A dichotomy for countable unions of smooth Borel equivalence relations

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Classification of definable equivalence relations

Definition

Let E and F be two equivalence relations on sets X and Y, respectively. A mapping $f \colon X \to Y$ is a reduction from E to F if it induces an injection $X/E \to Y/F$.

Equivalentely, for all $x, x' \in X$, $x \, E \, x' \Leftrightarrow f(x) \, F \, f(x')$.

This definition is of no interest in ZFC when there are no constraints on the mapping f. Most of the time, we will assume that X is a Polish space and that f is Borel, or sometimes Baire measurable.

Definition

Let E and F be two equivalence relations on Polish spaces X and Y, respectively. We say that E Borel reduces to F, denoted by $E \leqslant_B F$, if there is a Borel reduction from E to F.

Classification of definable equivalence relations

The relation $\leqslant_B$ is a quasi-ordering. We denote by $<_B$ and $\equiv_B$ the associated strict ordering and equivalence relation. This defines a hierarchy of complexity classes, which can be seen as Borel cardinals.

A motivation: Under AD, all mappings between Polish spaces are Baire measurable. Hence, Baire measurable cardinals are simply cardinals. Thus, saying that E does not Baire measurably reduce to F implies that no injection $X/E \to Y/F$ can be constructed without the use of AC.

However, Borel reducibility is more commonly used, since most existence results naturally give Borel reductions.

This theory can also be used to study the complexity of classification problems in different areas of mathematics.

Borel equivalence relations

We will restrict our attention to Borel equivalence relations. We have the following initial segment of the hierarchy (where Δ_X denotes equality on X):

$$\Delta_1 <_B \Delta_2 <_B \ldots <_B \Delta_{\mathbb{N}} <_B \Delta_{\mathbb{R}} <_B \mathbb{E}_0,$$

which is exhaustive in the sense that every Borel equivalence relation is either bireducible with one of the elements of this initial segment, or is strictly greater than $\mathbb{E}_0$. Above $\mathbb{E}_0$, there are incomparable equivalence relations.

Definition

The relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ is defined by $x \mathbb{E}_0 y$ iff $x(n) = y(n)$ eventually.

This equivalence relation is bireducible to the equality modulo $\mathbb{Q}$ on $\mathbb{R}$.

Borel equivalence relations

The exhaustivity comes from:

Theorem (Silver)

Let E be a coanalytic equivalence relation on a Polish space. Then either $E \leqslant_B \Delta_{\mathbb{N}}$ or $\Delta_{\mathbb{R}} \leqslant_B E$.

Theorem (Harrington–Kechris–Louveau)

Let E be a Borel equivalence relation on a Polish space. Then either $E \leqslant_B \Delta_{\mathbb{R}}$ or $\mathbb{E}_0 \leqslant_B E$.

Definition

An equivalence relation on a Polish space is said to be smooth if it is Borel reducible to $\Delta_{\mathbb{R}}$.

Group actions

Given a Borel action $G \curvearrowright X$ of a Polish group on a Polish space, we can consider the orbit equivalence relation associated to this action, i.e. the analytic equivalence relation E_G^X on X defined by:

$$x \, E_G^X \, x' \Leftrightarrow (\exists g \in G)(g \cdot x = x').$$

Definition

An equivalence relation is said to be:

1. countable if all of its classes are countable.
2. essentially countable if it is Borel reducible to a countable Borel equivalence relation.

For instance, the equality relations and $\mathbb{E}_0$ are countable.

Theorem (Feldman–Moore)

Let E be a countable Borel equivalence relation on a Polish space X. Then there is a Borel action $\Gamma \curvearrowright X$ of a countable discrete group such that $E = E_\Gamma^X$.

The equivalence relation $\mathbb{E}_1$

Define the equivalence relation $\mathbb{E}_1$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ by $x \ \mathbb{E}_1 \ y$ iff $x(n) = y(n)$ eventually.

