Weakly Ramsey ultrafilters

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A partition relation

This talk is about nonprincipal ultrafilters on ω (or on other countable sets).

Given $\mathcal{U}$ such an ultrafilter, and $n, k, h \in \omega$, we will consider the following partition relation:

$$\omega \longrightarrow (\mathcal{U})^n_{k,h},$$

meaning that for every coloring $c\colon [\omega]^n \to k$, there exists $U \in \mathcal{U}$ such that $|c\,[[U]^n]| \leqslant h$.

The goal of this talk is to link the properties of this partition relation with the position of $\mathcal{U}$ in the Rudin-Keisler ordering.

The Rudin-Keisler ordering

Definition

Let X, Y be sets, $\mathcal{U}$ be an ultrafilter on X, and $f\colon X \to Y$. The ultrafilter $f(\mathcal{U})$ on Y is defined by $V \in f(\mathcal{U}) \Leftrightarrow f^{-1}[V] \in \mathcal{U}$.

Definition

Let $\mathcal{U}$, $\mathcal{V}$ be ultrafilters on X, Y respectively. We say that $\mathcal{V} \leqslant_{RK} \mathcal{U}$ if there exists $f\colon X \to Y$ such that $f(\mathcal{U}) = \mathcal{V}$. This defines a quasi-ordering on the class of all ultrafilters. The associated equivalence relation and strict relation are denoted by $\equiv_{RK}$ and $<_{RK}$.

Principal ultrafilters are pairwise RK-equivalent and are RK-minimum. From now, we forget them.

Lemma

Let $\mathcal{U}$ be an ultrafilter on a set X, and let $U \in \mathcal{U}$. Let $\mathcal{U} \upharpoonright U = \mathcal{U} \cap \mathcal{P}(U)$. Then $\mathcal{U} \equiv_{RK} \mathcal{U} \upharpoonright U$.

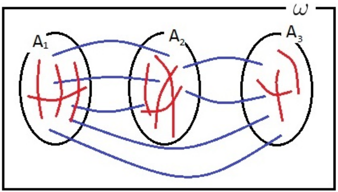
Selective ultrafilters

Theorem

Let $\mathcal{U}$ be an ultrafilter. The following are equivalent:

- *$\mathcal{U}$ is RK-minimal (above principal ultrafilters);*
- *Every $f : \omega \to \omega$ is either 1-1, or constant, on an element of $\mathcal{U}$;*
- *Every partition $\omega = \bigsqcup_{i \in I} A_i$ has either a member in $\mathcal{U}$, or a selector in $\mathcal{U}$;*
- $\omega \longrightarrow (\mathcal{U})^2_{2,1}$
- $\forall k, n < \omega, \ \ \omega \longrightarrow (\mathcal{U})^n_{k,1}$

An ultrafilter satisfying these properties is called selective.



An implication

What remains of this equivalence when we relax the partition relation?

Definition (Blass)

$\mathcal{U}$ is (n, h)-weakly Ramsey if for every $k < \omega$, $\omega \longrightarrow (\mathcal{U})^n_{k,h}$.

Definition

An RK-chain of length n below $\mathcal{U}$ is a chain of the form $\mathcal{U}_0 <_{RK} \mathcal{U}_1 <_{RK} \ldots <_{RK} \mathcal{U}_{n-1} = \mathcal{U}$.

Theorem

If $\mathcal{U}$ is (n, h)-weakly Ramsey, then every RK-chain below $\mathcal{U}$ has length at most $\lfloor \sqrt[n-1]{h} \rfloor$.

This result was already known for rapid P-points, it was proved by Laflamme. He also showed that it is optimal. Note that it implies that $(n, 2^{n-1} - 1)$-weakly Ramsey ultrafilters are selective.

