Local Banach-space dichotomies

Noé de Rancourt[1]
Joint work with Wilson Cuellar Carrera[2] and Valentin Ferenczi[2]

[1]Kurt Gödel Research Center, University of Vienna
[2]Universidade de São Paulo

Banach space webinar

June 12, 2020

Terminology

In this talk, all Banach spaces will be infinite-dimensional, and by a subspace, I will mean an infinite-dimensional, closed subspace.

A space is said to be Hilbertian if it is isomorphic to a Hilbert space.

We will work in real spaces, but everything works as well in complex spaces.

Gowers' classification program

It is well known that there is no hope classifying separable Banach spaces up to isomorphism.

In his famous paper *An infinite Ramsey theorem and some Banach-space dichotomies* (Ann. Math. '02), Gowers suggests a weak classification program for separable Banach spaces, up to subspaces. The goal is to build a list of classes of separable Banach spaces (usually called a Gowers list), as fine as possible, and satisfying the following conditions:

(1) the classes are hereditary: if X belongs to a class $\mathcal{C}$ then all subspaces of X also belong to $\mathcal{C}$ (or, in the case of classes defined by properties of bases, all block-subspaces of X belong to $\mathcal{C}$);

(2) the classes are pairwise disjoint;

(3) knowing that a space belongs to a class gives much information about the structure of this space;

(4) every Banach space contains a subspace belonging to one of the classes.

Gowers' first dichotomy

Property (4) is usually the one that is the most difficult to prove; it requires to prove Banach-space dichotomies

> **Theorem ("The first dichotomy", Gowers, '96)**
>
> *Every Banach space X contains a subspace Y such that:*
>
> - *either Y has an unconditional basis;*
>
> - *or Y is hereditarily indecomposable (HI), that is, it contains no topological direct sum of two subspaces.*

Gowers' first dichotomy provides a Gowers list with two classes: the class of spaces having an unconditional basis, and the class of HI spaces. The fact that HI spaces have very few operators justifies property (3) for this class.

Ferenczi–Rosendal's third dichotomy

Definition

A Banach space is *minimal* if it embeds into all of its subspaces.

If $(x_i)_{i \in I}$ is a family of vectors in a Banach space, we denote by $[x_i \mid i \in I]$ the closed subspace it spans.

Definition

- A Banach space Y is *tight in* a basis (e_i) if there exists a sequence of successive intervals of integers $I_0 < I_1 < ...$ such that for every infinite $A \subseteq \mathbb{N}$, Y cannot be embedded into $[e_i \mid i \notin \bigcup_{n \in A} I_n]$.

- A basis (e_i) is *tight* if every Banach space is tight in it.

- A Banach space X is *tight* if it admits a tight basis.

For instance, Tsirelson's space is tight.

Ferenczi–Rosendal's third dichotomy

Theorem ("The third dichotomy", Ferenczi–Rosendal, '09)

Every Banach space either has a minimal subspace, or has a tight subspace.

This dichotomy, combined with the first one, provides a Gowers list with three classes:

- minimal spaces with an unconditional basis;

- spaces having an unconditional and tight basis;

- tight HI spaces.

Other dichotomies by Gowers and Ferenczi–Rosendal extend this list to 6 classes (all of whose are known to be nonempty) and 19 possible subclasses.

A non-example: James' space

Maybe the first ever (?) example of a "pathological" Banach space was James' quasi-reflexive space $\mathcal{J}$.

James' space is ℓ_2 saturated; hence, a Gowers list will never give any relevant information about it.

Hence the idea to prove local dichotomies, i.e., dichotomies that are similar to the latter ones but where the outcome space can always be found "locally", that is, in a fixed family of subspaces. The word "local" used here was stolen to Ramsey-theorists.

Local properties

Consider a local property (P) of Banach spaces, that is, a property such that:

- a Banach space X has (P) with constant C if and only if all of its finite-dimensional subspaces have (P) with constant C;

- if a space X has (P) with constant C, and if Y is K-isomorphic to X, then Y has (P) with constant KC.

This includes, for example, being Hilbertian, having type p, cotype q...

Informal fact

Usual Banach-space dichotomies can be localized to spaces failing property (P).

We can actually generalize this to certain properties that are not local; such properties can be defined via the notion of degree.

Degrees

A pair (X, F) where X is a (finite- or infinite-dimensional) Banach space, and F is a finite-dimensional subspace of X will be called an approximation pair. Denote by AP the class of all approximation pairs.

Definition

A degree is a mapping $d \colon AP \to \mathbb{R}_+$ such that for every operators S, T making the following diagram commute :

$$
\begin{array}{ccc}
F & \overset{\iota}{\hookrightarrow} & X \\
S \uparrow & & \downarrow T \\
G & \overset{\iota}{\hookrightarrow} & Y
\end{array},
$$

we have $d(Y, G) \leqslant \|S\| \cdot \|T\| \cdot d(X, F)$.