Theorem (Kechris–Louveau)

The relation $\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation.

Conjecture (Folklore?)

Let E be a Borel equivalence relation on a Polish space. Then either E is Borel reducible to an orbit equivalence relation or $\mathbb{E}_1 \leqslant_B E$.

Hypersmooth equivalence relations

Definition

Say that a Borel equivalence relation E on a Polish space is hypersmooth if it can be written as a countable increasing union of smooth equivalence relations.

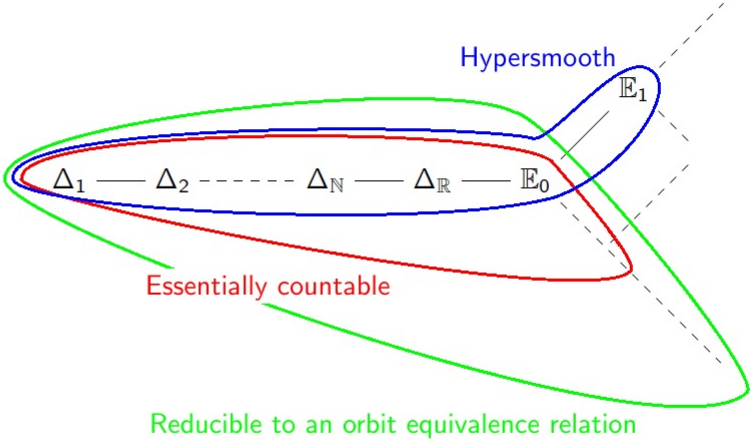
This is equivalent to the statement that $E \leqslant_B \mathbb{E}_1$.

Theorem (Kechris–Louveau)

Let E be a Borel hypersmooth equivalence relation. Then either $E \leqslant_B \mathbb{E}_0$ or $\mathbb{E}_1 \equiv_B E$.

Equivalently, $\mathbb{E}_1$ is an immediate successor of $\mathbb{E}_0$ under $<_B$.

A picture



Hypersmooth
$\mathbb{E}_1$
Δ_1 — Δ_2 - - - - $\Delta_\mathbb{N}$ — $\Delta_\mathbb{R}$ — $\mathbb{E}_0$
Essentially countable
Reducible to an orbit equivalence relation

σ-smoothness

Definition

A Borel equivalence relation E on a Polish space is said to be σ-smooth if it can be written as a countable union of smooth Borel subequivalence relations.

Hypersmooth Borel equivalence relations are obviously σ-smooth. We also have:

Lemma

Essentially countable equivalence relations are σ-smooth.

There are other examples, for instance the disjoint union of $\mathbb{E}_1$ and of a non-hypersmooth countable Borel equivalence relation.

The main theorem

Theorem

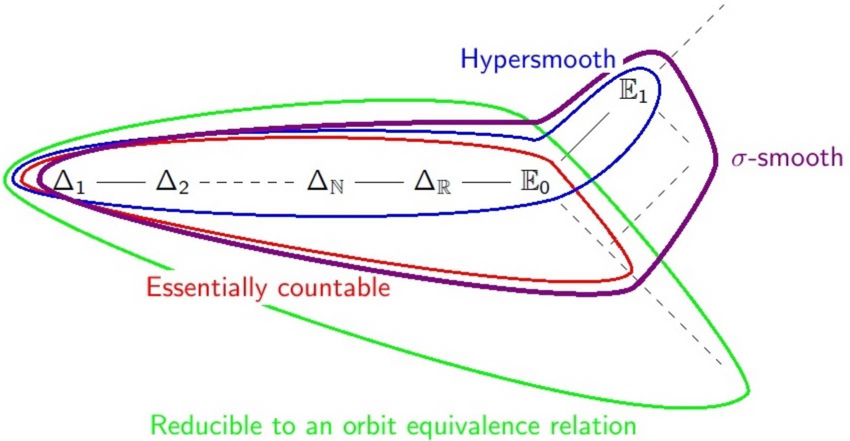
Let E be σ-smooth Borel equivalence relation on a Polish space. Then exactly one of the following conditions holds.

