# Approximate Gowers spaces

Noé de Rancourt

Université Paris VII, IMJ-PRG

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section Set theory & Topology

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# Infinite-dimensional Ramsey theory and the pigeonhole principle

*Infinite-dimensional Ramsey theory* is about coloring infinite sequences of objects.

**Theorem (Mathias-Silver)**

*Let $\mathcal{X}$ be an analytic set of infinite subsets of $\mathbb{N}$. Then there exists $M \subseteq \mathbb{N}$ infinite such that:*

- *either for every infinite $A \subseteq M$, we have $A \in \mathcal{X}$;*
- *or for every infinite $A \subseteq M$, we have $A \notin \mathcal{X}$.*

Here, the set $A$ can be viewed as an increasing sequence of integers.

# Infinite-dimensional Ramsey theory and the pigeonhole principle

Let FIN denote the set of finite subsets of $\mathbb{N}$. Given a sequence $A_0 < A_1 < A_2 < \ldots$ of nonempty elements of FIN, a *block-sequence* is a sequence of the form $\bigcup_{i \in B_0} A_i, \bigcup_{i \in B_1} A_i, \bigcup_{i \in B_2} A_i, \ldots$ where $B_0 < B_1 < B_2 < \ldots$ is a sequence of nonempty elements of FIN.

**Theorem (Milliken)**

*Let $\mathcal{X}$ be an analytic set of increasing sequences of nonempty elements* FIN*. Then there exists a sequence $A_0 < A_1 < A_2 < \ldots$ of nonempty elements of* FIN *such that:*

- *either every block-sequence of $(A_i)$ is in $\mathcal{X}$;*
- *or no block-sequence of $(A_i)$ is in $\mathcal{X}$.*

# Infinite-dimensional Ramsey theory and the pigeonhole principle

A *pigeonhole principle* is a one-dimensional Ramsey result, i.e. a Ramsey result where you color objects. Every infinite-dimensional Ramsey result has an associated pigeonhole principle, which is obtained by coloring sequences according to their first term.

The pigeonhole principle associated to Mathias-Silver's theorem is the following: for every coloring of the integers with two colors, there exists an infinite monochromatic subset.

The pigeonhole principle associated to Milliken's theorem is:

**Theorem (Hindman)**

*For every coloring of the nonempty elements of* FIN, *there exists a sequence* $A_0 < A_1 < A_2 < \ldots$ *of nonempty elements of* FIN *such that all the sets of the form* $\bigcup_{i \in B} A_i$, *for* $B \in \mathrm{FIN} \setminus \{\varnothing\}$, *have the same color.*

Can we still get something interesting without pigeonhole principle?

# Gowers' Ramsey-type theorems for Banach spaces

The first Ramsey-type result without pigeonhole principle is by Gowers, for Banach spaces.

Fix $E$ a Banach space. Recall that a (normalized) sequence $(e_i)_{i\in\mathbb{N}}$ of $E$ is called a *Schauder basis* if for every $x \in E$, there exists a unique sequence $(x^i)_{i\in\mathbb{N}}$ of scalars such that $x = \sum_{i=0}^{\infty} x^i e_i$.

A *block-sequence* of $(e_i)$ is a sequence $(x_i)_{i\in\mathbb{N}}$ of (normalized) vectors of $E$ with $\mathrm{supp}(x_0) < \mathrm{supp}(x_1) < \mathrm{supp}(x_2) < \ldots$. A *block-subspace* is a (closed) subspace spanned by a block-sequence.

We want a Ramsey result where we color block-sequences and where we find a monochromatic block-subspace.

## Notation

1. For $A \subseteq S_E$ and $\delta > 0$, let $(A)_\delta = \{y \in S_E \mid \exists x \in A \; \|x - y\| \leqslant \delta\}$.
2. For $\mathcal{X}$ a set of block-sequences and $\Delta$ a sequence of positive numbers, let $(\mathcal{X})_\Delta$ be the set of block-sequences $(y_n)$ for which there exists $(x_n) \in \mathcal{X}$ with $\forall n \in \mathbb{N} \; \|x_n - y_n\| \leqslant \Delta_n$.