The proof

The result follows from the following proposition:

Proposition

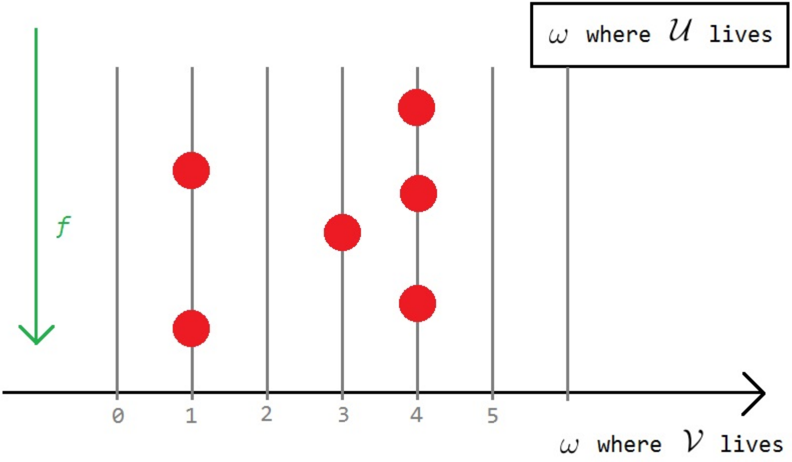
Suppose that $\mathcal{U}$ is at the top of a RK-chain of length m. Then for every $n \geqslant 1$, there exists a coloring of $[\omega]^n$ with m^{n-1} colors, such that every element of $\mathcal{U}$ meets every color.

The result is proved by induction on m. It is obvious for $m = 1$.

We fix $m \geqslant 1$ and we fix $\mathcal{U}$ at the top of a RK-chain of length $m + 1$. So there is $\mathcal{V} <_{RK} \mathcal{U}$ at the top of a RK-chain of length m. We fix $f : \omega \to \omega$ witnessing $\mathcal{V} \leqslant_{RK} \mathcal{U}$. So there is no $U \in \mathcal{U}$ on which f is 1-1.

For every $n \geqslant 1$, fix a coloring $\tilde{c} \colon [\omega]^n \to m^{n-1}$ such that every $V \in \mathcal{V}$ meets every color.

In the ω where $\mathcal{U}$ lives, we color a set $s \in [\omega]^n$ according to its pattern and the color of its image by f.

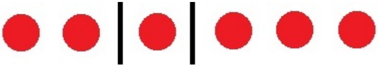


Red dots represent elements of s. Here, the pattern of s is $(2, 1, 3)$.

Formally, a n-pattern is an uple $p = (p_0, \dots, p_{r-1})$ of nonzero integers such that $p_0 + \dots + p_{r-1} = n$. Given $s \in [\omega]^n$ the pattern of s is $p(s) = (|f^{-1}[\{j_0\}] \cap s|, \dots, |f^{-1}[\{j_{r-1}\}] \cap s|)$, where $\{j_0 < \dots < j_{r-1}\} = f[s]$.

We define a coloring c of $[\omega]^n$ as follows: $c(s) = (p(s), \tilde{c}(f[s]))$.

For a fixed k, there are $\binom{n-1}{k-1}$ n-patterns of length k, and m^{k-1} possible colors for $f[s]$ when $|f[s]| = k$.



So the total number of colors is:

$$\sum_{k=1}^{n} \binom{n-1}{k-1} m^{k-1} = (m+1)^{n-1}.$$

Fix $U \in \mathcal{U}$. We now want to show that U meets every color. We restrict our attention to colors of the form (p, l) where the pattern p has length k.

Claim

There exists $V \in \mathcal{V}$ such that for every $i \in V$, $|f^{-1}[\{i\}] \cap U| \geqslant n$.

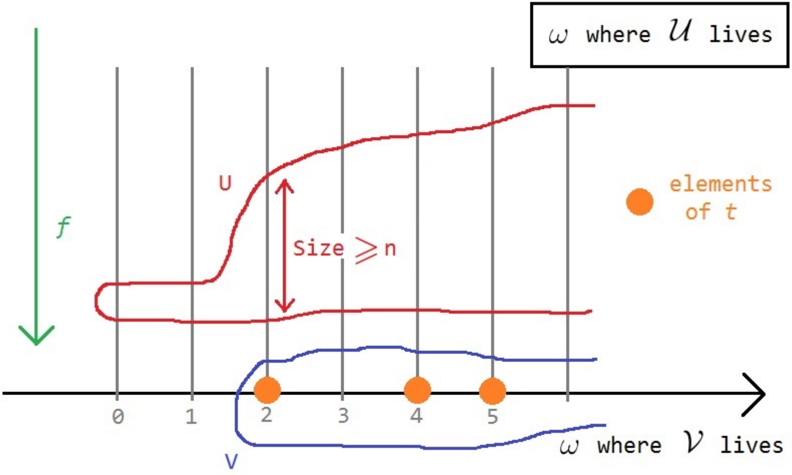
Proof.

