How to classify things?

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Let's go back to primary school!

Problem 1

Léa buys six cacti. Each cactus costs eight euros. How much does she pay?

Problem 2

I rent a spacious apartment in the center of Paris. It is three meters wide and four meters long. What is the surface of my apartment?

To solve these two very different concrete problems, we use the same abstract mathematical notions: numbers and multiplication.

Mathematics are a story of structures

First example: Euclidean rings

Theorem

Every positive integer can be decomposed, in an essentially unique way, as a product of prime numbers.

To prove this, we use:

Lemma (Existence of the Euclidean division)

For every $a \in \mathbb{N}$ and $b \in \mathbb{N}^*$, there exist $q \in \mathbb{N}$ and $0 \leqslant r < b$ with $a = bq + r$.

Mathematics are a story of structures

First example: Euclidean rings

Recall that a polynomial is an expression of the form $P(X) = a_n X^n + \ldots + a_1 X + a_0$, and that its degree is $\deg(P) = n$. A polynomial P is irreducible if for every Q, R such that $P(X) = Q(X)R(X)$, then either Q or R is constant.

Theorem

Every nonzero polynomial can be decomposed, in an essentially unique way, as a product of irreducible polynomials.

To prove this, we use:

Lemma (Existence of the Euclidean division)

For every polynomials A and $B \neq 0$, there exist polynomials Q and R with $A = BQ + R$ and $\deg(R) < \deg(B)$.

The proof of the theorem from the lemma is exactly the same in both cases.

Mathematics are a story of structures

First example: Euclidean rings

To avoid making the same proof several times, we define a kind of abstract structures, Euclidean rings. These are sets with two operations an addition and a multiplication in which there exist a Euclidean division.

In these structures, we can define a notion of irreducible elements, and prove that every nonzero element of the structure can be decomposed, in an essentially unique way, as a product of irreducibles.

The set of integers, and the set of polynomials, are particular cases of Euclidean rings. From this fact immediately follow the two previous theorems.

Mathematics are a story of structures

Second example: vector spaces

- Vectors of the plane can be summed and multiplied by a real number.

  If R is a rotation of the plane, $\vec{u}$ and $\vec{v}$ two vectors, and λ a real number, then $R(\vec{u}+\vec{v}) = R(\vec{u}) + R(\vec{v})$, and $R(\lambda\vec{u}) = \lambda R(\vec{u})$.

- $\mathcal{C}^\infty$ functions $\mathbb{R} \longrightarrow \mathbb{R}$ can be summed and multiplied by a real number.

  We can derivate these functions. If f and g two such functions, and λ a real number, then $(f+g)' = f' + g'$, and $(\lambda f)' = \lambda f'$.

We can define the abstract notion of a vector space, a set with two operation: an internal addition and a multiplication by scalar numbers; and the notion of a linear mapping, a mapping between vector spaces that preserves these operations. The two last examples are particular cases of vector spaces and of linear mappings.

Isomorphism

Sometimes, two structures arising in different contexts are "the same". That doesn't mean that their elements are the same, but rather that, if we forget the nature of their elements, they behave exactly in the same way. We say that they are isomorphic.

- $(\mathbb{R}, +)$ and $((0, +\infty), \times)$ are isomorphic, because the exponential function maps bijectively $\mathbb{R}$ to $(0, +\infty)$ and it "transforms" the addition into the multiplication.

- As vector spaces, the plane and the set of solutions of the differential equation $f'' = -f$ are isomorphic. Each element of the second has the form $a \cdot \cos + b \cdot \sin$, so has two "coordinates" a and b, and as for vectors of the plane, adding functions means adding coordinates.

If we prove a theorem for one of these structure, then it is also true for the other one.

What is classification?

Classifying a certain type of structures, it's finding a list of structures $\mathcal{L}$ of this type such that:

- the structures in $\mathcal{L}$ are pairwise non-isomorphic;
- every structure of this type is isomorphic to a structure in $\mathcal{L}$;
- given a structure, we can easily "test" to which structure in $\mathcal{L}$ it is isomorphic.

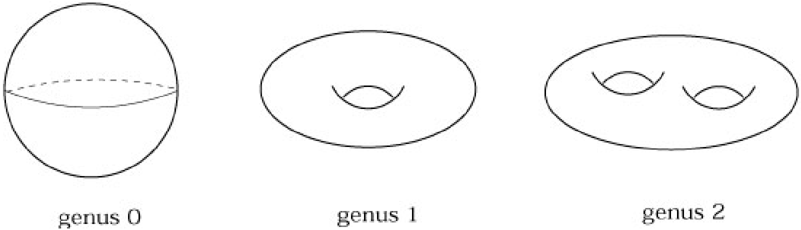
It enables to:

- prove easily that properties are true for all structures;
- test easily if two structures are isomorphic.

Sometimes we can study weaker notions of equivalence between structures than isomorphism.

Examples of classification

First example: orientable compact surfaces



Orientable compact surfaces are surfaces that can be embedded in the euclidean space.

They can be classified by their genus, i.e. the number of holes. Two surfaces are isomorphic if and only if they have the same genus.