# Gowers' Ramsey-type theorems for Banach spaces

**Definition**

Say that $E$ satisfies the *approximate pigeonhole principle* if for every $A \subseteq S_E$, for every (block) subspace $X \subseteq E$ and for every $\delta > 0$, there exists a (block) subspace $Y \subseteq X$ such that either $S_Y \subseteq (A)_\delta$, or $S_Y \subseteq (A^c)_\delta$.

**Theorem**

*$E$ satisfies the approximate pigeonhole principle iff it is $c_0$-saturated.*

$\Leftarrow$ is by Gowers, $\Rightarrow$ comes from a combination of a result by Milman, and another by Odell and Schlumprecht.

**Theorem (Gowers' Ramsey-type theorem for $c_0$)**

*Let $E$ be a $c_0$-saturated Banach space with a Schauder basis $(e_i)$. Let $\Delta$ be a sequence of positive numbers, and $\mathcal{X}$ be an analytic set of block-sequences. Then there exists a block-subspace $X$ such that:*

- *either no block-sequence of $X$ is in $\mathcal{X}$;*
- *or every block-sequence of $X$ is in $(\mathcal{X})_\Delta$.*

# Gowers' Ramsey-type theorems for Banach spaces

To remedy to the lack of pigeonhole principle, we introduce Gowers' game:

**Definition**

Let $E$ be a Banach space with a Schauder basis $(e_i)$, let $X$ be a block-subspace, and let $\mathcal{X}$ be a set of block-sequences of $(e_i)$. *Gowers' game* $G_X(\mathcal{X})$ is defined as follows:

| | | | | | |
|---|---|---|---|---|---|
| **I** | $Y_0$ | | $Y_1$ | | $\ldots$ |
| **II** | | $y_0 \in Y_0$ | | $y_1 \in Y_1$ | $\ldots$ |

where the $Y_i$'s are block-subspaces of $X$, and the $y_i$'s are normalized vectors. Player **II** wins the game iff $(y_i)_{i \in \mathbb{N}}$ is a block-sequence that belongs to $\mathcal{X}$.

# Gowers' Ramsey-type theorems for Banach spaces

**Theorem (Gowers' Ramsey-type theorem)**

*Let $E$ be a Banach space with a Schauder basis $(e_i)$. Let $\Delta$ be a sequence of positive numbers, and $\mathcal{X}$ be an analytic set of block-sequences. Then there exists a block-subspace $X$ such that:*

- *either no block-sequence of $X$ is in $\mathcal{X}$;*
- *or* **II** *has a winning strategy in $G_X((\mathcal{X})_\Delta)$.*

It turns out that this result has nothing to do with Banach spaces.

# Approximate Gowers spaces

Let $P$ be a set (the set of *subspaces*) and $\leqslant$ and $\leqslant^*$ be two quasi-orderings on $P$, satisfying:

1. for every $p, q \in P$, if $p \leqslant q$, then $p \leqslant^* q$;
2. for every $p, q \in P$, if $p \leqslant^* q$, then there exists $r \in P$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;
3. for every $\leqslant$-decreasing sequence $(p_i)_{i \in \mathbb{N}}$ of elements of $P$, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leqslant p_i$;

Write $p \lessapprox q$ for $p \leqslant q$ and $q \leqslant^* p$.

Let $(X, d)$ be Polish metric space (the set of *points*) and $\lhd \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \lhd p$.
5. for every $x \in X$ and every $p, q \in P$, if $x \lhd p$ and $p \leqslant q$, then $x \lhd q$.

The sextuple $\mathcal{G} = (P, X, d, \leqslant, \leqslant^*, \lhd)$ is called an *approximate Gowers space*.

