Ramsey determinacy of the adversarial Gowers games

Noé de Rancourt

Université Paris VII, IMJ-PRG

Descriptive set theory in Paris
December 8, 2015

Gowers' theorems

Theorem (Gowers' first dichotomy, 1996)

Every infinite-dimensional Banach space has an infinite-dimensional closed subspace which either has a unconditional basis, or is hereditarily indecomposable.

To proves this, Gowers uses a Galvin-Prikry-like theorem in separable Banach spaces: here, we color (some) infinite sequences of vectors and want to get an infinite-dimensional closed subspace which is "almost" monochromatic.

This theorem is an *approximate* and *strategical* Ramsey result.

Rosendal's version of Gowers' theorem

E: countable-dimensional vector space over an at most countable field K. In what follows, all subspaces will be infinite-dimensional.

Definition

For a subspace $X \subseteq E$, we define:

- The *Gowers game* G_X:

I	U_0		U_1		$\ldots$
II		$u_0 \in U_0$		$u_1 \in U_1$	$\ldots,$

  where the U_i's are subspaces of X and the u_i's are vectors of E. The *outcome* of the game is the sequence $(u_i)_{i \in \mathbb{N}} \in E^{\mathbb{N}}$.
- the *asymptotic game* F_X is the same as G_X, except that the U_i's are moreover required to have finite codimension in X.

Properties

Let $\mathcal{X} \subseteq E^{\mathbb{N}}$.

- *If* **I** *has a strategy to play in $\mathcal{X}^c$ in F_X, then he also has one in G_X;*
- *If* **II** *has a strategy to play in $\mathcal{X}$ in G_X, then she also has one in F_X.*

Definition

$\mathcal{X} \subseteq E^{\mathbb{N}}$ is *strategically Ramsey* if for every subspace $X \subseteq E$, there exists a further subspace $Y \subseteq X$ such that:

- either **I** has a strategy to play in $\mathcal{X}^c$ in F_Y;
- or **II** has a strategy to play in $\mathcal{X}$ in G_Y.

Endow E with the discrete topology. Then $E^{\mathbb{N}}$ is a Polish space.

Theorem (Rosendal, 2008)

Every analytic subset of $E^{\mathbb{N}}$ is strategically Ramsey.

Definition (Gowers and asymptotic games in $\mathbb{N}$)

For $M \subseteq \mathbb{N}$ infinite:

I	N_0		N_1		$\dots$
II		$n_0 \in N_0$		$n_1 \in N_1$	$\dots,$

In G_M, the N_i's are infinite subsets of M, and in F_M, they are cofinite subsets of M. The outcome is the sequence $(n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

Identifying subsets of $\mathbb{N}$ with sequences, we have, for $\mathcal{X} \subseteq \mathbb{N}^{\mathbb{N}}$:

Proposition

*(1) If **I** has a strategy in F_M to play in $\mathcal{X}^c$, then there is $N \subseteq [M]^\omega$ such that $[N]^\omega \cap \mathcal{X} = \varnothing$;*

*(2) If **II** has a strategy in G_M to play in $\mathcal{X}$, then there is $N \subseteq [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$.*

(1) holds approximately in Banach spaces, and (2) holds approximately in c_0-saturated Banach spaces.

The adversarial Gowers games

Definition

For $X \subseteq E$ a subspace, we define:

- The game A_X:

I		$u_0 \in U_0,\ V_0$		$u_1 \in U_1,\ V_1$	$\dots,$
II	U_0		$v_0 \in V_0,\ U_1$		$\dots$

  where the U_i's are infinite-dimensional subspaces of X, the V_i are finite-codimensional subspaces of X, and the u_i's and the v_i's are vectors of E. The outcome of the game is the pair of sequences $((u_i)_{i \in \mathbb{N}}, (v_i)_{i \in \mathbb{N}})$.

- The game B_X is defined in the same way as A_X, except that this time, the U_i's are required to be finite-codimensional in X whereas the V_i's can be arbitrary infinite-dimensional subspaces of X.

In A_X, **I** plays the role of **I** in F_X and the role of **II** in G_X. In B_X, it's the opposite.

Definition

$\mathcal{X} \subseteq E^{\mathbb{N}}$ is *adversarially Ramsey* if for every subspace $X \subseteq E$, there exists a further subspace $Y \subseteq X$ such that :

- either **I** has a strategy in A_Y to play in $\mathcal{X}^c$;
- or **II** has a strategy in B_Y to play in $\mathcal{X}$.

Each player has a winning strategy in the game which is the most difficult for him.

Theorem (Rosendal, 2012)

Every $\mathbf{\Sigma}_0^3$ *or* $\mathbf{\Pi}_0^3$ *subset of* $E^{\mathbb{N}}$ *is adversarially Ramsey.*

Let Γ be a suitable class of subsets of Polish spaces. If every Γ-subset of $E^{\mathbb{N}}$ is adversarially Ramsey, then:

- every Γ-subset of $E^{\mathbb{N}}$ is strategically Ramsey (take the projection on the v_i's);
- every Γ-set of sequences of integers is determined (play with the norms of the vectors).

In particular, in ZFC, we can't go beyond Borel.

Theorem (d.R., 2014)

Every Borel subset of $E^{\mathbb{N}}$ is adversarially Ramsey.

Sketch of proof

For a subspace $X \subseteq E$, we define the Kastanas game K_X :

$$\begin{array}{lllllll} \textbf{I} & & u_0 \in U_0,\ V_0 & & u_1 \in U_1,\ V_1 & & \dots, \\ \textbf{II} & U_0 & & v_0 \in V_0,\ U_1 & & \dots & \end{array}$$

where the u_i's and the v_i's are vectors of E, and the U_i's and the V_i's are infinite-dimensional subspaces of X such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$
The outcome of the game is the pair of sequences $((u_i)_{i\in\mathbb{N}}, (v_i)_{i\in\mathbb{N}})$.

Proposition

- *If* **I** *has a strategy in K_X to play in $\mathcal{X}^c$, then there exists a subspace $Y \subseteq X$ such that* **I** *has a strategy in A_Y to play in $\mathcal{X}^c$;*
- *If* **II** *has a strategy in K_X to play in $\mathcal{X}$, then there exists a subspace $Y \subseteq X$ such that* **II** *has a strategy in B_Y to play in $\mathcal{X}$.*

Question

Can we deduce the adversarial Ramsey property for a suitable class of subsets of Polish spaces from the determinacy of Gale-Stewart games on integers for the same class ?

If yes, this would:

- provide a direct proof that, under PD, every projective set of infinite sets of integers (resp. every projective subset of $E^{\mathbb{N}}$) is Ramsey (resp. strategically Ramsey);
- prove that under AD, every set of infinite sets of integers is Ramsey.