for every $p \in P$, $s \in X^{<\omega}$ and $A \subseteq X$, there exists $q \leqslant p$ such that either $q \subseteq_s A$, or $q \subseteq_s A^c$.

The pigeonhole principle holds in the Mathias–Silver space: there, it is the trivial fact that every subset of an infinite set is either infinite, or has infinite complement. It also holds in the Rosendal space over the field $\mathbb{F}_2$: this is Hindman's theorem I.6. However, it does not hold in the Rosendal space over K, for $K \neq \mathbb{F}_2$: to see this, take for example for A the set of all vectors whose first nonzero coordinate is 1. Note that apart from this trivial obstruction, the pigeonhole principle does not hold in the Rosendal space for much more intrinsic reasons. Indeed, consider the *projective Rosendal space*, i.e. the forgetful Gowers space $\mathcal{PR}_K = (P, \mathbb{P}(E), \subseteq, \subseteq^*, \subseteq)$, where $\mathbb{P}(E)$ is a countably infinite-dimensional projective space over the field K (that is, the set of vector lines of some countably infinite-dimensional K-vector space E), P is the set of block subspaces of E relative to a fixed basis $(e_i)_{i\in\omega}$ of E, $\subseteq^*$ is the inclusion up to finite codimension as in the definition of the Rosendal space, and where since the space is forgetful, the relation usually denoted by $\lhd$ is viewed as a relation between points and subspaces, here the inclusion. The definition of this space is made to avoid the previous obstruction to the pigeonhole principle and other possible ones of the same kind. However, for $K \neq \mathbb{F}_2$, the pigeonhole principle still does not hold in $\mathcal{PR}_K$: take for example for A the set of all vector lines Kx, where the first and the last non-zero coordinates of x are equal.

Under the pigeonhole principle, we will show a weak converse to lemma II.17:

Proposition II.20. *Suppose that the Gowers space $\mathcal{G}$ satisfies the pigeonhole principle. Let $\mathcal{X} \subseteq X^\omega$ and $p \in P$. If player **II** has a strategy in G_p to reach $\mathcal{X}$, then there exists $q \leqslant p$ such that **I** has a strategy in F_q to reach $\mathcal{X}$.*

Before proving this proposition, let us make some remarks. Firstly, proposition II.20 immediately implies the following corollary:

Corollary II.21. *Suppose that the Gowers space $\mathcal{G}$ satisfies the pigeonhole principle. Let $\mathcal{X} \subseteq X^\omega$ be a strategically Ramsey set. Then for all $p \in P$, there exists $q \leqslant p$ such that in F_q, player **I** has a strategy either to reach $\mathcal{X}$, or to reach $\mathcal{X}^c$.*

This corollary has some kind of converse. Indeed, for every $s \in X^{<\omega}$, consider the Gowers space $\mathcal{G}^s = (P, X, \leqslant, \leqslant^*, \lhd^s)$, where P, X, $\leqslant$ and $\leqslant^*$ are the same as in $\mathcal{G}$ and where $t \lhd^s p \Leftrightarrow s \frown t \lhd p$. Then if $\mathcal{G}$ satisfies the pigeonhole principle, all of the $\mathcal{G}^s$'s do so, so strategically Ramsey sets in these spaces satisfy the conclusion of the last corollary. Remark that conversely, if the conclusion of this corollary holds for sets of the form $\{(x_n)_{n\in\omega} \mid x_0 \in A\}$ (where $A \subseteq X$), in the space $\mathcal{G}^s$ for every s, then $\mathcal{G}$ satisfies the pigeonhole principle. Indeed, let $p \in P$, $s \in X^{<\omega}$, and $A \subseteq X$. Consider the set $\mathcal{X} = \{(x_n)_{n\in\omega} \in X^\omega \mid x_0 \in A\}$. By assumption, there exists $q \leqslant p$ such that in the space $\mathcal{G}^s$, either **I** has a strategy in F_q to reach $\mathcal{X}$, or he has one to reach $\mathcal{X}^c$. In the first case, his strategy tells him, at the first turn of F_q, to play some $q_0 \lessapprox q$; then, whatever the answer $x_0 \lhd^s q_0$ of player **II** is, if player **I** continues to play according to his strategy, the outcome of the game will be some sequence $(x_0, x_1, \ldots)$ belonging to $\mathcal{X}$, what means that $x_0 \in A$; so $q_0 \subseteq_s A$. In the second case, we show in the same way that there exists $q_0 \lessapprox q$ such that $q_0 \subseteq_s A$, what concludes. Thus, the satisfaction of the conclusion of corollary II.21 for clopen sets in $\mathcal{G}^s$ for every $s \in X^{<\omega}$ is equivalent to the pigeonhole principle in $\mathcal{G}$. Remark that if $\mathcal{G}$ is a forgetful space, then for every $s \in X^{<\omega}$, we have $\mathcal{G}^s = \mathcal{G}$; so for such a space, the pigeonhle principle is actually equivalent to the fact that the conclusion of corollary II.21 holds for sets of the form $\{(x_n)_{n\in\omega} \mid x_0 \in A\}$.

Also remark that corollary II.21 applied to the Mathias–Silver space, combined with lemma II.18, gives that a set $\mathcal{X} \subseteq [\omega]^\omega$ is Ramsey (in the sense of Mathias–Silver's theorem) if and only if it is strategically Ramsey in the Mathias–Silver space (when seen as a subset of ω^ω). In particular, Mathias–Silver's theorem is a consequence of the abstract Rosendal's theorem II.14.

We now prove proposition II.20.

Proof of proposition II.20. Fix τ a strategy for **II** in G_p to reach $\mathcal{X}$. We call a *state* a partial play of G_p either empty or ending with a move of **II**, during which **II** always plays according to her strategy. We say that a state *realises* a sequence $(x_0, \ldots, x_{n-1}) \in X^{<\omega}$ if it has the form $(p_0, x_0, \ldots, p_{n-1}, x_{n-1})$. We define in the same way the notion of a *total state* (which is a total play of G_p) and of realisation for a total state; if an infinite sequence is realised by some total state, then it belongs to $\mathcal{X}$. We say that a point $x \in X$ is *reachable* from a state $\mathscr{s}$ if there exists $r \leqslant p$ such that $\tau(\mathscr{s} \frown r) = x$. Denote by $A_{\mathscr{s}}$ the set of all points that are reachable from the state $\mathscr{s}$. We will use the following fact.

Fact II.22. *For every state $\mathscr{s}$ realising a finite sequence s, and for every $q \leqslant p$, there exists $r \leqslant q$ such that $r \subseteq_s A_{\mathscr{s}}$.*

Proof. Otherwise, by the pigeonhole principle, there would exist $r \leqslant q$ such that $r \subseteq_s (A_{\mathscr{I}})^c$. But then **I** could play r after the partial play $\mathscr{I}$, and **II** would answer, according to her strategy, by $x = \tau(\mathscr{I}^\frown r)$ that should satisfy $s^\frown x \lhd r$. Since $r \subseteq_s (A_{\mathscr{I}})^c$, this would imply that $x \in (A_{\mathscr{I}})^c$. But we also have, by definition of $A_{\mathscr{I}}$, that $x \in A_{\mathscr{I}}$, a contradiction.

□

Now let $(s_n)_{n\in\omega}$ be an enumeration of $X^{<\omega}$ such that if $s_m \subseteq s_n$, then $m \leqslant n$. We define, for some $n \in \omega$, a state $\mathscr{I}_n$ realising s_n, by induction in the following way: $\mathscr{I}_0 = \varnothing$ and for $n \geqslant 1$, letting $s_n = s_m {}^\frown x$ for some $m < n$ and some $x \in X$,

- if $\mathscr{I}_m$ has been defined and if x is reachable from $\mathscr{I}_m$, then choose a $r \leqslant p$ such that $x = \tau(\mathscr{I}_m {}^\frown r)$ and put $\mathscr{I}_n = \mathscr{I}_m {}^\frown (r, x)$,
- otherwise, $\mathscr{I}_n$ is not defined.

Remark that if $\mathscr{I}_n$ is defined and if $s_m \subseteq s_n$, then $\mathscr{I}_m$ is defined and $\mathscr{I}_m \subseteq \mathscr{I}_n$.

We now define a $\leqslant$-decreasing sequence $(q_n)_{n\in\omega}$ of elements of P in the following way: $q_0 = p$ and

- if $\mathscr{I}_n$ is defined, then q_{n+1} is the result of the application of fact II.22 to $\mathscr{I}_n$ and q_n;
- $q_{n+1} = q_n$ otherwise.

Finally, let $q \leqslant p$ be such that for every $n \in \omega$, $q \leqslant^* q_n$. We will show that **I** has a strategy in F_q to reach $\mathcal{X}$. We describe this strategy on the following play of F_q:

I	u_0		u_1		$\dots$
II		x_0		x_1	$\dots$

We actually show that **I** can always play preserving the fact that, if $n_i \in \omega$ is such that $s_{n_i} = (x_0, \dots, x_{i-1})$, then $\mathscr{I}_{n_i}$ is defined. This will be enough to conclude: indeed, $\bigcup_{i\in\omega} \mathscr{I}_{n_i}$ will be a total state realising the sequence $(x_i)_{i\in\omega}$, showing that this sequence belongs to $\mathcal{X}$.

Suppose that the i^{th} turn of the play has just been played, so the sequence $s_{n_i} = (x_0, \dots, x_{i-1})$ has been defined, in such a way that $\mathscr{I}_{n_i}$ is defined. Then by construction of q_{n_i+1}, we have that $q_{n_i+1} \subseteq_{s_{n_i}} A_{\mathscr{I}_{n_i}}$. We let **I** play some u_i such that $u_i \lessapprox q$ and $u_i \leqslant q_{n_i+1}$. Then $u_i \subseteq_{s_{n_i}} A_{\mathscr{I}_{n_i}}$, so whatever is the x_i that **II** answers with, this x_i is reachable from $\mathscr{I}_{n_i}$. So if $s_{n_{i+1}} = s_{n_i} {}^\frown x_i$, then $\mathscr{I}_{n_{i+1}}$ has been defined, and the wanted property is preserved.

□

Remark that this proof can be done in $ZF + DC$, even if the space $\mathcal{G}$ is not supposed effective.

II.3 The strength of the adversarial Ramsey principle

In section II.1, we proved the adversarial Ramsey property for Borel sets using Borel determinacy, and we saw on the trivial example of the space with only one subspace that, given Γ a suitable class of subsets of Polish spaces, the adversarial Ramsey property for Γ-sets implied the determinacy for Γ-games on integers. This had two consequences: on one hand, the use of a sufficiently large fragment of ZFC is necessary to prove the adversarial Ramsey property for Borel sets, and on the other hand, it is not possible to prove it for analytic or coanalytic sets in ZFC. However, the space we used to make this remark is quite artificial. Of course, we made the same remark in the introduction of this thesis using the Rosendal space, however we did it by making players play according to the norms of the vectors, which is quite artificial too (we would not do that, for example, in the applications to Banach-space geometry, where we usually restrict our attention to normalized vectors). Therefore, is it natural to ask in which cases using a large fragment of ZFC is necessary to prove the adversarial Ramsey property for Borel sets, or in which cases this property could be provable in ZFC for analytic and coanalytic sets; the aim of this section is to give an answer to this question. We will see, in particular, that Gowers spaces where the pigeonhole principle holds, and those where it does not hold, behave very differently.

In this section and the next one, we fix Γ a suitable class of subsets of Polish spaces. Given a Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$, we denote by $\mathrm{Adv}_{\mathcal{G}}(\Gamma)$ the statement "every Γ-subset of X^ω is adversarially Ramsey", and by $\mathrm{Strat}_{\mathcal{G}}(\Gamma)$ the statement "every Γ-subset of X^ω is strategically Ramsey". We let $\mathrm{Adv}(\Gamma)$ be the statement "for every analytic Gowers space $\mathcal{G}$, $\mathrm{Adv}_{\mathcal{G}}(\Gamma)$ holds", and $\mathrm{Strat}(\Gamma)$ be the statement "for every analytic Gowers space $\mathcal{G}$, $\mathrm{Strat}_{\mathcal{G}}(\Gamma)$ holds". We proved in the two previous sections the following implications:

$$\begin{array}{ccccc} \mathrm{Det}_{\mathbb{R}}(\Gamma) & \Longrightarrow & \mathrm{Adv}(\Gamma) & \Longrightarrow & \mathrm{Det}_\omega(\Gamma) \\ & & \Downarrow & & \\ & & \mathrm{Strat}(\exists\Gamma) & & \end{array}$$

In the rest of this section, we fix a Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$. We begin our analysis with making some remarks about games. In the games A_p, B_p, F_p and G_p, say that the *turn* n is the sequence of two moves, consisting in one move of each player, where one player plays a subspace and just after, the other player plays the element of index n in the outcome. For instance, in a run $(p_0, x_0, p_1, x_1, \ldots)$ of the game F_p or G_p, the turn n is (p_n, x_n); and in a run $(p_0, x_0, q_0, y_0, p_1, x_1, \ldots)$ of the game A_p, or B_p, the turn $2n$ is (p_n, x_n) and the turn $2n+1$ is (q_n, y_n). We say that a turn of a game played under the subspace p is an *asymptotic turn* if it has the form (p_n, x_n) where $p_n \lessapprox p$, player **I** plays p_n and player **II** plays x_n, an *anti-asymptotic turn* if it has the form (p_n, x_n) where $p_n \lessapprox p$, player **II** plays p_n and player **I** plays x_n, a *Gowers turn* if it has the form (p_n, x_n) where $p_n \leqslant p$, player **I** plays p_n and player **II** plays x_n, and an *anti-Gowers turn* if it has the form (p_n, x_n) where $p_n \leqslant p$, player **II** plays p_n and player **I** plays x_n. In this way, F_p is a sequence of asymptotic turns, G_p is a sequence of Gowers turn, A_p alternates between one anti-Gowers turn and one asymptotic turn, and B_p alternates between one anti-asymptotic turn and one Gowers turn. Given a game H, we will denote by H^* the same game, but where the roles of players **I** and **II** are reversed.

Recall lemma II.17, where we proved that if player **I** had a strategy in F_p to reach some set $\mathcal{X}$, then **II** had a strategy in G_p to reach $\mathcal{X}$. This lemma can be rephrased in the following way: if player **I** has a strategy in F_p to reach $\mathcal{X}$, then he has a strategy in G_p^* to reach $\mathcal{X}$. And the proof of this lemma actually show the following stronger result: if, in a game H, player **I** has a strategy to reach some set $\mathcal{X}$, then, in a game obtained from H by replacing some asymptotic turns by anti-Gowers turns, player **I** still have a strategy to reach $\mathcal{X}$. Now remark that if we replace turns with even index (resp. odd index) in F_p with anti-Gowers turns, we get A_p (resp. B_p^*) and that if we replace turns with odd index in A_p (resp. turns with even index in B_p^*), that are asymptotic turns, with anti-Gowers turns, then we get G_p^*. Thus, we have the following lemma:

Lemma II.23. *Let $\mathcal{X} \subseteq X^\omega$. Consider the following four assertions:*

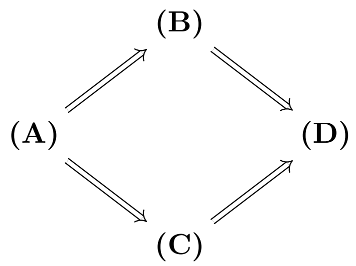
(A) *Player **I** has a strategy to reach $\mathcal{X}$ in F_p;*

(B) *Player **I** has a strategy to reach $\mathcal{X}$ in A_p;*

(C) *Player **II** has a strategy to reach $\mathcal{X}$ in B_p;*

(D) *Player **II** has a strategy to reach $\mathcal{X}$ in G_p;*

Then we have the following implications:



An interesting consequence is the following result:

Proposition II.24. *Suppose that the Gowers space $\mathcal{G}$ satisfies the pigeonhole principle. Then a set $\mathcal{X} \subseteq X^\omega$ is strategically Ramsey if and only if it is adversarially Ramsey.*

Proof. Suppose that $\mathcal{X}$ is strategically Ramsey, and let $p \in P$. By corollary II.21, there exists $q \leqslant p$ such that either player **I** has a strategy in F_q to reach $\mathcal{X}$, or he has one to reach $\mathcal{X}^c$. By lemma II.23, we deduce that either **I** has a strategy in A_q to reach $\mathcal{X}$, or **II** has a strategy in B_q to reach $\mathcal{X}^c$. So $\mathcal{X}$ is adversarially Ramsey.

Now suppose that $\mathcal{X}$ is adversarially Ramsey, and let $p \in P$. Then there exists $q \leqslant p$ such that either **I** has a strategy in A_q to reach $\mathcal{X}$, or **II** has a strategy in B_q to reach $\mathcal{X}^c$; so by lemma II.23, either **II** has a strategy in G_q to reach $\mathcal{X}$, **II** has a strategy in G_q to reach $\mathcal{X}^c$. In the second case, using proposition II.20, we get the existence of $r \leqslant q$ such that **I** has a strategy in F_r to reach $\mathcal{X}^c$. So $\mathcal{X}$ is strategically Ramsey.

□

In particular, the proof of the adversarial Ramsey property for Borel sets in spaces where the pigeonhole principle holds can be carried out in ZC, and the adversarial Ramsey property is provable, in ZC, for analytic and coanalytic sets. In these spaces, this property is actually useless. We will now see that in spaces where the pigeonhole principle does not hold, the situation is the opposite. We will need, here, to restrict our attention to forgetful Gowers spaces.

Proposition II.25. *Suppose that the Gowers space $\mathcal{G}$ is forgetful and does not satisfies the pigeonhole principle. Then* $\mathrm{Adv}_{\mathcal{G}}(\Gamma) \Rightarrow \mathrm{Det}_{\omega}(\Gamma)$.

Proof. Suppose $\mathrm{Adv}_{\mathcal{G}}(\Gamma)$. We show that every Γ-subset of 2^ω is determined. This implies that every Γ-subset of ω^ω is determined; see for example [47], exercise 6A.8. So we let $\mathcal{Y} \in 2^\omega$ be a Γ-set. Recall that the Gale-Stewart game over 2, that is, the game where **I** and **II** alternate playing elements of 2 and whose outcome is the sequence of these elements, is denoted by $\mathcal{G}(2^{<\omega})$. We have to prove that either **I** has a strategy to reach $\mathcal{Y}$ in this game, or that **II** has a strategy to reach $\mathcal{Y}^c$ in it.

Since $\mathcal{G}$ is forgetful, we will consider $\lhd$ as a binary relation between elements of X and elements of P. Since $\mathcal{G}$ does not satisfy the pigeonhole principle, there exists $p \in P$ and $A \subseteq X$ such that for every $q \leqslant p$, there exists $x, y \lhd q$ with $x \in A$ and $y \in A^c$. We let $f : X^\omega \longrightarrow 2^\omega$ be the function mapping a sequence $(x_n)_{n\in\omega}$ to the sequence $(\alpha_n)_{n\in\omega}$ defined by $\forall n \in \omega\ (\alpha_n = 1 \Leftrightarrow x_n \in A)$; and we let $\mathcal{X} = f^{-1}(\mathcal{Y})$. Then $\mathcal{X}$ is in Γ, so it is adversarially Ramsey; let $q \leqslant p$ such that either **I** has a strategy in A_q to reach $\mathcal{X}$, or **II** has a strategy in B_q to reach $\mathcal{X}^c$.

Suppose that **I** has a strategy τ to reach $\mathcal{X}$ in A_q. We show that **I** has a strategy to reach $\mathcal{Y}$ in the Gale-Stewart game $\mathcal{G}(2^{<\omega})$ by simulating a play $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ of this game by a play $(q_0, x_0, q_1, x_1, q_2, x_2, \ldots)$ of A_q during which **I** always plays according to τ (here, we use a slightly different notation than usual: the subspaces played by **I** are the q_i's, for i odd, and the points played by **I** are the x_i's, for i even). Suppose that the x_i's, the q_i's and the α_i's have been played for every $i < 2n$. In the game A_q, we make **II** play $q_{2n} = q$. According to the strategy τ, **I** answers with x_{2n} and q_{2n+1}. If $x_{2n} \in A$, then we make **I** play $\alpha_{2n} = 1$ in $\mathcal{G}(2^{<\omega})$; otherwise, we make him play $\alpha_{2n} = 0$. In this game, player **II** answers with α_{2n+1}. If $\alpha_{2n+1} = 1$, then, in A_q, we make **II** play $x_{2n+1} \in A$ such that $x_{2n+1} \lhd q_{2n+1}$; otherwise, we make her play $x_{2n+1} \in A^c$ such that $x_{2n+1} \lhd q_{2n+1}$. Remark that this is always possible by the definition of A, since $q_{2n+1} \leqslant q \leqslant p$. And the plays can continue.

At the end of the plays, the outcome of $\mathcal{G}(2^{<\omega})$ is $(\alpha_n)_{n\in\omega} = f((x_n)_{n\in\omega})$. Due to the use of the strategy τ by **I**, we have that $(x_n) \in \mathcal{X}$, so $(\alpha_n) \in \mathcal{Y}$ as wanted.

In the same way, if **II** has a strategy in B_q to reach $\mathcal{X}$, then we can deduce that **II** has a strategy in $\mathcal{G}(2^{<\omega})$ to reach $\mathcal{Y}^c$; this concludes the proof.

□

This proof does not work in spaces that are not forgetful. In these spaces, we need a slight strengthening of the negation of the pigeonhole principle, for example the fact that there exists $p \in P$ such that for every $s \in X^{<\omega}$, there exists $A_s \subseteq X$ such that for every $q \leqslant p$, we do not have $q \subseteq_s A_s$ nor $q \subseteq_s A_s^c$. In this case, we can define the function f in the following way: f maps a sequence $(x_n)_{n \in \omega}$ to the unique sequence $(\alpha_n)_{n \in \omega}$ such that for every n, $\alpha_n = 1$ iff $x_n \in A_{(x_0,\ldots,x_{n-1})}$, and carry out the proof in the same way.

A consequence of proposition II.25 is that if $\mathcal{G}$ is a forgetful Gowers space where the pigeonhole principle does not hold, then you cannot prove $\mathrm{Adv}_{\mathcal{G}}(\mathbf{\Delta}^1_1)$ in ZC: you need to use the powerset axiom and the replacement scheme to prove it. This holds, for instance, in the projective Rosendal space over a field with at least three elements, showing that "playing on the norm" is not the only way to get back determinacy from the adversarial Ramsey property. Also, in these spaces, $\mathrm{Adv}_{\mathcal{G}}(\mathbf{\Sigma}^1_1)$ and $\mathrm{Adv}_{\mathcal{G}}(\mathbf{\Pi}^1_1)$ are not provable in ZFC and even, are false in $ZFC + V = L$. This is a first major difference between spaces with and without a pigeonhole principle; we will see another one in the next section.

II.4 Closure properties and limitations for strategically Ramsey sets

In this section, we show the same kind of difference of behavior between spaces with and without a pigeonhole principle as in the previous section, but this kind for strategically Ramsey sets. We fix, in the whole section, a Gowers space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ and a suitable class Γ of subsets of Polish spaces. The first thing to remark is that if $\mathcal{G}$ satisfies the pigeonhole principle, then by corollary II.21, the class of strategically Ramsey sets is closed under taking complements: $\mathcal{X} \subseteq X^\omega$ is strategically Ramsey if and only if $\mathcal{X}^c$ is so. In particular, in ZFC, every $\mathbf{\Pi}^1_1$ subset of X^ω is strategically Ramsey. In spaces where the pigeonhole principle does not hold, the situation is very different; we firstly state the two main results of this section and present their consequences, before proving them.

The first result is a generalisation of a theorem proved by López-Abad [37] in the context of strategically Ramsey sets in Banach spaces. This result only holds for forgetful Gowers spaces, and to prove it, we need the negation of a slight weakening of the pigeonhole principle. We will say that the forgetful space $\mathcal{G}$ satisfies the *weak pigeonhole principle* if for every $A \subseteq X$, there exists $p \in P$ such that either $p \subseteq A$, or $p \subseteq A^c$ (where $p \subseteq A$ abusively denotes the fact that for every $x \in X$, if $x \lhd p$, then $x \in A$). Of course, in most of the concrete spaces we consider, P has a maximum $\mathbb{1}$ that is isomorphic to every subspace (meaning, here, that for every $p_0 \in P$, there are bijections

$\Phi : P \longrightarrow \{p \in P \mid p \leqslant p_0\}$ and $\varphi : X \to \{x \in X \mid x \lhd p_0\}$ that preserve the relations $\leqslant$, $\leqslant^*$ and $\lhd$); this is, for example, the case of the Mathias–Silver space or of the Rosendal space. In these spaces, the weak pigeonhole principle is equivalent to the pigeonhole principle. Our result is the following:

Proposition II.26. *Suppose that $\mathcal{G}$ is forgetful and does not satisfy the weak pigeonhole principle. Then* $\mathrm{Strat}_{\mathcal{G}}(\Gamma) \Rightarrow \mathrm{Strat}_{\mathcal{G}}(\exists\Gamma)$.

For the second result we need to ensure the fact that the space $\mathcal{G}$ is non-trivial enough. We say that the space $\mathcal{G}$ is *standard* if $|P| \leqslant \mathfrak{c}$ and if $\mathcal{G}$ satisfies the following property: for every $s \in X^{<\omega}$ and for every $p \in P$, there exists $q, r \leqslant p$ such that no $x \in X$ satisfies at the same time $s ^\frown x \lhd q$ and $s ^\frown x \lhd r$. This property is for instance satisfied by the Mathias–Silver space and by the Rosendal space. Our second result is the following:

Proposition II.27. *Suppose that the Gowers space $\mathcal{G}$ is standard. Then there exists $\mathcal{X} \in X^\omega$ satisfying the following property: for every $p \in P$, player **II** has no strategy in G_p to reach $\mathcal{X}$, and no strategy in G_p to reach $\mathcal{X}^c$. In particular, $\mathcal{X}$ is not strategically Ramsey. Moreover, if $V = L$, then such a set $\mathcal{X}$ can be choosen $\mathbf{\Sigma}^1_2$.*

Let us discuss the consequences of these two propositions. Firstly, we deduce immediately that if $\mathcal{G}$ is forgetful, standard, and does not satisfy the weak pigeonhole principle, then if $V = L$, there exist $\mathbf{\Pi}^1_1$-sets that are not strategically Ramsey in this space. In particular, in this space, the class of strategically Ramsey sets is not closed under complements in general. On the other hand, if $\mathcal{G}$ is standard and satisfies the pigeonhole principle, then $\mathrm{Strat}_{\mathcal{G}}(\Gamma)$ does not imply $\mathrm{Strat}_{\mathcal{G}}(\exists\Gamma)$ in general, since ZFC proves that every $\mathbf{\Pi}^1_1$-set in $\mathcal{G}$ is strategically Ramsey, but does not prove it for $\mathbf{\Sigma}^1_2$-sets. So, roughly speaking, we have a dichotomy between, on one side, spaces with a pigeonhole principle where the class of strategically Ramsey sets is closed under complements but not projections, and spaces without a pigeonhole principle where the class of strategically Ramsey sets is closed under projections but not complements.

We finish this section by giving the proof of propositions II.26 and II.27.

Proof of proposition II.26. As usual, since $\mathcal{G}$ is forgetful, we will consider $\lhd$ as a relation between points and subspaces. As in the proof of theorem II.14, we let $X' = X \times \{0,1\}$ and we define a relation $\lhd' \subseteq X' \times P$ by $(x, \varepsilon) \lhd p \Leftrightarrow x \lhd p$. Then $\mathcal{G}' = (P, X', \leqslant, \leqslant^*, \lhd')$ is a forgetful Gowers space. To avoid confusion, the asymptotic game and Gowers' game will be respectively denoted by F_p and G_p is the space $\mathcal{G}$, an by F'_p and G'_p in the space $\mathcal{G}'$. The proof of theorem II.14 actually show that $\mathrm{Strat}_{\mathcal{G}'}(\Gamma) \Rightarrow \mathrm{Strat}_{\mathcal{G}}(\exists\Gamma)$, so it remains to prove that $\mathrm{Strat}_{\mathcal{G}}(\Gamma) \Rightarrow \mathrm{Strat}_{\mathcal{G}'}(\Gamma)$.

So suppose $\mathrm{Strat}_{\mathcal{G}}(\Gamma)$. Since $\mathcal{G}$ does not satisfy the weak pigeonhole principle, there exists $A \subseteq X$ such that for every $p \in P$, there exists $x, y \lhd p$ such that $x \in A$ and $y \in A^c$. We define a mapping $f : X \to \{0,1\}$ by $f(x) = 1 \Leftrightarrow x \in A$, and a mapping $F : X^\omega \to X'^\omega$ by $F((x_n)_{n\in\omega}) = (x_0, f(x_1), x_2, f(x_3), x_4, f(x_5), \ldots)$. We show that, for

$\mathcal{X}' \subseteq X'^\omega$, if $F^{-1}(\mathcal{X}')$ is strategically Ramsey in $\mathcal{G}$, then $\mathcal{X}'$ is strategically Ramsey in $\mathcal{G}'$. Since the mapping F is continuous, it will be enough to conclude.

So we let $\mathcal{X}' \subseteq X'^\omega$, and we suppose that $\mathcal{X} = F^{-1}(\mathcal{X}')$ is strategically Ramsey. Let $p \in P$. There exists $q \leqslant P$ such that either **I** has a strategy in F_q to reach $\mathcal{X}^c$, or **II** has a strategy in G_q to reach $\mathcal{X}$.

*First case: **I** has a strategy in F_q to reach $\mathcal{X}^c$.* We show that **I** has a strategy in F'_q to reach $\mathcal{X}'^c$ by simulating a play of this game by a play of F_q where **I** uses a strategy to reach $\mathcal{X}^c$. Suppose that the first n turns of F'_q and the first $2n$ turns of F_q have been played. What happens during the $(n+1)^{\text{th}}$ turn of F'_q and during the $(2n+1)^{\text{th}}$ and the $(2n+2)^{\text{th}}$ turns of F_q is represented in the diagrams below:

F_q	**I**	$\ldots$	q_n		r_n		$\ldots$	
	II		$\ldots$	x_n		y_n		$\ldots$

F'_q	**I**	$\ldots$	q_n		$\ldots$	
	II		$\ldots$	x_n, ε_n		$\ldots$

According to his strategy in F_q, **I** plays $q_n \lessapprox q$. His strategy in F'_q will consist in copying this move. In F'_q, **II** answers with $(x_n, \varepsilon_n) \lhd' q_n$. Since we have that $x_n \lhd q_n$, we can make **II** play x_n in F_q. According to his strategy, **I** will answer with $r_n \lessapprox q$. Then, by definition of A, there exists $y_n \lhd r_n$ such that $f(y_n) = \varepsilon_n$. We make **II** play y_n in F_q, and the games can continue.

At the end of the games, the outcome $(x_0, \varepsilon_0, x_1, \varepsilon_1, \ldots)$ of the game F'_q will be the image by F of the outcome $(x_0, y_0, x_1, y_1, \ldots)$ of F_q. By the choice of the strategy of **I** in F_q, the outcome of this game is in $\mathcal{X}^c$, so the outcome of F'_q is in $\mathcal{X}'^c$ as wanted.

