An abstract formalism for strategical Ramsey theory

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Infinite-dimensional Ramsey theory and the pigeonhole principle

Infinite-dimensional Ramsey theory is about coloring infinite sequences of objects, and finding monochromatic subspaces.

Theorem (Silver)

Let $\mathcal{X}$ be an analytic set of infinite subsets of $\mathbb{N}$. Then there exists $M \subseteq \mathbb{N}$ infinite such that:

- *either for every infinite $A \subseteq M$, we have $A \in \mathcal{X}$;*
- *or for every infinite $A \subseteq M$, we have $A \notin \mathcal{X}$.*

Here, the set M is generally viewed as a element of a forcing poset, whereas the set A is viewed as an increasing sequence of integers.

Infinite-dimensional Ramsey theory and the pigeonhole principle

Fix k an at most countable field. Let $E = k^{(\mathbb{N})}$ be the countably infinite-dimensional vector space over k, with canonical basis $(e_i)_{i\in\mathbb{N}}$. Recall that a block-sequence of E is a sequence $(x_n)_{n\in\mathbb{N}}$ of nonzero successive vectors of E, i.e. such that $\mathrm{supp}(x_0) < \mathrm{supp}(x_1) < \ldots$ (where $\mathrm{supp}(\sum_{i\in\mathbb{N}} a_i e_i) = \{i \in \mathbb{N} \mid a_i \neq 0\}$).

Theorem (Milliken)

Suppose $k = \mathbb{F}_2$. Let $\mathcal{X}$ be an analytic set of block-sequences of E. Then there exists an infinite-dimensional subspace F of E such that:

- *either every block-sequence of F belongs to $\mathcal{X}$;*
- *or every block-sequence of F belongs to $\mathcal{X}^c$.*

A pigeonhole principle is a one-dimensional Ramsey result, i.e. a Ramsey result where you color objects. Every infinite-dimensional Ramsey result has an associated pigeonhole principle, which is obtained by coloring sequences according to their first term.

The pigeonhole principle associated to Silver's theorem is the following: for every coloring of the integers with two colors, there exists an infinite monochromatic subset.

The pigeonhole principle associated to Milliken's theorem is:

Theorem (Hindman)

Suppose $k = \mathbb{F}_2$. For every $A \subseteq E \backslash \{0\}$, there exists an infinite-dimensional subspace F of E such that either $F \backslash \{0\} \subseteq A$, or $F \backslash \{0\} \subseteq A^c$.

Can we still get something interesting without pigeonhole principle?

The formalism of Gowers spaces

Let P be a set (the set of subspaces) and $\leqslant$ and $\leqslant^*$ be two quasi-orderings on P, satisfying:

1. for every $p, q \in P$, if $p \leqslant q$, then $p \leqslant^* q$;
2. for every $p, q \in P$, if $p \leqslant^* q$, then there exists $r \in P$ such that $r \leqslant p$, $r \leqslant q$ and $p \leqslant^* r$;
3. for every $\leqslant$-decreasing sequence $(p_i)_{i \in \mathbb{N}}$ of elements of P, there exists $p^* \in P$ such that for all $i \in \mathbb{N}$, we have $p^* \leqslant^* p_i$;

Write $p \lessapprox q$ for $p \leqslant q$ and $q \leqslant^* p$.

Let X be an at most countable set (the set of points) and $\lhd \subseteq X \times P$ a binary relation, satisfying:

4. for every $p \in P$, there exists $x \in X$ such that $x \lhd p$.
5. for every $x \in X$ and every $p, q \in P$, if $x \lhd p$ and $p \leqslant q$, then $x \lhd q$.

The quintuple $\mathcal{G} = (P, X, \leqslant, \leqslant^*, \lhd)$ is called a Gowers space.

1. The Silver space:
   - $X = \mathbb{N}$;
   - P is the set of infinite subsets of $\mathbb{N}$;
   - $\leqslant$ is the inclusion;
   - $\leqslant^*$ is the inclusion-by-finite;
   - $\lhd$ the membership relation.