Suppose not. Then there exists $V \in \mathcal{V}$ such that for every $i \in V$, $|f^{-1}[\{i\}] \cap U| < n$. So $U \cap f^{-1}[V] \in \mathcal{U}$, and on this set, f is n-to-1. So it can be partitioned into n sets on which f is 1-to-1, and one of them, say W, is in $\mathcal{U}$. We have:

$$\mathcal{U} \equiv_{RK} \mathcal{U} \upharpoonright W \equiv_{RK} \mathcal{V} \upharpoonright f(W) \equiv_{RK} \mathcal{V},$$

a contradiction. □

Now, given a c-color (p, l) where the pattern p has length k, we can find $t \in [V]^k$ such that $\tilde{c}(t) = l$, and $s \in [U]^n$ having pattern p and such that $f[s] = t$. Then $c(s) = (p, l)$.



What about the converse?

Theorem

Consistently, there exists a P-point ultrafilter $\mathcal{U}$ on ω having a unique strict RK-predecessor, up to equivalence, and such that for all $n \geqslant 2$ and $h \geqslant 1$, $\mathcal{U}$ fails to be (n, h)-weakly Ramsey.

Definition

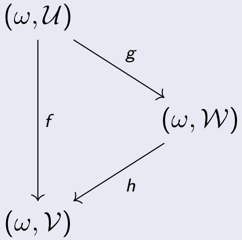
- A coideal on ω is the complement of an ideal. Equivalently, it is a nonempty proper subset $\mathcal{H} \subseteq \mathcal{P}(\omega)$, upwards closed, and such that if $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$, or $B \in \mathcal{H}$.
- A P^+-coideal is a coideal $\mathcal{H}$ such that for every decreasing sequence $(A_n)_{n<\omega}$ of elements of $\mathcal{H}$, there exists $A^* \in \mathcal{H}$ such that for every n, $A^* \subseteq^* A_n$.
- A P-point is an ultrafilter which is also a P^+-coideal.

Equivalently, an ultrafilter $\mathcal{U}$ is a P-point iff every $f : \omega \to \omega$ is either constant, or finite-to-one, on an element of $\mathcal{U}$.

A technical lemma

Lemma

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be ultrafilters on ω, where $\mathcal{U}$ is a P-point and $\mathcal{V}$ is selective. Suppose that $\mathcal{V} \leqslant_{RK} \mathcal{U}$ and $\mathcal{W} \leqslant_{RK} \mathcal{U}$, respectively witnessed by $f, g\colon \omega \to \omega$. Then there exists $h\colon \omega \to \omega$ such that the following diagram commutes on an element of $\mathcal{U}$:



In particular, h witnesses that $\mathcal{V} \leqslant_{RK} \mathcal{W}$.

Proof (sketch).

Let, for all $i, p \in \omega$, $A_i = f^{-1}[\{i\}]$ and $B_p = g^{-1}[\{p\}]$. Passing to an element of $\mathcal{U}$ if necessary, we can assume that the A_i's and the B_p's are finite. We would like to find an element of $\mathcal{U}$ on which (the trace of) each B_p is contained in one of the A_i's.

Define a coloring of the $c\colon [\omega]^2 \to 2$ as follows: $c(\{i,j\}) = 1$ iff there is a B_p intersecting both A_i and A_j. For a fixed i, there are only finitely B_p's intersecting A_i. And for each of these B_p, there are only finitely many A_j's intersecting it. To summarize, there are only finitely many j's such that $c(\{i,j\}) = 1$.

So there cannot be $V \in \mathcal{V}$ such that $c \restriction [V]^2 \equiv 1$. Since $\mathcal{V}$ is selective, there exists $V \in \mathcal{V}$ such that $c \restriction [V]^2 \equiv 0$. Let $U = f^{-1}[V]$; then $U \in \mathcal{U}$.

Restricting our attention to U, the trace of each B_p cannot intersect two different A_i's. So each one of them is contained in at most one A_i, as wanted.