*Second case: **II** has a strategy in G_q to reach $\mathcal{X}$.* We show that **II** has a strategy in G'_q to reach $\mathcal{X}'$ by simulating a play of this game by a play of G_q where **II** uses a strategy to reach $\mathcal{X}$. Suppose that the first n turns of G'_q and the first $2n$ turns of G_q have been played. What happens during the $(n+1)^{\text{th}}$ turn of G'_q and during the $(2n+1)^{\text{th}}$ and the $(2n+2)^{\text{th}}$ turns of G_q is represented in the diagrams below:

G_q	**I**	$\ldots$	q_n		q_n		$\ldots$	
	II		$\ldots$	x_n		y_n		$\ldots$

G'_q	**I**	$\ldots$	q_n		$\ldots$	
	II		$\ldots$	$x_n, f(y_n)$		$\ldots$

In G'_q, **I** plays $q_n \leqslant q$. We make him repeat this moves two times in G_q and we denote by x_n and y_n the two successive answers of **II** in this game, according to her strategy. In

G'_q, the strategy of **II** will consist in playing $(x_n, f(y_n)) \in X'$. In this way, the outcome of the game G'_q will be the image by F of the outcome of G_q, which is in $\mathcal{X}$; so the outcome of G'_q is in $\mathcal{X}'$.

$\square$

In order to prove proposition II.27, we will need to perform a diagonal argument for which we need to have at most continuum-many strategies in Gowers' game. For this reason, we will need to give a countable version of this game. Exceptionnaly, for technical reasons, we will define this game with a winning condition rather than an outcome.

Definition II.28. Let $p \in P$ and $\mathcal{X} \subseteq X^\omega$. The *countable Gowers' game* under p with target set $\mathcal{X}$, denoted by $CG_p(\mathcal{X})$, is the following two-players game:

I	$x_0^0, x_0^1, \ldots, x_0^{n_0}$		$x_1^0, x_1^1, \ldots, x_1^{n_1}$		$\ldots$
II		y_0		y_1	$\ldots$

It is played in the following way. **I** begins with playing a sequence $(x_0^0, x_0^1, \ldots)$ of elements of X. At some point, **II** can choose to interrupt him at some point $x_0^{n_0}$ and to choose $y_0 \in \{x_0^0, x_0^1, \ldots, x_0^{n_0}\}$. In this case, **I** begins back to choose points $x_1^0, x_1^1, \ldots$, and again, **II** can choose to interrupt him at some point $x_1^{n_1}$ by choosing $y_1 \in \{x_1^0, x_1^1, \ldots, x_1^{n_1}\}$, etc. Two cases can occur:

- If **II** always chooses to interrupt **I** after some time, then at the end of the game, **II** will have produced an infinite sequence $(y_i)_{i\in\omega}$. In this case, **II** wins if and only if this sequence belongs to $\mathcal{X}$.
- If, at some point, **II** chooses not to interrupt **I**, then **I** will continue to play points indefinitely and the game will stop after ω points have been played. In this case, **II** will have produced a finite sequence $s = (y_0, \ldots, y_{i-1})$, and after that, **I** will have produced an infinite sequence $(x_i^n)_{n\in\omega}$; we can let $A = \{x_i^n \mid n \in \omega\}$. Then **II** wins if and only if for no $q \leqslant p$, we have that $A = \{x \in X \mid s^\frown x \lhd q\}$.

In some situations, specifying the target set, or even the subspace under which the game is played, will be useless (as only the winning condition depend on this information); in this case, the countable Gowers' game will be denoted by CG_p, or simply CG.

The interest of this game is that it is in fact equivalent to Gowers' game. More precisely, we have:

Lemma II.29. *Let $p \in P$ and $\mathcal{X} \subseteq X$. Then:*

1. *player **I** has a strategy in G_p to reach $\mathcal{X}^c$ if and only if he has a winning strategy in $CG_p(\mathcal{X})$;*
2. *player **II** has a strategy in G_p to reach $\mathcal{X}$ if and only if she has a winning strategy in $CG_p(\mathcal{X})$.*

Proof. 1. ($\Rightarrow$) Suppose that **I** has a strategy to reach $\mathcal{X}^c$ in G_p. Then he can use the same strategy to win $CG_p(\mathcal{X})$, but, instead of playing the subspace p_i, he plays a (non-necessarily injective) enumeration $x_i^0, x_i^1, \ldots$ of the set $\{x \in X \mid s_i \frown x \lhd p_i\}$, where $s_i = (y_0, \ldots, y_{i-1})$ is the sequence of points already played by **II**. If **II** never interrupts him, then according to the rules of $CG_p(\mathcal{X})$, **I** will win this game. So we can suppose that **II** interrputs him to play a y_i such that $(y_0, \ldots, y_i) \lhd p_i$, and the play can continue exactly as in G_p.

($\Leftarrow$) We simulate a play $(p_0, y_0, p_1, y_1, \ldots)$ of G_p with a play $(x_0^0, \ldots, x_0^{n_0}, y_0, x_1^0, \ldots, x_1^{n_1}, y_1, \ldots)$ where **I** plays using a winning strategy. Suppose that, for $j < i$, all the p_j and the y_j have been played in G_p, and that the last move in $CG_p(\mathcal{X})$ is **II** playing y_{i-1}. In $CG_p(\mathcal{X})$, we let **I** play $x_i^0, x_i^1, \ldots$, according to his strategy. If **II** never interrupts it, he will have produced an infinite sequence $(x_i^n)_{n \in \omega}$, and with $A = \{x_i^n \mid n \in \omega\}$ and $s = (y_0, \ldots, y_{i-1})$, knowing that **I** is winning, we will get that there exists $p_i \leqslant p$ such that $A = \{x \in X \mid s \frown x \lhd p_i\}$. Then we make **I** play p_i in G_p; **II** will answer by y_i such that $s \frown y_i \in A$, so by construction, we will have that $y_i = x_i^{n_i}$ for some $n_i \in \omega$, and in $CG_p(\mathcal{X})$, **II** could have interrupted **I** after he played $x_i^{n_i}$ to play y_i. We will suppose that **II** did that, and the games can continue. At the end, the outcome of G_p is an infinite sequence of points played by **II** in $CG_p(\mathcal{X})$ while **I** was using his winning strategy, so it belongs to $\mathcal{X}^c$ as wanted.

2. ($\Rightarrow$) We simulate a play $(x_0^0, \ldots, x_0^{n_0}, y_0, x_1^0, \ldots, x_1^{n_1}, y_1, \ldots)$ of $CG_p(\mathcal{X})$ with a play $(p_0, y_0, p_1, y_1, \ldots)$ of G_p where **II** uses a strategy to reach $\mathcal{X}$. Suppose that the last move in both games is y_{i-1}, played by **II**. We say that $y \in X$ is *reachable* if there exists $p_i \leqslant p$ such that, if **I** plays p_i in G_p, then the strategy of **II** tells him to answer with y. The strategy of **II** in $CG_p(\mathcal{X})$ will be the following: she watches **I** playing a sequence of points $(x_i^0, x_i^1, \ldots)$, until he plays a reachable point. If, for some $n_i \in \omega$, $x_i^{n_i}$ is reachable, then **II** interrupts him and plays $y_i = x_i^{n_i}$. Then, in G_p, by assumption there exists a p_i that **I** can play and such that **II** will answer, according to her strategy, with y_i, and both games can continue. In the opposite case, if none of the points x_i^n's played by **I** in $CG_p(\mathcal{X})$ is reachable, then **II** never interrupts him. In this way, **I** will produce a sequence $(x_i^n)_{n \in \omega}$, and we will see that he loses the game $CG_p(\mathcal{X})$. Suppose not. Then, denoting by $s = (y_0, \ldots, y_{i-1})$ the sequence of points already played by **II** in $CG_p(\mathcal{X})$, there exists $p_i \leqslant p$ such that the set $\{x_i^n \mid n \in \omega\}$ is equal to the set $\{x \in X \mid s \frown x \lhd p_i\}$. Then, **I** can play p_i in the game G_p, and **II** will answer, according to her strategy, with a y_i belonging to these sets. This y_i is reachable, so this contradict the fact that no term of the sequence $(x_i^n)_{n \in \omega}$ was reachable.

In the case were **II** plays only finitely many points in $CG_p(\mathcal{X})$, we just saw that she wins this game. If she produces an infinite sequence $(y_i)_{i \in \omega}$, then this sequence is exactly the outcome of the auxiliary game G_p, so it belongs to $\mathcal{X}$ and **II** wins.

($\Leftarrow$) The proof is the same as for the direction ($\Leftarrow$) of 1.. If **II** has a strategy to win $CG_p(\mathcal{X})$, then she can use this strategy in G_p by believing that player **I** plays, instead of the p_i's, the points of the sets $\{y \in X \mid (y_0, \ldots, y_{i-1}, y) \lhd p_i\}$ successively. Her strategy will always make her interrupt **I**, because if it did not, then **I** could enumerate a set of this form and would win immediately.

□

Now let τ be a strategy for **II** in the countable Gowers' game CG (we do not need to specify the subspace under which the game is played, nor the target set to define the notion of a strategy in this game). Such a strategy can be seen as a function $\tau : \mathrm{Seq}(X) \longrightarrow X \cup \{*\}$: after **I** has played $z_0, \ldots, z_{k-1}$ in CG, if $\tau(z_0, \ldots, z_{k-1}) = y \in X$, this means that **II** has to interrupt **I** and to play y, and if $\tau(z_0, \ldots, z_{k-1}) = *$, then **II** has to wait and to let **I** play another point. (In particular, there are at most continuum-many such strategies.) If τ is such a strategy, we let $[\tau]$ be the set of sequences $(y_i)_{i \in \omega}$ that can be produced by **II** in plays of CG where she interrupts **I** infinitely many times and always plays according to her strategy τ. We say that the strategy τ is *good* if $|[\tau]| = \mathfrak{c}$. Given a subspace $p \in P$, we say that a strategy τ is *p-correct* if whenever, during a play of CG_p, **II** always plays according to τ and only interrupt **I** finitely many times, then **I** loses this play. (In this context, saying that **I** loses the play has a sense even without specifying the target set, since the winning condition for **II** when **II** only interrupts **I** finitely many times only depends on p.) Remark that a strategy τ is winning for **II** in the game $CG_p(\mathcal{X})$ if and only if it is p-correct and $[\tau] \subseteq \mathcal{X}$.

Lemma II.30. *Suppose that the Gowers space $\mathcal{G}$ is standard. Let τ be a strategy for **II** in CG. If there exists $p \in P$ such that τ is p-correct, then τ is good.*

Proof. Suppose that there exists $p \in P$ such that τ is p-correct and fix such a p. Let $\mathcal{X} = [\tau]$. Then by the previous remark, **II** has a strategy winning strategy in $CG_p(\mathcal{X})$, so by lemma II.29, she has a strategy σ in G_p to reach $\mathcal{X}$. As usual, we define a *state* as a partial play of G_p ending with a move of **II** and during which **II** always plays according to σ; this play *realises* a sequence $(x_0, \ldots, x_{n-1})$ if it has the form $(p_0, x_0, \ldots, p_{n-1}, x_{n-1})$. We build inductively, for $\alpha \in 2^{<\omega}$, a state $\mathcal{s}_\alpha$ realising a sequence s_α of length $|\alpha|$, in such a way that for $\alpha, \beta \in 2^{<\omega}$, we have $\alpha \subseteq \beta \Rightarrow \mathcal{s}_\alpha \subseteq \mathcal{s}_\beta$, and if $|\alpha| = |\beta|$, then $\alpha \neq \beta \Rightarrow s_\alpha \neq s_\beta$. This will be enough to conclude: letting $f(x) = \bigcup_{n<\omega} s_{x \upharpoonright n}$ will define a one-to-one mapping $f : 2^\omega \to \mathcal{X}$.

We let $\mathcal{s}_\varnothing = s_\varnothing = \varnothing$. Let $\alpha \in 2^\omega$ and suppose that $\mathcal{s}_\alpha$ and s_α have been built. Then, since $\mathcal{G}$ is standard, then there exists $q, r \leqslant p$ such that no $x \in X$ satisfies simultaneously $s_\alpha \frown x \lhd q$ and $s_\alpha \frown x \lhd r$. In particular, $x = \sigma(\mathcal{s}_\alpha \frown q)$ and $y = \sigma(\mathcal{s}_\alpha \frown r)$ are distinct so we can let $\mathcal{s}_{\alpha \frown 0} = \mathcal{s}_\alpha \frown (q, x)$, $s_{\alpha \frown 0} = s_\alpha \frown x$, $\mathcal{s}_{\alpha \frown 1} = \mathcal{s}_\alpha \frown (r, y)$, and $s_{\alpha \frown 1} = s_\alpha \frown y$, and this achieves the construction.

□

Lemma II.31. *Let $\mathcal{X} \subseteq X^\omega$. If, for some $p \in P$, **II** has a strategy in G_p to reach $\mathcal{X}$, then there is a good strategy τ for **II** in CG such that $[\tau] \subseteq \mathcal{X}$.*

Proof. By lemma II.29, if **II** has a strategy in G_p to reach τ, then she has a winning strategy τ in the game $CG_p(\mathcal{X})$. In particular, this strategy has to be satisfy $[\tau] \subseteq \mathcal{X}$. Moreover, it has to be p-correct, so by lemma II.30, it is good.

□

We can now prove the "ZFC" part of proposition II.27. For the "$V = L$" part, we will need some more lemmas and we will do that later. In the rest of this section, we will use the letters u, v, and w to denote elements of Polish spaces (as ω^ω or X^ω).

Proof of proposition II.27, first part. Suppose that the space $\mathcal{G}$ is standard; we build a set $\mathcal{X} \subseteq X$ such that for every $p \in P$, **II** has no strategy in G_p to reach $\mathcal{X}$, and she has no strategy in G_p to reach $\mathcal{X}^c$. By lemma II.31, we only have to ensure that for every good strategy τ for **II** in CG, we have $[\tau] \cap \mathcal{X} \neq \varnothing$ and $[\tau] \cap \mathcal{X}^c \neq \varnothing$. Let $(\tau_\alpha)_{\alpha<\mathfrak{c}}$ be a (non-necessarily injective) enumeration of good strategies for **II** in CG. We can build inductively two sequences $(u_\alpha)_{\alpha<\mathfrak{c}}$ and $(v_\alpha)_{\alpha<\mathfrak{c}}$ of elements of X^ω such that for every α, $u_\alpha \neq v_\alpha$ and $u_\alpha, v_\alpha \in [\tau] \backslash \{u_\xi, v_\xi \mid \xi < \alpha\}$. Then the set $\mathcal{X} = \{u_\alpha \mid \alpha < \mathfrak{c}\}$ is as wanted.

Of course, the $\mathcal{X}$ we built cannot be strategically Ramsey: indeed, by lemma II.17, we get that for no $p \in P$, **I** can have a strategy in F_p to reach $\mathcal{X}^c$.

□

For the "$V = L$" part of proposition II.27, we will use a well-known result by Gödel. We begin with a definition.

Definition II.32. A well-ordering $\prec$ of a Polish space U is said to be $\mathbf{\Sigma}^1_2$-*good* if it has order-type ω_1, if it is a $\mathbf{\Sigma}^1_2$-subset of U^2, and if the relation $R_\prec \subset U^\omega \times U$ defined by $(u_n)_{n\in\omega} \, R_\prec \, v \leftrightarrow \{u_n \mid n \in \omega\} = \{w \in \omega^\omega \mid w \prec v\}$ is $\mathbf{\Sigma}^1_2$.

Gödel's result is the following (for a proof, see for example [29], lemma 25.27):

Theorem II.33 (Gödel)**.** *Suppose $V = L$. Then there exists a $\mathbf{\Sigma}^1_2$-good well-ordering on ω^ω.*

Obviously, it follows that if $V = L$, such an ordering exists on every Polish space. Remark that if $\prec$ is a $\mathbf{\Sigma}^1_2$-good well-ordering on a Polish space U, then it is actually a $\mathbf{\Delta}^1_2$-subset of U^2: indeed, $u \prec v$ can be written $\neg(u = v \vee v \prec u)$, which is a $\mathbf{\Pi}^1_2$ definition. In the same way, the relation $R_\prec$ is in fact $\mathbf{\Delta}^1_2$, since $(u_n)_{n\in\omega} \, R_\prec \, v$ can be written $\forall u \in U \; (u \prec v \Leftrightarrow \exists n \in \omega \; u = u_n)$, which is a $\mathbf{\Pi}^1_2$ definition.

Also remark that if U and V are Polish spaces, if $\prec$ is a $\mathbf{\Sigma}^1_2$-good well ordering on V, and if A is a $\mathbf{\Delta}^1_2$-subset of $U \times V$, then the set $B \subseteq U \times V$ defined by $(u, v) \in B$ if and only if the set $\{w \in \omega^\omega \mid (u, w) \in A\}$ is nonempty and v is its $\prec$-least element, is $\mathbf{\Delta}^1_2$. Indeed, the fact that $(u, v) \in B$ can be written $(u, v) \in A \wedge \forall w \in \omega^\omega \, (w \prec v \Rightarrow (u, w) \notin A)$, which is a $\mathbf{\Pi}^1_2$-definition; and it can also be written $(u, v) \in A \wedge (v = v_0 \vee \exists (w_n)_{n\in\omega} \in V^\omega \, ((w_n)_{n\in\omega} \, R_\prec \, v \wedge \forall n \in \omega \; (u, w_n) \notin A))$, which is a $\mathbf{\Sigma}^1_2$-definition (here, v_0 denotes the $\prec$-least element of V). We will refer to this fact later by saying that *minimisation preserves* $\mathbf{\Delta}^1_2$*-sets.*

Our proof of the "$V = L$" part of proposition II.27 will be the same as this of the "ZFC" part, but we will replace the use of the axiom of choice by a careful use of a $\mathbf{\Sigma}^1_2$-good well ordering, enabling us to ensure that the set $\mathcal{X}$ we build is definable enough. The only difficulty here is to compute complexities.

We denote by $Strat$ the sets of strategies for **II** in the game CG.

Lemma II.34.

1. *Strat is a closed subset of the set of mappings* $\mathrm{Seq}(X) \longrightarrow X \cup \{*\}$*. In particular, it is a Polish space.*

2. *The set* $\{(\tau, u) \in Strat \times X^\omega \mid u \in [\tau]\}$ *is an analytic subset of* $Strat \times X^\omega$*.*

3. *The set of good strategies is a* $\mathbf{\Delta}^1_2$*-subset of* $Strat$*.*

Proof. 1. Let $\tau : \mathrm{Seq}(X) \longrightarrow X \cup \{*\}$ such that $\tau \notin Strat$. Then there exist a finite sequence $(x_0^0, \ldots, x_0^{n_0}, x_1^0, \ldots x_1^{n_1}, \ldots, x_i^0, \ldots, x_i^{n_i}) \in \mathrm{Seq}(X)$ such that for every $j \leqslant i$ and for every $n < n_j$, we have $\tau(x_0^0, \ldots, x_0^{n_0}, \ldots, x_{j-1}^0, \ldots x_{j-1}^{n_{j-1}}, x_j^0, \ldots, x_j^n) = *$, for every $j < i$ we have $\tau(x_0^0, \ldots, x_0^{n_0}, \ldots, x_j^0, \ldots, x_j^{n_j}) \in \{x_j^0, \ldots, x_j^{n_j}\}$, and $\tau(x_0^0, \ldots, x_0^{n_0}, \ldots, x_i^0, \ldots, x_i^{n_i}) \notin \{x_i^0, \ldots, x_i^{n_i}\}$. Any $\tau' : \mathrm{Seq}(X) \longrightarrow X \cup \{*\}$ satisfying the same conditions is not in $Strat$, showing that the complement of Strat is open.

2. For $\tau \in Strat$, and for $v \in X^\omega$, let $\tau \cdot v \in X^{\leqslant\omega}$ be the sequence of the points played by **II** in a play of CG where he always plays according to τ, and where **I** plays the sequence v. Denote by Inf the set of pairs $(\tau, v) \in Strat \times X^\omega$ such that $\tau \cdot v$ is an infinite sequence. The fact that $(\tau, (x_n)_{n\in\omega}) \notin Inf$ can be written "eventually, $\tau(x_0, \ldots, x_k) = *$", so Inf is a G_σ-subset of $Strat \times X^\omega$, so a Polish space. Moreover, the mapping $(\tau, v) \mapsto \tau \cdot v$ from Inf to X^ω is clearly continuous. The property $u \in [\tau]$, for $\tau \in Strat$ and $u \in X^\omega$, can be written as $\exists v \in Inf\ (\tau \cdot v = u)$, so this property in analytic, as wanted.

3. For $\tau \in Strat$, we have that τ is good if and only if for every $(u_n)_{n\in\omega} \in (X^\omega)^\omega$, there exists $v \in \mathcal{X}^\omega$ such that $(\tau, v) \in Inf$ and $\forall n \in \omega\ (\tau \cdot v \neq u_n)$; this is a $\mathbf{\Pi}^1_2$-definition. We now have to find a $\mathbf{\Sigma}^1_2$-definition. For $\tau \in Strat$, we denote by Fin_τ the set of $v \in X^\omega$ such that $(\tau, v) \in Fin$. We define the equivalence relation E_τ on Fin_τ by $v\, E_\tau\, w$ iff $\tau \cdot v = \tau \cdot w$, in such a way that τ is good if and only if E_τ has uncountably many classes. The relation E_τ is Borel, so by Silver's dichotomy theorem I.27, we get that τ is good if and only if there exists a continuous mapping $f : 2^\omega \longrightarrow Fin_\tau$ such that for every $w, w' \in 2^\omega$, $w \neq w' \Rightarrow (f(w), f(w')) \notin E_\tau$. Knowing that the set of continuous mappings $2^\omega \longrightarrow X^\omega$, with the uniform metric, is Polish (see [32], theorem 4.19), we see that this characterisation of goodness is $\mathbf{\Sigma}^1_2$.

□

We are now ready to prove the "$V = L$" part of proposition II.27.

Proof of proposition II.27, second part. We suppose $V = L$. We fix a $\mathbf{\Sigma}^1_2$-good well ordering $\prec_S$ on *Strat* and another one $\prec_X$ on $(X^\omega)^2$. We define two sequences $(u_\tau)_{\tau \in Strat}$ and $(v_\tau)_{\tau \in Strat}$, simultaneously by induction on the relation $\prec_S$ in the following way. Suppose that the u_σ and v_σ have been defined for all $\sigma \prec_S \tau$. Then, if τ is good, we let (u_τ, v_τ) be the $\prec_X$-least pair $(u, v) \in (X^\omega)^2$ such that $u, v \in [\tau] \backslash \{u_\sigma, v_\sigma \mid \sigma <_S \tau\}$, and $u \neq v$. Otherwise, we let (u_τ, v_τ) be the $\prec_X$-least pair $(u, v) \in (X^\omega)^2$ such that $u \neq v$ and $u, v \notin \{u_\sigma, v_\sigma \mid \sigma <_S \tau\}$. By construction, we have that all the u_τ's and the v_τ's, for $\tau \in Strat$, are pairwise distinct, and that for every good $\tau \in Strat$, we have $u_\tau, v_\tau \in [\tau]$. So if we let $\mathcal{X} = \{u_\tau \mid \tau \in Strat\}$, then by lemma II.31, for every $p \in P$, **II** has no strategy in G_p to reach $\mathcal{X}$, nor to reach $\mathcal{X}^c$, as wanted. It remains to compute the complexity of $\mathcal{X}$.

We say that a sequence $(\tau_n, u'_n, v'_n)_{n \in \omega} \in (Strat \times X^\omega \times X^\omega)^\omega$ is *nice* is it satisfies the following properties:

(1) The set $\{\tau_n \mid n \in \omega\}$ is an initial segment of *Strat* for the ordering $\prec_S$;

(2) For every $n \in \omega$, we have $u'_n = u_{\tau_n}$ and $v'_n = v_{\tau_n}$.

Then for $u \in X^\omega$, the fact that $u \in \mathcal{X}$ can be written "there exists a nice sequence $(\tau_n, u'_n, v'_n)_{n \in \omega} \in (Strat \times X^\omega \times X^\omega)^\omega$ and $n \in \omega$ such that $u = u'_n$". So to prove that $\mathcal{X}$ is $\mathbf{\Sigma}^1_2$, it is enough to prove that the set of nice sequences is $\mathbf{\Sigma}^1_2$.

Property (1) in the definition of a nice sequence can be written $\exists \tau \in Strat\ ((\tau_n)_{n \in \omega}\, R_{\prec_S}\, \tau)$, so it is $\mathbf{\Sigma}^1_2$.

If we know that property (1) is satisfied, then property (2) can be written in the following way: "for every $n \in \omega$, (u_n, v_n) is the $\prec_X$-least pair $(u, v) \in X^\omega \times X^\omega$ satisfying the following properties:

(a) $u \neq v$, and for every $m \in \omega$, if $\tau_m \prec_S \tau_n$, then $u \neq u_m$, $u \neq v_m$, $v \neq u_m$ and $v \neq v_m$;

(b) if τ_n is good, then $u, v \in [\tau_n]$".

Property (a) is $\mathbf{\Delta}^2_1$, and by lemma II.34, property (b) is also $\mathbf{\Delta}^1_2$. Since minimisation preserves $\mathbf{\Delta}^1_2$ set, we deduce that this writing of property (2) is $\mathbf{\Delta}^1_2$. So the set of nice sequences is $\mathbf{\Sigma}^1_2$, as wanted.

□

II.5 The adversarial Ramsey property under large cardinal assumptions

As we already saw, if Γ is a class of subsets of Polish spaces, then $\mathrm{Adv}(\Gamma)$ is implied by $\mathrm{Det}_{\mathbb{R}}(\Gamma)$ and implies $\mathrm{Det}_\omega(\Gamma)$, and an interesting question is to know where $\mathrm{Adv}(\Gamma)$ lies between these two determinacy statements, both in terms of implication and of consistency strength. We do not know much about this question; in this section, we discuss the consequences of some usual large cardinal assumptions on the adversarial Ramsey property in order to have a better idea of its strength. In particular, we will give an answer to Rosendal's question I.17. As usual, we fix $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ a Gowers space.

Recall that, for κ an uncountable cardinal, an ultrafilter on a set X is *κ-complete* if it is closed under intersections of size $< \kappa$ (if $\kappa = \aleph_1$, such an ultrafilter will also be said *σ-complete*). A *measurable cardinal* is an uncountable cardinal κ on which there exists a non-principal, κ-complete ultrafilter. Such cardinals are inaccessible, and it can be shown that the existence of a measurable cardinal is equivalent to the existence of a set X with a non-principal, σ-complete ultrafilter on X (see [29], lemmas 10.2 and 10.4).

The first determinacy result under large cardinal assumptions was proved by Martin [38]. We recall that, unless otherwise specified, if X is a set and $T \subseteq X^{<\omega}$ a tree, we put the discrete topology on X, and the topology induced by the product topology on $[T]$.

Theorem II.35 (Martin)**.** *Suppose that there exists a measurable cardinal κ. Let X be a set with $|X| < \kappa$ and $T \subseteq X^{<\omega}$ be a tree. Then every $\mathbf{\Sigma}^1_1$-subset of $[T]$ is determined.*

In particular, if there exists a measurable cardinal above $|P|$ and if $\mathcal{X} \in \mathbf{\Sigma}^1_1(X^\omega)$ then in Kastanas' game, either player **I** has a strategy to reach $\mathcal{X}$, or **II** has one to reach $\mathcal{X}^c$. So proposition II.6, and the proof of theorem II.14, give:

Theorem II.36. *If there exists a measurable cardinal above $|P|$, then every analytic subset of X^ω is adversarially Ramsey, and every $\mathbf{\Sigma}^1_2$-subset of X^ω is strategically Ramsey.*

In particular, this gives an answer to Rosendal's question I.17.

Determinacy results for higher levels of the projective hierarchy were then proved, based on the notion of *Woodin cardinals*. We will not give the definition of a Woodin cardinal, since it is quite sophisticated and would have no interest here. Woodin cardinals are inaccessible, they are not necessarily measurable but contain a stationary set of measurable cardinals. For more details, see [29], section 34. The first determinacy results assuming the existence of Woodin cardinals were proved by Martin and Steel [41, 42]:

Theorem II.37 (Martin–Steel)**.** *Suppose that there exist n Woodin cardinals, and a measurable cardinal above them. Let X be a set with cardinality strictly less than the Woodins, and $T \subseteq X^{<\omega}$ be a tree. Then every $\mathbf{\Sigma}^1_{n+1}$-subset of $[T]$ is determined.*

(The proof given by Martin and Steel is for $X = \omega$, but a proof of the general case can be found in [50]). Then, Woodin proves the following result (see [42]):

Theorem II.38 (Woodin)**.** *Suppose that there exist ω Woodin cardinals and a measurable above them. Then every subset of ω^ω that belongs to $L(\mathbb{R})$ is determined.*

$L(\mathbb{R})$ is the class of sets constructible from reals, see [29], section 13. An interesting consequence of the last theorem is that, under the same hypotheses, AD holds in $L(\mathbb{R})$. Indeed strategies for games on ω can be coded by reals, so are in $L(\mathbb{R})$; moreover, the sentence, for instance, "τ is a strategy for $\mathbf{I}$ in $\mathcal{G}(\omega^{<\omega})$ to reach A", only quantifies over sequences of integers, so is absolute for $L(\mathbb{R})$. This shows that "being determined", for a game on integers, is absolute for Ł$(\mathbb{R})$. Since $ZF + DC$ also holds in $L(\mathbb{R})$, this give the consistency of the theory $ZF + DC + AD$ relatively to large cardinal axioms.

Given Y an uncountable Polish space, we will denote by $\widetilde{L(\mathbb{R})}(Y)$ the set of $A \subseteq Y$ such that there exist a Borel mapping $\varphi : Y \to \omega^\omega$ and $B \in \mathcal{P}(\omega^\omega) \cap L(\mathbb{R})$ such that $A = \varphi^{-1}(B)$. Since Borel subsets of ω^ω and Borel mappings from ω^ω to itself can be coded by real numbers, these sets and mappings are in $L(\mathbb{R})$. So we deduce that $\widetilde{L(\mathbb{R})}$ is a suitable class of subsets of Polish spaces. Neeman confirmed to the author that theorem II.38 was also true for games on real numbers. From this result, and from theorem II.37, we can deduce the following results:

Theorem II.39.

1. *If there are n Woodin cardinals above $|P|$ and a measurable cardinal above them, then every $\mathbf{\Sigma}^1_{n+1}$-subset of X^ω is adversarially Ramsey, and every $\mathbf{\Sigma}^1_{n+2}$-subset of X^ω is strategically Ramsey.*

2. *Suppose that there are ω Woodin cardinals, and a measurable cardinal above them. Suppose that the space $\mathcal{G}$ is analytic. Then every $\widetilde{L(\mathbb{R})}$-subset of X^ω is adversarially Ramsey and strategically Ramsey.*

Proof. The proof of *1.* is exactly the same as the proof of theorem II.36, using Martin and Steel's result. For *2.*, we use Woodin's result for games on reals with outcome in $\widetilde{L(\mathbb{R})}$, corollary II.8 and the fact that every $\widetilde{L(\mathbb{R})}$-subset of $\mathbb{R}^\omega$ (with its Polish topology) is in $L(\mathbb{R})$ (this is due to the fact that Borel mappings $\mathbb{R}^\omega \longrightarrow \omega^\omega$ can be coded by real numbers).