2. The Rosendal space over an at most countable field k:
   - $X = E$ is a countably-infinite-dimensional vector space over k;
   - P is the set of infinite-dimensional subspaces of E;
   - $\leqslant$ is the inclusion;
   - $\leqslant^*$ is the inclusion up to finite dimension ($F \leqslant^* G$ iff $F \cap G$ has finite codimension in F);
   - $\lhd$ is the membership relation.

Definition

The space $\mathcal{G}$ is said to satisfy the pigeonhole principle if for every $A \subseteq X$ and every $p \in P$, there exists $q \leqslant p$ such that either for all $x \lhd q$, we have $x \in A$, or for all $x \lhd q$, we have $x \in A^c$.

Asymptotic games

Definition

Let $p \in P$. The asymptotic game below p, denoted by F_p, is the following two-players game:

I	$p_0 \lessapprox p$		$p_1 \lessapprox p$		$\dots$	
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$		$\dots,$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

Saying that **I** has a strategy to reach $\mathcal{X} \subseteq X^{\mathbb{N}}$ in F_p means that "almost every" sequence below p belongs to $\mathcal{X}$.

In the Silver space, we have the following:

Proposition

If $\mathcal{X} \subseteq \mathbb{N}^{\mathbb{N}}$ is such that **I** *has a strategy to reach $\mathcal{X}$ in F_M, then there exists $N \subseteq M$ infinite such that every increasing sequence of elements of N belongs to $\mathcal{X}$.*

The abstract Silver's theorem

So this is an equivalent formulation of Silver's theorem:

Theorem

For every analytic $\mathcal{X} \subseteq \mathbb{N}^{\mathbb{N}}$, there exists $M \subseteq \mathbb{N}$ infinite such that:

- *either* **I** *has a strategy in F_M to reach $\mathcal{X}^c$;*
- *or* **I** *has a strategy in F_M to reach $\mathcal{X}$.*

In general, we have:

Theorem (Abstract Silver's)

Suppose that the space $\mathcal{G}$ satisfies the pigeonhole principle. Let $p \in P$ and $\mathcal{X} \subseteq X^{\mathbb{N}}$ be analytic. Then there exists $q \leqslant p$ such that:

- *either* **I** *has a strategy in F_q to reach $\mathcal{X}^c$;*
- *or* **I** *has a strategy in F_q to reach $\mathcal{X}$.*

Gowers' games and the abstract Rosendal's theorem

Definition

Let $p \in P$. Gowers' game below p, denoted by G_p, is the following two-players game:

I	$p_0 \leqslant p$		$p_1 \leqslant p$		$\ldots$	
II		$x_0 \lhd p_0$		$x_1 \lhd p_1$		$\ldots,$

The outcome of the game is the sequence $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

We have the following implication : if **I** has a strategy to reach $\mathcal{X}$ in F_p, then **II** has a strategy to reach $\mathcal{X}$ in G_p. Under the pigeonhole principle, the converse is true up to taking a subspace.

Theorem (Abstract Rosendal's)

Let $p \in P$ and $\mathcal{X} \subseteq X^{\mathbb{N}}$ be analytic. Then there exists $q \leqslant p$ such that:

- *either* **I** *has a strategy in F_q to reach $\mathcal{X}^c$;*
- *or* **II** *has a strategy in G_q to reach $\mathcal{X}$.*

Local Ramsey theory in Gowers spaces

Gowers spaces are great for doing local Ramsey theory. If X is an (algebraic) structure with a natural notion of subspaces, then you can define a Gowers space by taking for P more or less any subfamily of the family of subspaces provided we can diagonalize among this subfamily.

Definition

Let $\mathcal{F}$ be a nonempty family of infinite subsets of $\mathbb{N}$. We say that:

- $\mathcal{F}$ is a p-family if it is $\mathbf{E_0}$-invariant and if for every decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{F}$, there exists $A^* \in \mathcal{F}$ such that for every $n \in \mathbb{N}$, $A^* \subseteq^* A_n$;
- $\mathcal{F}$ is selective if it is a p-family and if moreover, the set A^* can be choosen in such a way that for every $n \in A^*$, $A^*/n \subseteq A_n$ (where $A^*/n = \{k \in A^* \mid k > n\}$).

Local Ramsey theory in Gowers spaces

Fix $\mathcal{F}$ a p-family of subsets of $\mathbb{N}$. Then $(\mathcal{F}, \mathbb{N}, \subseteq, \subseteq^*, \in)$ is a Gowers space.