The degree d is said to be local if $d(X, F)$ only depends on F (in which case it is simply denoted by $d(F)$).

Degrees

Definition

Let d be a degree. A Banach space is said to be d-small if $sup_{F \subseteq X} d(X, F) < \infty$, and d-large otherwise.

A local property and a local degree are the same thing. If (P) is a local property, let $d(F)$ be the infimum of constants C such that F has property (P) with constant C. Then d is a local degree, and being d-small is equivalent to having property (P).

Degrees: examples

(1) $d(F) = \dim(F)$ is a local degree. d-small spaces are exactly finite-dimensional spaces.

(2) $d_2(F) = d_{BM}(F, \ell_2^{dim(F)})$ is a local degree, the Hilbertian degree. d_2-small spaces are exactly Hilbertian spaces.

(3) Fix $1 \leqslant p \leqslant 2$. Then $d(F) = T_p(F)$, the type constant, is a degree. d-small spaces are exactly those having type p. The same works for cotype.

Degrees: examples

(4) Fix $1 \leqslant p \leqslant \infty$. Define $d(X, F)$ as the infimum of the M's for which the canonical inclusion of F into X M-factorizes through some ℓ_p^n, meaning that there exists $n \in \mathbb{N}$ and operators $U \colon F \to \ell_p^n$ and $V \colon \ell_p^n \to X$ with $\|U\| \cdot \|V\| = M$, making the following diagram commute:

Then d is a non-local degree. By a result by Lindenstrauss and Rosenthal, we have that:

- if $1 < p < \infty$, a space is d-small if and only if it is either a $\mathcal{L}_p$-space, or a Hilbertian space;
- if $p = 1$ or $p = \infty$, a space is d-small if and only if it is a $\mathcal{L}_p$-space.

Degrees: examples

(5) Let $d(X, F)$ be the infimum of the M's for which the canonical inclusion of F into X M-factorizes through some space with a 1-unconditional basis. Then d is a non-local degree. A space is d-small iff it has Gordon-Lewis local unconditional structure.

(6) More generally, recall that an normed operator ideal $\mathfrak{U}$ is given by, for all Banach spaces X and Y, a vector subspace $\mathfrak{U}(X, Y) \subseteq \mathcal{L}(X, Y)$ and a complete norm N on $\mathfrak{U}(X, Y)$, such that, whenever $S \in \mathcal{L}(X, Y)$, $T \in \mathfrak{U}(Y, Z)$ and $U \in \mathcal{L}(Z, W)$, we have $UTS \in \mathfrak{U}(X, W)$ and $N(UTS) \leqslant \|U\| \cdot N(T) \cdot \|S\|$, and such that moreover $\mathrm{Id}_{\mathbb{R}} \in \mathfrak{U}(\mathbb{R}, \mathbb{R})$ with $N(\mathrm{Id}_{\mathbb{R}}) = 1$. Then $d(X, F) = N(\iota_{F,X})$ is a degree.

(7) Let $d(X, F)$ be the infimum of the K's such that F is K-complemented in X. Then d is not a degree. This is somewhat surprising, since if it were one, d-small spaces would exactly be Hilbertian spaces.

Properties of degrees

Proposition

Let d be a degree.

- *Finite-dimensional spaces are d-small.*

- *The properties of being d-small and d-large are preserved by isomorphism.*

- *A complemented subspace of a d-small space is d-small.*

- *If d is local, every subspace of a d-small space is d-small.*

- *If (X_n) is a decreasing sequence of d-large spaces, then there is a d-large space X_∞ such that $X_\infty \subseteq^* X_n$ for every n (meaning that X_n contains a finite-codimensional subspace of X_∞).*

Better FDD's

Given a degree d, it is not clear whether every d-large Banach space should contain a d-large subspace with a basis (this is open for $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$). Hence we can't prove local dichotomies between classes defined by properties of bases. We use finite-dimensional decompositions (FDD's) instead.

Definition
An FDD (F_n) of a Banach space X is said to be d-better if $d(X, F_n) \longrightarrow \infty$.

If a space has a d-better FDD, then it is d-large. Conversely:

Lemma
Let X be a d-large Banach space. Then X has a subspace spanned by a d-better FDD.

Standard results about bases, such as Bessaga–Pełczyński selection principle, still hold for better FDD's.

A local version of Gowers' first dichotomy

We fix a degree d. We will localize Gowers' first dichotomy to the class of d-large spaces, that is, we will ensure that the outcome space is d-large. This strenghtening has a cost: the two conclusions have to be weakened in some other way.

Definition

A space is said to be d-HI if it is d-large and contains no topological direct sum of two d-large subspaces.