□

Corollary II.40. *If the theory $ZFC+$ "there exist ω Woodin cardinals and a measurable above them" is consistent, then the following theory is also consistent: $ZF + DC+$ "in every analytic Gowers spaces, every set is adversarially and strategically Ramsey".*

Proof. We suppose the existence of ω Woodin cardinals and a measurable above them, and we show that the sentence "in every analytic Gowers spaces, every set is adversarially Ramsey" is satisfied in $L(\mathbb{R})$. The case of strategically Ramsey sets will follow since if every set is adversarially Ramsey, then every set is strategically Ramsey. Let $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd) \in L(\mathbb{R})$ such that $L(\mathbb{R})$ satisfies "$\mathcal{G}$ is an analytic Gowers space". Then in V, $\mathcal{G}$ is an analytic Gowers space, and subsets of X^ω that are in $L(\mathbb{R})$ are in

$\widetilde{L(\mathbb{R})}(X^\omega)$, so by theorem II.39, they are adversarially Ramsey in V. It remains to prove that the property of being adversarially Ramsey relativizes to $L(\mathbb{R})$. For this, we have to prove that given $p \in P$ and $\mathcal{X} \in \mathcal{P}(X^\omega) \cap L(\mathbb{R})$, the two notions "**I** has a strategy in A_p to reach $\mathcal{X}$" and "**II** has a strategy in B_p to reach $\mathcal{X}$" relativize to $L(\mathbb{R})$. Since both proofs are the same, we show it for A_p.

As well as we did for Gowers games in the the last section (see definition II.28), we will here define a countable version of the game A_p. For $p \in P$, and for $\mathcal{X} \subseteq X^\omega$, we define the game $CA_p(\mathcal{X})$ in the following way:

I		$y_0, z_0^0, z_0^1, \ldots, z_0^{n_0}$		$\ldots$
II	$x_0^0, x_0^1, \ldots, x_0^{m_0}$		$t_0, x_1^0, x_1^1, \ldots, x_1^{m_1}$	$\ldots$

It is played in the following way. **II** begins with playing a sequence $(x_0^0, x_0^1, \ldots)$ of elements of X. At some point, **I** can choose to interrupt her at some point $x_0^{m_0}$ and to choose $y_0 \in \{x_0^0, x_0^1, \ldots, x_0^{m_0}\}$. If he does, then after playing y_0, **I** plays a sequence $(z_0^0, z_0^1, \ldots)$ of elements f X, and **II** can choose to interrupt him at some point $z_0^{n_0}$ by choosing $t_0 \in \{z_0^0, z_0^1, \ldots, z_0^{n_0}\}$. If she does, then **II** begins back playing a sequence $(x_1^0, x_1^1, \ldots)$, etc.. Three cases can occur:

- First case: both of the player never let the other one play infinitely many consecutive times without interrupting him. Then at the end of the game, the players will have produced an infinite sequence $(y_0, t_0, y_1, t_1, \ldots) \in X^\omega$. In this case, **I** wins if and only if this sequence belongs to $\mathcal{X}$.

- Second case: at some point, **I** chooses not to interrupt **II** and to let her play infinitely many successive times. Then **II** will continue to play points indefinitely and the game will stop after ω points have been played. In this case, the players will have produced a finite sequence $s = (y_0, t_0, y_1, t_1, \ldots, y_{i-1}, t_{i-1})$, and after that, **II** will have produced an infinite sequence $(x_i^n)_{n\in\omega}$; we can let $A = \{x_i^n \mid n \in \omega\}$. Then **I** wins if and only if for no $q \leqslant p$, we have that $A = \{x \in X \mid s^\frown x \lhd q\}$.

- Third case: at some point, **II** chooses not to interrupt **I** and to let him play infinitely many successive times. Then **I** will continue to play points indefinitely and the game will stop after ω points have been played. In this case, the players will have produced a finite sequence $s = (y_0, t_0, y_1, t_1, \ldots, t_{i-1}, y_i)$, and after that, **I** will have produced an infinite sequence $(z_i^n)_{n\in\omega}$; we can let $A = \{z_i^n \mid n \in \omega\}$. Then **I** wins if and only if for some $q \lessapprox p$, we have that $A = \{x \in X \mid s^\frown x \lhd q\}$.

Using exactly the same proof as in lemma II.29, we can show the following:

- player **I** has a strategy in A_p to reach $\mathcal{X}$ if and only if he has a winning strategy in $CA_p(\mathcal{X})$;

- player **II** has a strategy in A_p to reach $\mathcal{X}^c$ if and only if she has a winning strategy in $CA_p(\mathcal{X})$.

In particular, it is sufficient to show that for $p \in P$ and $\mathcal{X} \in \mathcal{P}(X^\omega) \cap L(\mathbb{R})$, the notion "**I** has a winning strategy in $CA_p(\mathcal{X})$" relativizes to $L(\mathbb{R})$. But this is true, since strategies in this game are points of a Polish space and thus can be coded by real numbers.

□

Our results theorem II.36, theorem II.39 and corollary II.40 are certainly not optimal, since the statements on the adversarial Ramsey property they give are not enough to recover the large cardinal assumptions used to deduce them, even in terms of consistency. And they are not enough to compare the stength of $\mathrm{Adv}(\Gamma)$ with this of $\mathrm{Det}_\omega(\Gamma)$ (for the classes Γ that are studied here), because the statements $\mathrm{Det}_\omega(\Gamma)$ have already been shown to be equiconsistent to large cardinal assumptions that are strictly weaker as those used in our results. For instance, Harrington showed [26] that $\mathrm{Det}_\omega(\mathbf{\Sigma}^1_1)$ is equivalent to the existence of a *sharp* for every real number, an hypothesis that is weaker in consistency than the existence of a measurable cardinal and that is not enough to deduce $\mathrm{Det}_\mathbb{R}(\mathbf{\Sigma}^1_1)$ (and thus, to deduce $\mathrm{Adv}(\mathbf{\Sigma}^1_1)$ using our methods). This particular case will be discussed at the end of this section. Then, Woodin showed that the determinacy of games on ω with payoff in $L(\mathbb{R})$ had the same consistency strength as the existence of ω Woodin cardinals (see [33] for a proof of the direction from determinacy to large cardinals, and [50] for the other direction). However, it seems that ω Woodin cardinals are not enough to get the determinacy of games on real numbers with payoff in $L(\mathbb{R})$, so to get $\mathrm{Adv}(\widetilde{L(\mathbb{R})})$ using our methods. The same occur for the case of $\mathbf{\Sigma}^1_n$-sets for $n \geqslant 2$, for which $\mathrm{Det}_\omega(\mathbf{\Sigma}^1_n)$ has been shown to be equivalent in consistency strength to large cardinal assumptions by Woodin (see [48]). So the question of the comparison between $\mathrm{Adv}(\Gamma)$ and $\mathrm{Det}_\omega(\Gamma)$ remains widely open. However, $\mathrm{Adv}(\Gamma)$ seem, in general, to be quite close to $\mathrm{Det}_\omega(\Gamma)$, and to illustrate this, in the rest of this section, we will study the link between the adversarial Ramsey property and the property of being *homogeneously Souslin*, a property of sets of sequences closely linked to determinacy.

In what follows, if X and K are sets, and we consider a tree T on $X \times K$ as a subset of $X^{<\omega} \times K^{<\omega}$, whose elements are pairs of finite sequences of the same length. Given $s \in X^{<\omega}$, we will let $T_s = \{t \in K^{|s|} \mid (s,t) \in T\}$. We will often identify the sets $(X \times K)^\omega$ and $X^\omega \times K^\omega$, and thus consider $[T]$ as a subset of $X^\omega \times K^\omega$. We denote by $p : X^\omega \times K^\omega \to X^\omega$ the first projection. If $m \leqslant n$ are integers, and if $\mathcal{U}$ is an ultrafilter on K^n, we will denote by $\pi^n_m(\mathcal{U})$ the ultrafilter on K^m defined by $A \in \pi^n_m(\mathcal{U}) \Leftrightarrow \{(k_0, \ldots, k_{n-1}) \in K^n \mid (k_0, \ldots, k_{m-1}) \in A\} \in \mathcal{U}$.

Definition II.41.

1. Let X, K be sets and T be a tree on $X \times K$. We say that T is *homogeneous* if there exists a family $(\mathcal{U}_s)_{s \in X^{<\omega}}$, where $\mathcal{U}_s$ is a $\max(|X|^+, \aleph_1)$-complete ultrafilter on $K^{|s|}$, satisfying the following properties:

   (a) for every $s \in X^{<\omega}$, $T_s \in \mathcal{U}_s$;

   (b) for every $s, t \in X^{<\omega}$ with $s \subseteq t$, we have $\mathcal{U}_s = \pi^{|t|}_{|s|}(\mathcal{U}_t)$;

(c) for every $x \in p([T])$, and for every sequence of sets $(A_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{U}_{x\restriction n}$, there exists $k \in K^\omega$ such that for every $n \in \omega$, $k_{\restriction n} \in A_n$.

2. Let X be a set and $\mathcal{X} \subseteq X^\omega$. We say that $\mathcal{X}$ is *homogeneously Souslin* if there exists a set K and an homogeneous tree T on $X \times K$ such that $\mathcal{X} = p([T])$.

This is a classical fact that homogeneously Souslin sets are determined (see [50], section 4). This fact is often used in proofs of determinacy from large cardinals; for example, the results of Martin from a measurable cardinal, or of Martin and Steel, and of Woodin, supposing the existence of Woodin cardinals with a measurable above them, actually show the fact that the studied sets are homogeneously Souslin. Our result will be the following:

Theorem II.42. *Let $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ be a Gowers space and suppose that there is no measurable cardinal $\leqslant |P|$. Then every homogeneously Souslin subset of X^ω is determined.*

This result is interesting because unlike previous results, it does not deduce the adversarial Ramsey property for a set $\mathcal{X}$ from an assumption on a set of real numbers, but from an assumption on the set $\mathcal{X}$ himself. Before proving it, we recall an usual fact about measurable cardinals: if $\mathcal{U}$ is a σ-complete, nonprincipal ultrafilter on a set K, then there exists a measurable cardinal κ such that $\mathcal{U}$ is actually κ-complete (this is a conseqence of the proof of lemma 10.2 in [29]).

Proof of theorem II.42. Let $\mathcal{X} \subseteq X^\omega$ be a homogeneously Souslin set, K a set, T a homogeneous tree on $X \times K$ such that $\mathcal{X} = p([T])$, and $(\mathcal{U}_s)_{s\in X^{<\omega}}$ a family of ultrafilters witnessing that T is a homogeneous tree. Given $s \in X^{<\omega}$, if $\mathcal{U}_s$ is nonprincipal, then by the previous remark, it is κ-complete for a measurable cardinal κ, so in particular it is $|P|^+$-complete; and this is obviously also true if $\mathcal{U}_s$ is principal.

Let $p \in P$; we show that either **I** has a strategy in Kastanas' game K_p to reach $\mathcal{X}$, or **II** has one to reach $\mathcal{X}^c$. For this, we consider the following game K_p^*:

I		x_0, q_0, k_0		l_0, x_1, q_1, k_1		$\dots$	
II	p_0		y_0, p_1		y_1, p_2		$\dots$

where the x_i's and the y_i's are elements of X, the p_i's and the q_i's are elements of P, and the k_i's and the l_i's are elements of K. The rules are the following:

- for **I**: for all $i \in \omega$, $(x_0, y_0, \dots, x_{i-1}, y_{i-1}, x_i) \lhd p_i$ and $q_i \leqslant p_i$;
- for **II**: $p_0 \leqslant p$, and for all $i \in \omega$, $(x_0, y_0 \dots, x_i, y_i) \lhd q_i$ and $p_{i+1} \leqslant q_i$.

The outcome of the game is the pair of sequences $((x_0, y_0, x_1, y_1, \dots), (k_0, l_0, k_1, l_1, \dots)) \in X^\omega \times K^\omega$.

Since $[T]$ is a closed subset of $X^\omega \times K^\omega$, then either **I** has a strategy to reach $[T]$ in K_p^*, or **II** has one to reach $[T]^c$. So the conclusion will follow from the following fact:

Fact II.43.

1. *If **I** has a strategy in K_p^* to reach $[T]$, then he has a strategy in K_p to reach $\mathcal{X}$.*

2. *If **II** has a strategy in K_p^* to reach $[T]^c$, then she has a strategy in K_p to reach $\mathcal{X}^c$.*

Proof. 1. If **I** has a strategy in K_p^* to reach $[T]$, then the same strategy used in K_p, but omitting to display the k_i's and the l_i's, will enable him to reach $\mathcal{X}$.

2. Let τ^* be a strategy for **II** in K_p^* to reach $[T]^c$. Let $e = (p_0, x_0, q_0, y_0, \ldots, p_n, x_n, q_n)$ be a partial play of K_p ending with a move of **I**, and let $s = (x_0, y_0, \ldots, x_n)$. Since $\mathcal{U}_s$ is $|P|^+$-complete, there exist an unique pair $(y_n, p_{n+1}) \in X \times P$ such that $\{(k_0, l_0, \ldots, k_n) \in K^{2n+1} \mid \tau^*(p_0, x_0, q_0, k_0, y_0, p_1, l_0, \ldots, p_n, l_{n-1}, x_n, q_n, k_n) = (y_n, p_{n+1})\} \in \mathcal{U}_s$; let call this pair $\tau(e)$. This defines a strategy τ for **II** in K_p; we will show that this strategy enables her to reach $\mathcal{X}^c$.

   Suppose not. Then there exists a play $(p_0, x_0, q_0, y_0, p_1, \ldots)$ of K_p during which **II** always plays according to τ and such that $(x_0, y_0, x_1, y_1, \ldots) \in \mathcal{X}$. For every $n \geqslant 1$, let $A_{2n+1} = \{(k_0, l_0, \ldots, k_n) \in K^{2n+1} \mid \tau^*(p_0, x_0, q_0, k_0, y_0, p_1, l_0 \ldots, p_n, l_{n-1}, x_n, q_n, k_n) = (y_n, p_{n+1})\}$. This is an element of $\mathcal{U}_{(x_0, y_0, \ldots, x_n)}$, so $B_{2n+1} = A_{2n+1} \cap T_{(x_0, y_0, \ldots, x_n)}$ is also in $\mathcal{U}_{(x_0, y_0, \ldots, x_n)}$. Since $(x_0, y_0, x_1, y_1, \ldots) \in p([T])$, then by the definition of a homogeneous tree, we get that there exists a sequence $(k_0, l_0, k_1, l_1, \ldots) \in K^\omega$ such that for every $n \in \omega$, $(k_0, l_0, \ldots, k_n) \in B_{2n+1}$. This shows that $(p_0, x_0, q_0, k_0, y_0, p_1, l_0, x_1, q_1, k_1, \ldots)$ is a play of K_p^* during which **II** always plays according to τ^*, so $((x_0, y_0, \ldots), (k_0, l_0, \ldots)) \in [T]^c$. But on the other hand, we have for every $n \in \omega$, $(k_0, l_0, \ldots, k_n) \in B_{2n+1} \subseteq T_{(x_0, y_0, \ldots, x_n)}$, so $((k_0, l_0, \ldots, k_n), (x_0, y_0, \ldots, x_n)) \in T$, and thus $((x_0, y_0, \ldots), (k_0, l_0, \ldots)) \in [T]$, a contradiction.

□

□

Though being determined is not so far from being homogeneously Souslin, theorem II.42 still does not enable us to compare $\mathrm{Det}_\omega(\Gamma)$ and $\mathrm{Adv}(\Gamma)$, since the minimal hypotheses to get consistantly $\mathrm{Det}_\omega(\Gamma)$ do not enable to prove that, consistently, every Γ-subset of ω^ω is homogeneously Souslin. We can illustrate this on the case $\Gamma = \mathbf{\Sigma}^1_1$. As we already mentioned, $\mathrm{Det}_\omega(\mathbf{\Sigma})^1_1$ is equivalent to the existence of a *sharp* for every real (for a definition of the sharps, see [30], section 9). The proof of $\mathrm{Det}_\omega(\mathbf{\Sigma})^1_1$ assuming the existence of sharps is very similar to the proof of theorem II.42 (see [30], theorem 31.2), however everything is done in $L[x]$, for some real number x. This does not enable to generalize to the determinacy of Kastanas' game: indeed, in this game, players play elements of P, that are in general reals, and for this reason they do not necessarily belong to $L[x]$. In general, it seems that the main obstacle to prove the equivalence between $\mathrm{Adv}(\Gamma)$ and $\mathrm{Det}_\omega(\Gamma)$ is the fact that the parameters played by players in Kastanas' game do not belong to the inner models with large cardinals given by $\mathrm{Det}_\omega(\Gamma)$. This should be taken into account when trying to prove, either that $\mathrm{Det}_\omega(\Gamma)$ and $\mathrm{Adv}(\Gamma)$ have the same consistency strenght, or that they do not.

Chapter III

Ramsey theory in uncountable spaces

In the setting of Gowers spaces defined in the last chapter, the set of points is always countable: this is necessary to perform the diagonal arguments in the proof of our dichotomies. As it will be shown in the first section of this chapter, this hypothesis is necessary: when X in uncountable, we can find very simple sets that are neither adversarially Ramsey nor strategically Ramsey (see proposition III.1). Thus this chapter is devoted to present weak versions of the results of the last chapter in the case where X is uncountable.

The results we will present are inspired by Gowers' theorems I.8 and I.11: they are based on metrical approximation. In section III.2, we will define the setting of *approximate Gowers spaces*, where the set of points is a Polish space. In such a space, analogs of theorem II.4, theorem II.14 and corollary II.21 involving approximation, will be shown (these are theorem III.6 and corollary III.11). The proof of the first one is based on the corresponding result without approximation.

In section III.3, we present a general method to get, from statements involving a strategy for **I** in the asymptotic game, non-strategical Ramsey conclusions as in Mathias–Silver's theorem, Milliken's theorem, or one of the conclusions of Gowers' theorem I.8. Our method enables as well to get such results in Gowers spaces, without approximation, and in approximate Gowers spaces, with approximation (actually, our results are stated for structures more general than approximate Gowers spaces, that are called *approximate asymptotic spaces*). Our central result, theorem III.16, can be seen as a generalization of lemma II.18. From this and from the results of section III.2, we can deduce an abstract version of Gowers' theorem (corollary III.17) that immediately implies as well Gowers' theorems I.8 and I.11 and Mathias–Silver's theorem.

III.1 A counterexample

In this section, we present a counterexample showing the necessity of the hypothesis that the set of points is countable in the definition of a Gowers space: without this hypothesis, theorems II.4 and II.14 are not true in general.

Let X be the $\mathbb{R}$-vector space $\mathbb{R}^\omega$, endowed with the product topology. This makes it a Polish vector space. For $x = (x^i)_{i\in\omega} \in X$, we let $\mathrm{supp}(x) = \{i \in \omega \mid x^i \neq 0\}$, and we let $N(x) = x^{\min \mathrm{supp}(x)}$ if $x \neq 0$, and $N(0) = 0$. A *block sequence* is an infinite sequence $(x_n)_{n\in\omega}$ of nonzero vectors of X such that $\mathrm{supp}(x_0) < \mathrm{supp}(x_1) < \mathrm{supp}(x_2) < \ldots$. The closed linear span of a block sequence is called *block subspace*. Remark that if Y is a block subspace generated by a block sequence $(y_n)_{n\in\omega}$, then for $(a^n)_{n\in\omega} \in \mathbb{R}^\omega$, the sum $\sum_{n=0}^{\infty} a^n y_n$ is always convergent, and the elements of Y are exactly the vectors of X that can be expressed as such a sum. We denote by P the set of all block sequences. For $(x_n), (y_n) \in P$, we say that $(x_n) \leqslant (y_n)$ if for every $n \in \omega$, x_n is a (finite) linear combination of the y_m's; and we say that $(x_n) \leqslant^* (y_n)$ if there exists $n_0 \in \omega$ such that $(x_{n+n_0})_{n\in\omega} \leqslant (y_n)_{n\in\omega}$. Finally, for $x \in X$ and $(x_n) \in P$, we say that $x \lhd (x_n)$ if x belongs to the block subspace generated by (x_n).

It is easy to verify that the space $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ satisfies all the axioms defining a forgetful Gowers space, apart from the fact that X is not countable (here, we defined $\lhd$ as a subset of $X \times P$); to verify the diagonalisation axiom, use a similar method as for the Rosendal space. Remark that, for $(x_n), (y_n) \in P$ with $(x_n) \leqslant (y_n)$, we have $(x_n) \lessapprox (y_n)$ if and only of there exists $n_0 \in \omega$ such that for every n large enough, x_n and y_{n+n_0} are colinear. We can define, for $\mathcal{G}$, the notions of strategically Ramsey sets and of adversarially Ramsey sets exactly in the same way as for a genuine Gowers space. We equip X^ω with the product topology. We will show the following:

Proposition III.1. *There exist a Borel set $\mathcal{X} \subseteq X^\omega$ that is not strategically Ramsey.*

Remark that the set $\mathcal{X}$ we will build has the form $\{(x_n)_{n\in\omega} \in X^\omega \mid (x_0, x_1) \in \mathcal{Y}\}$ for some set $\mathcal{Y} \subseteq X^2$; so if we endow X with the discrete topology and X^ω with the product topology, then $\mathcal{X}$ is actually clopen.

Also recall that $\mathcal{X}$ is strategically Ramsey if and only if $\{(x_n)_{n\in\omega} \in X^\omega \mid (x_0, x_2, \ldots) \in \mathcal{X}\}$ is adversarially Ramsey. So we deduce the following corollary:

Corollary III.2. *Not all Borel subsets X^ω are adversarially Ramsey.*

Proof of proposition III.1. The set P can be seen as a subset of X^ω with the product topology; it is a G_δ-subset, so a Polish space. Therefore, there is a Borel isomorphism $\varphi : \mathbb{R} \longrightarrow P$. We define the set $\mathcal{Y} \subseteq X^2$ in the following way: $(x, y) \in \mathcal{Y}$ if y is equal to a term of the block sequence $\varphi(N(x))$. This is a Borel subset of X^2. Let $\mathcal{X} = \{(x_n)_{n\in\omega} \in X^\omega \mid (x_0, x_1) \in \mathcal{Y}\}$. We show that $\mathcal{X}$ is not strategically Ramsey.

Firstly suppose that there exists $p \in P$ such that player **II** has a strategy in G_p to reach $\mathcal{X}$ and consider the following play of G_p, where **II** uses her strategy:

I	p		q	
II		x		y

Player **I** starts the game by playing p (his move actually does not matter). According to her strategy, **II** answers by some vector x. Let $(x_n)_{n\in\omega} = \varphi(N(x))$. Let $A = \{n \in \omega \mid x_n \lhd p\}$. There are two cases.

First case: A is finite. Then let $q = (y_n)_{n\in\omega}$ be a final segment of the sequence p such that $\forall n \in A\ \operatorname{supp}(x_n) < \operatorname{supp}(y_0)$. We make **I** play q. Then, whatever is the answer $y \lhd q$ of **II**, we have $\operatorname{supp}(x_n) < \operatorname{supp}(y)$ for every $n \in A$, so y is different from all the x_n's, $n \in \omega$. So $(x, y) \notin \mathcal{Y}$ and **II** loses the game, a contradiction.

Second case: A is infinite. Then let $(n_i)_{i\in\omega}$ be an increasing enumeration of A and let $q = (x_{n_0} + x_{n_1}, x_{n_2} + x_{n_3}, x_{n_4} + x_{n_5}, \ldots)$. We make **I** play q. Then, whatever is the answer $y \lhd q$ of **II**, y is different from all the x_n's, $n \in \omega$, so $(x, y) \notin \mathcal{Y}$ and **II** loses the game, a contradiction.

Now suppose that there exists $p = (x_n)_{n\in\omega} \in P$ such that player **I** has a strategy in F_p to reach $\mathcal{X}^c$ and consider the following play of F_p, where **I** uses his strategy:

I	q		r	
II		x		x_k

Player **I** starts by playing some $q \lessapprox p$ according to his strategy. Now consider a real number u such that $\varphi(u) = p$. **II** can always answer by an $x \lhd q$ such that $N(x) = u$. Then, according to his strategy, **I** answers by $r = (y_n)_{n\in\omega}$. Since $(y_n) \lessapprox (x_n)$, there exists $k, l \in \omega$ such that x_k and y_l are colinear, so $x_k \lhd r$. We make **II** play x_k, which is a term of the block sequence $(x_n)_{n\in\omega} = \varphi(N(x))$, so $(x, x_k) \in \mathcal{Y}$ and **I** loses the game, a contradiction.

□

III.2 Approximate Gowers spaces

The counterexample given in the last section shows that the formalism of Gowers spaces is not sufficient if we want to work with uncountable spaces, like Banach spaces. In this section, following an idea introduced by Gowers for his Ramsey-type theorem I.8, we introduce an approximate version of Gowers spaces, allowing us to get approximate Ramsey-type results in situations where the set of points is uncountable. The results of this section, along with these of the next section, will allow us to directly recover results like Gowers' theorems I.8 and I.11. The interest of the spaces we introduce here is more practical that theoretical: their main aim is to allow applications, for instance in Banach-space geometry.

Definition III.3. An *approximate Gowers space* is a sextuple $\mathcal{G} = (P, X, d, \leqslant, \leqslant^*, \lhd)$, where P is a nonempty set, X is a nonempty Polish space, d is a compatible distance on X, $\leqslant$ and $\leqslant^*$ are two quasiorders on P, and $\lhd \subseteq X \times P$ is a binary relation, satisfying the same axioms 1. – 3. as in the definition of a Gowers' space and satisfying moreover the two following axioms:

4. for every $p \in P$, there exists $x \in X$ such that $x \lhd p$;

5. for every $x \in X$ and every $p, q \in P$, if $x \lhd p$ and $p \leqslant q$, then $x \lhd q$.

The relation $\lessapprox$ and the compatibility relation on P are defined in the same way as for a Gowers space.

For $p \in P$, we define the games A_p, B_p, F_p, and G_p exactly in the same way as for Gowers spaces (see definitions II.2 and II.12), except that we naturally replace the rules $(x_0, y_0, \ldots, x_{i-1}, y_{i-1}, x_i) \lhd p_i$ and $(x_0, y_0, \ldots, x_i, y_i) \lhd q_i$ in the definition of A_p and B_p and the rule $(x_0, \ldots, x_i) \lhd p_i$ in the definition of F_p and G_p, respectively by $x_i \lhd p_i$, $y_i \lhd q_i$ and $x_i \lhd p_i$. The outcome is, there, an element of X^ω.

Remark that, with this definition, approximate Gowers spaces are always forgetful, that is, we define the relation $\lhd$ as a subset of $X \times P$ and not as a subset of $\mathrm{Seq}(X) \times P$. Indeed, for technical reasons, to be able to get the results we want (in particular theorem III.6), we can only make depend the range of possible choices of points of a player in the games on the subspace played just before by the other player (for example, the range of possible choices of x_i in G_p can only depend on p_i). That is not a real problem since all interesting examples we currently know satisfy this requirement.

In the rest of this section, we fix an approximate Gowers space $\mathcal{G} = (P, X, d, \leqslant, \leqslant^*, \lhd)$. An important notion in the setting of approximate Gowers spaces is that of *expansion*.

Definition III.4.

1. Let $A \subseteq X$ and $\delta > 0$. The *δ-expansion* of A is the set $(A)_\delta = \{x \in X \mid \exists y \in A\ d(x, y) \leqslant \delta\}$;

2. Let $\mathcal{X} \subseteq X^\omega$ and $\Delta = (\Delta_n)_{n\in\omega}$ be a sequence of positive real numbers. The *Δ-expansion* of $\mathcal{X}$ is the set $(\mathcal{X})_\Delta = \{(x_n)_{n\in\omega} \in X^\omega \mid \exists (y_n)_{n\in\omega} \in \mathcal{X}\ \forall n \in \omega\ d(x_n, y_n) \leqslant \Delta_n\}$.

We can now define the notions of adversarially Ramsey sets and of strategically Ramsey sets in an approximate Gowers space:

Definition III.5. Let $\mathcal{X} \subseteq X^\omega$.

1. We say that $\mathcal{X}$ is *adversarially Ramsey* if for every sequence Δ of positive real numbers and for every $p \in P$, there exists $q \leqslant p$ such that either player **I** has a strategy in A_q to reach $(\mathcal{X})_\Delta$, or player **II** has a strategy in B_q to reach $(\mathcal{X}^c)_\Delta$.

2. We say that $\mathcal{X}$ is *strategically Ramsey* if for every sequence Δ of positive real numbers and for every $p \in P$, there exists $q \leqslant p$ such that either player **I** has a strategy in F_q to reach $\mathcal{X}^c$, or player **II** has a strategy in G_q to reach $(\mathcal{X})_\Delta$.

Remark that if $\mathcal{G}_0 = (P, X, \leqslant, \leqslant^*, \lhd)$ is a forgetful Gowers space (where we consider $\lhd$ as a subset of $X \times P$), then we can turn it into an approximate Gowers space $\mathcal{G}'_0 = (P, X, d, \leqslant, \leqslant^*, \lhd)$ by taking for d the discrete distance on X ($d(x, y) = 1$ for $x \neq y$). In this way, for $0 < \delta < 1$ and $A \subseteq X$ we have $(A)_\delta = A$, and for Δ a sequence of positive real numbers strictly lower than 1 and for $\mathcal{X} \subseteq X^\omega$, we have $(\mathcal{X})_\Delta = \mathcal{X}$. So for a set $\mathcal{X} \subseteq X^\omega$, the definition of being adversarially or strategically Ramsey in $\mathcal{G}_0$ and in $\mathcal{G}'_0$ coincide. Therefore, we will consider forgetful Gowers spaces as particular cases of approximate Gowers spaces.

Another interesting family of examples of approximate Gowers spaces is the following. Given a Banach space E with a Schauder basis $(e_i)_{i\in\omega}$, we can consider the *canonical approximate Gowers space over* E, $\mathcal{G}_E = (P, S_E, d, \subseteq, \subseteq^*, \in)$, where P is the set of all block subspaces of E, S_E is the unit sphere of E, d the distance given by the norm, and $X \subseteq^* Y$ if and only if Y contains some finite-codimensional block subspace of X. We will see in the next section how to get Gowers' theorems I.8 and I.11 from the study of this space.

The results that generalize theorems II.4 and II.14 to adversarial Gowers spaces are the following:

Theorem III.6.

1. *Every Borel subset of X^ω is adversarially Ramsey;*

2. *Every analytic subset of X^ω is strategically Ramsey.*

Proof. Remark that to prove *2.*, it is actually sufficient to prove the following apparently weaker result: for every $\mathcal{X} \subseteq X^\omega$ analytic, for every sequence Δ of positive real numbers and for every $p \in P$, there exists $q \leqslant p$ such that either player **I** has a strategy in F_q to reach $(\mathcal{X}^c)_\Delta$, or player **II** has a strategy in G_q to reach $(\mathcal{X})_\Delta$. Indeed, if $\mathcal{X}$ is analytic, then $(\mathcal{X})_{\frac{\Delta}{2}}$ is analytic too; so applying the last result to $(\mathcal{X})_{\frac{\Delta}{2}}$ and to the sequence $\frac{\Delta}{2}$, and using the fact that $\left(\left((\mathcal{X})_{\frac{\Delta}{2}}\right)^c\right)_{\frac{\Delta}{2}} \subseteq \mathcal{X}^c$ and $\left((\mathcal{X})_{\frac{\Delta}{2}}\right)_{\frac{\Delta}{2}} \subseteq (\mathcal{X})_\Delta$, we get that $\mathcal{X}$ is strategically Ramsey.

