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Report on the doctoral thesis of Noé de Rancourt,

I am writing to report on the doctoral thesis of Noé de Rancourt entitled *Théorie de Ramsey sans principe des tiroirs et application à la preuve de dichotomies d'espaces de Banach*. Since de Rancourt was able to respond to a question of mine already several years ago, I am quite familiar with many of the results of his thesis and the general subject area. Therefore, I feel quite confident when stating that the results presented in his thesis are absolutely first rate and certainly merit a doctoral diploma at any university worldwide. So I am very strongly in favour of the acceptance of the thesis for the doctoral degree in mathematics at the *École Doctorale de Sciences Mathématiques de Paris Centre*.

The thesis lies at the intersection of infinite-dimensional Ramsey theory and functional analysis, in particular, Banach space theory. While this line of research has a very long prehistory, the main impetus to the current interest in the area comes from the seminal results of Gowers on Banach's homogeneous space problem. In his work, Gowers proved a game theoretical Ramsey statement for block sequences in Banach spaces that, when combined with work of Komorowski and Tomczak-Jaegermann, showed that Hilbert space is the only separable Banach space isomorphic to all of its infinite-dimensional subspaces. Since this initial result, several dichotomies for Banach spaces have been discovered by Gowers and other researchers using either Gowers' original Ramsey theorem or strengthenings thereof. Thus apart from the intrinsic interest in infinite-dimensional Ramsey theory, the research is highly motivated by its potential applications in functional analysis.

One particular motivating problem here is the question of whether every non-Hilbertian Banach space must contain a continuum of non-isomorphism subspaces. Today, the only lower bound is 2 as furnished by the solution to Banach's homogeneous space problem, though positive results exists for large classes of spaces.

In his thesis, de Rancourt develops an abstract theory of *Gowers spaces*. This, on the one hand, continuous a line of investigation initiated by Todorcevic, who studied so called abstract Ramsey spaces as a sort of systematisation and axiomatisation for a grand number of distinct infinite-dimensional Ramsey theorems from the literature. On the other hand, it utilises a framework for block sequences in abstract vector spaces that I had developed in order to avoid the messier analytical details involved in Gowers' framework by avoiding approximations of vectors.

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With this, de Rancourt expands the area of applicability and, in particular, is able fit the Galvin–Prikry, Mathias and Silver theory into the same setting as block sequences. More importantly, de Rancourt is able to use ideas from Kastanas' proof of the Ramsey property for Borel sets using determinacy to respond to a question of mine regarding the adversarial Ramsey property for Borel sets. This is a beautiful result that at the same time generalises Borel determinacy for games on $\mathbb{N}$, while simultaneously generalising the game theoretical Ramsey result due to Gowers and the Galvin–Prikry theorem. Nevertheless, the exact status of his result remains unclear and its force is only known to lie somewhere in between determinacy for Borel games on $\mathbb{N}$ and on $\mathbb{R}$.

The last part of de Rancourt's thesis is dedicated to establishing new dichotomies for Banach spaces based on some ideas due to Ferenczi. The central underlying fact here is that, by a result of Anisca, if a separable Banach space X has only two classes of non-isomorphic infinite-dimensional Banach spaces, then one of these must be Hilbertian. This means that X would have to sit minimally above Hilbert space, which seems highly implausible. However, on the face of it, Ramsey theory seems to have little to say in such a setup, since the homogeneous object it would produce would simply be a Hilbertian subspace. One is therefore led to finding dichotomies for Banach spaces that would produce other homogenous sets, i.e., non-Hilbertian subspaces.

De Rancourt obtains two such new dichotomies, one with a flavour of Gowers' dichotomy for containing an unconditional basis or an HI subspace and the other analogous to the dichotomy for minimality. With these, de Rancourt arrives at giving very strong results for Banach spaces close to and potentially saturated with Hilbert spaces, by obtaining, for example, spaces minimal above Hilbert spaces inside of these. This is undoubtedly a big step towards solving the conjecture that every non-Hilbertian space must contain a continuum of non-isomorphic subspaces.

As should be clear from the report, I am very impressed by de Rancourt's thesis, in particular, his result that Borel sets are adversarially Ramsey and his Hilbert avoiding dichotomies. This represents a significant step forward both in abstract Ramsey theory and in its applications to functional analysis. For this reason, I very strongly recommend the acceptance of the dissertation for a doctoral degree.

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