Examples of classification

Example 2: finite-dimensional vector spaces

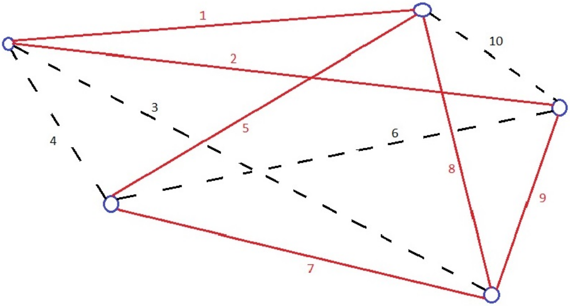
- Finite-dimensional vector spaces (over $\mathbb{R}$) can be classified by their dimension. Two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

- All standard Borel spaces with a non-atomic probability measure are isomorphic. Thus, when you want to prove a result on these spaces, you only need to prove it on you favourite example, $[0, 1]$ with the Lebesgue measure.

In general, classifying a class of structures means associating to each structure a characteristic (which is a real number or a sequence of real numbers), such that two structures are isomorphic iff they have the same characteristic.

An example of bad classification

A countably infinite graph is an infinite graph whose vertices can be enumerated by integers. In this case, pairs of vertices can also be enumerated by integers.

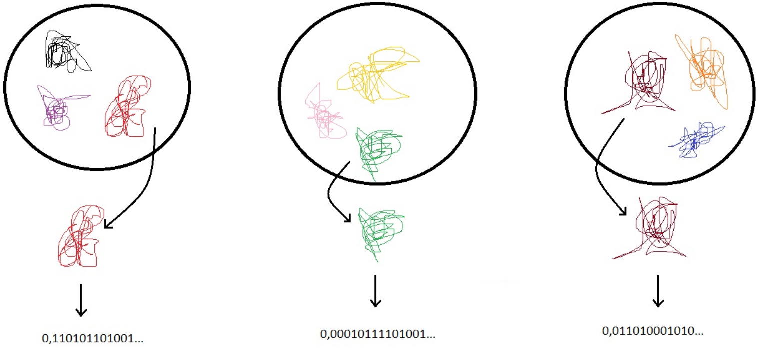


To such a graph, we can associate a real number between 0 and 1 in the following way: its n^{th} digit (in base 3) is 1 if the n^{th} pair of vertices is linked by an edge, and 0 otherwise.

For this graph we get $0,1100101110\ldots$

An example of bad classification

Then it is easy to get a classification:



1. Gather graphs in isomorphism classes;
2. Pick one graph in each class;
3. Associate to the whole class the number of this graph.

NO DON'T DO THAT, THIS IS BAD :-(((((

This is not a good way to classify because:

- The choices of graphs in each class are arbitrary: you cannot effectively choose, and it is impossible to compute the characteristic!
- In reality you classified nothing: you just said that each graph is isomorphic to... Another graph.

A good classification associate to each structure a characteristic in a definable way.

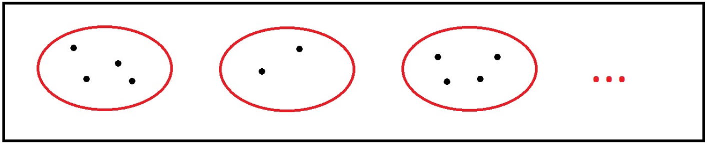
This seems to be impossible for countably infinite graphs. How to prove it?

A formalism

To prove something, we need to formalise it.

An equivalence relation is a binary relation which is reflexive, symmetric and transitive. The equality is an equivalence relation, and being isomorphic is another one.

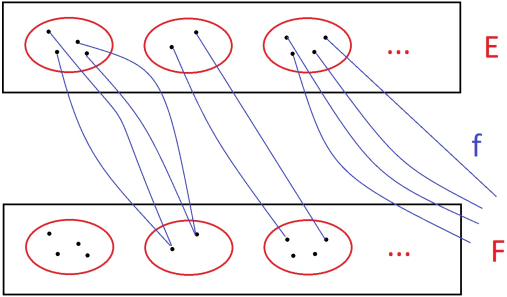
The set of elements that are equivalent to a given element x is called an equivalence class.



Here, we consider equivalence relations on standard Borel spaces. These are spaces where we can define a continuous analogue of calculability. The morphisms between these spaces are called Borel mappings; they can be seen as computable functions, in a continuous way. Classes of infinite structures can often be endowed with a structure of standard Borel space.

A formalism

We say that an equivalence relation E on a standard Borel space X is reducible to an equivalence relation F on a standard Borel space Y if there exists a mapping $f : X \to Y$ that maps each class of E to exactly one class of F.



The idea is that if we know F, then we can compute E.

Classifying classification

Then a class of structures is classifiable if and only if the isomorphism relation on this class is reducible to the equality on real numbers.

There exists an equivalence relation E_0 which is not reducible to the equality on $\mathbb{R}$. We say that two reals number x and y are E_0-equivalents if and only if they have the same writing in basis 2 from some rank.

- $11,10100101101101$
- $10,10000101101101$

Theorem (Folklore)

The isomorphism relation between countably infinite graphs has the highest possible complexity among all isomorphism relations between countably infinite structures. In particular, it is not reducible to the equality on the real numbers.

Thank you for your attention!

"C'est bien plus beau lorsque c'est inutile !"

"It's much more beautiful when it's useless!"

Cyrano de Bergerac, acte V, scène 6