Now let $D \subseteq X$ be a countable dense subset, and Δ be a sequence of positive real numbers. Consider the Gowers space $\mathcal{G}_\Delta = (P, D, \leqslant, \leqslant^*, \lhd_\Delta)$, where $\lhd_\Delta$ is defined by $(y_0, \ldots, y_n) \lhd_\Delta p$ if there exists $x_n \in X$ with $x_n \lhd p$ and $d(x_n, y_n) < \Delta_n$. To avoid confusion, we denote by A_p, B_p, F_p and G_p the games in the space $\mathcal{G}$, and by A_p^Δ, B_p^Δ, F_p^Δ and G_p^Δ the games in the space $\mathcal{G}_\Delta$.

If $\mathcal{X}$ is Borel (resp. analytic) then the set $\mathcal{X} \cap D^\omega$ is Borel (resp. analytic) too (when D is endowed by the discrete topology), so it is adversarially (resp. strategically) Ramsey in $\mathcal{G}_\Delta$. So to prove the theorem, it is enough to show that for every $p \in P$, we have that:

(i) if player **I** has a strategy in F_p^Δ to reach $\mathcal{X}^c$, then he has a strategy in F_p to reach $(\mathcal{X}^c)_\Delta$;

(ii) if player **II** has a strategy in G_p^Δ to reach $\mathcal{X}$, then she has a strategy in G_p to reach $(\mathcal{X})_\Delta$;

(iii) if player **I** has a strategy in A_p^Δ to reach $\mathcal{X}$, then he has a strategy in A_p to reach $(\mathcal{X})_\Delta$;

(iv) if player **II** has a strategy in B_p^Δ to reach $\mathcal{X}^c$, then she has a strategy in B_p to reach $(\mathcal{X}^c)_\Delta$.

We only prove (i) and (ii); the proofs of (iii) and (iv) are naturally obtained by combining the proofs of (i) and (ii).

(i) As usual, we fix a strategy for **I** in F_p^Δ, enabling him to reach $\mathcal{X}^c$, and we describe a strategy for **I** in F_p to reach $(\mathcal{X}^c)_\Delta$ by simulating a play $(p_0, x_0, p_1, x_1, \ldots)$ of F_p by a play $(p_0, y_0, p_1, y_1, \ldots)$ of F_p^Δ in which **I** always plays using his strategy; we suppose moreover that the same subspaces are played by **I** in both games.

Suppose that in both games, the first n turns have been played, so the p_i's, the x_i's and the y_i's are defined for $i < n$. According to his strategy, in F_p^Δ, **I** plays some $p_n \lessapprox p$. Then we let **I** play the same p_n in F_p, and in this game, **II** answers with $x_n \in X$ such that $x_n \lhd p_n$. Then we choose $y_n \in D$ such that $d(x_n, y_n) < \Delta_n$; by the definition of $\lhd_\Delta$, we have that $(y_0, \ldots, y_n) \lhd_\Delta p_n$, so we can let **II** play y_n in F_p^Δ, and the games can continue!

Due to the choice of the strategy of **I** in F_p^Δ, we get that $(y_n)_{n\in\omega} \in \mathcal{X}^c$, so $(x_n)_{n\in\omega} \in (\mathcal{X}^c)_\Delta$ as wanted.

(ii) We simulate a play $(p_0, x_0, p_1, x_1, \ldots)$ of G_p by a play $(p_0, y_0, p_1, y_1, \ldots)$ of G_p^Δ where **II** uses a strategy to reach $\mathcal{X}$, and we suppose moreover that **I** plays the same subspaces in both games. Suppose that the first n turns of boths games have been played. In G_p, **I** plays p_n. We make **I** copy this move in G_p^Δ, and according to her strategy, **II** answers, in this game, by a $y_n \in D$ such that $(y_0, \ldots, y_n) \lhd_\Delta p_n$. We can find $x_n \in X$ such that $x_n \lhd p_n$ and $d(x_n, y_n) < \Delta_n$; we let **II** play this x_n in G_p and the games continue. At the end, we have that $(y_n)_{n\in\omega} \in \mathcal{X}$, so $(x_n)_{n\in\omega} \in (\mathcal{X})_\Delta$ as wanted.

□

Say that the approximate Gowers space $\mathcal{G}$ is *analytic* if P is an analytic subset of a Polish space, if the relation $\leqslant$ is a Borel subset of P^2, and if for every open set $U \subseteq X$, the set $\{p \in P \mid \exists x \in U \; x \lhd p\}$ is a Borel subset of P. Also recall that if Y is a Polish space, and if $\mathcal{F}(Y)$ is the set of all closed subsets of Y, the *Effros Borel structure* on $\mathcal{F}(Y)$ is the σ-algebra generated by the sets $\{F \in \mathcal{F}(Y) \mid F \cap U \neq \varnothing\}$ where U varies over open subsets of Y. If P is an analytic subset of $\mathcal{F}(X)$ endowed with the Effros Borel structure, and if $\subseteq$ and $\lhd$ are respectively the inclusion and the membership relation, then $\mathcal{G}$ is an analytic approximate Gowers space. This is, for instance, the case of the canonical approximate Gowers space $\mathcal{G}_E$ over a Banach space E with a basis: indeed, the fact that $F \in \mathcal{F}(S_E)$ is the unit sphere of a block subspace of E can be written "there exists a block sequence $(x_i)_{i\in\omega}$ such that for every U in a countable basis of open subsets of S_E, $F \cap U \neq \varnothing$ if and only if there exists $n \in \omega$ and $(a_i)_{i<n} \in \mathbb{Q}^n\backslash\{0\}$ with $\frac{\sum_{i<n} a_i x_i}{\|\sum_{i<n} a_i x_i\|} \in U$".

Remark that if $\mathcal{G}$ is an analytic approximate Gowers space and Δ a sequence of positive real numbers, then the Gowers space $\mathcal{G}_\Delta$ defined in the proof of theorem III.6 is analytic. So this proof, combined with corollaries II.8 and II.16, gives us the following:

Corollary III.7. *Let Γ be a suitable class of subsets of Polish spaces. Suppose that every Γ-subset of $\mathbb{R}^\omega$ is determined. Then for every analytic approximate Gowers space $\mathcal{G} = (P, X, d, \leqslant, \leqslant^*, \lhd)$, we have that:*

1. *every Γ-subset of X^ω is adversarially Ramsey;*
2. *every $\exists\Gamma$-subset of X^ω is strategically Ramsey.*

However, it is not straightforward, in the setting of approximate Gowers spaces, to get results in $ZF + DC + AD_\mathbb{R}$, because the proof of III.6 uses the full axiom of choice. Indeed, since there is, in general, an uncountable number of subspaces, in the proof of (ii) (and the same will happen in the proofs of (iii) and (iv)), player **II** needs AC to choose x_n such that $d(x_n, y_n) < \Delta_n$ and $x_n \lhd p_n$. However, under a slight restriction, we can get a positive result. Define the notion of an *effective approximate Gowers space* exactly in the same way as for effective Gowers spaces. Effective forgetful Gowers spaces are obviously effective when seen as approximate Gowers spaces, but also, the canonical approximate Gowers space $\mathcal{G}_E$ is effective (this can be shown in the same way as for the Rosendal space). If $\mathcal{G}$ is an effective approximate Gowers space and Δ a sequence of positive real numbers, then the Gowers space $\mathcal{G}_\Delta$ defined in the proof of theorem III.6 is also effective. And we have:

Corollary III.8 ($ZF + DC + AD_\mathbb{R}$)**.** *Let $\mathcal{G} = (P, X, d, \leqslant, \leqslant^*, \lhd)$ be an effective approximate Gowers space such that P is a subset of a Polish space, and such that for every $p \in P$, the set $\{x \in X \mid x \lhd p\}$ is closed in X. Then every subset of X^ω is adversarially Ramsey and strategically Ramsey.*

Proof. We follow the proof of theorem III.6, using corollaries II.10 and II.16 to get that the set $\mathcal{X} \cap D^\omega$ is adversarially Ramsey and strategically Ramsey in $\mathcal{G}_\Delta$. The only thing to do is to verify that the proofs of (i)–(iv) can be carried out with only DC instead of AC; as previously, we only do it for (i) and (ii). In the proof of (i), we have to be able to choose $y_n \in D$ such that $d(x_n, y_n) < \Delta_n$; this can be done by fixing, at the beginning of the proof, a well-ordering of D, and by choosing, each time, the least such y_n. In the proof of (ii), the difficulty is to choose x_n; so we have to prove that given $p \in P$, $n \in \omega$, and $y \in D$, if there exists $x \in X$ with $x \lhd p$ and $d(x, y) < \Delta_n$, then we are able to choose such an x without using AC.

Using countable choices, for every $y \in D$ and $n \in \omega$, we choose $f_{y,n} : \omega^\omega \longrightarrow B(y, \Delta_n)$ a continuous surjection. Given p, n and y as in the previous paragraph, we can let $F = \{u \in \omega^\omega \mid f_{y,n}(u) \lhd p\}$, a closed subset of ω^ω. Consider $T \subseteq \omega^{<\omega}$ the unique pruned tree such that $F = [T]$. Then we can let u be the leftmost branch of T and let $x = f_{y,n}(u)$.

□

Remark that in the proof of theorem III.6, the most important hypothesis on X is its separableness, and the only interest of its Polishness is the fact that if $\mathcal{X}$ is analytic, then $(\mathcal{X})_{\frac{\Delta}{2}}$ is analytic too. Thus, if we only suppose X separable, then the *1.* of this theorem remains true, and the *2.* can be replaced with "for every $\mathbf{\Sigma}^1_1$-subset $\mathcal{X}$ of X^ω, for every sequence Δ of positive real numbers and for every $p \in P$, there exists $q \leqslant p$ such that either player **I** has a strategy in F_q to reach $(\mathcal{X}^c)_\Delta$, or player **II** has a strategy in G_q to reach $(\mathcal{X})_\Delta$". In the same way, given a suitable class Γ of subsets of Polish spaces, say that a subset $\mathcal{Y}$ of a topological space Y is *potentially* Γ if for every Polish space Z and every continuous mapping $f : Z \longrightarrow Y$, $f^{-1}(\mathcal{Y})$ is a Γ-subset of Z. Then corollary III.7 remains true for X only assumed separable, if we modify the conclusion of *2.* in the same way as for theorem III.6, and if in *1.* and *2.*, we replace Γ-subsets and $\exists\Gamma$-subsets respectively by potentially Γ-subsets and potentially $\exists\Gamma$-subsets. However, the proof of corollary III.8 does not adapt to arbitrary separable metric spaces; but it remains true if we only suppose that X is an analytic subset of a Polish space. All of these extensions can be combined to the other results of this section and of the next section, since their proof will only use the separableness of X (or the fact that X is an analytic subset of a Polish space, if we work in $ZF + DC$).

We now introduce the pigeonhole principle in an approximate Gowers space and its consequences. We actually only need an approximate pigeonhole principle in this setting. For $q \in P$ and $A \subseteq X$, we write abusively $q \subseteq A$ to say that $\forall x \in X\ (x \lhd q \Rightarrow x \in A)$.

Definition III.9. The approximate Gowers space $\mathcal{G}$ is said to satisfy the *pigeonhole principle* if for every $p \in P$, $A \subseteq X$, and $\delta > 0$ there exists $q \leqslant p$ such that either $q \subseteq A^c$, or $q \subseteq (A)_\delta$.

For example, by theorem I.10, the canonical approximate Gowers space $\mathcal{G}_E$ satisfies the pigeonhole principle if and only if E is c_0-saturated.

As for Gowers spaces, we have the following proposition:

Proposition III.10. *Suppose that the approximate Gowers space $\mathcal{G}$ satisfies the pigeonhole principle. Let $\mathcal{X} \subseteq X^\omega$, $p \in P$ and Δ be a sequence of positive real numbers. If player **II** has a strategy in G_p to reach $\mathcal{X}$, then there exists $q \leqslant p$ such that player **I** has a strategy in F_q to reach $(\mathcal{X})_\Delta$.*

Before proving this proposition, let us make some remarks. Using again the fact that $\left((\mathcal{X})_{\frac{\Delta}{2}}\right)_{\frac{\Delta}{2}} \subseteq (\mathcal{X})_\Delta$, we deduce from proposition III.10 the following corollary:

Corollary III.11. *Suppose that the approximate Gowers space $\mathcal{G}$ satisfies the pigeonhole principle. Let $\mathcal{X} \subseteq X^\omega$ be a strategically Ramsey set. Then for every $p \in P$ and every sequence Δ of positive real numbers, there exists $q \leqslant p$ such that in F_q, player **I** either has a strategy to reach $\mathcal{X}^c$, or has a strategy to reach $(\mathcal{X})_\Delta$.*

Conversely, if the conclusion of corollary III.11 holds for sets of the form $\{(x_n)_{n\in\omega} \in X^\omega \mid x_0 \in F\}$, where $F \subseteq X$ is closed, then the space $\mathcal{G}$ satisfies the pigeonhole principle. Indeed, let $p \in P$, $A \subseteq X$ and $\delta > 0$. Let $F = \{x \in X \mid \forall y \in A \ d(x, y) \geqslant \delta\}$, and $\mathcal{X} = \{(x_n)_{n\in\omega} \in X^\omega \mid x_0 \in F\}$. Then by assumption, there exists $q \leqslant p$ such that **I** either has a strategy to reach $\mathcal{X}^c$, or has a strategy to reach $(\mathcal{X})_\Delta$, in F_q, where $\Delta = (\frac{\delta}{2}, \frac{\delta}{2}, \ldots)$. As in the case of Gowers spaces, in the first case we find $q_0 \lessapprox q$ with $q_0 \subseteq F^c \subseteq (A)_\delta$, and in the second case we get $q_0 \lessapprox q$ with $q_0 \subseteq (F)_{\frac{\delta}{2}} \subseteq A^c$.

Also remark that if $\mathcal{G}_0$ is a forgetful Gowers space, and if $\mathcal{G}_0'$ is the associated approximate Gowers space, then the pigeonhole principle in $\mathcal{G}_0$ is equivalent to the pigeonhole principle in $\mathcal{G}_0'$, and proposition III.10 and corollary III.11 are respectively the same as proposition II.20 and corollary II.21.

We now prove proposition III.10.

Proof of proposition III.10. Unlike the previous results about approximate Gowers spaces, here we cannot deduce this result from its exact version; thus, we adapt the proof of proposition II.20. To save notation, we show that there exists $q \leqslant p$ such that **I** has a strategy in F_q to reach $(\mathcal{X})_{3\Delta}$.

We fix τ a strategy for **II** in G_p to reach $\mathcal{X}$. We call a *state* a partial play of G_p either empty or ending with a move of **II**, during which **II** always plays according to her strategy. We say that a state *realises* a sequence $(x_0, \ldots, x_{n-1}) \in X^{<\omega}$ if it has the form $(p_0, x_0, \ldots, p_{n-1}, x_{n-1})$. The *length* of the state $\mathscr{s}$, denoted by $|\mathscr{s}|$, is the length of the sequence it realises. We define in the same way the notion of a *total state* (which is a total play of G_p) and of realisation for a total state; if an infinite sequence is realised by a total state, then it belongs to $\mathcal{X}$. We say that a point $x \in X$ is *reachable* from a state $\mathscr{s}$ if there exists $r \leqslant p$ such that $\tau(\mathscr{s} ^\frown r) = x$. Denote by $A_\mathscr{s}$ the set of all points that are reachable from the state $\mathscr{s}$. We will use the following fact.

Fact III.12. *For every state $\mathscr{s}$ and for every $q \leqslant p$, there exists $r \leqslant q$ such that $r \subseteq (A_\mathscr{s})_{\Delta_{|\mathscr{s}|}}$.*

Proof. Otherwise, by the pigeonhole principle, there would exist $r \leqslant q$ such that $r \subseteq (A_\mathscr{s})^c$. But then **I** could play r after the partial play $\mathscr{s}$, and **II** would answer, according to her strategy, by $x = \tau(\mathscr{s} ^\frown r)$ that should satisfy $x \lhd r$. Since $r \subseteq (A_\mathscr{s})^c$, this would imply that $x \in (A_\mathscr{s})^c$. But we also have, by the definition of $A_\mathscr{s}$, that $x \in A_\mathscr{s}$, a contradiction.

□

For two sequences $s, t \in X^{\leqslant\omega}$ of the same length, we denote by $d(s, t) \leqslant \Delta$ the fact that for every $i < |s|$, we have $d(s_i, t_i) \leqslant \Delta_i$. Let $D \subseteq X$ be a countable dense set and let $(s_n)_{n\in\omega}$ be an enumeration of $D^{<\omega}$ such that if $s_m \subseteq s_n$, then $m \leqslant n$. We define, for some $n \in \omega$, a state $\mathcal{S}_n$ realising a sequence t_n satisfying $d(s_n, t_n) \leqslant 2\Delta$, by induction in the following way: $\mathcal{S}_0 = \varnothing$ and for $n \geqslant 1$, letting $s_n = s_m {}^\frown y$ for some $m < n$ and some $y \in X$,

- if $\mathcal{S}_m$ has been defined and if there exists $z \in X$ reachable from $\mathcal{S}_m$ such that $d(y, z) \leqslant 2\Delta_{|\mathcal{S}_m|}$, then choose a $r \leqslant p$ such that $z = \tau(\mathcal{S}_m {}^\frown r)$ and put $t_n = t_m {}^\frown z$ and $\mathcal{S}_n = \mathcal{S}_m {}^\frown (r, z)$,
- otherwise, $\mathcal{S}_n$ is not defined.

Remark that if $\mathcal{S}_n$ is defined and if $s_m \subseteq s_n$, then $\mathcal{S}_m$ is defined, and we have $\mathcal{S}_m \subseteq \mathcal{S}_n$ and $t_m \subseteq t_n$.

We now define a $\leqslant$-decreasing sequence $(q_n)_{n\in\omega}$ of elements of P in the following way: $q_0 = p$ and

- if $\mathcal{S}_n$ is defined, then q_{n+1} is the result of the application of fact III.12 to $\mathcal{S}_n$ and q_n;
- $q_{n+1} = q_n$ otherwise.

Finally, let $q \leqslant p$ be such that for every $n \in \omega$, $q \leqslant^* q_n$. We will show that **I** has a strategy in F_q to reach $(\mathcal{X})_{3\Delta}$. We describe this strategy on the following play of F_q:

I	u_0		u_1		$\dots$	
II		x_0		x_1		$\dots$

We moreover suppose that at the same time as this game is played, we build a sequence $(n_i)_{i\in\omega}$ of integers, with $n_0 = 0$ and n_i being defined during the i^{th} turn, such that $(s_{n_i})_{i\in\omega}$ is increasing and for every $i \in \omega$, $|s_{n_i}| = i$, $\mathcal{S}_{n_i}$ is defined, and $d(s_{n_i}, (x_0, \dots, x_{i-1})) \leqslant \Delta$. This will be enough to conclude: indeed, $\bigcup_{i\in\omega} \mathcal{S}_{n_i}$ will be a total state realising the sequence $\bigcup_{i\in\omega} t_{n_i}$, showing that this sequence belongs to $\mathcal{X}$; and since $d\left(\bigcup_{i\in\omega} t_{n_i}, (x_i)_{i\in\omega}\right) \leqslant d\left(\bigcup_{i\in\omega} t_{n_i}, \bigcup_{i\in\omega} s_{n_i}\right) + d\left(\bigcup_{i\in\omega} s_{n_i}, (x_i)_{i\in\omega}\right) \leqslant 3\Delta$, we will have that $(x_i)_{i\in\omega} \in (\mathcal{X})_{3\Delta}$.

Suppose that the i^{th} turn of the game has just been played, so the sequence $(x_0, \dots, x_{i-1})$ and the integers $n_0, \dots, n_i$ has been defined. Then by construction of q_{n_i+1}, we have that $q_{n_i+1} \subseteq (A_{\mathcal{S}_{n_i}})_{\Delta_{|\mathcal{S}_{n_i}|}}$. We let **I** play some u_i such that $u_i \lessapprox q$ and $u_i \leqslant q_{n_i+1}$. Then $u_i \subseteq (A_{\mathcal{S}_{n_i}})_{\Delta_{|\mathcal{S}_{n_i}|}}$. Now, suppose that **II** answers by x_i. Then we choose a $y_i \in D$ such that $d(x_i, y_i) \leqslant \Delta_i$ and we choose n_{i+1} in such a way that $s_{n_{i+1}} = s_{n_i} {}^\frown y_i$. So we have that $y_i \in (A_{\mathcal{S}_{n_i}})_{2\Delta_{|\mathcal{S}_{n_i}|}}$; this shows that $\mathcal{S}_{n_{i+1}}$ has been defined. Moreover we have $d(s_{n_{i+1}}, (x_0, \dots, x_i)) \leqslant \Delta$ as wanted, what ends the proof.

□

Again, this proof can be done in $ZF + DC$, even if the space $\mathcal{G}$ is not supposed effective.

III.3 Eliminating the asymptotic game

Unlike Mathias–Silver's theorem which ensures that in some subspace, all of the increasing sequences have the same color, and unlike Gowers' theorem, one of whose possible conclusions says that all block sequences in some subspace have the same color, all the results we proved by now only have game-theoretical conclusions. The aim of this section is to provide a tool to deduce, from a statement of the form "player **I** has a strategy in F_p to reach $\mathcal{X}$", a conclusion of the form "in some subspace, every sequence satisfying some structural condition is in $\mathcal{X}$". This tool can be seen as a generalization of lemma II.18. It will allow us to get, from Ramsey results with game-theoretical conclusions, stronger results having the same form as Mathias–Silver's theorem or Gowers' theorem.

We will actually not add any structure on the set of points, but rather provide a tool enabling, in each concrete situation, to build this structure in the way we want. Our result could be stated in the setting of approximate Gowers spaces, but we prefer to state it in the more general setting of *approximate asymptotic spaces*, since it could be useful in itself in situations where we have no natural Gowers space structure.

Definition III.13. An *approximate asymptotic space* is a quintuple $\mathcal{A} = \{P, X, d, \lessapprox, \lhd\}$, where P is a nonempty set, (X, d) is a nonempty separable metric space, $\lessapprox$ is a quasiorder on P, and $\lhd \subseteq X \times P$ is a binary relation, satisfying the following properties:

1. for every $p, q, r \in P$, if $q \lessapprox p$ and $r \lessapprox p$, then there exists $u \in P$ such that $u \lessapprox q$ and $u \lessapprox r$;
2. for every $p \in P$, there exists $x \in X$ such that $x \lhd p$;
3. for every every $x \in X$ and every $p, q \in P$, if $x \lhd p$ and $p \lessapprox q$, then $x \lhd q$.

Every approximate Gowers space has a natural structure of approximate asymptotic space. In an approximate asymptotic space, we can define the notion of expansion, and the asymptotic game, in the same way as in an approximate Gowers space.

In the rest of this section, we fix $\mathcal{A} = \{P, X, d, \lessapprox, \lhd\}$ an approximate asymptotic space. To be able to get the result we want, we need some more structure. Recall that a subset of X is said to be *precompact* if its closure in X is compact. In what follows, for $K \subseteq X$ and $p \in P$, we abusively write $K \lhd p$ to say that the set $\{x \in K \mid x \lhd p\}$ is dense in K.

Definition III.14. A *system of precompact sets* for $\mathcal{A}$ is a set $\mathcal{K}$ of precompact subsets of X, equipped with an associative binary operation $\oplus$, satisfying the following property: for every $p \in P$, and for every $K, L \in \mathcal{K}$, if $K \lhd p$ and $L \lhd p$, then $K \oplus L \lhd p$.

If $(\mathcal{K}, \oplus)$ is a system of precompact sets for $\mathcal{A}$ and if $(K_n)_{n\in\omega}$ is a sequence of elements of $\mathcal{K}$, then:

- for $A \subseteq \omega$ finite, we denote by $\bigoplus_{n\in A} K_n$ the sum $K_{n_1} \oplus \ldots \oplus K_{n_k}$, where $n_1, \ldots, n_k$ are the elements of A taken in increasing order;

- a *block sequence* of (K_n) is, by definition, a sequence $(x_i)_{i\in\omega} \in X^\omega$ for which there exists an increasing sequence of nonempty sets of integers $A_0 < A_1 < A_2 < \ldots$ such that for every $i \in \omega$, we have $x_i \in \bigoplus_{n\in A_i} K_n$.

We denote by $\mathrm{bs}((K_n)_{n\in\omega})$ the set of all block sequences of (K_n).

We can already give some examples. For the Mathias–Silver space $\mathcal{N}$, let $\mathcal{K}_\mathcal{N}$ be the set of all singletons, and define the operation $\oplus_\mathcal{N}$ by $\{m\} \oplus_\mathcal{N} \{n\} = \{max(m,n)\}$. Then $(\mathcal{K}_\mathcal{N}, \oplus_\mathcal{N})$ is a system of precompact sets. If $(m_i)_{i\in\omega}$ is an increasing sequence of integers, then the block sequences of $(\{m_i\})_{i\in\omega}$ are exactly the subsequences of (m_i).

Now, for a Banach space E with a basis, consider the canonical approximate Gowers space $\mathcal{G}_E$. Let $\mathcal{K}_E$ be the set of all unit spheres of finite-dimensional subspaces of E. We define the operation $\oplus_E$ on $\mathcal{K}_E$ by $S_F \oplus_E S_G = S_{F+G}$. Then $(\mathcal{K}_E, \oplus_E)$ is a system of precompact sets for $\mathcal{G}_E$. If $(x_n)_{n\in\omega}$ is a (normalized) block sequence of E, then for every n, $S_{\mathbb{R}x_n} = \{x_n, -x_n\}$ is in $\mathcal{K}_E$, and the block sequences of $(S_{\mathbb{R}x_n})_{n\in\omega}$ in the sense of $\mathcal{K}$ are exactly the (normalized) block sequences of (x_n) in the Banach-theoretical sense. More generally, it is often useful to study the block sequences of sequences of the form $(S_{F_n})_{n\in\omega}$, where $(F_n)_{n\in\omega}$ is a FDD of a closed, infinite-dimensional subspace F of E (that is, a sequence such that every $x \in F$ can be written in a unique way as a sum $\sum_{n=0}^{\infty} x_n$, where for every n, $x_n \in F_n$).

In general, in an asymptotic space, a sequence $(K_n)_{n\in\omega}$ of elements of a system of precompact sets can be seen as another kind of subspace. Sometimes, some subspaces of the type $(K_n)_{n\in\omega}$ can be represented as elements of P; that is, for example, the case in the Mathias–Silver space and in the canonical approximate Gowers space over a Banach space with a basis, as we just saw. We now introduce a theorem enabling us to build sequences $(K_n)_{n\in\omega}$ such that $\mathrm{bs}((K_n)_{n\in\omega}) \subseteq \mathcal{X}$, knowing that player **I** has a strategy in an asymptotic game to reach $\mathcal{X}$. Firstly, we have to define a new game.

Definition III.15. Let $(\mathcal{K}, \oplus)$ be a system of precompact sets for the space $\mathcal{A}$, and $p \in P$. The *strong asymptotic game below* p, denoted by SF_p, is defined as follows:

I	p_0		p_1		$\ldots$	
II		K_0		K_1		$\ldots$

where the K_n's are elements of $\mathcal{K}$, and the p_n's are elements of P. The rules are the following:

- for **I**: for all $n \in \omega$, $p_n \lessapprox p$;
- for **II**: for all $n \in \omega$, $K_n \lhd p_n$.

The outcome of the game is the sequence $(K_n)_{n\in\omega} \in \mathcal{K}^\omega$.

Theorem III.16. *Let $(\mathcal{K}, \oplus)$ be a system of precompact sets on the space $\mathcal{A}$, $p \in P$, $\mathcal{X} \subseteq X^\omega$, and Δ be a sequence of positive real numbers. Suppose that player **I** has a strategy in F_p to reach $\mathcal{X}$. Then he has a strategy in SF_p to build a sequence $(K_n)_{n\in\omega}$ such that* $\mathrm{bs}((K_n)_{n\in\omega}) \subseteq (\mathcal{X})_\Delta$.

Proof. For each $K \in \mathcal{K}$, each $q \in P$ such that $K \lhd q$, and each $i \in \omega$, let $\mathfrak{N}_{i,q}(K)$ be a Δ_i-net in K (that is, a finite subset of K such that $K \subseteq (\mathfrak{N}_{i,q}(K))_{\Delta_i}$), such that for every $x \in \mathfrak{N}_{i,q}(K)$, we have $x \lhd q$. We fix τ a strategy for **I** in F_p, enabling him to reach $\mathcal{X}$. As in the proofs of fact II.15 and lemma II.18, we consider that in F_p, **II** is allowed to play against the rules, but that she immedately loses if she does; so we will view τ as a mapping from $X^{<\omega}$ to P, such that for every $s \in X^{<\omega}$, we have $\tau(s) \lessapprox p$.

Let us describe a strategy for **I** in SF_p on a play $(p_0, K_0, p_1, K_1, \ldots)$ of this game. Suppose that the first n turns have been played, so the p_j's and the K_j's, for $j < n$, are defined. Moreover suppose that the sequence $(p_j)_{j<n}$ is $\lessapprox$-decreasing. Let $S_{(K_0,\ldots,K_{n-1})} \subseteq X^{<\omega}$ be the set of all finite sequences $(y_0, \ldots, y_{k-1})$ satisfying the following property: there exists an increasing sequence $A_0 < \ldots < A_{k-1}$ of nonempty subsets of n such that for every $i < k$, we have $y_i \in \mathfrak{N}_{i,p_{\min(A_i)}}(\oplus_{j\in A_i} K_j)$. Then $S_{(K_0,\ldots,K_{n-1})}$ is finite and for every $s \in S_{(K_0,\ldots,K_{n-1})}$, we have $\tau(s) \lessapprox p$, so by iterating the axiom 1. in the definition of an approximate asymptotic space, we can find $p_n \lessapprox p$ such that for every $s \in S_{(K_0,\ldots,K_{n-1})}$, we have $p_n \lessapprox \tau(s)$. Moreover, if $n \geqslant 1$, we can choose p_n such that $p_n \lessapprox p_{n-1}$. The strategy of **I** will consist in playing this p_n.