Theorem (The first dichotomy, local version)

Let X be a d-large Banach space. Then X has a d-large subspace Y such that:

- *either Y is spanned by a d-better UFDD;*

- *or Y is d-HI.*

When d is the dimension, we recover the original dichotomy.

A local version of Ferenczi–Rosendal's third dichotomy

Definition

A Banach space is said to be *d-minimal* if it is d-large and embeds into all of its d-large subspaces.

Definition

- A Banach space Y is tight in a FDD (F_i) if there exists a sequence of successive intervals of integers $I_0 < I_1 < ...$ such that for every infinite $A \subseteq \mathbb{N}$, Y cannot be embedded into $[F_i \mid i \notin \bigcup_{n \in A} I_n]$.

- A FDD (F_i) is *d-tight* if every d-large Banach space is tight in it.

- A Banach space X is *d-tight* if it admits a d-better, d-tight FDD.

Theorem (The third dichotomy, local version)

Every d-large Banach space either has a d-minimal subspace, or has a d-tight subspace.

Growth gap for d-minimal spaces

Lemma

Fix d a local degree, and X a d-minimal Banach space. Then there is a mapping $\Gamma \colon \mathbb{N} \to \mathbb{R}_+$, tending to infinity, such that for every d-large subspace Y of X and every n, we have:

$$\sup_{\substack{F \subseteq Y \\ \dim(F) = n}} d(F) \geqslant \Gamma(n).$$

In other words, d-large subspaces of a d-minimal space are uniformly d-large. If X is not minimal (or equivalently, is saturated with d-small subspaces), then there is a gap in the possible growth rates of the function

$$\sup_{\substack{F \subseteq Y \\ \dim(F) = n}} d(F),$$

when Y ranges over all subspaces of X: either it is bounded, or it grows at least at the same rate as $\Gamma(n)$. This is a very surprising local property.

Question

Does there exist local degrees d such that all d-minimal subspaces are minimal? Does there exist some for which d-minimality does not coincide with minimality?

A more general setting

In all generality, our local dichotomies can be extended to D-families, a more general class of families of Banach spaces. In particular, given a degree d, the family of d-large spaces is a D-family.

Unlike degrees, D-families have good closure properties: they are closed under countable intersection and finite unions. So, given a sequence (d_n) of degrees, we can prove local dichotomies for the family of all spaces that are d_n-large for every n.

For instance, if $1 \leqslant p < 2$, we can localize dichotomies to the class of spaces that don't have any type $> p$. Such a dichotomy could be used on spaces having type p.

The Hilbertian degree

We study in more details the Hilbertian degree $d_2(F) = d_{BM}(F, \ell_2^{\dim(F)})$, for which d-small spaces are Hilbertian spaces. For this degree, our local dichotomies become:

Theorem

Let X be a non-Hilbertian Banach space. Then X has a non-Hilbertian subspace Y satisfying one of the following propertties:

(1) *Y is d_2-minimal and has an unconditional basis;*

(2) *Y has a d_2-better, d_2-tight UFDD;*

(3) *Y is d_2-HI and d_2-minimal;*

(4) *Y is d_2-HI and d_2-tight.*

This theorem gives no new information when X is not ℓ_2-saturated. However, in the case of ℓ_2-saturated spaces, this can be seen as a Gowers list for non-Hilbertian, ℓ_2-saturated Banach spaces.

The Hilbertian degree

Lemma

Consider $X = \left(\bigoplus_{n \in \mathbb{N}} \ell_{p_n}^{k_n} \right)_{\ell_2}$, where $p_n \longrightarrow 2$ and $k_n \longrightarrow \infty$ are well-chosen. Then X is ℓ_2-saturated and has a subspace in class (2) of the latter list.

This class (2) is the only one we know to be nonempty, when we restrict our attention to ℓ_2-saturated Banach spaces.

Question

Does there exist a Banach space which is simultaneously d_2-minimal and d_2-HI?

Such a space should be ℓ_2-saturated. We don't even know whether ℓ_2 saturated d_2-minimal spaces, or ℓ_2-saturated d_2-HI spaces, exist.

The Hilbertian degree

Question

Does James' space belong to one of the classes? If not, in what classes does it have subspaces? What about other classical ℓ_2-saturated spaces?

A few words on ergodicity

Definition (Ferenczi–Rosendal, '05)

A separable Banach space X is said to be *ergodic* if $\mathbb{E}_0$ reduces to the isomorphism relation between its subspaces.

This is a little bit stronger than saying that X has continuum-many pairwise non-isomorphic subspaces.

Conjecture (Ferenczi–Rosendal, '05)

Every non-Hilbertian separable Banach space is ergodic.

Theorem

Let X be a counterexample to the above conjecture. Then X has a non-Hilbertian subspace Y (that is itself a counterexample) that is d_2-minimal. Moreover, it can be ensured that Y either has an unconditional basis, or is d_2-HI.

Thank you for your attention!