□

Ramsey theory on graphs

Definition

A k-colored graph is a finite set G equipped with a mapping $\gamma_G \colon [G]^2 \to k$.

We denote by $\binom{B}{A}$ the set of isomorphic copies of A in B. We consider the following partition relation for colored graphs:

$$C \longrightarrow (B)^A_p,$$

meaning that for every mapping $f \colon \binom{C}{A} \to p$, there exists $B' \in \binom{C}{B}$ such that f is constant on $\binom{B'}{A}$.

Theorem (Nešetřil–Rödl)

For every $k, p \geqslant 1$ and for every k-colored graphs A and B, there exists a k-colored graph C such that $C \longrightarrow (B)^A_p$.

Canonical Ramsey theory

For $I \subseteq n$, define $p_I : [\omega]^n \to [\omega]^I$ by $p_I(\{s_0 < \ldots < s_{n-1}\}) = \{s_i \mid i \in I\}$.

Theorem (Erdös-Rado's canonical Ramsey theorem)

For every partition $f : [\omega]^n \to \omega$, there exists an infinite $M \subseteq \omega$ and $I \subseteq n$ such that for every $s, t \in [M]^n$, we have $f(s) = f(t) \Leftrightarrow p_I(s) = p_I(t)$.

For $n = 1$, we get the fact that every mapping $\omega \to \omega$ is either constant, or 1-1, on some infinite set.

Theorem (Prömel–Voigt)

Let $k \geqslant 1$ and let G be a k-colored graph. Then there exists a k-colored graph F having the following property: every $f : F \to \omega$ is either constant, or 1-1, on a copy of G.

The construction of the counterexample

Reminder: We are looking for an ultrafilter $\mathcal{U}$, having only one strict RK-predecessor, up to equivalence, and failing to be (n, h)-weakly Ramsey for every $n \geqslant 2$ and $h \geqslant 1$. We can actually restrict our attention to $n = 2$.

We start by fixing a selective ultrafilter $\mathcal{V}$ on ω which will be the only strict RK-predecessor of $\mathcal{U}$.

Using Prömel–Voigt's theorem, we can define a sequence $(G_n)_{n\in\omega}$, where G_n is a $(n+1)$-colored graph, having the following properties:

- $\gamma_{G_n} \colon [G_n]^2 \to (n+1)$ is surjective;
- every $f \colon G_{n+1} \to \omega$ is either 1-1 or constant on a copy of G_n;
- $|G_0| \geqslant 3$.

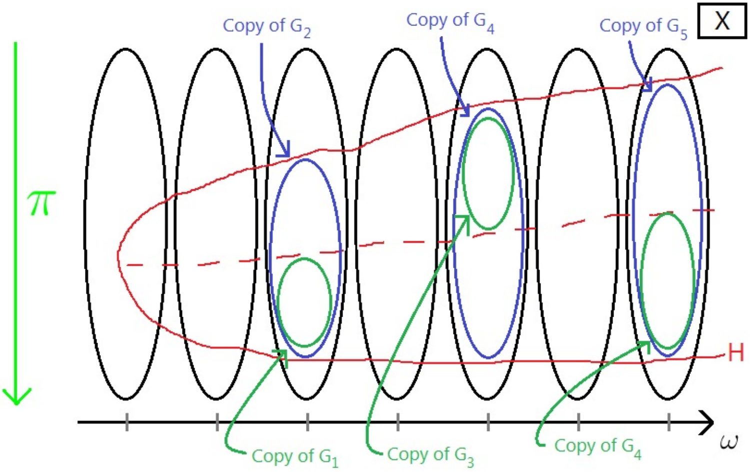
This implies that for every $n \in \mathbb{N}$, $G_{n+1} \longrightarrow (G_n)^{\bullet}_2$.

We let $X = \bigcup_{n\in\omega} \{n\} \times G_n$, and we denote by $\pi \colon X \to \omega$ the projection. The ultrafilter $\mathcal{U}$ will be an ultrafilter on X, the mapping π witnessing that $\mathcal{V} \leqslant_{RK} \mathcal{U}$.