Now suppose that this play has been played completely; we show that $\mathrm{bs}((K_n)_{n\in\omega}) \subseteq (\mathcal{X})_\Delta$. Let $(x_i)_{i\in\omega}$ be a block sequence of (K_n) and $A_0 < A_1 < \ldots$ be a sequence of nonempty subsets of ω such that for every i, we have $x_i \in \bigoplus_{n\in A_i} K_n$. For every $i \in \omega$, we have $\left(\bigoplus_{n\in A_i} K_n\right) \lhd p_{\min(A_i)}$, so $\mathfrak{N}_{i,p_{\min(A_i)}}\left(\bigoplus_{n\in A_i} K_n\right)$ has been defined and we can choose a y_i in it such that $d(x_i, y_i) \leqslant \Delta_i$. We have to show that $(x_i)_{i\in\omega} \in (\mathcal{X})_\Delta$, so it is enough to show that $(y_i)_{i\in\omega} \in \mathcal{X}$. Knowing that τ is a strategy for **I** in F_p to reach $\mathcal{X}$, it is enough to show that, letting $q_i = \tau(y_0, \ldots, y_{i-1})$ for all i, in the following play of F_p, **II** always respects the rules:

I	q_0		q_1		$\ldots$	
II		y_0		y_1		$\ldots$

In other words, we have to show that for all $k \in \omega$, we have $y_k \lhd q_k$.

So let $k \in \omega$. We let $n_0 = \min A_k$. Since the sets $A_0, \ldots, A_{k-1}$ are subsets of n_0, we have that $(y_0, \ldots, y_{k-1}) \in S_{(K_0,\ldots,K_{n_0-1})}$, and therefore $p_{n_0} \lessapprox \tau(y_0, \ldots, y_{k-1}) = q_k$. But $y_k \in \mathfrak{N}_{k,p_{n_0}}\left(\bigoplus_{n\in A_k} K_n\right)$, so $y_k \lhd p_{n_0}$, so $y_k \lhd q_k$, as wanted. ☐

Again, under slight restrictions, we can prove theorem III.16 without using the full axiom of choice. Say that the approximate asymptotic space $\mathcal{A}$ is *effective* if there exist a function $f : P^2 \longrightarrow P$ such that for every $q, r \in P$, if there exist $p \in P$ such that $q \lessapprox p$ and $r \lessapprox p$, then we have $f(q,r) \lessapprox q$ and $f(q,r) \lessapprox r$. Effective approximate Gowers spaces, when seen as approximate asymptotic spaces, are effective. We will show that if $\mathcal{A}$ is an effective approximate asymptotic space, if X is an analytic subset of a Polish space, if for every $p \in P$, the set $\{x \in X \mid x \lhd p\}$ is closed in X, and if every element of $\mathcal{K}$ is compact, then theorem III.16 for $\mathcal{A}$ and $\mathcal{K}$ can be shown in $ZF + DC$. In the proof of theorem III.16, AC is only used:

- to choose p_n such that for every $s \in S_{(K_0,\ldots,K_{n-1})}$, we have $p_n \lessapprox \tau(s)$, and such that $p_n \lessapprox p_{n-1}$ if $n \geqslant 1$;

- to choose the nets $\mathfrak{N}_{i,q}(K)$;
- and to choose $y_i \in \mathfrak{N}_{i,p_{\min(A_i)}}\left(\bigoplus_{n\in A_i} K_n\right)$ such that $d(x_i, y_i) \leqslant \Delta_i$.

The choice of the p_n's can be done without AC as soon as the space $\mathcal{A}$ is effective. For the choice of the nets and of the y_i's, firstly remark that, given $K \in \mathcal{K}$ and $q \in P$, since $\{x \in X \mid x \lhd q\}$ is closed in X, we have that $K \lhd q$ if and only if $K \subseteq \{x \in X \mid x \lhd q\}$; so $\mathfrak{N}_{i,q}(K)$ can actually be an arbitrary Δ_i-net in K, and does not need to depend on q. Thus, to be able to chose these nets and the y_i's without AC, it is enough to show that we can choose, without AC, a Δ_i-net $\mathfrak{N}_i(K)$ in K and a wellordering $\prec_{i,K}$ on it, for every $K \in \mathcal{K}$ and every $i \in \omega$. This can be done in the following way. Let $\varphi : \omega^\omega \to X$ be a continuous surjection. If $K \in \mathcal{K}$, then $\varphi^{-1}(K)$ has the form $[T_K]$, where T_K is a pruned tree on ω. We can easily build, without choice, a countable dense subset of $[T_K]$, for example the set of all the u_s's where for every $s \in T_K$, u_s is the leftmost branch of T_K satisfying $s \subseteq u_s$. Since T_K can naturally be wellordered, then this dense subset can also be wellordered. Pushing forward by φ, this enables us to get, for every $K \in \mathcal{K}$, a countable dense subset $D_K \subseteq K$ with a wellordering $\prec_K$. From this we can naturally wellorder the set of all finite subsets of D_K, take for $\mathfrak{N}_i(K)$ the least finite subset of D_K that is a Δ_i-net in K and take for $\prec_{i,K}$ the restriction of $\prec_K$ to $\mathfrak{N}_i(K)$.

Theorem III.16, combined with the results of the last section and with the last remark, gives us the following corollary:

Corollary III.17 (Abstract Gowers' theorem)**.** *Let $\mathcal{G} = (P, X, d, \leqslant, \leqslant^*, \lhd)$ be an approximate Gowers space, equipped with a system of precompact sets $(\mathcal{K}, \oplus)$. Let $\mathcal{X} \subseteq X^\omega$, and suppose that one of the following conditions holds:*

- *$\mathcal{X}$ is analytic;*
- *$\mathcal{G}$ is analytic and $\mathcal{X}$ is $\exists\Gamma$, for some suitable class Γ of subsets of Polish spaces such that every Γ-subset of $\mathbb{R}^\omega$ is determined;*
- *$AD_{\mathbb{R}}$ holds, the space $\mathcal{G}$ is effective, P is a subset of Polish space, for every $p \in P$, the set $\{x \in X \mid x \lhd p\}$ is closed in X, and every element of $\mathcal{K}$ is compact.*

Let $p \in P$ and Δ be a sequence of positive real numbers. Then there exists $q \leqslant p$ such that:

- *either player **I** has a strategy in SF_q to build a sequence $(K_n)_{n\in\omega}$ such that $\mathrm{bs}((K_n)_{n\in\omega}) \subseteq \mathcal{X}^c$;*
- *or player **II** has a strategy in G_q to reach $(\mathcal{X})_\Delta$.*

*Moreover, if $\mathcal{G}$ satisfies the pigeonhole principle, then the second conclusion can be replaced with the following stronger one: player **I** has a strategy in SF_q to build a sequence $(K_n)_{n\in\omega}$ such that $\mathrm{bs}((K_n)_{n\in\omega}) \subseteq (\mathcal{X})_\Delta$.*

Now see how to deduce Mathias–Silver's theorem, Gowers' theorem I.8, and Gowers' theorem for c_0 (theorem I.11) from corollary III.17.

- For Mathias–Silver's theorem, work in the Mathias–Silver space $\mathcal{N}$ with the system $(\mathcal{K}_\mathcal{N}, \oplus_\mathcal{N})$ of precompact sets introduced before. Let M be an infinite set of integers, and $\mathcal{X} \subseteq [\omega]^\omega$ be analytic, that we will consider as a subset of ω^ω by identifying infinite subsets of ω with increasing sequences of integers. Applying corollary III.17 to $\mathcal{X}$, to M, and to the constant sequence equal to $\frac{1}{2}$, we get an infinite $N \subseteq M$ such that either **I** has a strategy in SF_N to build $(\{n_i\})_{i\in\omega}$ with $\mathrm{bs}((\{n_i\})_{i\in\omega}) \subseteq \mathcal{X}$, or he has one to build $(\{n_i\})_{i\in\omega}$ with $\mathrm{bs}((\{n_i\})_{i\in\omega}) \subseteq \mathcal{X}^c$. Remark that in SF_N, **II** can always play in such a way that the sequence $(n_i)_{i\in\omega}$ is increasing. So in the first case, we get an increasing sequence $(n_i)_{i\in\omega}$ of elements of N such that every block sequence of $(\{n_i\})_{i\in\omega}$ belongs to $\mathcal{X}$, or in other words, such that every infinite subset of $\{n_i \mid i \in \omega\}$ belongs to $\mathcal{X}$; and in the second case, in the same way, we get an infinite subset of N every infinite subset of whose belongs to $\mathcal{X}^c$.
- For Gowers' theorem, let E be a Banach space with a Schauder basis and work in the canonical approximate Gowers space $\mathcal{G}_E$ with the system $(\mathcal{K}_E, \oplus_E)$ of precompact sets introduced before. Given $Y \in P$, in SF_Y, whatever **I** plays, **II** can always ensure that the outcome will have the form $(S_{\mathbb{R}y_n})_{n\in\omega}$, where $(y_n)_{n\in\omega}$ is a block sequence. So given $\mathcal{X} \subseteq [E]$ analytic, $X \subseteq E$ a block subspace, and Δ a sequence of positive real numbers, corollary III.17 gives us either a block sequence $(y_n)_{n\in\omega}$ in X such that $\mathrm{bs}((S_{\mathbb{R}y_n})_{n\in\omega}) \subseteq \mathcal{X}^c$, or a block subspace $Y \subseteq X$ such that **II** has a strategy in G_Y to reach $(\mathcal{X})_{\frac{\Delta}{2}}$. In the first case, denoting by Y the block subspace generated by the sequence (y_n), this precisely means that $[Y] \subseteq \mathcal{X}^c$. In the second case, we have to be careful because the Gowers' game of the space $\mathcal{G}_E$ is not exactly the same as this defined in the introduction: in the one of the introduction, player **II** is required to play vectors with finite support forming a block sequence, while in the one of $\mathcal{G}_E$, she she can play any vector in the unit sphere of the subspace played by **I**. This is not a real problem as, by perturbating a little bit the vectors given by her strategy, player **II** can reach $\mathcal{X}_\Delta$ playing vectors with finite support; and without loss of generality, we can assume that the subspace Y_n played by **I** at the $(n+1)^{\text{th}}$ turn is choosen small enough to force **II** to play a y_n such that $\mathrm{supp}(y_{n-1}) < \mathrm{supp}(y_n)$.
- To deduce Gowers' theorem for c_0, the method is the same except that this time, $\mathcal{G}_E$ satisfies the pigeonhole principle so corollary III.17 will give us a conclusion with a strong asymptotic game in both sides.

To finish this section, let us show on an example that the hypothesis "**I** has a strategy in F_p to reach $\mathcal{X}$" does not always imply that for some subspace q, every sequence below q satisfying some natural structural condition (for instance, being block) is in $\mathcal{X}_\Delta$. To see this, consider the Rosendal space $\mathcal{R}_K = (P, E\backslash\{0\}, \subseteq, \subseteq^*, \in)$ over a field K. We have the following fact:

Fact III.18. *Suppose that K is a finite field. Let $\mathcal{X} \subseteq (E\backslash\{0\})^\omega$ and $X \in P$, and suppose that **I** has a strategy in F_X to reach $\mathcal{X}$. Then there exists a block subspace $Y \subseteq X$ such that every block sequence of Y is in $\mathcal{X}$.*

Proof. Let $\mathcal{K}$ be the set of all sets of the form $F\backslash\{0\}$, where F is a finite-dimensional subspace of E. Since the field K is finite, the elements of $\mathcal{K}$ are finite too. For $F, G \subseteq E$ finite-dimensional, we let $(F\backslash\{0\}) \oplus (G\backslash\{0\}) = (F+G)\backslash\{0\}$. Then $(\mathcal{K}, \oplus)$ is a system of precompact sets. The conclusion follows from theorem III.16 applied to this system, using the same method as previously.

□

Remark that this proof does not work when K is infinite, and actually, this result is false. Let us give a counterexample. Let $(e_i)_{i<\omega}$ be the basis of E with respect to whose block subspaces are taken, and let $\varphi : K^* \to \omega$ be a bijection. For $x \in E\backslash\{0\}$, let $N(x)$ be the first nonzero coordinate of x. We let $\mathcal{Y} = \{(x,y) \in (E\backslash\{0\})^2 \mid \varphi(N(x)) < \min \operatorname{supp}(y)\}$ and $\mathcal{X} = \{(x_n)_{n\in\omega} \in (E\backslash\{0\})^\omega \mid (x_0, x_1) \in \mathcal{Y}\}$. Then player **I** has a strategy in F_E to reach $\mathcal{X}$; this strategy is illustrated on the following diagram:

I	E		$\operatorname{span}(\{e_i \mid i > \varphi(N(x))\})$	
II		x		y

But there is no block subspace Y of E such that every block sequence in Y belongs to $\mathcal{X}$. Indeed, given $Y \subseteq E$ a block subspace generated by a block sequence $(y_n)_{n\in\omega}$, we can take $\lambda \in K$ such that $\varphi(N(\lambda y_0)) = \min \operatorname{supp}(y_1)$, and we have $(\lambda y_0, y_1, y_2, \ldots) \notin \mathcal{X}$.

Just like the counterexample to the pigeonhole principle presented in section II.2, this counterexample could be avoided by working in the projective Rosendal space $\mathcal{PR}_K = (P, \mathbb{P}(E), \subseteq, \subseteq^*, \subseteq)$ (where we recall that $\mathbb{P}(E)$ is the set of all vector lines in E). However, even in this space, counterexamples to the natural analogue of fact III.18 can be found. For example, for $Kx \in \mathbb{P}(E)$, denote by $N'(Kx)$ the quotient of the last nonzero coordinate of x by its first nonzero coordinate (which does not depend of the choice of the representative x); and let $\mathcal{X} = \{(l_i)_{i\in\omega} \in \mathbb{P}(E)^\omega \mid \varphi(N'(l_0)) < \min \operatorname{supp}(l_1)\}$. Then $\mathcal{X}$ is a counterexample as well.

Therefore, many cases, the "subspaces" of the form $(K_n)_{n\in\omega}$, where the K_n's are elements of a system of precompact sets, cannot always be identified with "genuine" subspaces (i.e. elements of P): we always need a form of compactness for that.

Chapter IV

Hilbert-avoiding dichotomies and ergodicity

Recall that Johnson's problem ask whether there exists a separable Banach space with exactly two subspaces, up to isomorphism (a *Johnosn space*), and that Ferenczi and Rosendal's ergodic conjecture ask whether there exists a non-ergodic separable Banach space non-isomorphic to ℓ_2, where a space is *ergodic* if $\mathbf{E_0}$ reduces to the isomorphism relation between its subspaces. In this chapter, we try to answer the following question: if counterexamples to these conjectures exist, do there necessarily exist such counterexamples having an unconditional basis? More precisely, we will work on the following conjectures:

Conjecture IV.1. *Let E be a separable Banach space, non-ergodic and non-isomorphic to ℓ_2. Then E has a subspace with an unconditional basis that is non-isomorphic to ℓ_2.*

Conjecture IV.2. *Every Johnson space has an unconditional basis.*

Remark that conjecture IV.2 is a consequence of conjecture IV.1: indeed, a result by Anisca [3] implies that a Johnson space necessarily has a subspace isomorphic to ℓ_2.

We do not manage to solve these conjecture, but we prove results that should help for them. The basic idea is the following. Recall that Rosendal [55] proved that HI spaces cannot be ergodic; so if a space E is non-ergodic, then by Gowers' first dichotomy (theorem I.21), it must have a subspace with an unconditional basis. However, this does not give us anything interesting, since this space could be isomorphic to ℓ_2. So what we will do is to prove *Hilbert-avoiding dichotomies*, i.e. dichotomies ensuring that the subspace obtained is non-isomorphic to ℓ_2.

The basic ideas to prove such dichotomies was given to the author by Ferenczi. The fact that a Banach space is isomorphic to ℓ_2 can be verified only on its finite-dimensional subspaces, and this implies that we can diagonalize among subspaces that are not isomorphic to ℓ_2. Thus, a Banach space E non-isomorphic to ℓ_2 can be made an approximate Gowers space by taking for subspaces only subspaces of E that are not isomorphic to ℓ_2. In this manner, we will be able to prove Hilbert-avoiding versions of

Gowers' first dichotomy, and of Ferenczi–Rosendal's dichotomy between minimal spaces and tight spaces (theorem I.25). Obviously, Gowers' game and the adversarial Gowers' games change in our new approximate Gowers space, and the consequence of this is that the possible conclusions in our Hilbert-avoiding dichotomies will be weaker than in their "original versions".

Using these dichotomies, we get interesting consequences about conjectures IV.1 and IV.2. In particular, we define the class of *hereditarily Hilbert-primary (HHP)* spaces as follows: a Banach space E is HHP if there is no topological direct sum of subspaces of E that are both non-isomorphic to ℓ_2. Then we get that, to prove conjecture IV.1, it would be enough to prove that an HHP space cannot be embedded in any subspace of itself that is not isomorphic to ℓ_2, and to prove conjecture IV.2, it would be enough to prove that an HHP space must at least have two non-isomorphic subspaces that are non-isomorphic to ℓ_2. The two last statements are quite similar to Gowers–Maurey's result IV.33 that an HI space cannot be isomorphic to a proper subspace of itself; thus, it is tempting to try to prove them using the same methods.

This chapter is organized as follows. In section IV.1, we recall some facts and prove some preliminary results about finite-dimensional decompositions. In section IV.2, we introduce our Hilbert-avoiding approximate Gowers space and we use it to prove a Hilbert-avoiding version of Gowers Ramsey-type theorem, theorem IV.9. Then, we use this theorem to prove our first dichotomy, the Hilbert-avoiding version of Gowers' first dichotomy (theorem IV.12). In section IV.3, we prove our Hilbert-avoiding version of Ferenczi–Rosendal's minimal-tight dichotomy (theorem IV.14). Note that here, since the argument is quite technical, we will not use approximate Gowers spaces, but rather a Gowers space and apply the results of chapter II. In section IV.4, using, among others, recent unpublished results by Ferenczi, we set the consequences of our two dichotomies for non-ergodic spaces and Johnson spaces; in particular, we get the results stated in the last paragraph. Finally, in section IV.5, we give a new and simple proof of Gowers–Maurey's result that HI spaces are isomorphic to no proper subspaces. This proof is only based on Fredholm theory and works as well in the real and the complex case. We hope that the method used here could help to finish to solve conjectures IV.1 and IV.2, combined with our dichotomies.

IV.1 Preliminaries

In this section, we recall some preliminary results that will be useful in the next sections.

A *finite-dimensional decomposition (FDD)* of a Banach space E is a sequence $(E_n)_{n\in\omega}$ of nonzero finite-dimensional subspaces of E such that every $x \in E$ can be decomposed in a unique way as a convergent sum $x = \sum_{n=0}^{\infty} x_n$, where for every $n \in \omega$, $x_n \in E_n$. With these notation, we let $P_n(x) = \sum_{i<n} x_i$; this defines a linear projection $P_n : E \longrightarrow \bigoplus_{i<n} E_i$. As for Schauder bases, we can show that the P_n's are uniformily bounded; the number $C = \sup_{n\in\omega} \|\|P_n\|\|$ is called the *constant* of the FDD. FDDs are

a generalisation of Schauder bases: given $(x_n)_{n\in\omega}$ a normalized sequence in E, we have that $(x_n)_{n\in\omega}$ is a basis of E if and only if $(\mathbb{R}x_n)_{n\in\omega}$ is an FDD of E, and in this case, the constants are the same.

If a sequence $(F_n)_{n\in\omega}$ of finite-dimensional subspaces of E is a FDD of $\overline{\bigoplus_{n\in\omega} F_n}$, then $(F_n)_{n\in\omega}$ will simply be called a *FDD*. We have the same characterisation for FDDs as for basic sequences: if $(F_n)_{n<\omega}$ is a sequence of nonzero finite-dimensional subspaces of E and if there exists a constant C such that, for every $m \leqslant n$ and for every $(x_i)_{i<n} \in \prod_{i<n} F_i$, we have $\|\sum_{i<m} x_i\| \leqslant C \|\sum_{i<n} x_i\|$, then $(F_n)_{n\in\omega}$ is a FDD. Moreover, the constant of this FDD is the least C satisfying this property.

A *block-FDD* of an FDD $(F_n)_{n\in\omega}$ is a sequence $(G_i)_{i\in\omega}$ of nonzero finite-dimensional subspaces of E such that there exists a sequence $A_0 < A_1 < \ldots$ of finite subsets of ω such that for every $i \in \omega$, $G_i \subseteq \oplus_{n\in A_i} F_n$. By the previous characterisation, a block-FDD of $(F_n)_{n\in\omega}$ is an FDD and its constant is less or equal to than the constant of $(F_n)_{n\in\omega}$. For $x = \sum_{n=0}^{\infty} x_n$, where $\forall n \in \omega\ x_n \in F_n$, the *support of* x on the FDD $(F_n)_{n\in\omega}$ is $\mathrm{supp}(x) = \{n \in \omega \mid x_n \neq 0\}$. A *block-sequence* of $(F_n)_{n\in\omega}$ is a sequence $(x_n)_{n\in\omega}$ of normalized vectors of $\oplus_{n\in\omega} F_n$ such that $\mathrm{supp}(x_0) < \mathrm{supp}(x_1) < \ldots$. Remark that a normalized sequence $(x_n)_{n\in\omega}$ is a block-sequence of $(F_n)_{n\in\omega}$ if and only if $(\mathbb{R}x_n)_{n\in\omega}$ is a block-FDD of $(F_n)_{n\in\omega}$. In particular, a block-sequence of $(F_n)_{n\in\omega}$ is a basic sequence with constant less or equal to than the constant of $(F_n)_{n\in\omega}$.

An *unconditional finite-dimensional decomposition (UFDD)* is an FDD $(F_n)_{n\in\omega}$ such that for every $(x_n)_{n\in\omega} \in \prod_{n\in\omega} F_n$, if the series $\sum_{n=0}^{\infty} x_n$ converges, then for every $A \subseteq \omega$, the series $\sum_{n\in A} x_n$ also converges. If this holds, it can be shown that for every $a = (a_n)_{n\in\omega} \in \ell_\infty$ and, the series $T_a\left(\sum_{n=0}^{\infty} x_n\right) = \sum_{n=0}^{\infty} a_n x_n$ converges. Moreover, letting $F = \overline{\bigoplus_{n\in\omega} F_n}$, this defines a bounded operator $T_a : F \longrightarrow F$, and $K := \sup_{a\in S_{\ell_\infty}} \|T_a\| < \infty$. The constant K is called the *unconditional constant* of the FDD.

A sequence $(F_n)_{n\in\omega}$ of nonzero finite-dimensional subspaces of E is a UFDD if and only if there exists a constant K such that for every $n \in \omega$, for every $(\varepsilon_0, \ldots, \varepsilon_{n-1}) \in \{-1, 1\}^n$, and for every $(x_i)_{i<n} \in \prod_{i<n} F_n$, we have $\|\sum_{i<n} \varepsilon_i x_i\| \leqslant K \|\sum_{i<n} x_i\|$. In this case, the unconditional constant of $(F_n)_{n\in\omega}$ is the least K satisfying this property. This characterisation shows that a block-FDD of $(F_n)_{n\in\omega}$ is a UFDD with unconditional constant less or equal than the unconditional constant of $(F_n)_{n\in\omega}$. We can also show that a sequence $(F_n)_{n\in\omega}$ of nonzero finite-dimensional subspaces of E is a UFDD if and only if there exists a constant K' such that for every $n \in \omega$, for every $A \subseteq n$, and for every $(x_i)_{i<n} \in \prod_{i<n} F_n$, we have $\|\sum_{i\in A} x_i\| \leqslant K' \|\sum_{i<n} x_i\|$.

Before going further, let us recall some facts about the *equivalence* of sequences. Here, α will denote an integer or ω. Two sequences $(x_n)_{n<\alpha}$ and $(y_n)_{n<\alpha}$ of elements of a Banach space E are said to be *C-equivalent*, for some constant $C \geqslant 1$, if there exist $A, B \geqslant 1$ such that $AB \leqslant C$ and for every $(a_n)_{n<\alpha} \in \mathbb{R}^\alpha$ with finite support, we have $\frac{1}{A} \|\sum_{n<\alpha} a_n y_n\| \leqslant \|\sum_{n<\alpha} a_n x_n\| \leqslant B \|\sum_{n<\alpha} a_n y_n\|$. Two sequences are *equivalent* if they are C-equivalent for some C. If two normalized sequences $(x_n)_{n<\alpha}$ and $(y_n)_{n<\alpha}$

are C-equivalent, and if X and Y denote the respective closed subspaces spanned by the x_n's and the y_n's, then there exists a unique C-isomorphism T from X to Y, such that for every $n < \alpha$, $T(x_n) = y_n$. Moreover, if $(x_n)_{n\in\omega}$ is a basic sequence with constant M, then $(y_n)_{n\in\omega}$ is a basic sequence with constant less or equal than CM.

A classical result says that a small perturbation of a basic sequences is still a basic sequence, equivalent to the first one. We will later need a generalization of this result; we state it now. Recall that, given a finite-dimensional normed space F, a normalized basis $(f_i)_{i<d}$ of F is said to be an *Auerbach basis* if all of the biorthogonal functionals $f_i^* \in F^*$, for $i < d$, have norm 1. Auerbach's lemma says that such bases always exist; for a proof see problem 12.1 in [2]. Here, we will say that a sequence $(x_i)_{i<\alpha}$ of normalized vectors in a Banach space is *M-Auerbach*, for $M \geqslant 1$, if for every sequence $(a_i)_{i<\alpha} \in \mathbb{R}^\alpha$ with finite support, and for every $n < \alpha$, we have $|a_n| \leqslant M\|\sum_{i<\alpha} a_i x_i\|$. Remark that if two sequences $(x_n)_{n<\alpha}$ and $(y_n)_{n<\alpha}$ are C-equivalent, and if $(x_n)_{n<\alpha}$ is M-Auerbach, then $(y_n)_{n<\alpha}$ is CM-Auerbach. Obviously, Auerbach bases are 1-Auerbach, and basic sequences with constant M are $2M$-Auerbach. But there also exist other examples of Auerbach sequences. For exemple, take $(F_i)_{i\in\omega}$ a FDD with constant C, let $n_i = \sum_{j<i} \dim(F_j)$, and for every $i \in \omega$, let $(x_n)_{n_i \leqslant n < n_{i+1}}$ be a normalized basis of F_i which is M-Auerbach, for a fixed M. Then the sequence $(x_n)_{n\in\omega}$ is $2CM$-Auerbach; however, this is not necessarily a basic sequence.

The principle of small perturbations we will use here is the following.

Lemma IV.3. *Let $(x_i)_{i<\alpha}$ be a C-Auerbach sequence, and let $(y_i)_{i<\alpha}$ be a normalized sequence in the same Banach space. Let $\varepsilon < \frac{1}{C}$, and suppose that $\sum_{i<\alpha} \|x_i - y_i\| \leqslant \varepsilon$. Then the sequences $(x_i)_{i<\alpha}$ and $(y_i)_{i<\alpha}$ are $\frac{1+C\varepsilon}{1-C\varepsilon}$-equivalent.*

Proof. Let $(a_i)_{i<\alpha} \in \mathbb{R}^\alpha$ be a sequence with finite support. We have:

$$\begin{aligned}\left\|\sum_{i<\alpha} a_i y_i\right\| &\leqslant \left\|\sum_{i<\alpha} a_i x_i\right\| + \sum_{i<\alpha} |a_i| \|y_i - x_i\| \\ &\leqslant \left\|\sum_{i<\alpha} a_i x_i\right\| + C\left\|\sum_{i<\alpha} a_i x_i\right\| \cdot \sum_{i<\alpha} \|y_i - x_i\| \\ &\leqslant (1 + C\varepsilon)\left\|\sum_{i<\alpha} a_i x_i\right\|.\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}\left\|\sum_{i<\alpha} a_i x_i\right\| &\leqslant \left\|\sum_{i<\alpha} a_i y_i\right\| + \sum_{i<\alpha} |a_i| \|y_i - x_i\| \\ &\leqslant \left\|\sum_{i<\alpha} a_i y_i\right\| + C\varepsilon\left\|\sum_{i<\alpha} a_i x_i\right\|,\end{aligned}$$

so:

$$(1 - C\varepsilon)\left\|\sum_{i<\alpha} a_i x_i\right\| \leqslant \left\|\sum_{i<\alpha} a_i y_i\right\|.$$

The result immediately follows.

□

We now turn back to FDDs and introduce a method for constructing them. The idea is the same as the usual method for building basic sequences: each term has to be choosen "far enough" from the previous ones. We give here a formulation of this criterion that will be quite convenient for our work. We start by giving a new version of the asymptotic game.

Definition IV.4. Let E be a Banach space. The *subspace-asymptotic game* below E, denoted by $SubF_E$, is the following two-players game:

I	X_0		X_1		$\dots$
II		F_0		F_1	$\dots$

where the X_n's are finite-codimensional subspaces of E, and the F_n's are finite-dimensional subspaces of E, with the constraint for **II** that for all $n \in \omega$, $F_n \subseteq X_n$. The outcome of the game is the sequence $(F_n)_{n\in\omega}$.

Our criterion will be the following.

Lemma IV.5. *Let E be a separable Banach space and $\varepsilon > 0$. Then player **I** has a strategy in $SubF_n$ to build a FDD with constant less or equal than $1+\varepsilon$.*

Proof. Recall that $\mathcal{C}([0,1])$, the space of continuous functions $[0,1] \longrightarrow \mathbb{R}$ with the sup norm, has a Schauder basis $(e_i)_{i\in\omega}$ with constant 1 (see [2], theorem 1.2.1). We denote by P_i, $i \in \omega$, the projections relative to this basis. Recall also Banach-Mazur's theorem (theorem 1.4.3. in [2]), saying that every separable Banach space can be isometrically embedded in $\mathcal{C}([0,1])$. So we can assume that $E \subseteq \mathcal{C}([0,1])$. Remark that a strategy for **I** in the subspace-asymptotic game into $\mathcal{C}([0,1])$ to reach some target immediately gives a strategy for **I** in the same game played in E to reach the same target: **I** can play in E the intersection of E and of the subspace he would play in $\mathcal{C}([0,1])$. So we can assume that $E = \mathcal{C}([0,1])$.

Consider the approximate asymptotic space $(P, S_E, d, \lessapprox, \in)$ where P is the set of all infinite-dimensional subspaces of E, d is the distance of the norm on S_E, and $X \lessapprox Y$ if X is a finite-codimensional subspace of Y. On this space, we can consider the system of compact sets $(\mathcal{K}, \oplus)$ where K is the set of balls of nonzero finite-dimensional subspaces of E and $S_F \oplus S_G = S_{F+G}$. In this space, the strong asymptotic game below E is exactly the same as the subspace-asymptotic game below E. Moreover, denote by $\mathcal{X}_C$ the set of basic sequences in E with constant less or equal than C. Then, for a sequence $(F_n)_{n<\omega}$ of nonzero finite-dimensional subspaces of E, if $\mathrm{bs}((S_{F_n})_{n\in\omega}) \subseteq \mathcal{X}_{1+\epsilon}$, then $(F_n)_{n\in\omega}$ is a FDD with constant less or equal than $1+\varepsilon$. However, by lemma IV.3, we have that for a well-chosen sequence Δ of positive real numbers, $(\mathcal{X}_{1+\frac{\varepsilon}{2}})_\Delta \subseteq \mathcal{X}_{1+\varepsilon}$. So by theorem III.16, it is enough to show that player **I** has a strategy in the asymptotic game F_E to build a basic sequence with constant $1+\frac{\varepsilon}{2}$.

