Hilbert-avoiding dichotomies and ergodicity

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The homogeneous space problem

In this talk, we will only consider separable Banach spaces.

Question (Banach's homogeneous space problem, 1932)

Say that a space is homogeneous if it is isomorphic to all of its subspaces. Is every homogeneous space isomorphic to ℓ_2 ?

The answer is yes ; it relies on three results. Recall that a space X is hereditarily indecomposable (HI) if no two subspaces of X are in topological direct sum.

The homogeneous space problem

Theorem (Gowers – Maurey, 1992)

An HI space is isomorphic to no proper subspace of itself.

Theorem (Komorowski – Tomczak-Jaegermann, 1995)

Every Banach space has either a subspace with no unconditional basis, or a subspace isomorphic to ℓ_2.

Theorem (Gowers' first dichotomy, 1995)

Every Banach space has either a subspace with an unconditional basis, or an HI subspace.

"Counting" subspaces

Question (Godefroy)

How many pairwise non-isomorphic subspaces can have a Banach space non-isomorphic to ℓ_2 ?

Question (Johnson)

Does there exist a space with exactly two subspaces, up to isomorphism?

Even this question is still open.

The right setting to study Godefroy's question is that of the complexity of equivalence relations.

Definition

Let (X, E) and (Y, F) be to nonempty standard Borel spaces endowed with equivalence relations. We say that (X, E) is Borel reducible to (Y, F), denoted by $(X, E) \leqslant_B (Y, F)$, if there is a Borel map $f : X \longrightarrow Y$ such that $\forall x, y \in X\ (x \mathrel{E} y \Leftrightarrow f(x) \mathrel{F} f(y))$.

“Counting” subspaces

If $(X, E) \leqslant_B (Y, F)$, then $|X/E| \leqslant |Y/F|$ (if E is Borel and X/E is countable, this is an equivalence). So knowing the complexity of an equivalence relation gives (strictly) more information that knowing the number of its classes.

Define the equivalence relation $\mathbf{E_0}$ on $2^{\mathbb{N}}$ by $x \, \mathbf{E_0} \, y$ iff $\{n \in \mathbb{N} \mid x_n \neq y_n\}$ is finite. We have the following hierarchy of Borel equivalence relations:

$$(1, =) <_B (2, =) <_B (3, =) <_B \ldots <_B (\mathbb{N}, =) <_B (2^{\mathbb{N}}, =) <_B (2^{\mathbb{N}}, \mathbf{E_0}),$$

which is exhaustive in the sense that if E is a Borel equivalence relation on a space X, then either (X, E) is bireducible with one member of the hierarchy, or $(2^{\mathbb{N}}, \mathbf{E_0}) <_B (X, E)$ (Silver '80, Harrington–Kechris–Louveau '90). This is not true when E is only supposed analytic.

“Counting” subspaces

For X a separable Banach space, denote by $\mathrm{Sub}(X)$ the set of subspaces of X. We endow $\mathrm{Sub}(X)$ with the Effros Borel structure, i.e. the σ-algebra generated by the sets $\{Y \in \mathrm{Sub}(X) \mid Y \cap U \neq \varnothing\}$, where U ranges over open subsets of X. This makes it a standard Borel space ; moreover, the relation $\simeq$ on $\mathrm{Sub}(X)$ is analytic.

Question (Godefroy, rephrasing)

If $X \not\simeq \ell_2$, what is the complexity of $(\mathrm{Sub}(X), \simeq)$?

Ergodic Banach spaces

Definition (Ferenczi – Rosendal, 2003)

A space X is ergodic if $(2^{\mathbb{N}}, E_0) \leqslant_B (\mathrm{Sub}(X), \simeq)$.

Non-ergodic spaces have nice regularity properties. For instance, if X is non-ergodic and has an unconditional basis, then:

- X is isomorphic to its hyperplanes;
- $X \simeq X \oplus Y$ for every Y generated by a subsequence of the basis.

(Ferenczi – Rosendal, 2003)

Moreover, every non-ergodic space has a minimal subspace, i.e. a subspace Y that can be embedded in every further subspace (Ferenczi, 2005).

Conjecture (Ferenczi – Rosendal)

Every Banach space non-isomorphic to ℓ_2 is ergodic.

Ergodic Banach spaces

Some progress has been done on this conjecture, even if it is still open:

Theorem (Anisca, 2009)

Every asymptotically Hilbertian space which is not isomorphic to ℓ_2 is ergodic.

Theorem (Cuellar, 2016)

Every non-ergodic Banach space is near Hilbert (that is, has type p and cotype q for every $p < 2 < q$).

A dummy reasoning

Questions

(1) *Let X be a counterexample to Johnson's question. Does X necessarily have an unconditional basis?*

(2) *Let X be a non-ergodic space, non-isomorphic to ℓ_2. Does X necessarily have a subspace non-isomorphic to ℓ_2 with an unconditional basis?*

A positive answer to (2) would also provide a positive answer to (1), since by a result by Anisca (2007), a ℓ_2 can be embedded in every counterexample to Johnson's question.