For $A \subseteq X$ and $n \in \omega$, we let $(A)_n = \{x \in G_n \mid (n, x) \in A\}$. We let

$$\mathcal{H} = \{H \subseteq X \mid (\forall m)(\mathcal{V}n)\ (H)_n \text{ contains a copy of } G_m\}.$$

Since $(\forall n) G_{n+1} \longrightarrow (G_n)^{\bullet}_2$, it follows that $\mathcal{H}$ is a coideal.



Lemma

$\mathcal{H}$ is a P^+-coideal.

Proof.

Let (H_n) be a decreasing sequence of elements of $\mathcal{H}$. Let $V_n = \{m \in \omega \mid (H_n)_m$ contains a copy of $G_n\}$. We have $V_n \in \mathcal{V}$. $(V_n)_{n\in\omega}$ is decreasing and has empty intersection. Since $\mathcal{V}$ is selective, there exists $V \in \mathcal{V}$ such that $V \subseteq V_0$ and for every $n \in \omega$, $V \subseteq^* V_n$.

Define $H \subseteq X$ in the following way: for $m \notin V$, $(H)_m = \varnothing$, and for $m \in (V_n \setminus V_{n+1}) \cap V$, $(H)_m = (H_n)_m$. For such an m, $(H)_m$ contains a copy of G_n.

As a consequence, for every $m \in V_n \cap V$, $(H)_m$ contains a copy of G_p for some $p \geqslant n$, so contains a copy of G_n. This shows that $H \in \mathcal{H}$.

Finally, for every $m \in V_n \cap V$, $(H)_m \subseteq (H_n)_m$. So $H \setminus H_n \subseteq \bigcup_{m\in V\setminus V_n} \{m\} \times (H)_m$. Since $V \setminus V_n$ is finite, we deduce that $H \setminus H_n$ is finite, too. So $H \subseteq^* H_n$.

□

Force with $(\mathcal{H}, \subseteq)$. Note that $(\mathcal{H}, \subseteq)$ is forcing-equivalent to $(\mathcal{H}/\operatorname{FIN}, \subseteq^*)$, which is σ-closed, so this forcing does not add new reals. In particular, $\mathcal{V}$ is still an ultrafilter in the extension.

Denote by $\mathcal{U}$ the generic. Then $\mathcal{U}$ is a P-point ultrafilter on X.

Lemma

$\pi(\mathcal{U}) = \mathcal{V}$.

Proof.

Let $A \subseteq \omega$. If $A \notin V$, then $\pi^{-1}[A] \notin \mathcal{H}$, so $\pi^{-1}[A] \notin \mathcal{U}$.

If $A \in \mathcal{V}$, then for every $H \in \mathcal{H}$, we have $H \cap \pi^{-1}[A] \in \mathcal{H}$. In particular, $\{H \in \mathcal{H} \mid H \subseteq \pi^{-1}[A]\}$ is dense and open in $\mathcal{H}$, so contains an element of $\mathcal{U}$. Thus, $\pi^{-1}[A] \in \mathcal{U}$.

□

Lemma

For every $h \geqslant 1$, $\mathcal{U}$ fails to be $(2, h)$-weakly Ramsey.

Proof.

Recall that $X = \bigcup_{n \in \mathbb{N}} \{n\} \times G_n$, and that each $[G_n]^2$ comes with a given coloring with range in $(n+1)$. So part of $[X]^2$ is already colored, *a priori* with ω colors. Fixing h, we can turn this partial ω-coloring into a total h-coloring c as follows: we replace all edges colored in a color $\geqslant h$ by the color 0, and we color all the edges that are not yet colored in color 0. The rest remains as it is.

Let $U \in \mathcal{U}$; we show that U meets every color. We have $U \in \mathcal{H}$, so there is an $n \in \omega$ such that $(U)_n$ contains a copy of G_{h-1}. The natural coloring of this graph meets all colors from 0 to $h-1$. Moreover, none of these colors have been modified when definig c. This concludes.

□

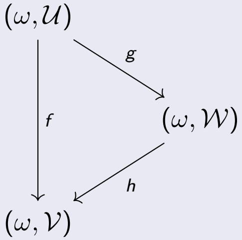
Lemma

The unique strict RK-predecessor of $\mathcal{U}$ is $\mathcal{V}$.