- *E is essentially countable.*
- $\mathbb{E}_1 \leqslant_B E$.

Corollary

Let $G \curvearrowright X$ be a Borel action of a Polish group on a Polish space. If E_G^X can be expressed as a countable union of essentially countable Borel equivalence relations, then it is essentially countable.

A (more complete) picture



Idealisticity

A Borel equivalence relation E on a Polish space X is said to be idealistic (resp. strongly idealistic) if there is an E-invariant assignment $x \mapsto \mathcal{I}_x$ mapping each point in X to a σ-ideal on X in such a way that:

- $\forall x \in X,\ [x]_E \notin \mathcal{I}_x$;
- For every Borel set $R \subseteq X \times X$, the set $\{x \in X \mid R_x \in \mathcal{I}_x\}$ is Borel (resp. for every Polish space Y and every Borel set $R \subseteq X \times Y \times X$, the set $\{(x, y) \in X \times Y \mid R_{x,y} \in \mathcal{I}_x\}$ is Borel).

The equivalence relation E is said to be ccc idealistic (resp. strongly ccc idealistic) if for every $x \in X$ and every uncountable family $(B_i)_{i \in I}$ of pairwise disjoint Borel subsets of X, one of the B_i's is in $\mathcal{I}_x$.

Idealisticity

Theorem (Kechris–Louveau)

$\mathbb{E}_1$ *is not Borel reducible to any ccc idealistic Borel equivalence relation.*

Proposition (essentially Kechris)

Let $G \curvearrowright X$ *be a Borel action of a Polish group on a Polish space. Suppose that* E_G^X *is Borel. Then it is strongly ccc idealistic.*

Still a few definitions...

Definition

An equivalence relation E on a Polish space X is said to be potentially F_σ if it is Borel reducible to an F_σ equivalence relation on a Polish space.

Definition

Let $E \subseteq F$ be two equivalence relations on the same set X. Say that F has countable index over E if each F-class is a countable union of E-classes.

If $\mathcal{F}$ is a family of Borel equivalence relations on Polish spaces, denote by $\mathcal{F}^{\leqslant_B}$ the family of all equivalence relations on Polish spaces that are Borel reducible to an element of $\mathcal{F}$, and by $\sigma(\mathcal{F})$ the class of all equivalence relations on Polish spaces that can be expressed as countable unions of subequivalence relations belonging to $\mathcal{F}$.

Our most general result

Theorem

Let $\mathcal{F}$ be a class of strongly idealistic potentially F_σ equivalence relations on Polish spaces. Suppose that $\mathcal{F}$ is closed under countable disjoint unions and countable index Borel superequivalence relations. Let $E \in \sigma(\mathcal{F}^{\leqslant_B})$. Then at least one of the following conditions holds:

- $E \in \mathcal{F}^{\leqslant_B}$;
- $\mathbb{E}_1 \leqslant_B E$.

Moreover, if elements of $\mathcal{F}$ are ccc idealistic, then these two conditions are mutually exclusive.

Our dichotomy for σ-smooth equivalence relations is the special case when $\mathcal{F}$ is the class of all countable Borel equivalence relations.

Consequences

Corollary

Let $\mathcal{F}$ be a class of strongly idealistic potentially F_σ equivalence relations on Polish spaces. Suppose that $\mathcal{F}$ is closed under countable disjoint unions and countable index Borel superequivalence relations. Let E be a ccc idealistic Borel equivalence relation on a Polish space. If $E \in \sigma(\mathcal{F}^{\leqslant_B})$, then $E \in \mathcal{F}^{\leqslant_B}$.

Corollary

Let E be an equivalence relation on a Polish space. Suppose that E can be expressed as a countable union of subequivalence relations that are Borel reducible to strongly ccc idealistic potentially F_σ equivalence relations on Polish spaces. Then exactly one of the following conditions hold:

- *E is Borel reducible to a strongly ccc idealistic potentially F_σ equivalence relation on a Polish space.*
- $\mathbb{E}_1 \leqslant_B E$.