Since an HI space contains no minimal subspace, it has to be ergodic. Hence, by Gowers' dichotomy, a non-ergodic space must contain a subspace with an unconditional basis...But this subspace could be isomorphic to ℓ_2. How to avoid this case?

A not so dummy reasonning, finally

The "only" ingredient we need in the proof of Gowers' dichtomy is the fact that, given a space X and a decreasing sequence of subspaces $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$, there exists a subspace X_∞ such that $\forall n \in \mathbb{N}\, X_\infty \subseteq^* X_n$ (where $Y \subseteq^* Z$ means that $Y \cap Z$ has finite codimension in Y).

But... This also works if we restrict our attention to subspaces non-isomorphic to ℓ_2 !

The first dichotomy

Say that a FDD (E_n) is good if $d_{BM}(E_n, \ell_2^{\dim(E_n)}) \longrightarrow \infty$.

Definition

A space X is Hereditarily Hilbert-primary (HHP) if for every subspaces $Y, Z \subseteq X$, if Y and Z are in topological direct sum, then either Y or Z is isomorphic to ℓ_2.

Theorem

Let X be a Banach space non-isomorphic to ℓ_2. Then there exists a subspace Y of X, non-isomorphic to ℓ_2, such that:

- *either Y has an good UFDD;*
- *or Y is HHP.*

Moreover, these two cases are mutually exclusive.

What about non-ergodic spaces ?

The first dichotomy

Proposition

Every non-ergodic space non-isomorphic to ℓ_2 with a UFDD has a subspace non-isomorphic to ℓ_2 with an unconditional basis.

Corollary

Let X be a non-ergodic space, non-isomorphic to ℓ_2. Then either exists a subspace Y of X, non-isomorphic to ℓ_2, such that:

- *either Y has an unconditional basis;*
- *or Y is HHP.*

In particular, if we manage to prove that an HHP space must have at least three pairwise non-isomorphic subspaces, then our first conjecture is true, and if all HHP spaces are ergodic, then our second conjecture is true.

HHP spaces

Known proofs that HI spaces are not isomorphic to any of their proper subspaces don't seem to adapt easily to the case of HHP spaces. We are still not able to prove that such spaces have at least three three pairwise non-isomorphic subspaces.

The following spaces are the only examples of HHP spaces we currently know:

- Trivial ones: ℓ_2, HI spaces.
- $X \oplus \ell_2$, for X an HI space.
- A HI sum of spaces isomorphic to ℓ_2 (Argyros–Raikoftsalis, 2008). The space $\mathfrak{X}_2$ they construct actually admits a unique decomposition as $\mathfrak{X}_2 \oplus \ell_2$.

All these spaces contain an HI subspace, so they are ergodic.

Question

Does there exist ℓ_2-saturated HHP spaces that are not isomorphic to ℓ_2?

The second dichotomy

In order to reduce the question of the ergodicity of HHP spaces to something simpler, we introduce a second dichotomy. This is a Hilbert-avoiding version of the minimal/tight dichotomy by Ferenczi and Rosendal.

Definition (Ferenczi–Rosendal)

Let (e_i) be a basis.

- A space Y is tight in (e_i) if there are successive intervals of integers $I_0 < I_1 < \ldots$ such that for every infinite $A \subseteq \mathbb{N}$, Y does not embed into $\overline{\text{span}}(e_i \mid i \notin \bigcup_{j \in A} I_j)$.
- The basis (e_i) is tight if every space is tight in it. A space is tight if it has a tight basis.

Theorem (Ferenczi–Rosendal, 2009)

Every Banach space either contains a minimal subspace, or a tight subspace.

The second dichotomy

Definition

A space X is minimal among non-Hilbertian spaces (MNH) if it is non-isomorphic to ℓ_2 and if it embeds in all of its subspaces non-isomorphic to ℓ_2.

Definition

Let (E_i) be a FDD.

- A space Y is tight in (E_i) if there are successive intervals of integers $I_0 < I_1 < \ldots$ such that for every infinite $A \subseteq \mathbb{N}$, Y does not embed into $\overline{\bigoplus_{i \notin \bigcup_{j \in A} I_j} E_i}$.
- The FDD (E_i) is tight for non-Hilbertian spaces (TNH) if every space non-isomorphic to ℓ_2 is tight in it. A space is tight for non-Hilbertian spaces (TNH) if it has good FDD which is tight.

The second dichotomy

Theorem

Every space non-isomorphic to ℓ_2 contains either a MNH subspace, or a TNH subspace.

Proposition

A TNH space is ergodic.

Corollary

A non-ergodic space non-isomorphic to ℓ_2 has a MNH subspace.

So to prove our second conjecture, it would be enough to show that an HHP space cannot be MNH.

Question

Are there non-trivial (ℓ_2-saturated) MNH spaces?

Thank you for your attention!