We recall a previous result:

Lemma

Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be ultrafilters on ω, where $\mathcal{U}$ is a P-point and $\mathcal{V}$ is selective. Suppose that $\mathcal{V} \leqslant_{RK} \mathcal{U}$ and $\mathcal{W} \leqslant_{RK} \mathcal{U}$, respectively witnessed by $f, g\colon \omega \to \omega$. Then there exists $h\colon \omega \to \omega$ such that the following diagram commutes on an element of $\mathcal{U}$:



In particular, h witnesses that $\mathcal{V} \leqslant_{RK} \mathcal{W}$.

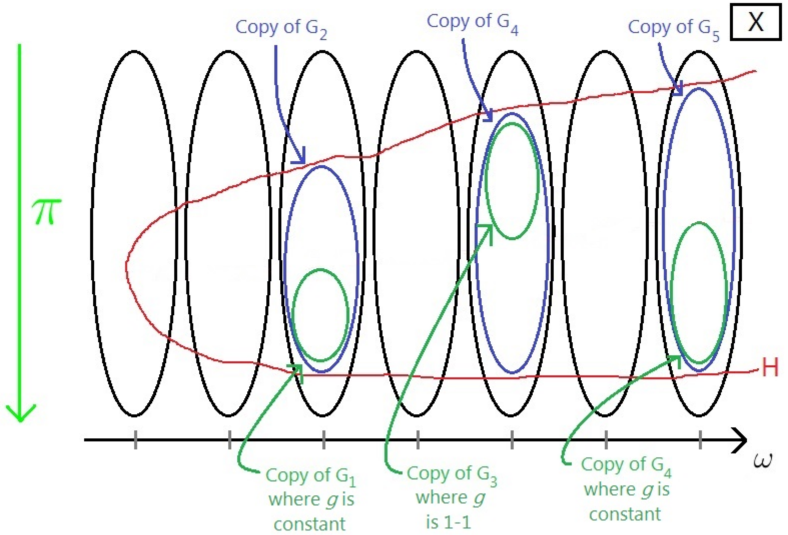
Let $\mathcal{W}$ be a RK-predecessor of $\mathcal{U}$, witnessed by $g\colon \omega \to \omega$. The lemma allows us to assume the existence of $h\colon \omega \to \omega$ such that $\pi = h \circ g$; in particular, h witnesses $\mathcal{V} \leqslant_{RK} \mathcal{W}$. We want to prove that either g is 1-1 on an element of $\mathcal{U}$, or h is 1-1 on an element of $\mathcal{W}$. So we let:

$$\mathcal{D} = \{H \in \mathcal{H} \mid \text{either } g_{\restriction H} \text{ is 1-1, or } h_{\restriction g[H]} \text{ is 1-1}\}.$$

We show that $\mathcal{D}$ is dense; this is enough to conclude.

Recall that if Γ_{n+1} is a copy of G_{n+1}, then there is a copy of G_n in Γ_{n+1} on which either g is 1-1, or it is constant.

Fix $H \in \mathcal{H}$. We can find $H' \in \mathcal{H} \upharpoonright H$ such that either for every n, g is constant on $(H')_n$, or for every n, g is 1-1 on $(H')_n$.



Suppose that for every n, g is 1-1 on $(H')_n$. For $x \neq y \in H'$, either $x, y \in (H')_n$ for the same n, so $g(x) \neq g(y)$; or $x \in (H')_m$ and $y \in (H')_n$ for $m \neq n$. In this case, $\pi(x) = m \neq n = \pi(y)$, and since π factorizes through g, we get that $g(x) \neq g(y)$. So g is 1-1 on H', and $H' \in \mathcal{D}$.

Suppose that for every n, g is constant on $(H')_n$, and denote by z_n its value. Since $\pi = h \circ g$, then $h(z_n)$ is equal to the value of π on $(H')_n$, which is n. So $h_{\restriction g[H']}$ is 1-1, so $H' \in \mathcal{D}$. This finishes the proof!

Note that only $\mathfrak{c}$ dense sets have been used in this proof. Since our forcing is ω_1-closed, this implies that under CH, a sufficiently generic ultrafilter can be built without passing to an extension. In particular, the result we proved is true in $ZFC + CH$.

Thank you for your attention!