Fix $\delta \in \left(0, \frac{1}{2}\right)$ be such that $\frac{1+2\delta}{1-2\delta} \leqslant 1+\frac{\varepsilon}{2}$. We describe a strategy for **I** in F_E on a play $(X_0, x_0, X_1, x_1, \dots)$ of this game. We suppose moreover that at the same time as each x_i, a vector y_i is built such that $\|x_i - y_i\| \leqslant \frac{\delta}{2^{i+1}}$ and such that $(y_i)_{i\in\omega}$ is a block-sequence of $(e_n)_{n\in\omega}$. This will be enough to conclude: indeed, $(y_i)_{i\in\omega}$ will be a basic sequence with constant 1, so by lemma IV.3, $(x_i)_{i\in\omega}$ will be a basic sequence with constant $1 + \frac{\varepsilon}{2}$.

At the first turn of the game, **I** plays $X_0 = E$. **II** answers with x_0. Then, for $n_0 \in \omega$ large enough, we can let $y_0 = \frac{P_{n_0}(x_0)}{\|P_{n_0}(x_0)\|}$ and we have $\|x_0 - y_0\| \leqslant \frac{\delta}{2}$. Suppose now that the first i turns of the game have been played, so $x_0, \ldots, x_i$ and $y_0, \ldots, y_i$ have been built. Let $m_i = \max\operatorname{supp}(y_i)$. Player **I** plays $X_{i+1} = \operatorname{Ker} P_{m_i}$. Then **II** answers by $x_{i+1} > y_i$; for $n_{i+1} \in \omega$ large enough, letting $y_{i+1} = \frac{P_{n_{i+1}}(x_{i+1})}{\|P_{n_0}(x_{i+1})\|}$, we have $\|x_{i+1} - y_{i+1}\| \leqslant \frac{\delta}{2^{i+2}}$. We have $y_{i+1} > y_i$ as wanted, what finishes the proof.

□

It will be very important, in the following work, to be able to characterise separable spaces that are not isomorphic to ℓ_2. Recall that it is a well-known fact that if a separable Banach space X is not isomorphic to ℓ_2, then for every $C \geqslant 1$, there exists a finite-dimensional subspace F of X which is not C-isomorphic to $\ell_2^{\dim(F)}$. We state here a little stronger result.

Lemma IV.6 (Folklore)**.** *Let X be a Banach space and $(F_n)_{n\in\omega}$ be an increasing sequence of finite-dimensional subspaces of X such that $\bigcup_{n\in\omega} F_n$ is dense in X. Then $d_{BM}(X, \ell_2) = \sup_{n\in\omega} d_{BM}(F_n, \ell_2^{\dim(F_n)})$.*

Proof. Let $C \geqslant 1$ and suppose that for every n, F_n is C-isomorphic to ℓ_2. We need to show that X is C-isomorphic to ℓ_2. For every n, let $\varphi_n : F_n \longrightarrow E_n$ an isomorphism, where E_n is a subspace of ℓ_2, $\|\varphi_n\| \leqslant C$, and $\|\varphi_n^{-1}\| \leqslant 1$. By composing successively the φ_n's by isometries between finite-dimensional subspaces of ℓ_2, we can moreover assume that $E_0 \subseteq E_1 \subseteq \ldots$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. For every $x \in \bigcup_{i\in\omega} F_i$, we let $\varphi(x) = \lim_{i\longrightarrow\mathcal{U}} \varphi_i(x)$. As, if $x \in \overline{B}_{F_n}(R)$, we have for every i large enough, $\varphi_i(x) \in \overline{B}_{E_n}(CR)$, this limit is well-defined. This defines a linear mapping $\varphi : \bigcup_{n\in\omega} F_n \longrightarrow \ell_2$ with, for every x, $\|x\| \leqslant \|\varphi(x)\| \leqslant C\|x\|$. So φ can be extended to an C-isomorphism between X and a subspace Y of ℓ_2, and since Y is isometric to ℓ_2, this concludes.

□

We can now state a characterisation of non-isomorphism to ℓ_2 based on FDDs. This characterisation will be central in the following work.

Lemma IV.7. *Let E be a separable Banach space. Then E is non-isomorphic to ℓ_2 if and only if there is a FDD $(F_n)_{n\in\omega}$ in E such that for every $n \in \omega$, we have $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \geqslant n$. Moreover, if such a FDD exist, it can be choosen with constant as close as 1 as we want.*

Proof. It is immediate that if there is a FDD $(F_n)_{n\in\omega}$ in E with $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \geqslant n$ for every n, then E is not isomorphic to ℓ_2. Now suppose that E is not isomorphic to ℓ_2. Then no finite-codimensional subspace of E is isomorphic to ℓ_2, so by lemma IV.6, when playing the subspace-asymptotic game in E, player **II** can, at the n^{th} turn, play F_n with $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \geqslant n$. Lemma IV.5 concludes immediately.

□

An FDD satisfying the conclusion of lemma IV.7 will be called a *good FDD* in the rest of this manuscript.

To finish this section, we recall some simple facts about directs sums and HI spaces. Recall that two subspaces Y, Z of a Banach space X are in *topological direct sum* if $Y \cap Z = \{0\}$ and if the natural projection $Y \oplus Z \longrightarrow Y$ is bounded. This is equivalent to say that the mapping $Y \times Z \longrightarrow Y \oplus Z$ defined by $(y, z) \mapsto y + z$ is an isomorphism; thus, by the open mapping theorem, saying that Y and Z are in topological direct sum is equivalent to say that $Y \cap Z = \{0\}$ and $Y + Z$ is closed in X. In particular, a space X is HI if and only if no pair of subspaces of X are in topological direct sum. Also recall that Y and Z are not in topological direct sum if and only if $d(S_Y, S_Z) = 0$: indeed, saying that the projection $Y \oplus Z \longrightarrow Y$ is unbounded is equivalent to say that we can find $y \in S_Y$ and $z \in Z$ such that $\|y + z\|$ is arbitrarily small, so that y and z are arbitrarily close.

IV.2 The first dichotomy

In this section, we give a Hilbert-avoiding version of Gowers' first dichotomy I.21. We fix E a separable Banach space non-isomorphic to ℓ_2. We let P be the set of its subspaces that are not isomorphic to ℓ_2, and on P, we put the usual quasi-order $\subseteq^*$ defined by $X \subseteq^* Y$ if $X \cap Y$ has finite codimension in X. We let d be the distance on S_E induced by the norm.

The results presented in this section take their roots in an idea of Valentin Ferenczi. He remarked that lemma IV.6 has the following corollary:

Proposition - definition IV.8. *The space $\mathcal{HA}_E = (P, S_E, \subseteq, \subseteq^*, \in)$, called the Hilbert-avoiding space over E, is an approximate Gowers space.*

Proof. The verification of the axioms 1., 4., and 5. in the definition of a Gowers space are straightforward. The axiom 2. follows from the fact that if a space X has a finite-codimensional subspace isomorphic to ℓ_2, then X is itself isomorphic to ℓ_2. We now verify 3.. Let $(X_n)_{\in\omega}$ be a $\subseteq$-decreasing sequence of elements of P. Since X_n is non-isomorphic to ℓ_2, then by the lemma there exist a finite-dimensional subspace $F_n \subseteq X_n$ such that F_n is not n-isomorphic to $\ell_2^{\dim(F_n)}$. Then we let $X^* = \overline{\sum_{n\in\omega} F_n}$. Since X contains all the F_n's, it is infinite-dimensional and non-isomorphic to ℓ_2. Moreover, for every $n \in \omega$, $X^* \subseteq X_n + \sum_{i<n} F_i$, so $X^* \subseteq^* X_n$ as wanted.

□

In the same way as we deduced Gowers' Ramsey-type theorem from its abstract version theorem III.17, we can, using this space, give a Hilbert-avoiding version of this theorem:

Theorem IV.9. *Let $\mathcal{X} \in (S_E)^\omega$ be an analytic set, Δ be a sequence of positive real numbers, and $\varepsilon > 0$. Then:*

- *either there exists a good FDD $(F_n)_{n\in\omega}$ in E, with constant at most $1+\varepsilon$, such that no block-sequence of this FDD belongs to $\mathcal{X}$;*
- *or there exists a subspace X of E, non-isomorphic to ℓ_2, such that **II** has a strategy in Gowers' game below X to reach $(\mathcal{X})_\Delta$.*

Beware: here, when talking about *Gowers' game*, we talk about the version of Gowers' game corresponding to the approximate Gowers space $\mathcal{HA}_E$. This means that in this game, player **I** is only allowed to play subspaces of X that are not isomorphic to ℓ_2.

Proof of theorem IV.9. Let $\mathcal{K}$ be the set of unit spheres of nonzero finite-dimensional subspaces of E. For $F, G \subseteq E$ finite-dimensional, let $S_F \oplus S_G = S_{F+G}$. This defines a system $(\mathcal{K}, \oplus)$ of compact sets on $\mathcal{HA}_E$. Apply the abstract Gowers' theorem III.17 to $\mathcal{HA}_E$, to this system, to the set $\mathcal{X}$, the subspace E, and the sequence Δ. It gives us a subspace X of E, non-isomorphic to ℓ_2, such that:

- either player **I** has a strategy τ in SF_X to build a sequence $(S_{F_n})_{n\in\omega}$ such that $\mathrm{bs}((S_{F_n})_{n\in\omega}) \subseteq \mathcal{X}^c$;
- or player **II** has a strategy in G_X to reach $(\mathcal{X})_\Delta$.

In the second case we are done, so suppose now that we are in the first case. By lemma IV.5, player **I** has also a strategy σ in $SubF_X$ to build a FDD with constant at most $(1+\varepsilon)$. Remark that in this case, the games SF_X and $SubF_X$ can be identified. We let **I** play to this unique game using both of the strategies τ and σ *at the same time*, that is, at each turn, he plays the intersection of the subspace given by σ and of the subspace given by τ, which is still finite-codimensional in X. This ensures that the outcome $(F_n)_{n\in\omega}$ will be a FDD with constant at most $1+\varepsilon$ such that $\mathrm{bs}((S_{F_n})_{n\in\omega}) \subseteq \mathcal{X}^c$. On her side, since **I** always plays subspaces that are non-isomorphic to ℓ_2, **II** can play at the n^{th} turn a subspace F_n such that $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \geqslant n$. This ensures that the outcome will be a good FDD. To finish, block-sequences of $(S_{F_n})_{n\in\omega}$ are exactly the block-sequences of the FDD $(F_n)_{n\in\omega}$, so the fact that $\mathrm{bs}((S_{F_n})_{n\in\omega}) \subseteq \mathcal{X}^c$ ensures that the outcome will have the wanted property.

□

We can now turn to our dichotomy. Recall that a Banach space X is said to be *primary* if for every subspaces Y, Z of X, if $X = Y \oplus Z$, then either Y or Z is isomorphic to X. This motivates the following definition, that can be seen as a variant of primary spaces, or as a weakening of HI spaces:

Definition IV.10.

1. A separable Banach space X is said to be *Hilbert-primary* if for every subspaces Y, Z of X, if $X = Y \oplus Z$, then either Y or Z is isomorphic to ℓ_2.
2. The space X is *hereditarily Hilbert-primary (HHP)* if every subspace of X is Hilbert-primary.

Remark that, in the same way as we did for HI spaces in the last section, HHP spaces can be characterized as spaces X such that no pair of subspaces $Y, Z \subseteq X$ non-isomorphic to ℓ_2 is in topological direct sum. Obviously, ℓ_2 is HHP, and every HI space is HHP. The following proposition gives us another example of an HHP space.

Lemma IV.11. *If X is a separable HI space, then $X \oplus \ell_2$ is HHP.*

Proof. Suppose not. Then $X \oplus \ell_2$ has two subspaces Y and Z, non-isomorphic to ℓ_2, and whose sum is a topological direct sum. We denote respectively by P_{ℓ_2} and P_X the projections of $X \oplus \ell_2$ onto X and ℓ_2 and we suppose that the norm on $X \oplus \ell_2$ has been choosen in such a way that these projections have norm 1.

We describe a play $(U_0, \mathbb{R}u_0, U_1, \mathbb{R}u_1, \ldots)$ of the game $SubF_{X\oplus\ell_2}$ where $(u_i)_{i\in\omega}$ is a normalized sequence and where **I** plays using his strategy to build a FDD with constant at most 2. Describe how **II** plays. Suppose that we are at turn i, so player **I** just played U_i; and suppose that i is even. Since U_i is a finite-codimensional subspace of $X \oplus \ell_2$, we have that $U_i \cap Y$ is not isomorphic to ℓ_2, so in particular, $P_{\ell_2}\restriction(U_i \cap Y)$ is not an isomorphism onto its image. In particular, there exists $u_i \in S_{U_i\cap X}$ such that $\|P_{\ell_2}(u_i)\| \leqslant \frac{1}{2^{n+4}}$. We let **II** play $\mathbb{R}u_i$ in $SubF_{X\oplus\ell_2}$. If i is odd, we do the same but with Z instead of Y.

In this way we have built a basic sequence $(u_i)_{i\in\omega}$ with constant at most 2 such that $u_i \in Y$ for i even and $u_i \in Z$ for i odd. Let U be the closed subspace spanned by the u_i's, and let $x \in U$ with norm 1. We write $x = \sum_{i=0}^{\infty} x_i u_i$. Then for every $i \in \omega$, $|x_i| \leqslant 4$. And we have:

$$\|P_{\ell_2}(x)\| = \left\|\sum_{i=0}^{\infty} x_i P_{\ell_2}(u_i)\right\| \leqslant 4 \left\|\sum_{i=0}^{\infty} P_{\ell_2}(u_i)\right\| \leqslant \frac{1}{2}.$$

So $\|P_X(x)\| = \|x - P_{\ell_2}(x)\| \geqslant \frac{1}{2}$. In particular, $P_X\restriction U$ is an isomorphism between U and its image. Since both $Y \cap U$ and $Z \cap U$ are infinite-dimensional, and are in topological direct sum, the same holds for their images by P_X. But $P_X(Y \cap U)$ and $P_X(Z \cap U)$ are subspaces of X which is HI, so this is a contradiction.

□

By now, we do not know any other example of an HHP space. It would be particularly interesting for us to know if there exist HHP spaces that are non-isomorphic to ℓ_2 and that do not have any HI subspace; such spaces should be ℓ_2-saturated (i.e. ℓ_2 can be embedded in every subspace of such a space).

Our dichotomy is the following.

Theorem IV.12. *Let E be a separable Banach space, non-isomorphic to ℓ_2. Then there exists a subspace X of E, non-isomorphic to ℓ_2, such that:*

- *either X has a good UFDD;*
- *or X is HHP.*

This is a dichotomy between two classes that are, in some sense, hereditary. The second one is hereditary with respect to taking subspaces that are non-isomorphic to ℓ_2, and the first one is hereditary with respect to good block-FDDs: a block-FDD of a UFDD is a UFDD. Moreover, these classes are disjoint: if $(F_i)_{i\in\omega}$ is a good UFDD of X, then for every infinite and coinfinite $A \subseteq \omega$, we have a decomposition of X in a direct sum of two subspaces that are not isomorphic to ℓ_2, $\overline{\bigoplus_{i\in A} F_i}$ and $\overline{\bigoplus_{i\in A^c} F_i}$. Thus, we have a genuine dichotomy of spaces non-isomorphic to ℓ_2 in the sense of Gowers; we know how to build lots of operators on a space X with a good UFDD, the only missing thing would be a better understanding of the operators on a HHP space that is not isomorphic to ℓ_2.

Proof of theorem IV.12. Fix Δ a sequence of positive real numbers that will be determined at the end of the proof. For every integer $N \geqslant 1$, let $\mathcal{X}_N$ be the set of sequences $(x_i)_{i\in\omega} \in (S_E)^\omega$ such that there exists $n \in \omega$ and a sequence $(a_i)_{i<n} \in \mathbb{R}^n$ such that $\left\|\sum_{\substack{i<n\\ i \text{ even}}} a_i x_i\right\| > N \left\|\sum_{i<n} a_i x_i\right\|$. The $\mathcal{X}_N$'s are open subsets of $(S_E)^\omega$. Firstly suppose that the following property $(*)$ holds:

$(*)$ There exists $N \in \omega$ and a good FDD $(F_n)_{n\in\omega}$ in E such that no block-sequence of $(F_n)_{n\in\omega}$ belongs to $\mathcal{X}_N$.

We then show that $(F_n)_{n\in\omega}$ is a UFDD. More precisely, we will show that given $m \in \omega$, $A \subseteq m$, and $(y_i)_{i<m} \in \prod_{i<m} F_i$, we have $\|\sum_{i\in A} y_i\| \leqslant (N+1)\|\sum_{i<m} y_i\|$; by the criterion given in the last section, it will be enough to conclude.

Let $B = \{i < m \mid y_i \neq 0\}$. If $B = \varnothing$, then there is noting to prove, so we suppose $B \neq \varnothing$. If $\min B \in A$, then we can build a sequence $A_0 < A_1 < \ldots < A_{n-1}$ of subsets of B such that $B = \bigcup_{i<n} A_i$ and $A \cap B = \bigcup_{\substack{i<n\\ i \text{ even}}} A_i$. Then, we let, for $i < n$, $a_i = \left\|\sum_{j\in A_i} y_j\right\|$ and $x_i = \frac{1}{a_i}\sum_{j\in A_i} y_j$. In this way, we have $\sum_{i\in A} y_i = \sum_{\substack{i<n\\ i \text{ even}}} a_i x_i$ and $\sum_{i<m} y_i = \sum_{i<n} a_i x_i$. Moreover, $(x_i)_{i<n}$ is a finite block-sequence of $(F_n)_{n\in\omega}$, so it can be prolonged to an (infinite) block-sequence, that will belong to $\mathcal{X}_N^c$. Therefore, we have that $\left\|\sum_{\substack{i<n\\ i \text{ even}}} a_i x_i\right\| \leqslant N \left\|\sum_{i<n} a_i x_i\right\|$, or in other words $\|\sum_{i\in A} y_i\| \leqslant N\|\sum_{i<m} y_i\|$, as

wanted. Now, if $\min B \notin A$, then we can apply the previous result to A^c and get that $\|\sum_{i\in A^c} y_i\| \leqslant N \|\sum_{i<m} y_i\|$, so $\|\sum_{i\in A} y_i\| \leqslant \|\sum_{i<m} y_i\| + \|\sum_{i\in A^c} y_i\| \leqslant (N+1) \|\sum_{i<m} y_i\|$, as wanted.

We now suppose that the property $(*)$ is not satisfied. We build a decreasing sequence $(X_N)_{N\in\omega}$ of subspaces of E, non-isomorphic to ℓ_2, in the following way. We let $X_0 = E$. If X_N has been constructed, knowing that $(*)$ is not satisfied and applying theorem IV.9 to the space X_N, the sequence Δ and the set $\mathcal{X}_{N+1}$, we get $X_{N+1} \subseteq X_N$ non-isomorphic to ℓ_2 such that player **II** has a strategy in $G_{X_{N+1}}$ to reach $(\mathcal{X}_{N+1})_\Delta$. The sequence $(X_N)_{N\in\omega}$ being built, there exists a subspace $X \subseteq E$ non-isomorphic to ℓ_2 such that for every n, we have $X \subseteq^* X_N$. This show that for every $N \geqslant 1$, player **II** has a strategy in G_X to reach $(\mathcal{X}_N)_\Delta$.

We now show that X is HHP. Suppose not, then there exists two subspaces Y, Z of X, non-isomorphic to ℓ_2, such that $Y \oplus Z$ is a topological direct sum. We let P be the projection from $Y \oplus Z$ to Y and we choose an integer $N \geqslant \|\!|P|\!\|$. We consider a play of F_X and a play of G_X played simultaneously, and having the same outcome $(x_i)_{i\in\omega}$, as represented on the diagrams below:

F_X	**I**	U_0		U_1		U_2		U_3		$\dots$	
	II		x_0		x_1		x_2		x_3		$\dots$

G_X	**I**	$U_0 \cap Y$		$U_1 \cap Z$		$U_2 \cap Y$		$U_3 \cap Z$		$\dots$	
	II		x_0		x_1		x_2		x_3		$\dots$

This is how these games are played:

- In F_X, **I** plays using a strategy enuring that the outcome is a basic sequence with constant at most 2. Such a strategy exists by lemma IV.5 (here, the games F_X and $SubF_X$ can be identified, since **II** only plays vectors). We denote by $(U_i)_{i\in\omega}$ the sequence of his moves.
- At the turn i of G_X, if i is even, **I** plays $U_i \cap Y$, and if i is odd, he plays $U_i \cap Z$.
- In G_X, **II** plays using her strategy to reach $(\mathcal{X}_{4N})_\Delta$. The sequence of her moves will be denoted by $(x_i)_{i\in\omega}$.
- At the turn i of F_X, **II** plays x_i. This is always a legal move: indeed, by the rules of G_X, we have $x_i \in U_i$.

This ensures that the sequence $(x_i)_{i\in\omega}$ built in this way is a basic sequence with constant at most 2, is in $(\mathcal{X}_{4N})_\Delta$, and that for i even, we have $x_i \in Y$, and for i odd, we have $x_i \in Z$.

We now choose Δ in such a way that if $(y_i)_{i\in\omega}$ is a basic sequence with constant at most 2, and if $(z_i)_{i\in\omega} \in (S_E)^\omega$ is a sequence such that for every i, $\|y_i - z_i\| \leqslant \Delta_i$, then $(y_i)_{i\in\omega}$ and $(z_i)_{i\in\omega}$ are 2-equivalent; such a Δ exists by lemma IV.3. Remark that if $(y_i)_{i\in\omega}$ and $(z_i)_{i\in\omega}$ are 2-equivalent, and if $(y_i)_{i\in\omega} \in \mathcal{X}_{4N}$, then $(z_i)_{i\in\omega} \in \mathcal{X}_N$. In particular, we deduce that $(x_i)_{i\in\omega} \in \mathcal{X}_N$. So there exists $n \in \omega$ and a sequence $(a_i)_{i<n} \in \mathbb{R}^n$ such that $\left\| \sum_{\substack{i<n \\ i \text{ even}}} a_i x_i \right\| > N \left\| \sum_{i<n} a_i x_i \right\|$. Now let $y = \sum_{\substack{i<n \\ i \text{ even}}} a_i x_i$ and $z = \sum_{\substack{i<n \\ i \text{ odd}}} a_i x_i$. We have $y \in Y$, $z \in Z$ and $\|y\| > N\|y + z\|$; this contradicts the fact that the projection from $Y \oplus Z$ to Y has norm less or equal than N.

□

IV.3 The second dichotomy

In this section, we give a Hilbert-avoiding version of Ferenczi and Rosendal's dichotomy between minimal subspaces and tight subspaces (theorem I.25). We begin with some definitions. Given a FDD $(F_i)_{n\in\omega}$ in some Banach space E, and $A \subseteq \omega$, we will denote by $[F_i \mid i \in A]$ the subspace $\overline{\bigoplus_{i\in A} F_i}$.

Definition IV.13.

1. A separable Banach space X non-isomorphic to ℓ_2 is *minimal among non-hilbertian spaces (MNH)* if it embeds in all of its subspaces that are not isomorphic to ℓ_2.
2. Let $(F_i)_{i\in\omega}$ be a FDD in some Banach space E. A Banach space X is *tight in* $(F_i)_{i\in\omega}$ if there is an infinite sequence of intervals $I_0 < I_1 < \ldots$ of integers such that for every infinite $A \subseteq \omega$, we have $X \not\sqsubseteq \left[F_i \middle| i \notin \bigcup_{j\in A} I_j\right]$.
3. A good FDD $(F_i)_{i\in\omega}$ is said to be *tight for non-hilbertian spaces (TNH)* if every Banach space non-isomorphic to ℓ_2 is tight in it. A Banach space X is *tight for non-hilbertian spaces (TNH)* if it has a good FDD which is TNH.

Some more properties of TNH spaces will be proved in the next setion. The dichotomy we will prove is the following:

Theorem IV.14. *Let E be a Banach space with a good FDD $(E_i)_{i\in\omega}$. Then $(E_i)_{i\in\omega}$ has a good block-FDD $(F_i)_{i\in\omega}$ such that:*

- *either $[F_i \mid i \in \omega]$ is MNH;*
- *or $(F_i)_{i\in\omega}$ is TNH.*

In particular, every separable Banach space non-isomorphic to ℓ_2 has either an MNH subspace, or a TNH subspace.

Again, this a genuine dichotomy in the sense of Gowers. A subspace of a MNH space that is not isomorphic to ℓ_2 is itself MNH; and a good block-FDD of a TNH FDD is itself TNH. Moreover, a TNH space cannot be MNH.

The rest of this section is devoted to prove this dichtomy. Remark that the "in particular" part of the theorem is a direct consequence of the first part, since every separable Banach space non-isomorphic to ℓ_2 contains a good FDD. So we prove the first part. We fix a Banach space E with a good FDD $(E_i)_{i\in\omega}$.

Since the proof is quite technical, it is inconvenient to deal with approximation, so we will work with vector spaces on a countable field. For every $i \in \omega$, we fix a basis $(e^i_j)_{j<d_i}$ of E_i. In this way, every $x \in E$ can be decomposed in a unique way as a sum $x = \sum_{i=0}^{\infty} x^i$ with $x^i \in E_i$ for every i, and every x^i can be decomposed in a unique way as a sum $x^i = \sum_{j<d_i} x^i_j e^i_j$. We fix K a countable subfield of $\mathbb{R}$ such that for every $x \in E$, if all the x^i_j are in K and if all them are zero except for a finite number, then $\|x\| \in K$. Such a field can be built inductively: begin with $K_0 = \mathbb{Q}$, and define K_{n+1} the subfield of $\mathbb{R}$ generated by K_n and all of the $\|x\|$'s, for $x \in E$ such that for every $x \in E$, all the x^i_j are in K and all them are zero except for a finite number; and then let $K = \bigcup_{n\in\omega} K_n$. In the rest of this section, vector spaces on K will be denoted by capital script roman letters, and closed subspaces of E (of finite or infinite dimension) will be denoted by capital printscript roman letters. We let $\mathscr{V}$ be the K-vector subspace of E generated by all the e^i_j's. For $\mathscr{R}$ a K-vector subspace of E, we let $\overline{\mathscr{R}}$ be its closure in E, and $S_{\mathscr{R}}$ be the set of its normalized vectors. Remark that $\overline{\mathscr{R}}$ is a $\mathbb{R}$-vector subspace of E, that $\overline{\mathscr{R}}$ is $\mathbb{R}$-finite-dimensional if and only $\mathscr{R}$ is K-finite-dimensional, and that in this case, their dimensions are equal. We have $\overline{\mathscr{V}} = E$. Also remark that since, for $x \in \mathscr{V}$, we have $\frac{x}{\|x\|} \in \mathscr{V}$, then for $\mathscr{R}$ a vector subspace of $\mathscr{V}$, $S_{\mathscr{R}}$ is always non-trivial.

We now define a Gowers space. For every $i \in \omega$, we let $\mathscr{E}_i$ be the K-vector subspace of E_i generated by the e^i_j's for $j < d_i$. Obviously we have $\overline{\mathscr{E}_i} = E_i$ and $\mathscr{V} = \bigoplus_{i\in\omega} \mathscr{E}_i$. We define a *block-FDD* of $(\mathscr{E}_i)_{i\in\omega}$ as a sequence $(\mathscr{F}_i)_{i\in\omega}$ of nonzero finite-dimensional K-vector subspaces of E such that there exists a sequence $A_0 < A_1 < \ldots$ of finite sets of integers such that for every i, we have $\mathscr{F}_i \subseteq \oplus_{j\in A_i} \mathscr{E}_j$. A block-FDD $(\mathscr{F}_i)_{i\in\omega}$ will often be denoted by the letter $\mathscr{F}$; thus, when we speak about a block-FDD $\mathscr{F}$ without further explanation, it will be supposed that its terms are denoted by $\mathscr{F}_i$. Remark that if $\mathscr{F}$ is a block-FDD of $\mathscr{E}$, then $\left(\overline{\mathscr{F}_i}\right)_{i\in\omega}$ is a block-FDD of $(E_i)_{i\in\omega}$. So we will say that $\mathscr{F}$ is *good* if and only if $\left(\overline{\mathscr{F}_i}\right)_{i\in\omega}$ is a good block-FDD of $(E_i)_{i\in\omega}$.

We let P be the set of good block-FDDs of $\mathscr{E}$. If $\mathscr{F}, \mathscr{G} \in P$, we let $\mathscr{F} \leqslant \mathscr{G}$ if $\mathscr{F}$ is the block-FDD of $\mathscr{G}$. We let $\mathscr{F} \leqslant^* \mathscr{G}$ if there exists $n \in \omega$ such that $(\mathscr{F}_i)_{i\geqslant n} \leqslant \mathscr{G}$. We let X be the set of pairs $(\mathscr{R}, x)$ where $\mathscr{R}$ is a finite-dimensional subspace of $\mathscr{V}$ and x an element of $S_{\mathscr{V}}$. For $\mathscr{F} \in P$ and $(\mathscr{R}, x) \in X$, we say that $(\mathscr{R}, x) \lhd \mathscr{F}$ if $\mathscr{R} \subseteq \bigoplus_{i\in\omega} \mathscr{F}_i$.

Lemma IV.15. *$\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ is a Gowers space.*

Proof. The only non-trivial property to verify is the diagonalization property. So, suppose that we have a $\leqslant$-decreasing sequence $(\mathscr{F}^i)_i \in \omega$ of elements of P (with for every i, $\mathscr{F}^i = (\mathscr{F}_j{}^i)_{j\in\omega}$). Then we can verify, by induction, that for every $k \in \omega$ and $i < j$, we have $\mathscr{F}_k{}^j \subseteq \bigoplus_{l\geqslant k} \mathscr{F}_l{}^i$. Letting $\mathscr{F}^* = (\mathscr{F}_i{}^i)_{i\in\omega}$, this proves that $\mathscr{F}^*$ is a good block-FDD and that for every $i \in \omega$, $(\mathscr{F}_l{}^*)_{l\geqslant i} \leqslant \mathscr{F}^i$, as wanted.

□

In this proof, we will use variants of the usual games $F_{\mathscr{F}}$, $G_{\mathscr{F}}$, $A_{\mathscr{F}}$, $B_{\mathscr{F}}$ of the Gowers space $\mathcal{G}$, but with additional rules. These games will be denoted with a prime: $F'_{\mathscr{F}}$, $G'_{\mathscr{F}}$, $A'_{\mathscr{F}}$, $B'_{\mathscr{F}}$. We define these games below.

Definition IV.16. Let $\mathscr{F} \in P$.

- The game $G'_{\mathscr{F}}$ is defined in the following way:

I	$\mathscr{F}^0$		$\mathscr{F}^1$		$\dots$
II		$\mathscr{R}^0, x_0$		$\mathscr{R}^1, x_1$	$\dots$

  where the $\mathscr{F}^i$'s are good block-FDDs of $\mathscr{F}$, the $\mathscr{R}^i$'s are finite-dimensional subspaces of $\mathscr{V}$, and the x_i's are elements of $S_{\mathscr{V}}$, with the constraints for **II** that for all $i < \omega$, $\mathscr{R}^i \subseteq \bigoplus_{j\in\omega} \mathscr{F}_j{}^i$, and $x_i \in \mathscr{R}^0 + \dots + \mathscr{R}^i$. The outcome of the game is the sequence $(x_i)_{i\in\omega} \in (S_{\mathscr{V}})^\omega$.