1. The Mathias-Silver space:
   - $X = \mathbb{N}$, and $\forall x, y \in \mathbb{N}\ (x \neq y \Rightarrow d(x, y) = 1)$;
   - $P$ is the set of infinite subsets of $\mathbb{N}$;
   - $\leqslant$ is the inclusion;
   - $\leqslant^*$ is the inclusion-by-finite;
   - $\lhd$ is the membership relation.
2. For $E$ a Banach space with a Schauder basis $(e_i)$, the standard Gowers space associated to $E$:
   - $X = S_E$ and $d$ is the usual distance;
   - $P$ is the set of block-subspaces of $E$;
   - $\leqslant$ is the inclusion;
   - $\leqslant^*$ is the inclusion up to finite dimension ($F \leqslant^* G$ iff $F \cap G$ has finite codimension in $F$);
   - $\lhd$ is the membership relation.

# Approximate Gowers spaces

## Notation

1. For $A \subseteq X$ and $\delta > 0$, let $(A)_\delta = \{y \in X \mid \exists x \in A \, d(x,y) \leqslant \delta\}$.
2. For $\mathcal{X} \subseteq X^\omega$ and $\Delta$ a sequence of positive numbers, let $(\mathcal{X})_\Delta = \{(y_n) \in X^\omega \mid \exists (x_n) \in \mathcal{X} \, \forall n \in \mathbb{N} \, d(x_n, y_n) \leqslant \Delta_n\}$.

## Definition

Say that $\mathcal{G}$ satisfies the *pigeonhole principle* if for every $A \subseteq X$, for every $p \in P$ and for every $\delta > 0$, there exists $q \leqslant p$ such that either $q \subseteq (A)_\delta$, or $q \subseteq (A^c)_\delta$. (Here, $q$ is identified with $\{x \in X \mid x \lhd q\}$.)

- The Mathias-Silver space satisfies the pigeonhole principle;
- The pigeonhole principle for the standard Gowers space associated to $E$ is precisely the approximate pigeonhole principle defined some slides ago.

# Approximate Gowers spaces

**Definition**

Given $p \in P$ and $\mathcal{X} \subseteq X^{\mathbb{N}}$, *Gowers' game* $G_p(\mathcal{X})$ is defined as follows:

| | | | | | |
|---|---|---|---|---|---|
| **I** | $p_0 \leqslant p$ | | $p_1 \leqslant p$ | | $\ldots$ |
| **II** | | $x_0 \lhd p_0$ | | $x_1 \lhd p_1$ | $\ldots$ |

Player **II** wins the game iff $(x_i)_{i \in \mathbb{N}} \in \mathcal{X}$.

# Approximate Gowers spaces

We now add some structure to compensate for the lack of ordering on $X$.

Consider a family $\mathcal{K}$ of compact subsets of $X$ and a binary operation $\oplus$ on $\mathcal{K}$, associative and commutative, satisfying the following conditions:

- $\forall K_1, K_2 \in \mathcal{K},\ K_1 \cup K_2 \subseteq K_1 \oplus K_2$;
- For all $p \in P$ and all $K_1, K_2 \in \mathcal{K}$, if $K_1 \lhd p$ and $K_2 \lhd p$, then $K_1 \oplus K_2 \lhd p$.

(Here, $K \lhd p$ means that $\forall x \in K\ x \lhd p$.)

## Definition

Given $p \in P$ and $\mathcal{X} \subseteq X^{\mathbb{N}}$, the *strong asymptotic game* $SF_p(\mathcal{X})$ is defined as follows:

| | | | | | |
|---|---|---|---|---|---|
| **I** | $p_0 \lessapprox p$ | | $p_1 \lessapprox p$ | | $\dots$ |
| **II** | | $K_0 \lhd p_0$ | | $K_1 \lhd p_1$ | $\dots$ |

where $K_i$'s are elements of $\mathcal{K}$. Player **I** wins the game iff for every sequence $A_0 < A_1 < \dots$ of subsets of $\mathbb{N}$, we have $\left(\bigoplus_{i \in A_0} K_i\right) \times \left(\bigoplus_{i \in A_1} K_i\right) \times \cdots \subseteq \mathcal{X}$.