Corollary

Let $\mathcal{X} \subseteq \mathbb{N}^{\mathbb{N}}$ be analytic. Then there exists $M \in \mathcal{F}$ such that:

- *either* **I** *has a strategy in F_M to reach $\mathcal{X}^c$;*
- *or* **II** *has a strategy in G_M to reach $\mathcal{X}$.*

Moreover, if $\mathcal{F}$ is selective, then the first possible conclusion can be replaced by "$[M]^\infty \subseteq \mathcal{X}^c$".

Beware, here in G_M, player **I** can only play elements of $\mathcal{F}$!

Corollary (Mathias)

Let $\mathcal{H}$ be a selective coideal on $\mathbb{N}$, and $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ be analytic. Then there exists $M \in \mathcal{H}$ such that either $[M]^\infty \subseteq \mathcal{X}^c$, or $[M]^\infty \subseteq \mathcal{X}$.

What about Banach spaces ?

What follows is part of a common work with W. Cuellar Carrera and V. Ferenczi.

On $\mathbb{N}$, F_σ ideals are p^+-ideals. The same phenomenon appears in Banach spaces.

Fix X a Banach space. We denote by $\mathrm{Sub}(X)$ the set of closed infinite-dimensional subspaces of X. We endow $\mathrm{Sub}(X)$ with the slice topology, i.e. the topology such that (Y_λ) converges to Y iff for every equivalent norm $|||\cdot|||$ and for every $x \in X$, the norm of x in the quotient $(X, |||\cdot|||)/Y_\lambda$ coverges to the norm of x in the quotient $(X, |||\cdot|||)/Y$.

Theorem

Let $P \subseteq \mathrm{Sub}(X)$ be a slice-G_δ subset, invariant under finite-dimensional modifications. Then $(P, S_X, \subseteq, \subseteq^*, \in)$ is an (uncountable) Gowers space.

What about Banach spaces ?

Definition

A finite-dimensional decomposition (FDD) of a Banach space Y is a sequence $(F_i)_{i\in\mathbb{N}}$ of finite-dimensional subspaces of Y such that every $x \in Y$ can be written in a unique way as a sum $x = \sum_{i=0}^{\infty} x_i$, where for every i, $x_i \in F_i$.

A block-sequence of the FDD (F_i) is a sequence $(x_n)_{n\in N}$ of normalized successive vectors for this FDD (i.e. there exists $A_0 < A_1 < A_2 < \ldots$ sets of integers such that for every n, $x_n \in \bigoplus_{i\in A_n} F_i$).

Definition

Given $\mathcal{X} \subseteq (S_X)^{\mathbb{N}}$ and $\Delta = (\Delta_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers, we let $(\mathcal{X})_\Delta = \{(y_n) \in (S_X)^{\mathbb{N}} \mid \exists (x_n) \in \mathcal{X}\ \forall n\ \|x_n - y_n\| \leqslant \Delta_n\}$.

What about Banach spaces ?

Corollary

Let $P \subseteq \mathrm{Sub}(X)$ be a slice-G_δ subset, invariant under finite-dimensional modifications. Let $\mathcal{X} \subseteq (S_X)^{\mathbb{N}}$ be analytic, and let Δ be a sequence of positive real numbers. Then there exists $Y \in P$ such that:

- *either Y has a FDD (F_n) such that every subsequence of (F_n) generates an element of P, and such that every block-sequence of (F_n) is in $\mathcal{X}^c$;*
- *or **II** has a strategy in G_Y to reach $(\mathcal{X})_\Delta$ (where in G_Y, player **I** is only allowed to play elements of P).*

An example

The condition of being slice-G_δ is typically satisfied for families of Banach spaces that can be defined by conditions on finite-dimensional subspaces.

Lemma

A Banach space X is non-Hilbertian iff for every $n \in \mathbb{N}$, there exists a finite-dimensional subspace $F \subseteq X$ that is not n-isomorphic to a Euclidean space. In particular, the family of non-Hilbertian spaces is slice-G_δ.

Question

Does there exist similar examples in other areas of mathematics?

Thank you for your attention!