- The game $F'_{\mathscr{F}}$ is defined in the same way as $G'_{\mathscr{F}}$ apart form the fact that this time, player **I** has to choose the $\mathscr{F}^i$ in such a way that $\mathscr{F}^i \lessapprox \mathscr{F}$.

- The game $A'_{\mathscr{F}}$ is defined in the following way:

I		$\mathscr{R}^0, x_0, \mathscr{G}^0$		$\mathscr{R}^1, x_1, \mathscr{G}^1$			$\dots$
II	$\mathscr{F}^0$		$\mathscr{S}^0, y_0, \mathscr{F}^1$		$\mathscr{S}^1, y_1, \mathscr{F}^2$		$\dots$

  where the $\mathscr{F}^i$'s and the $\mathscr{G}^i$'s are elements of P, the $\mathscr{R}^i$'s and the $\mathscr{S}^i$'s are finite-dimensional subspaces of $\mathscr{V}$, and the x_i's and the y_i's are elements of $S_{\mathscr{V}}$. The rules are the following:

  - for **I** : for all $i \in \omega$, $\mathscr{G}^i \lessapprox \mathscr{F}$, $\mathscr{R}^i \subseteq \bigoplus_{j\in\omega} \mathscr{F}_j{}^i$, and $x_i \in \mathscr{R}^0 + \dots + \mathscr{R}^i$;
  - for **II** : for all $i \in \omega$, $\mathscr{F}^i \leqslant \mathscr{F}$, $\mathscr{S}^i \subseteq \bigoplus_{j\in\omega} \mathscr{G}_j{}^i$, and $y_i \in \mathscr{S}^0 + \dots + \mathscr{S}^i$;

  and the outcome of the game is the pair of sequences $((x_i)_{i\in\omega}, (y_i)_{i\in\omega}) \in ((S_{\mathscr{V}})^\omega)^2$.

- The game $B'_{\mathscr{F}}$ is defined in the same way as $A'_{\mathscr{F}}$, except that this time the $\mathscr{F}^i$'s are required to satisfy $\mathscr{F}^i \lessapprox \mathscr{F}$, whereas the $\mathscr{G}^i$ are only required to satisfy $\mathscr{G}^i \leqslant \mathscr{F}$.

The starting point of this proof will be the following lemma.

Lemma IV.17. *There exists $\mathscr{F} \in P$ such that either player **I** has a strategy in $A'_{\mathscr{F}}$ to ensure that the sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are not equivalent, or player **II** has a strategy in $B'_{\mathscr{F}}$ to ensure that the sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are equivalent.*

Proof. The idea is that the games $A'_{\mathscr{F}}$ and $B'_{\mathscr{F}}$ can be seen as special cases of the games $A_{\mathscr{F}}$ and $B_{\mathscr{F}}$ by coding the rules in the target set. Let $\mathcal{X}$ be the set of sequences $(\mathscr{R}^0, x_0, \mathscr{S}^0, y_0, \mathscr{R}^1, x_1, \ldots) \in X^\omega$ satisfying one of the two following conditions:

- The sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are inequivalent, and for every $i \in \omega$, we have $x_i \in \mathscr{R}^0 + \ldots + \mathscr{R}^i$;
- There exists $i \in \omega$ such that $y_i \notin \mathscr{S}^0 + \ldots + \mathscr{S}^i$, and for every $j \leqslant i$, $x_j \in \mathscr{R}^0 + \ldots + \mathscr{R}^j$.

The first condition says that **I** reaches his target without cheating, and the second one says that **II** cheats, and is the first player to do so. Then we have that:

- If player **I** has a strategy in $A_{\mathscr{F}}$ to reach $\mathcal{X}$, then he has a strategy in $A'_{\mathscr{F}}$ to ensure that the sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are inequivalent;
- If player **II** has a strategy in $B_{\mathscr{F}}$ to reach $\mathcal{X}^c$, then he has a strategy in $B'_{\mathscr{F}}$ to ensure that the sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are equivalent.

Since the set $\mathcal{X}$ is a G_δ-subset of X^ω, the conclusion of the lemma immediately follows from the adversarial Ramsey property in the space $\mathcal{G}$ (theorem II.4).

□

In the rest of this proof, we fix the block-FDD $\mathscr{F}$ given by the last lemma. We say that a sequence $(u_i)_{i\in\omega} \in (S_{\mathscr{F}})^\omega$ is *$\mathscr{F}$-correct* if there exists $\mathscr{G} \leqslant \mathscr{F}$ and a partition of ω in successive intervals $I_0 < I_1 < \ldots$ such that for every $i \in \omega$, the finite sequence $(u_j)_{j\in I_i}$ is a basis of $\mathscr{G}_i$. The next proposition contains the combinatorial content of theorem IV.14.

Proposition IV.18. *One of the following statements is satisfied:*

(1) *For every $\mathscr{F}$-correct sequence $(u_i)_{i\in\omega}$, player **I** has a strategy in $F'_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\omega}$ that is not equivalent to $(u_i)_{i\in\omega}$;*

(2) *There exists a $\mathscr{F}$-correct sequence $(u_i)_{i\in\omega}$ such that player **II** has a strategy in $G'_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\omega}$ that is equivalent to $(u_i)_{i\in\omega}$.*

Proof. Suppose that **(1)** is not satisfied. For the rest of the proof, we fix a $\mathscr{F}$-correct sequence $(u_i)_{i\in\omega}$ such that player **I** has no strategy in $F'_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\omega}$ that is not equivalent to $(u_i)_{i\in\omega}$. By the determinacy of this game, player **II** has a strategy τ in $F'_{\mathscr{F}}$ to build a sequence which is equivalent to $(u_i)_{i\in\omega}$. By correctness of this sequence, we can also fix $\mathscr{G} \leqslant \mathscr{F}$ and a partition of ω in successive intervals $I_0 < I_1 < \ldots$ such that for every $i \in \omega$, $(u_j)_{j\in I_i}$ is a basis of $\mathscr{G}_i$.

Step 1. We prove that **II** has a strategy in $A'_{\mathscr{F}}$ to build two equivalent sequences. We describe this strategy on a play $(\mathscr{G}, \mathscr{R}^0, x_0, \mathscr{F}^0, \mathscr{S}^0, y_0, \mathscr{G}, \ldots)$ of $A'_{\mathscr{F}}$, in which the FDDs played by **II** will always be equal to $\mathscr{G}$ and that will be played at the same time as an auxiliary play $(\mathscr{H}^0, \mathscr{U}^0, z_0, \mathscr{H}^1, \mathscr{U}^1, z_1, \ldots)$ of $F'_{\mathscr{F}}$ during which player **II** always plays according to her strategy τ. Actually, the $\mathscr{R}^i$'s played by **I** in $A'_{\mathscr{F}}$ will not matter at all in this proof, so we will omit them in the notation. At the same time as the games are played, a sequence of integers $0 = n_0 < n_1 < \ldots$ will be constructed. The idea is that the turn i of $A'_{\mathscr{F}}$ will be played at the same time as the turns $n_i, n_i + 1, \ldots, n_{i+1} - 1$ of the game $F'_{\mathscr{F}}$. Suppose that we are just before the turn i of the game $A'_{\mathscr{F}}$, so the x_j's, the $\mathscr{F}^j$'s, the $\mathscr{S}^j$'s, and the y_j's have been defined for all $j < i$. Suppose also that the integers n_j have been defined for all $j \leqslant i$, and that we are just before the turn n_i of the game $F'_{\mathscr{F}}$, so the $\mathscr{H}^n$'s, the $\mathscr{U}^n$'s and the z_n's have been played for all $n < n_i$. We represent on the diagram below the turn i of the game $A'_{\mathscr{F}}$, and the turns $n_i, \ldots, n_{i+1} - 1$ of the game $F'_{\mathscr{F}}$.

$F'_{\mathscr{F}}$	**I**		$\mathscr{F}^i$	$\ldots$	$\mathscr{F}^i$		$\ldots$
	II	$\ldots$	$\mathscr{U}^{n_i}$, z_{n_i}	$\ldots$		$\mathscr{U}^{n_{i+1}-1}$, $z_{n_{i+1}-1}$	
$A'_{\mathscr{F}}$	**I**	$\ldots$	x_i, $\mathscr{F}^i$		$\ldots$		
	II	$\ldots$, $\mathscr{G}$		$\mathscr{S}^i$, y_i, $\ldots$			

We now describe how these turns are played. In $A'_{\mathscr{F}}$, player **II** plays $\mathscr{G}$. Then player **I** answers by a FDD $\mathscr{F}^i \lessapprox \mathscr{F}$ and a vector $x_i \in \bigoplus_{k\in\omega} \mathscr{G}_k$. Thus, x_i can be decomposed on the basis $(u_m)_{m\in\omega}$: we can find $n_{i+1} \in \omega$ and $(a_i^m)_{m<n_{i+1}} \in K^{n_{i+1}}$ such that $x_i = \sum_{m<n_{i+1}} a_i^m u_m$. Moreover, we can assume that $n_{i+1} > n_i$.

Now, during the $n_{i+1} - n_i$ following turns of the game $F'_{\mathscr{F}}$, we will let player **I** play $\mathscr{F}^i$ (So we will have, for every $n_i \leqslant m < n_{i+1}$, $\mathscr{H}^m = \mathscr{F}^i$). According to the strategy τ, player **II** will answer with $\mathscr{U}^{n_i}$, $z_{n_i}, \ldots, \mathscr{U}^{n_{i+1}-1}$, $z_{n_{i+1}-1}$. We now let $\mathscr{S}^i = \mathscr{U}^{n_i} + \ldots + \mathscr{U}^{n_{i+1}-1}$, and $y_i = \sum_{m<n_{i+1}} a_i^m z_m$. Since all the $\mathscr{U}^m$'s, for $n_i \leqslant m < n_{i+1}$ are finite-dimensional subspaces of $\bigoplus_{k\in\omega} \mathscr{F}_k{}^i$, then $\mathscr{S}^i$ is itself a finite-dimensional subspace of $\bigoplus_{k\in\omega} \mathscr{F}_k{}^i$. And since all the z_m, for $n_i \leqslant m < n_{i+1}$, are elements of $\mathscr{U}^0 + \ldots + \mathscr{U}^{n_{i+1}-1} = \mathscr{S}^0 + \ldots + \mathscr{S}^i$, then y_i is itself an element of $\mathscr{S}^0 + \ldots + \mathscr{S}^i$. So we can let **II** play $\mathscr{S}^i$ and y_i in $A'_{\mathscr{F}}$, what finishes the description of the strategy.

The fact that in $F'_{\mathscr{F}}$, player **II** always plays according to the strategy τ, ensures that the sequences $(u_m)_{m\in\omega}$ and $(z_m)_{m\in\omega}$ are equivalent. Remark that the sequence $(x_i)_{i\in\omega}$ is built from $(u_m)_{m\in\omega}$ in exactly the same way that the sequence $(y_i)_{i\in\omega}$ is built from $(z_m)_{m\in\omega}$; so this ensures that $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are equivalent, concluding this step of the proof.

Step 2. **II** has a strategy σ in $B'_{\mathscr{F}}$ to build two equivalent sequences. Indeed, by step 1, **I** has no strategy in $A'_{\mathscr{F}}$ to build inequivalent sequences; so the conclusion follows from lemma IV.17.

Step 3. We conclude, proving that player **II** has a strategy in $G'_{\mathscr{F}}$ to build a sequence $(y_i)_{i\in\omega}$ that is equivalent to $(u_i)_{i\in\omega}$. We describe this strategy on a play of $G'_{\mathscr{F}}$ that will played simultaneously with a play of $B'_{\mathscr{F}}$ where **II** will play according to her strategy σ, and a play of $F'_{\mathscr{F}}$ where **II** will play according to her strategy τ (for a fixed $i \in \omega$, the turn i of each game will be played at the same time). The moves of the players during the turn i of the games are described in the diagram below.

$F'_{\mathscr{F}}$	**I**		$\mathscr{F}^i$		$\ldots$
	II	$\ldots$		$\mathscr{R}^i$, x_i	
$B'_{\mathscr{F}}$	**I**			$\mathscr{R}^i$, x_i, $\mathscr{G}^i$	
	II	$\ldots$, $\mathscr{F}^i$			$\mathscr{S}^i$, y_i, $\ldots$
$G'_{\mathscr{F}}$	**I**		$\mathscr{G}^i$		$\ldots$
	II	$\ldots$		$\mathscr{S}^i$, y_i	

We describe more precisely these moves. Suppose that in $G'_{\mathscr{F}}$, player **I** plays $\mathscr{G}^i$. We look at the move $\mathscr{F}^i$ made by **II** in $B'_{\mathscr{F}}$ according to her strategy σ, and we let **I** copy this moves in $F'_{\mathscr{F}}$. In this game, according to her strategy τ, player **II** will answer with some $\mathscr{R}^i$ and x_i. Now, in $B'_{\mathscr{F}}$, we can let **I** answer with $\mathscr{R}^i$, x_i and $\mathscr{G}^i$. In this game, according to her strategy σ, player **II** answers with some $\mathscr{S}^i$ and some y_i. Then the strategy of player **II** in $G'_{\mathscr{F}}$ will consist in answering with $\mathscr{S}^i$ and y_i.

Let us verify that this strategy is as wanted. The outcome of the game $F'_{\mathscr{F}}$ is the sequence $(x_i)_{i\in\omega}$; the use of the strategy τ by **II** ensures that this sequence is equivalent to $(u_i)_{i\in\omega}$. The outcome of the game $B'_{\mathscr{F}}$ is the pair of sequences $((x_i)_{i\in\omega}, (y_i)_{i\in\omega})$; the use by **II** of her strategy σ ensures that these two sequences are equivalent. We deduce that the sequences $(u_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are equivalent, concluding the proof.

$\square$

We now let, for every $i \in \omega$, $F_i = \overline{\mathscr{F}_i}$. The sequence $(F_i)_{i\in\omega}$ is a good block-FDD of $(E_i)_{i\in\omega}$ and we can let $F = [F_i \mid i \in \omega]$. By proposition IV.18, theorem IV.14 will be proved once we have proved the two following lemmas:

Lemma IV.19. *Suppose that there exists a $\mathscr{F}$-correct sequence $(u_i)_{i\in\omega}$ such that player **II** has a strategy in $G'_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\omega}$ that is equivalent to $(u_i)_{i\in\omega}$. Then $(F_i)_{i\in\omega}$ has a good block-FDD $(G_i)_{i\in\omega}$ such that $G = [G_i \mid i \in \omega]$ is MNH.*

Lemma IV.20. *Suppose that for every $\mathscr{F}$-correct sequence $(u_i)_{i\in\omega}$, player **I** has a strategy in $F'_{\mathscr{F}}$ to build a sequence $(x_i)_{i\in\omega}$ that is not equivalent to $(u_i)_{i\in\omega}$. Then the FDD $(F_i)_{i\in\omega}$ is TNH.*

In order to prove these, we need two more lemmas. The first one is due to Ferenczi and Rosendal and its proof can be found in [18] (lemma 3).

Lemma IV.21 (Ferenczi – Rosendal)**.** *For every $n \in \omega$, there exists $c(n) \geqslant 1$ such that for every Banach space U, and every subspaces V and W having both codimension n, V and W are $c(n)$-isomorphic.*

Lemma IV.22. *Let $\mathscr{G} \in P$, U a subspace of $\left[\overline{\mathscr{G}_i}\middle| i \in \omega\right]$ non-isomorphic to ℓ_2, and $\varepsilon > 0$. Then there exists $\mathscr{H} \leqslant \mathscr{G}$ such that $\left[\overline{\mathscr{H}_i}\middle| i \in \omega\right]$ can be $1+\varepsilon$-embedded in U.*

Proof. Let C be the constant of the FDD $\left(\overline{\mathscr{G}_i}\right)_{i\in\omega}$. We fix Δ a sequence of positive real numbers that will be defined in the course of the proof. We build inductively the block-sequence $\mathscr{H}$. We will let, for every $i \in \omega$, $n_i = \sum_{j<i} \dim(\mathscr{H}_i)$, and we will build, at the same time as the block-sequence $\mathscr{H}$, two normalized sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$, with the property that for every i, the sequence $(x_n)_{n_i\leqslant n<n_{i+1}}$ will be a basis of $\mathscr{H}_i$, and for every $n \in \omega$, $\|x_n - y_n\| \leqslant \Delta_n$.

Fix $i \in \omega$ and suppose that the $\mathscr{H}_j$'s have been built for $j < i$, that the n_j' have been built for $j \leqslant i$, and that the x_i's and the y_j's have been built for $n < n_i$. Let $m_i \in \omega$ be such that for every $j < i$, $\mathscr{H}_j \subseteq \bigoplus_{m<m_i} \mathscr{G}_m$ (take for example for m_i the supremum of the supports of the x_n's for $n < n_i$). Then $\overline{\bigoplus_{m\geqslant m_i} \mathscr{G}_m} = \left[\overline{\mathscr{G}_m}\middle| m \geqslant m_i\right]$ has finite codimension in $\left[\overline{\mathscr{G}_m}\middle| m \in \omega\right]$, so $U \cap \overline{\bigoplus_{m\geqslant m_i} \mathscr{G}_m}$ is not isomorphic to ℓ_2, and contains a $\mathbb{R}$-finite-dimensional vector subspace H_i such that H_i is not $2e^i$-isomorphic to $\ell_2^{\dim(H_i)}$. We let $n_{i+1} = n_i + \dim(H_i)$ and we let $(y_n)_{n_i\leqslant n<n_{i+1}}$ be an Auerbach basis of H_i. We choose $x_{n_i}, \ldots, x_{n_{i+1}-1}$ normalized vectors in $\bigoplus_{m\geqslant m_i} \mathscr{G}_m$ such that for $n_i \leqslant n < n_{i+1}$, we have $\|x_n - y_n\| \leqslant \Delta_n$. We now let $\mathscr{H}_i$ be the vector subspace of $\mathscr{V}$ generated by the x_n's for $n_i \leqslant n < n_{i+1}$. This achieves the construction of $\mathscr{H}$. This is a good FDD: indeed, since $(x_n)_{n_i\leqslant n<n_{i+1}}$ is 1-Auerbach, we can choose Δ small enough to ensure that $(y_i)_{n_i\leqslant n<n_{i+1}}$ is 2-equivalent to it, so $\overline{\mathscr{H}_i}$ is 2-isomorphic to H_i and hence cannot be e^i-isomorphic to $\ell_2^{\dim(\mathscr{H}_i)}$.

Since all the $(x_n)_{n_i\leqslant n<n_{i+1}}$ are 2-Auerbach and since the FDD $\left(\overline{\mathscr{H}_i}\right)_{i\in\omega}$ has constant at most C, this ensures that $(x_n)_{n\in\omega}$ is $4C$-Auerbach. So if Δ has been choosen small enough, we can ensure that the sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ are $(1+\varepsilon)$-equivalent. Since the closed subspace of E generated by the x_i's is $\left[\overline{\mathscr{H}_i}\middle| i \in \omega\right]$, and since all the y_i's are in U, this ensures that $\left[\overline{\mathscr{H}_i}\middle| i \in \omega\right]$ can be $(1+\varepsilon)$-embedded in U.

□

Proof of lemma IV.19. Since the sequence $(u_i)_{i\in\omega}$ is $\mathscr{F}$*-correct* we can fix $\mathscr{G} \leqslant \mathscr{F}$ and a partition of ω in successive intervals $I_0 < I_1 < \ldots$ such that for every $i \in \omega$, the finite sequence $(u_j)_{j\in I_i}$ is a basis of $\mathscr{G}_i$. We let $G_i = \overline{\mathscr{G}_i}$. Then $(G_i)_{i\in\omega}$ is a good block-FDD of $(F_i)_{i\in\omega}$; we will show that $G = [G_i \mid i \in \omega]$ is MNH. By lemma IV.22, it is enough to show that for every $\mathscr{H} \leqslant \mathscr{G}$, the space G can be embedded in $H = \left[\overline{\mathscr{H}_i}\,\middle|\, i \in \omega\right]$. Fix such an $\mathscr{H}$ and consider a play of $G'_{\mathscr{F}}$ where **I** plays $\mathscr{H}$ at each turn, and **II** answers with her strategy to build a sequence $(x_i)_{i\in\omega}$ that is equivalent to $(u_i)_{i\in\omega}$. Since all the x_i's are in H and since the closed space generated by the u_i's is G, the mapping $u_i \mapsto x_i$ extends to an embedding of G into H.

□

Proof of lemma IV.20. We have to prove that every Banach G space non-isomorphic to ℓ_2 is tight in $(F_i)_{i\in\omega}$. By lemma IV.22, it is enough to prove it in the case where G has the form $\left[\overline{\mathscr{G}_i}\,\middle|\, i \in \omega\right]$, where $\mathscr{G} \leqslant \mathscr{F}$. So we fix such a $\mathscr{G}$. For every $i \in \omega$, we let $n_i = \sum_{j<i} \dim(\mathscr{G}_i)$, and we let $(u_n)_{n_i \leqslant n < n_{i+1}}$ be a normalized basis of $\mathscr{G}_i$ that is 2-Auerbach (this can be done by firstly, choosing an Auerbach basis of $\overline{\mathscr{G}_i}$ and then, perturbing it a little bit in order to have all the terms in $\mathscr{G}_i$). In this way, the sequence $(u_k)_{k\in\omega}$ is $\mathscr{F}$-correct. We let C be the constant of the FDD $\left(\overline{\mathscr{G}_i}\right)_{i\in\omega}$. We then have that the sequence $(u_k)_{k\in\omega}$ is $4C$-Auerbach. Since the proof is quite technical, we will proceed in several steps.

Step 1. The hypothesis of this lemma says that **I** has a strategy τ in $F'_{\mathscr{F}}$ to build a sequence that is inequivalent to $(u_k)_{k\in\omega}$. We reinterpret this statement using the asymptotic game of an approximate asymptotic space we now define. The space will be $\mathcal{A} = (\omega, Y, D, \lessapprox, \lhd)$, where:

- the set of subspaces is ω, and the order $\lessapprox$ is defined by $m \lessapprox n \Leftrightarrow n \leqslant m$;
- an element of Y is a pair (I, x) where I is a finite interval of ω and x is an element of S_F; the distance on Y is defined by $d((I,x),(J,y)) = \|x - y\|$ if $I = J$ and 1 otherwise;
- $(I, x) \lhd n$ if $n \leqslant I$, i.e. every element of I is greater or equal than n.

In this proof, we will denote by F''_n the asymptotic game of the space $\mathcal{A}$ under the subspace n of $\mathcal{A}$ in order to avoid confusion with $F'_{\mathscr{F}}$. We fix $K \geqslant 1$, and Δ a sequence of positive real numbers, less than 1, such that for every normalized sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$, if $(x_i)_{i\in\omega}$ is $16KC$-Auerbach and if for every $i \in \omega$, $\|x_i - y_i\| \leqslant 2\Delta_i$, then $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ are 2-equivalent. We let $\mathcal{X}$ be the set of sequences $(I_0, x_0, I_1, x_1, \ldots)$ of elements of Y such that if for every $i \in \omega$, we have $x_i \in \left(S_{\bigoplus_{j\leqslant i}\left(\bigoplus_{k\in I_j} F_k\right)}\right)_{\Delta_i}$, then $(x_i)_{i\in\omega}$ is not $4K$-equivalent to $(u_i)_{i\in\omega}$. The aim of this step is to show that **I** has a strategy to reach $\mathcal{X}$ in F''_0. For this, we describe a play $(n_0, I_0, x_0, n_1, I_1, x_1, \ldots)$ of F''_0

at the same time as a play $(\mathscr{F}^0, \mathscr{R}^0, y_0, \mathscr{F}^1, \mathscr{R}^1, y_1, \ldots)$ of $F'_{\mathscr{F}}$ during which **I** always plays according to his strategy τ and such that for every $i \in \omega$, $\|x_i - y_i\| \leqslant 2\Delta_i$. Suppose that the first i turns of both games have been played; at the turn i, in $F'_{\mathscr{F}}$, according to his strategy, player **I** plays $\mathscr{F}^i$. Since $\mathscr{F}^i \lessapprox \mathscr{F}$, there is $n_i \in \omega$ such that $(\mathscr{F}_k)_{k \geqslant n_i} \leqslant \mathscr{F}^i$; we let **I** play this n_i in F''_0. In this game, **II** answers with I_i and x_i, and we can suppose that $x_i \in \left(S_{\bigoplus_{j \leqslant i}\left(\bigoplus_{k \in I_j} F_k\right)}\right)_{\Delta_i}$ otherwise **II** has lost the game. So we can find a nomalized element $y_i \in \bigoplus_{j \leqslant i}\left(\bigoplus_{k \in I_j} \mathscr{F}_k\right)$ such that $\|x_i - y_i\| \leqslant 2\Delta_i$. In $F'_{\mathscr{F}}$, we let **II** play $\mathscr{R}^i = \bigoplus_{k \in I_i} \mathscr{F}_k$ and y_i; this finishes the description of the strategy.

Now verify that this strategy is as wanted, that is, that $(I_0, x_0, I_1, x_1, \ldots) \in \mathcal{X}$. Suppose not. Then $(x_i)_{i \in \omega}$ is $4K$-equivalent to $(u_i)_{i \in \omega}$, so $(x_i)_{i \in \omega}$ is $16KC$-Auerbach. By the choice of Δ, we get that $(x_i)_{i \in \omega}$ and $(y_i)_{i \in \omega}$ are equivalent, so $(y_i)_{i \in \omega}$ is equivalent to $(u_i)_{i \in \omega}$, thus contradicting the assumption on the strategy τ.

Step 2. We prove that for every $K \geqslant 1$, there exists a sequence of intervals of integers $I_0 < I_1 < \ldots$ such that for every infinite $A \subseteq \omega$ *containing* 0, we have $G \not\sqsubseteq_K \left[F_i \middle| i \notin \bigcup_{j \in A} I_j\right]$. We fix $K \geqslant 1$, and we keep the sequence Δ and the set $\mathcal{X}$ defined at the previous step relatively to K. We define a system of compact sets on $\mathcal{A}$. For J a nonempty finite interval of integers, let $K_J = \{J\} \times S_{\left(\bigoplus_{i \leqslant \max(J)} F_i\right)}$; this is a compact subset of Y. We let $\mathcal{K}$ be the set of all the K_J's, and for $K_{J_1}, K_{J_2} \in \mathcal{K}$, we let $K_{J_1} \oplus K_{J_2} = K_J$, where J is the smallest interval of ω containing J_1 and J_1. Then $(\mathcal{K}, \oplus)$ is a system of compact sets on $\mathcal{A}$. By step 1, player **I** has a strategy in F''_0 to reach $\mathcal{X}$; so by theorem III.16, he has a strategy in SF''_0, the strong asymptotic game of the space $\mathcal{A}$ under the subspace 0, to build a sequence $(K_{J_i})_{i \in \omega}$ with $\mathrm{bs}((K_{J_i})_{i \in \omega}) \subseteq (\mathcal{X})_\Delta$. In particular, there exists such a sequence with $\min(J_0) > 0$ and for every $i \geqslant 1$, $\max(J_{i-1}) + 1 < \min(J_i)$. We let $I_0 = [\![0, \min(J_0) - 1]\!]$ and for every $i \geqslant 1$, $I_i = [\![\max(J_{i-1}) + 1, \min(J_i) - 1]\!]$, in such a way that we have a partition of ω in intervals $I_0 < J_0 < I_1 < J_1 < \ldots$. We prove that the sequence $(I_i)_{i \in \omega}$ is as wanted.

Suppose not. Then there exists an infinite $A \subseteq \omega$ containing 0 such that $G \sqsubseteq_K \left[F_i \middle| i \notin \bigcup_{j \in A} I_j\right]$. In particular, in $\left[F_i \middle| i \notin \bigcup_{j \in A} I_j\right]$, there is a normalized sequence $(x_i)_{i \in \omega}$ that is K-equivalent to $(u_i)_{i \in \omega}$. We can then find a normalized sequence $(y_i)_{i \in \omega}$ that is close enough to $(x_i)_{i \in \omega}$ to be 2-equivalent to it, and such that moreover, every y_i has finite support on the FDD $(F_i)_{i \in \omega}$; so we can find integers $0 = n_0 < n_1 < \ldots$ in A such that for every i, $\mathrm{supp}(y_i) < \min I_{n_{i+1}}$. For every i, we let $L_i = J_{n_i} \cup I_{n_i+1} \cup J_{n_i+1} \cup \ldots \cup J_{n_{i+1}-2} \cup I_{n_{i+1}-1} \cup J_{n_{i+1}-1}$. In this way we have $K_{L_i} = \oplus_{n_i \leqslant n < n_{i+1}} K_{J_n}$, and $(L_i, y_i) \in K_{L_i}$. Since $\mathrm{bs}((K_{J_n})_{n \in \omega}) \subseteq (\mathcal{X})_\Delta$, we deduce that $(L_0, y_0, L_1, y_1, \ldots) \in (\mathcal{X})_\Delta$. So there exists $(z_i)_{i \in \omega} \in (S_F)^\omega$ such that for every i, $\|z_i - y_i\| \leqslant \Delta_i$, and such that $(L_0, z_0, L_1, z_1, \ldots) \in \mathcal{X}$. Since the n_i's are in A, and since $\mathrm{supp}(y_i) < \min(I_{n_{i+1}})$, we have that for every i, $y_i \in \left[F_k \middle| k < \min(I_{n_{i+1}}),\, k \notin \bigcup_{j<i} I_{n_j}\right]$.

Remark that the set of $k < \min(I_{n_{i+1}})$ such that $k \notin \bigcup_{j<i} I_{n_j}$ is exactly $\bigcup_{j\leqslant i} L_i$, so $y_k \in S_{\bigoplus_{j\leqslant i}\left(\bigoplus_{k\in L_j} F_k\right)}$. So for every i, $z_i \in \left(S_{\bigoplus_{j\leqslant i}\left(\bigoplus_{k\in L_j} F_k\right)}\right)_{\Delta_i}$. By the definition of $\mathcal{X}$, this implies that $(z_i)_{i\in\omega}$ and $(u_i)_{i\in\omega}$ are not $4K$-equivalent. But on the other hand, $(y_i)_{i\in\omega}$ is 2-equivalent to $(x_i)_{i\in\omega}$ which is K-equivalent to $(u_i)_{i\in\omega}$ which is $4C$-Auerbach. So $(y_i)_{i\in\omega}$ is $16KC$-Auerbach, and by the choice of Δ, we get that $(z_i)_{i\in\omega}$ is 2-equivalent to $(y_i)_{i\in\omega}$, so $4K$-equivalent to $(u_i)_{i\in\omega}$, a contradiction.

Step 3. We show that G is tight in $(F_n)_{n\in\omega}$. For this, for every $N \geqslant 1$, we consider a sequence of intervals of integers $I_0^N < I_1^N < \ldots$ given by step 2, such that for every infinite $A \subseteq \omega$ containing 0, we have $G \not\sqsubseteq_N \left[F_n \,\middle|\, n \notin \bigcup_{i\in A} I_i^N\right]$. For every $d \in \omega$, we denote by $c(d)$ the constant given by lemma IV.21 such that for every Banach space U, and every subspaces V and W having both codimension d, V and W are $c(d)$-isomorphic. We build a sequence $J_1 < J_2 < \ldots$ of intervals of integers in the following way. All the J_l's, for $l < k$, being defined, we can choose J_k such that:

- for every $N \leqslant k$, J_k contains at least one interval of the sequence $(I_i^N)_{i\in\omega}$;
- $\max(J_k) \geqslant d_k + \max(I_0^{N_k})$, where $d_k = \dim([F_n \mid n < \min(J_k)])$ and $N_k = \lceil kc(d_k)\rceil$.

