Isometrically homogeneous Banach spaces

Noé de Rancourt

KGRC, University of Vienna

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The usual warning

In this talk, all Banach spaces will be infinite-dimensional, and by a subspace, I will mean an infinite-dimensional, closed subspace.

Operators are bounded, and isometries are linear.

We will work in real spaces, but everything works as well in complex spaces.

Homogeneous and isometrically homogeneous spaces

Definition

A space is said to be homogeneous if it is isomorphic to all of its subspaces.

Question (Banach's homogeneous space problem, '32)

Is every homogeneous space isomorphic to ℓ_2?

This problem was solved positively in the 90s.

Definition

A space is said to be isometrically homogeneous if it is isometric to all of its subspaces.

Question (Cúth–Doležal–Doucha–Kurka '19)

Is every isometrically homogeneous space isometric to ℓ_2?

The solution of the homogeneous space problem

Definition

A Banach space is hereditarily indecomposable (HI) if it contains no topological direct sum of two subspaces.

HI spaces cannot contain unconditional basic sequences.

The solution of the homogeneous space problem comes from the three following results:

- An HI space is isomorphic to no proper subspace of itself (Gowers–Maurey '93);
- Every Banach space either contains a subspace without unconditional basis, or an isomorphic copy of ℓ_2; (Komorowski–Tomczak-Jaegermann '95);
- Every Banach space either contains an unconditional basic sequence, or an HI subspace (Gowers' first dichotomy, '96).

Quantitative versions

A natural idea to investigate the isometrically homogeneous space problem is to look for quantitative versions of the previous three results.

- Gowers' first dichotomy admits a quantitative version, proved by Gowers himself.
- For Komorowski–Tomczak-Jaegermann's result, I don't know.
- Gowers–Maurey' result admits a quantitative version, that's the goal of this talk.

Summarizing, we get:

Theorem (dR, '19)

Every isometrically homogeneous space has a $(1+\varepsilon)$-unconditional basis, for every $\varepsilon > 0$.

C-HI spaces

Two subspaces Y and Z of X are not in topological direct sum iff for every $C < \infty$ there are $y \in Y$ and $z \in Z$ such that $\|y + z\| > C\|y - z\|$.

Definition (Gowers)

X is said to be C-HI if for every subspaces $Y, Z \subseteq X$, there exists $y \in Y$ and $z \in Z$ such that $\|y + z\| > C\|y - z\|$.

A space is HI iff it is C-HI for every $C < \infty$. A C-HI space cannot contain any C-unconditional basic sequence.

Theorem (Gowers)

For every $\varepsilon > 0$, every space either contains either a C-unconditional basic sequence, or a $(C - \varepsilon)$-HI subspace.

Theorem (dR '19)

If X is $(1 + \varepsilon)$-HI for some $\varepsilon > 0$, then X is isometric to no proper subspace of itself.

Fredholm theory

Let $T \colon X \to Y$ be an operator. Let $\mathrm{n}(T) = \dim \ker(T)$ and $\mathrm{d}(T) = \mathrm{codim}_Y\, T(X)$.

Definition

- T is said to be semi-Fredholm if $T(X)$ is closed in Y and if at least one of the numbers $\mathrm{n}(T)$ and $\mathrm{d}(T)$ is finite. In this case, its Fredholm index is $\mathrm{ind}(T) = \mathrm{n}(T) - \mathrm{d}(T) \in \mathbb{Z} \cup \{-\infty, +\infty\}$.
- T is Fredholm if it is semi-Fredholm with finite index.

Let $\hat{\mathcal{F}}red(X, Y)$ be the set of semi-Fredholm operators, seen as a subset of $\mathcal{L}(X, Y)$ with the norm topology.

Theorem (Folklore)

$\mathrm{ind} \colon \hat{\mathcal{F}}red(X, Y) \to \mathbb{Z} \cup \{-\infty, +\infty\}$ *is locally constant.*

Fredholm theory

Definition

$T \colon X \to Y$ is finitely singular if it is semi-Fredholm and $\mathrm{ind}(T) < +\infty$. Otherwise, T is said to be infinitely singular.

Theorem (Folklore)

The following are equivalent:

- *T is infinitely singular;*
- *For every $\varepsilon > 0$, there exists a subspace $X_\varepsilon \subseteq X$ such that $\|T_{\restriction X_\varepsilon}\| \leqslant \varepsilon$.*

Definition

An infinitely singular value of an operator $T \colon X \to X$ is a $\lambda \in \mathbb{R}$ such that $T - \lambda \, \mathrm{Id}_X$ is infinitely singular.

The main lemma

Lemma

Let X be a Banach space and $T\colon X \to X$ be a non-surjective isometry. Then the infinitely singular values of T are exactly 1 and -1.

Proof. It is enough to show that the only positive infinitely singular value of T is 1.

Draw a line between T and $-\operatorname{Id}_X$. Since $\operatorname{ind}(T) < 0$ and $\operatorname{ind}(\operatorname{Id}_X) = 0$, then there exists someone on this line, $tT - (1-t)\operatorname{Id}_X$, which is not semi-Fredholm, so infinitely singular. So $\lambda = \frac{1-t}{t}$ is a positive infinitely singular value of T.

If $\lambda > 1$ then for every $x \in X$, $\|T(x) - \lambda x\| \geqslant \lambda\|x\| - \|T(x)\| \geqslant (\lambda - 1)\|x\|$. So $T - \lambda \operatorname{Id}_X$ is an isomorphism on its image, a contradiction.

If $\lambda < 1$ then for every $x \in X$, $\|T(x) - \lambda x\| \geqslant \|T(x)\| - \lambda\|x\| \geqslant (1-\lambda)\|x\|$, a contradiction again.

So $\lambda = 1$. □

The conclusion

Suppose that X is isometric to a proper subspace of itself. We want to prove that X is not $(1+\varepsilon)$-HI. So we want to find two subspaces $Y, Z \subseteq X$ such that:

$$(\forall y \in Y)\ (\forall z \in Z)\ \|y+z\| \approx \|y-z\|.$$

Let $T\colon X \to X$ be a non-surjective isometry. 1 and -1 are infinitely singular values of T, so there exists subspaces $Y, Z \subseteq X$ such that on Y, T is very close to Id_X and on Z, T is very close to $-\mathrm{Id}_X$. Thus, for $y \in Y$ and $z \in Z$:

$$\|y+z\| = \|T(y)+T(z)\| \approx \|y-z\|.$$

□

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Thank you for your attention!