F^+ and $F^\cap$

Definition

Let F be an equivalence relation on a Polish space X.

- For $A \subseteq X$, denote by $[A]_F$ the F-saturation of A, that is, the set $\{x \in X \mid (\exists x' \in A)(x \mathrel{F} x')\}$.
- The Friedman–Stanley jump of F is the equivalence relation F^+ on $X^{\mathbb{N}}$ defined by $x \mathrel{F^+} x'$ iff $[x(\mathbb{N})]_F = [x'(\mathbb{N})]_F$.
- The binary relation $F^\cap$ on $X^{\mathbb{N}}$ is defined by $x \mathrel{F^\cap} x'$ iff $[x(\mathbb{N})]_F \cap [x'(\mathbb{N})]_F$ is nonempty.

Definition

- A homomorphism from a binary relation R on a set X to a binary relation S on a set Y is a mapping $f \colon X \to Y$ such that $(f \times f)[R] \subseteq S$.
- A reduction from R to S is a mapping $f \colon X \to Y$ which is both a homomorphism from R to S and from $\sim R$ to $\sim S$.

F^+ and $F^\cap$

Proposition

Let E be an equivalence relation on a Polish space X, and F be a strongly idealistic Borel equivalence relation on a Polish space Y. Suppose that E Borel reduces to a countable-index superequivalence relation of F. Then there is a Borel homomorphism from $(E, \sim E)$ to $(F^+, \sim F^\cap)$. In particular, E Borel reduces to $F^\cap$.

Proposition

Let E and F be Borel equivalence relations on Polish spaces X and Y, respectively. The following are equivalent:

- *E Borel reduces to $(F \times \Delta_{\mathbb{N}})^\cap$;*
- *E is a countable union of subequivalence relations that are Borel reducible to $F \times \Delta_{\mathbb{N}}$.*

Another consequence

Theorem

Let E be an equivalence relation on a Polish space which is Borel reducible to a ccc idealistic Borel equivalence relation. Let F be a strongly idealistic potentially F_σ equivalence relation on a Polish space. The following are equivalent:

- *E is Borel reducible to a countable index superequivalence relation of $F \times \Delta_{\mathbb{N}}$;*
- *There is a Borel homomorphism from $(E, \sim E)$ to $((F \times \Delta_{\mathbb{N}})^+, \sim(F \times \Delta_{\mathbb{N}})^\cap)$;*
- *E Borel reduces to $(F \times \Delta_{\mathbb{N}})^\cap$;*
- *E is a countable union of subequivalence relations that are Borel reducible to $F \times \Delta_{\mathbb{N}}$.*

Our dichotomy for σ-smooth equivalence relations is the particular case when $F = \Delta_{\mathbb{R}}$.

The graph dichotomy

The equivalence relation $\mathbb{F}_n$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ is defined by:

$$x \; \mathbb{F}_n \; x' \Leftrightarrow (\forall i \geqslant n)(x(i) = x'(i)).$$

We have $\mathbb{E}_1 = \bigcup_{n \in \mathbb{N}} \mathbb{F}_n$.

Theorem

Let X be a Polish space, E be an analytic equivalence relation on X, and $(E_n)_{n \in \mathbb{N}}$ be a sequence of Borel equivalence relations on X such that $\bigcup_{n \in \mathbb{N}} E_n \subseteq E$. Then exactly one of the following two conditions holds.

1. *There exists a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel sets such that for every $n \in \mathbb{N}$, $E \restriction B_n$ has countable index over $E_n \restriction B_n$.*
2. *There is a continuous homomorphism $\varphi \colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{E}_1 \setminus \mathbb{F}_n)_{n \in \mathbb{N}}$ to $(E \setminus E_i)_{n \in \mathbb{N}}$.*

Thank you for your attention!