# Approximate Gowers spaces

**Theorem (dR.)**

*Let $p \in P$, $\Delta$ a sequence of positive numbers, and $\mathcal{X} \subseteq X^\omega$ analytic. Then there exists a $q \leqslant p$ such that:*

- *either* **I** *has a winning strategy in $SF_q(\mathcal{X}^c)$;*
- *or* **II** *has a winning strategy in $G_q((\mathcal{X})_\Delta)$.*

*Moreover, if $\mathcal{G}$ satisfies the pigeonhole principle, the second conclusion can be replaced with the following (stronger) one:*

- **I** *has a winning strategy in $SF_q((\mathcal{X})_\Delta)$*

# Approximate Gowers spaces

- To get Mathias-Silver's theorem, take for $\mathcal{K}$ the set of finite subsets of $\mathbb{N}$, and for $\oplus$ the union. Then, in $SF_q(\mathcal{X}^c)$, **II** just has to play a increasing sequence $\{n_0\}, \{n_1\}, \{n_2\}, \ldots$ of singletons, so any subsequence will be in $\mathcal{X}^c$.
- To get Gowers' theorem, take for $\mathcal{K}$ the set of $S_F$'s, where $F \subseteq E$ is a finite-dimensional block-subspace, where $S_F \oplus S_G = S_{F+G}$. Then in $SF_q(\mathcal{X}^c)$, **II** just has to play a sequence $K_0 = \{x_{i_0}, -x_{i_0}\}, K_1 = \{x_{i_1}, -x_{i_1}\}, \ldots$, where $i_0 < i_1 < \ldots$ are integers and $(x_i)$ is the canonical basis of the block-subspace $q$. Then every block-sequence of $(x_{i_0}, x_{i_1}, \ldots)$ will belong to a set of the form $\left(\bigoplus_{i \in A_0} K_i\right) \times \left(\bigoplus_{i \in A_1} K_i\right) \times \cdots$.

# Another interesting example

**Lemma**

*A separable Banach space $E$ is $C$-isomorphic to $\ell_2$ if and only if every finite-dimensional subspace $F \subseteq E$ is $C$-isomorphic to $\ell_2^{\dim(F)}$.*

**Corollary**

*Let $E$ be a separable Banach space, non-isomorphic to $\ell_2$. Let $P$ be the set of (closed, infinite-dimensional) subspaces of $E$ that are not isomorphic to $\ell_2$, $\subseteq^*$ be the inclusion of subspaces up to finie dimension, and $d$ be the usual distance on $S_E$. Then $(P, S_E, d, \subseteq, \subseteq^*, \in)$ is an approximate Gowers space.*

# Another interesting example

**Corollary (dR. – Ferenczi)**

*Let $E$ be a separable Banach space, non-isomorphic to $\ell_2$, $\Delta$ be a sequence of positive numbers, $\varepsilon > 0$ and $\mathcal{X} \subseteq S_E^\omega$ be analytic. Then there exists a closed, infinite-dimensional subspace $X \subseteq E$, non-isomorphic to $\ell_2$, such that:*

- *either $X$ has an FDD $(F_n)_{n\in\mathbb{N}}$ with constant $\leqslant 1+\varepsilon$, such that $d_{BM}(F_n, \ell_2^{\dim(F_n)}) \underset{n\to\infty}{\longrightarrow} \infty$ and such that every normalized block-sequence of $(F_n)$ is in $\mathcal{X}^c$;*
- *or **II** has a winning strategy in $G_X((\mathcal{X})_\Delta)$, when **I** only plays subspaces that are non-isomorphic to $\ell_2$.*

**Thank you for your attention!**