We show that for every infinite $A \subseteq \omega$, we have $G \not\sqsubseteq [F_n \,|\, n \notin \bigcup_{k\in A} J_k]$. Suppose not, and let A be witnessing it. Let $K \geqslant 1$ such that $G \sqsubseteq_K [F_n \,|\, n \notin \bigcup_{k\in A} J_k]$. Let $k_0 \in A$ such that $K \leqslant k_0$. Let $n_0 = \min J_{k_0}$. Since $\max(J_{k_0}) \geqslant d_{k_0} + \max(I_0^{N_{k_0}})$, we have in particular $d_{k_0} \leqslant \dim([F_n \mid \max(I_0^{N_{k_0}}) < n \leqslant \max(J_{k_0})])$, so we can find a subspace $H \subseteq [F_n \mid \max(I_0^{N_{k_0}}) < n \leqslant \max(J_{k_0})]$ of dimension d_{k_0}. Remark that $[F_n \,|\, (n < n_0) \vee (n \notin \bigcup_{k\in A} J_k)]$ and $[F_n \,|\, (n \geqslant n_0) \wedge (n \notin \bigcup_{k\in A} J_k)] \oplus H$ both have codimension d_{k_0} in their sum, so they are $c(d_{k_0})$-isomorphic. In particular, since G can be k_0-embedded in the first of these spaces, then it can be $N_{k_0} = \lceil k_0 c(d_{k_0})\rceil$-embedded in the second one. So in particular, $G \sqsubseteq_{N_{k_0}} \left[F_n \,\middle|\, n \notin I_0^{N_{k_0}} \cup \left(\bigcup_{\substack{k\in A\\ k>k_0}} J_k\right)\right]$. But the set $I_0^{N_{k_0}} \cup \left(\bigcup_{\substack{k\in A\\ k>k_0}} J_k\right)$ contains infinitely many of the $I_i^{N_{k_0}}$, $i \in \omega$, and in particular $I_0^{N_{k_0}}$, so this contradicts the hypothesis.

□

IV.4 Links with ergodicity and Johnson's problem

In this section, we discuss some consequences of the two previous dichotomies that could help for Johnson's problem and for Ferenczi and Rosendal's conjecture about ergodic spaces. We start by looking at the form that the first dichotomy takes for non-ergodic spaces. The result we will prove is the following:

Theorem IV.23. *Let E be a non-ergodic separable Banach space, non-isomorphic to ℓ_2. Then there exists a subspace X of E, non-isomorphic to ℓ_2, such that:*

- *either X has a unconditional basis;*
- *or X is HHP.*

This theorem is an immediate consequence of the first dichotomy and of the following proposition, which is an unpublished result by Ferenczi:

Proposition IV.24 (Ferenczi)**.** *Let E be a non-ergodic separable Banach space, non-isomorphic to ℓ_2, having a good UFDD. Then E has a subspace X, non-isomorphic to ℓ_2, with an unconditional basis.*

We reproduce here the proof of this proposition. We start by introducing two results that will be needed in the proof. The first one involves the following property, defined and studied by Pisier [53]:

Definition IV.25. A Banach space X is said to have the property (H) if for every $\lambda \geqslant 1$, there exists a constant $K(\lambda)$ such that for every finite sequence $(x_i)_{i<n} \in (S_X)^{<\omega}$, if $(x_i)_{i<n}$ is λ-unconditional, then $\frac{\sqrt{n}}{K(\lambda)} \leqslant \|\sum_{i<n} x_i\| \leqslant K(\lambda)\sqrt{n}$.

A Hilbert space has property (H): indeed, a λ-inconditional normalized sequence in a Hilbert space is λ^2-equivalent to an orthonormal sequence (see, for example, [36], page 71). So property (H) characterizes spaces that are, in some sense, "close" to ℓ_2. In [5] and [4], Anisca proved the following result:

Theorem IV.26 (Anisca)**.** *Every separable Banach space non-isomorphic to ℓ_2 and having the property (H) is ergodic.*

The second result we need is due to Rosendal ([55], theorem 15). Let $\mathbf{E_0'}$ be the equivalence relation on $[\omega]^\omega$ defined as follows: if $A, B \in [\omega]^\omega$, we say that $A\mathbf{E_0'}B$ if there exists $n \in \omega$ such that $|A \cap n| = |B \cap n|$ and $A\backslash n = B\backslash n$. Rosendal proved the following:

Proposition IV.27. *Let E be a meager equivalence relation on $[\omega]^\omega$, with $\mathbf{E_0'} \subseteq E$. Then $\mathbf{E_0} \leqslant_B E$.*

(In [55], this result is stated and proved for equivalence relations on $\mathcal{P}(\omega)$, however, the same proof works in the case of $[\omega]^\omega$.)

We now prove proposition IV.24. For $s \in 2^{<\omega}$, we denote by N_s the basic open subset $\{A \in [\omega]^\omega \mid \forall n < |s| \, (n \in A \Leftrightarrow s(n) = 1)\}$ of $[\omega]^\omega$. We begin with a lemma.

Lemma IV.28 (Ferenczi)**.** *Let X be a non-ergodic Banach space with an FDD $(F_i)_{i\in\omega}$. Then there exists a constant K such that for every $i \in \omega$, $[F_j \mid j < i] \sqsubseteq_K [F_j \mid j \geqslant i]$.*

Proof. For every $i \in \omega$, let $n_i = \sum_{j<i} \dim(F_j)$. Let $(x_n)_{n_i \leqslant n < n_{i+1}}$ be a normalized basis of F_i. For $A \in \mathcal{P}(\omega)$, let X_A be the closed subspace of X generated by the x_n's, for $n \in A$. Remark that for $A \in [\omega]^\omega$ and for a cofinite $B \subseteq A$, we have $X_A = X_B \oplus X_{A \backslash B}$: indeed, up to reducing B, we can suppose that there exists $i \in \omega$ such that $B \geqslant n_i$ and $A \backslash B < n_i$; but in this case we have $X_B \subseteq [F_j \mid j \geqslant i]$ and $X_{A \backslash B} \subseteq [F_j \mid j < i]$, so the answer follows. In particular, if $A, B \in [\omega]^\omega$ are such that $A \mathbf{E_0'} B$, then X_A and X_B have the same finite codimension in $X_{A \cup B}$, so by lemma IV.21, they are isomorphic.

We define an equivalence relation E on $[\omega]^\omega$ by $A \, E \, B$ if X_A and X_B are isomorphic. As we just saw, $\mathbf{E_0'} \subseteq E$. Also remark that the mapping $A \mapsto X_A$ from $[\omega]^\omega$ to $\mathrm{Sub}(X)$ with the Effros Borel structure is a Borel mapping. Indeed, if U is an open subset of X and if $X_A \cap U \neq \varnothing$, then there exists $m \in \omega$ such that $X_{A \cap m} \cap U \neq \varnothing$, so for $B \in [\omega]^\omega$, as soon as $A \cap m = B \cap m$, we have $X_B \cap U \neq \varnothing$. In particular, since X is non-ergodic, then $\mathbf{E_0}$ does not reduce to E. So by proposition IV.27, we deduce that E is non-meager.

E is analytic so has the Baire property, so by Kuratowski-Ulam theorem, there exists $A \in [\omega]^\omega$ such that the E-equivalence class of A, denoted by $[A]$, is non-meager. For $K \geqslant 1$, denote by $[A]_K$ the set of $B \in [\omega]^\omega$ such that X_A and X_B are K-isomorphic. Since $[A] = \bigcup_{K \geqslant 1} [A]_K$, then for some $K \geqslant 1$, $[A]_K$ is non-meager. So it is comeager in a basic open set N_s, for some $s \in 2^{<\omega}$. We let $N = |s|$ and $m = |\{n < |s| \mid s(n) = 1\}|$. We denote by $c(m)$ (resp. $c(N-m)$) the constant given by lemma IV.21 such that two subspaces of a Banach space having both codimension m (resp. $N-m$) are m-isomorphic (resp. $(N-m)$-isomorphic). We show that for $i \in \omega$ such that $n_i \geqslant N$, we have $[F_j \mid j < i] \sqsubseteq_{K^2 c(m) c(N-m)} [F_j \mid j \geqslant i]$; the conclusion will follow.

Let i be such that $n_i \geqslant N$. Consider $t_1 = s^\frown(0, \ldots, 0)$ and $t_2 = (0, \ldots, 0)^\frown s$, where in each definition, there are n_i 0's. Since $[A]_K$ is dense in N_s, there exists $B_1 \in N_{t_1} \cap [A]_K$. We define $B_2 \in [\omega]^\omega$ in the following way: for $n \geqslant N + n_i$, we let $n \in B_2$ iff $n \in B_1$, and for $n < N + n_i$, we let $n \in B_2$ iff $t_2(n) = 1$. The set B_2 has been obtained by shifting m 1's at the beginning of B_1. In particular, $|(B_1 \cup B_2) \backslash B_1| = |(B_1 \cup B_2) \backslash B_2| = m$ so X_{B_1} and X_{B_2} are $c(m)$-isomorphic. Thus, X_{B_2} is $Kc(m)$-isomorphic to A.

In the same way, we can consider $u_1 = s ^\frown (1, \ldots, 1)$ and $u_2 = (1, \ldots, 1) ^\frown s$, where in each definition, there are n_i 1's. Then there exists $C_1 \in N_{u_1}$ and $C_2 \in N_{u_2}$ such that $C_1 \backslash (N + n_i) = C_2 \backslash (N + n_i)$, and $C_1 \in [A_k]$. The set C_2 has been obtained by shifting $N-m$ 0's at the beginning of C_1. Thus, X_{C_1} and X_{C_2} are $c(N-m)$-isomorphic. Therefore, X_{C_2} and X_{B_2} are $K^2 c(m) c(N-m)$-isomorphic. Since $[F_j \mid j < i] \subseteq C_2$ and $B_2 \subseteq [F_j \mid j \geqslant i]$, the conclusion follows.

□

Proof of proposition IV.24. Let $(E_i)_{i \in \omega}$ be a good UFDD of E, and let K be its unconditional constant. If there exists a block-sequence of this UFDD that spans a subspace that is non-isomorphic to ℓ_2, then we can take for X this subspace and we are done. So we will suppose that every block-sequence of (E_i) spans a subspace isomorphic to ℓ_2.

Step 1. *We show that there exists a constant C such that every block-sequence of (E_i) spans a subspace C-isomorphic to ℓ_2.* If not, then for every $i \in \omega$ and every C, there exists a block-sequence $(x_n^{i,C})_{n\in\omega}$ of $[E_j \mid j \geqslant i]$ spanning a block-subspace that is not C-isomorphic to ℓ_2. By lemma IV.6, for every i and C, we can find an integer $n_{i,C}$ such that $(x_n^{i,C})_{n<n_{i,C}}$ spans a finite-dimensional subspace that is not C-isomorphic to $\ell_2^{n_{i,C}}$. Now we build an increasing sequence of integers $(m_N)_{N\geqslant 1}$ in the following way : $m_1 = 0$ and m_N having been built, let $m_{N+1} = \min \operatorname{supp}(x_{n_{m_N,N}}^{m_N,N})$. In this way, the sequence $(x_0^{0,1}, \ldots x_{n_{0,1}-1}^{0,1}, x_0^{m_2,2}, \ldots, x_{n_{m_2,2}-1}^{m_2,2}, x_0^{m_3,3}, \ldots)$ is a block-sequence of (E_i) having subsequences spanning finite-dimensional spaces that are arbitrarily far away from euclidean spaces; so the subspace spanned by this sequence is not isomorphic to ℓ_2, a contradiction.

Step 2. *We show that there is a constant M such that every block-sequence of (E_i) is M-equivalent to the canonical basis of ℓ_2.* Let $(x_n)_{n\in\omega}$ be such a sequence. It is K-unconditional and by the previous step, it spans a block-subspace that is C-isomorphic to ℓ_2, so it is C-equivalent to a sequence $(y_n)_{n\in\omega}$ in ℓ_2 that is KC-unconditional. Remark that (y_n) is not necessarily normalized, but by C-equivalence with the normalized sequence (x_n), we get that for every n, $\frac{1}{C} \leqslant \|y_n\| \leqslant C$. Let $z_n = \frac{y_n}{\|y_n\|}$. By KC-unconditionality of (y_n), we get that $(z_n)_{n\in\omega}$ is K^2C^4-equivalent to (y_i), so K^2C^5-equivalent to (x_i). Moreover, (z_i) is a normalized KC-unconditional sequence in ℓ_2, so it is K^2C^2-equivalent to the canonical basis of ℓ_2. So (x_i) is K^4C^7-equivalent to the canonical basis of ℓ_2. Hence, $M = K^4C^7$ is as wanted.

Step 3. *We show that there exists $\mu \geqslant 1$ such that for every $A \geqslant 1$ and $i_0 \in \omega$, there exists $j_0 \geqslant i_0$ and a μ-unconditional normalized sequence $(x_k)_{k<k_0} \in [E_i \mid i_0 \leqslant i < j_0]^{<\omega}$ spanning a subspace that is not A-isomorphic to $\ell_2^{k_0}$.* Since E is non-ergodic and non-isomorphic to ℓ_2, by theorem IV.26, it does not have property (H); so there exists $\lambda \geqslant 1$ such that for every $B \geqslant 1$, there exists a finite λ-unconditional sequence $(u_k)_{k<k_0} \in (S_E)^{k_0}$ with either $\|\sum_{i<n} x_i\| < \frac{\sqrt{n}}{\lambda^4B^7}$, or $\lambda^4B^7\sqrt{n} < \|\sum_{i<n} x_i\|$. In particular, this sequence is not λ^4B^7-equivalent to the orthonormal basis of a euclidean space, but it is λ-unconditional, so by the same method as in the previous step, we can show that $\operatorname{span}(\{u_k \mid k < k_0\})$ is not B-isomorphic to a $\ell_2^{k_0}$. We can take a sufficiently small perturbation $(v_k)_{k<k_0}$ of $(u_k)_{k<k_0}$, still normalized and whose elements have finite support, to ensure that $(v_k)_{k<k_0}$ is 2λ-unconditional and spans a space that is not $\frac{B}{2}$-isomorphic to a euclidean space. Now recall that lemma IV.28 gives a constant D such that for every $i \in \omega$, $[E_j \mid j < i] \sqsubseteq_D [E_j \mid j \geqslant i]$. Using these embeddings, we can find, given $i_0 \in \omega$, a sequence $(w_k)_{k<k_0}$ that is $2D\lambda$-unconditional and that spans a subspace that is not $\frac{B}{2D}$-isomorphic to a euclidean space, such that for every $k < k_0$, we have $w_k \in [E_i \mid i \geqslant i_0]$. Finally, we can choose a sufficiently small perturbation $(x_k)_{k<k_0}$ of $(w_k)_{k>k_0}$, still normalized, and such for every $k < k_0$, x_k is a vector of $[F_i \mid i \geqslant i_0]$ with finite support, such that $(x_k)_{k<k_0}$ is $4D\lambda$-unconditional and spans a subspace that is not $\frac{B}{4D}$-isomorphic to a euclidean space. So we can take $\mu = 4D\lambda$.

Step 4. *We conclude.* Using step 3, we can build a sequence $(x_n)_{n<\omega} \in (S_E)^\omega$ and integers $0 = n_0 < n_1 < \ldots$ such that, for every i, letting $F_i = \mathrm{span}(\{x_n \mid n_i \leqslant n < n_{i+1}\})$, we have that $(F_i)_{i\in\omega}$ is a good block-FDD of (E_i), and the sequence $(x_n)_{n_i\leqslant n<n_{i+1}}$ is μ-unconditional. Since (F_i) is good, we have that $F := [F_i \mid i \in \omega] = \overline{\mathrm{span}(\{x_n \mid n \in \omega\})}$ is not isomorphic to ℓ_2. So to conclude the proof, it is enough to show that the sequence $(x_n)_{n\in\omega}$ is unconditional. So let $(a_n)_{n\in\omega} \in \mathbb{R}^\omega$ be with finite support, and $(\varepsilon_n)_{n\in\omega} \in \{-1,1\}^\omega$, we will show that $\|\sum_{n\in\omega} \varepsilon_n a_n x_n\| \leqslant M^2\mu \|\sum_{n\in\omega} a_n x_n\|$. For $i \in \omega$, let $b_i = \left\|\sum_{n_i\leqslant n<n_{i+1}} a_n x_n\right\|$, $y_i = \frac{1}{b_i}\left(\sum_{n_i\leqslant n<n_{i+1}} a_n x_n\right)$, $c_i = \left\|\sum_{n_i\leqslant n<n_{i+1}} \varepsilon_n a_n x_n\right\|$, $z_i = \frac{1}{c_i}\left(\sum_{n_i\leqslant n<n_{i+1}} \varepsilon_i a_n x_n\right)$. Since the sequence $(x_n)_{n_i\leqslant n<n_{i+1}}$ is μ-unconditional, we have that $c_i \leqslant \mu b_i$. Also remark that $(y_i)_{i\in\omega}$ and $(z_i)_{i\in\omega}$ are normalized block-sequences of the FDD (e_i), so by step 2, they are M-equivalent to the canonical basis of ℓ_2. Thus, we have:

$$\begin{aligned}
\left\|\sum_{n\in\omega} \varepsilon_n a_n x_n\right\| &= \left\|\sum_{i\in\omega} c_i z_i\right\| \\
&\leqslant M\sqrt{\sum_{i\in\omega} c_i^2} \\
&\leqslant M\mu\sqrt{\sum_{i\in\omega} b_i^2} \\
&\leqslant M^2\mu\left\|\sum_{i\in\omega} b_i y_i\right\| \\
&= M^2\mu\left\|\sum_{n\in\omega} a_n x_n\right\|.
\end{aligned}$$

□

We now give an interesting consequence of theorem IV.23 for Johnson spaces. In [3], Anisca proves a result implying that a separable Banach space with a finite number of subspaces, up to isomorphism, must contain a subspace isomorphic to ℓ_2. In particular, a Johnson space must contain a subspace isomorphic to ℓ_2. So theorem IV.23 has the following corollary:

Corollary IV.29. *A Johnson space either has an unconditional basis, or is HHP.*

Thus, to prove that a Johnson space necessarily has an unconditional basis, it would be enough to prove that a non-trivial HHP space must have at least three subspaces, up to isomorphism. By similarity with Gowers–Maurey's result that an HI space is not isomorphic to any proper subspace of itself, this seems plausible. However, we did not manage to prove this conjecture. In the next section, a simple proof of Gowers–Maurey's theorem will be presented; this could be a good starting point to try to prove that non-trivial HHP spaces have many non-isomorphic subspaces.

We now turn to the consequences of the second dichotomy. In [15], Ferenczi and Godefroy studied the links between tightness and Baire-category. In particular, they proved that if $(e_i)_{i\in\omega}$ is a basis and $\overline{X \text{ a Banach space}}$, then X is tight in (e_i) if and only if the set of $A \subseteq \omega$ such that $X \sqsubseteq \overline{\operatorname{span}(\{e_i \mid i \in A\})}$ meager in $\mathcal{P}(\omega)$. Using the same ideas, and the result of Rosendal IV.27 linking ergodicity with Baire category, we get the following result:

Theorem IV.30. *Every TNH space is ergodic.*

Proof. Let X be a TNH space with a TNH FDD $(F_i)_{i\in\omega}$. As in the proof of lemma IV.28, we let, for every $i \in \omega$, $n_i = \sum_{j<i} \dim(F_j)$, and $(x_n)_{n_i \leqslant n < n_{i+1}}$ be a normalized basis of F_i. For $A \in [\omega]^\omega$, we let X_A be the closed subspace of X generated by the x_n's, for $n \in A$. And we define an equivalence relation E on $[\omega]^\omega$ by $A\,E\,B$ if X_A and X_B are isomorphic. We will show that $\mathbf{E_0} \leqslant_B E$. Since $\mathbf{E_0'} \subseteq E$, it is enough to show, by proposition IV.27, that E is meager, so by Kuratwski–Ulam's theorem, that for every $A \in [\omega]^\omega$, the E-equivalence class $[A]$ of A is meager. We distinguish two cases.

First case: X_A is isomorphic to ℓ_2. For $N \geqslant 1$, we let D_N the set of $B \in [\omega]^\omega$ such that X_B is not N-isomorphic to ℓ_2. This is an open set: indeed, if X_B is not N-isomorphic to ℓ_2, then by lemma IV.6, there exists $n \in \omega$ such that $X_{B\cap n}$ is not N-isomorphic to a euclidean space, so as soon as $C \in [\omega]^\omega$ satisfies $B \cap n = C \cap n$, we have $C \in D_N$. The set D_N is also dense: indeed, if $s \in 2^{<\omega}$, the set $B \in [\omega]^\omega$ defined by $n \in B \Leftrightarrow (n \geqslant |s| \vee s(n) = 1)$ is in N_s and X_B has finite codimension in X, so it is not isomorphic to ℓ_2 and thus, $B \in D_N$. Since $[A] = \left(\bigcap_{n\geqslant 1} D_N\right)^c$, we have that $[A]$ is meager.

Second case: X_A is not isomorphic to ℓ_2. In this case, since the FDD (F_i) is TNH, then there is an infinite sequence of intervals $I_0 < I_1 < \ldots$ of integers such that for every infinite $M \subseteq \omega$, we have $X_A \not\sqsubseteq \left[F_i \middle| i \notin \bigcup_{j\in M} I_j\right]$. We can let $J_i = \{n \in \omega \mid \exists j \in I_i\ n_j \leqslant n < n_{j+1}\}$, in such a way that $J_0 < J_1 < \ldots$ and that for $B \in [\omega]^\omega$, if for an infinite number of i, we have $B \cap J_i = \varnothing$, then X_B is not isomorphic to X_A. For $k \in \omega$, we let D_k the set of $B \in [\omega]^\omega$ such that there exists $i \geqslant k$ with $B \cap J_i = \varnothing$. Then D_k is an open dense set, and by the previous remark, $[A] \cap (\bigcap_{k\in\omega} D_k) = \varnothing$, so $[A]$ is meager.

□

Corollary IV.31. *Every separable Banach space, non-ergodic and non-isomorphic to ℓ_2, has a MNH subspace.*

This corollary, combined with theorem IV.23, show that to prove the conjecture IV.1, it is enough to prove the following conjecture, seeming much easier:

Conjecture IV.32. *A HHP space cannot be MNH.*

The methods presented in next section could help for this conjecture as well.

IV.5 A simple proof of Gowers–Maurey's theorem

In this section, we present a new proof of the following result by Gowers and Maurey [25]:

Theorem IV.33 (Gowers–Maurey)**.** *An HI space is not isomorphic to any proper subspace of itself.*

Recall that a bounded operator $T : X \longrightarrow Y$ between two Banach space is said to be *bounded below* if there is a constant $c > 0$ such that for every $x \in X$, we have $\|T(x)\| \geqslant c\|x\|$ (by the open mapping theorem, it is equivalent to say that it is one-to-one and has closed range), and *strictly singular* if no restriction of T to a subspace of X is bounded below. In [25], Gowers and Maurey prove theorem IV.33 in the following way: they prove, using spectral theory and Fredholm theory, that every bounded operator from a complex HI space to itself has the form $\lambda \operatorname{Id} + S$, where S is a strictly singular operator (this is not true for real HI spaces), and they deduce the theorem for complex and real HI spaces using Fredholm theory. Here, we present a simple proof using only Fredholm theory and working as well for real and complex spaces. We suppose here that the spaces we consider are real, but the proof is the same for complex spaces.

We start by recalling some basic Fredholm theory; for more details and for proofs, the reader can refer to [1], section 4.4.

Definition IV.34. Let $T : X \to Y$ be a bounded operator between two Banach spaces.

1. We denote by $n(T) \in \omega \cup \{+\infty\}$ the dimension of the kernel of T, and $d(T) \in \omega \cup \{+\infty\}$ the codimension of the range of T.
2. We say that T is *semi-Fredholm* if it has closed range and if at least one of the numbers $n(T)$ and $d(T)$ is finite.
3. We say that T is *Fredholm* if both numbers $n(T)$ and $d(T)$ are finite (this implies that T has closed range).
4. If T is semi-Fredholm, we define its *Fredholm index* as $i(T) = n(T) - d(T) \in \mathbb{Z} \cup \{-\infty, +\infty\}$.

We denote by $\mathcal{F}red(X, Y)$ and $\hat{\mathcal{F}}red(X, Y)$ respectively the set of Fredholm operators and of semi-Fredholm operators between X and Y. We equip $\mathbb{Z} \cup \{-\infty, +\infty\}$ with the topology such that $\mathbb{Z}$ is a discrete subset, the sets $[\![n, +\infty]\!]$ form a basis of neighborhoods of $+\infty$, and the sets $[\![-\infty, n]\!]$ form a basis of neighborhoods of $-\infty$. We have the following theorem:

Theorem IV.35. *$\hat{\mathcal{F}}red(X, Y)$ is an open subset of the space of bounded operators from X to Y, and the Fredholm index $i : \hat{\mathcal{F}}red(X, Y) \longrightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ is continuous.*

We now present the proof of theorem IV.33. Let X be a Banach space (at this point, we do not need to assume that X is HI). We say that a bounded operator $T : X \longrightarrow X$ is *infinitely singular* if for every $\varepsilon > 0$, there exists a subspace Y of X such that $|||T_{\upharpoonright Y}||| \leqslant \varepsilon$. We say that λ is an *infinitely singular value* of a bounded operator $T : X \longrightarrow X$ if $T - \lambda \operatorname{Id}_X$ is infinitely singular.

Lemma IV.36. *Let $T : X \longrightarrow X$ a bounded operator. We have equivalence between:*

(1) *T is not infinitely singular;*

(2) *There exists a finite-codimensional subspace Y of X such that $T_{\upharpoonright Y}$ is bounded below;*

(3) *T is semi-Fredholm and $i(T) < +\infty$.*

Proof. **(2)** $\Rightarrow$ **(1)** is obvious.

(3) $\Rightarrow$ **(2)** Since $i(T) < +\infty$, then $\ker(T)$ is finite-dimensional; let Y be a closed complement of $\ker(T)$. Then T is a bijection between Y and $\operatorname{im}(T)$ and $\operatorname{im}(T)$ is closed, by the open mapping theorem, $T_{\upharpoonright Y}$ is bounded below.

(2) $\Rightarrow$ **(3)** Letting F be a complement of Y in X, we have $\operatorname{im}(T) = T(Y) + T(F)$. Since T is bounded below on Y, we have that $T(Y)$ is closed; moreover $T(F)$ has finite dimension so $\operatorname{im}(T)$ is closed. Since $\ker(T)$ is finite-dimensional, the result follows.

(1) $\Rightarrow$ **(2)** Suppose that **(2)** is not satisfied, and let $\varepsilon > 0$. Then by lemma IV.5, there exists a normalized basic sequence $(f_n)_{n\in\omega}$ in X, with constant at most 2, such that, for every $n \in \omega$, $\|T(f_n)\| \leqslant \frac{\varepsilon}{2^{n+3}}$ (in the game $SubF_X$, player **I** plays with a strategy to build a FDD with constant at most 2, and in the subspace X_n played by **I** at the $(n+1)^{\text{th}}$ turn, **II** can always choose a convenient f_n by the assumption). We let Y be the closed subspace of X generated by the f_n's. Then for $x = \sum_{n=0}^{\infty} x_n f_n \in Y$, we have $\|T(x)\| \leqslant \sum_{n=0}^{\infty} |x_n| \|T(f_n)\| \leqslant \sum_{n=0}^{\infty} 4\|x\| \frac{\varepsilon}{2^{n+3}} = \varepsilon\|x\|$. So $|||T_{\upharpoonright Y}||| \leqslant \varepsilon$, and T is infinitely singular.

□

Lemma IV.37. *Let $T : X \to Y$ be an isomorphism, where Y is a proper subspace of X. Then T has at least two infinitely singular values, a positive one and a negative one.*

Proof. For $t \in [0, 1]$, define $T_t = tT + (1-t)\operatorname{Id}_X$. We show that there exists $t \in (0, 1)$ such that T_t is infinitely singular; this will imply that $\frac{t-1}{t}$ is a negative infinitely singular value of T. Suppose not. Then by lemma IV.36, for every $t \in [0, 1]$, T_t is semi-Fredholm. So letting $f(t) = i(T_t)$ we define a function $f : [0, 1] \longrightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$; by the continuity of Fredholm index, this function is continuous, so constant. This is a contradiction since $f(0) = 0$ and $f(1) < 0$.

We prove in the same way that T has a positive infinitely singular value, considering the operators $T'_t = tT - (1-t)\operatorname{Id}_X$.

□

Using lemma IV.37, the proof of Gowers–Maurey theorem will be complete once we prove the following lemma:

Lemma IV.38. *A bounded operator from an HI space into itself has at most one infinitely singular value.*

Proof. Suppose that X is HI and let $T : X \longrightarrow X$ be a bounded operator. Suppose that T has two infinitely singular values λ and μ. Let $\varepsilon > 0$. We can find subspaces $Y, Z \subseteq X$ such that $|||(T - \lambda \operatorname{Id}_X)_{\restriction Y}||| \leqslant \varepsilon$ and $|||(T - \lambda \operatorname{Id}_X)_{\restriction Z}||| \leqslant \varepsilon$. Since X is HI, Y and Z are not in topological direct sum, so we have $d(S_Y, S_Z) = 0$. In particular, we can find $y \in S_Y$ and $z \in S_Z$ with $\|y - z\| \leqslant \varepsilon$. So we have:

$$\begin{aligned} |\lambda - \mu| &= \|\lambda x\| - \|\mu y\| \\ &\leqslant \|\lambda y - \mu z\| \\ &\leqslant \|\lambda y - T(y)\| + \|T(y) - T(z)\| + \|T(z) - \mu z\| \\ &\leqslant \varepsilon + |||T||| \cdot \|y - z\| + \varepsilon \\ &\leqslant (2 + |||T|||)\varepsilon. \end{aligned}$$

So by making $\varepsilon \longrightarrow 0$, we get that $\lambda = \mu$.

□

We hope that this kind of methods could also apply to show that HHP spaces cannot be MNH, or at least that they must have two non-isomorphic subspace that are non-isomorphic to ℓ_2, thus respectively proving the conjectures IV.1 or IV.2. However, this seems quite difficult, since here, we should replace the use of infinitely singular operators with *ℓ_2-singular operators*, that is, operators $T : X \longrightarrow X$ such that for every $\varepsilon > 0$, there exists a subspace Y, non-isomorphic to ℓ_2, such that $|||T_{\restriction Y}||| \leqslant \varepsilon$. These operators are not Fredholm in general. Thus, an idea could be to define an analog of Fredholm index allowing us to deal with operators T such that $\ker(T)$ is isomorphic to ℓ_2, but not necessarily finite-dimensional.

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