Modification of Recourse Data for Integer Recourse Models

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Outline

- (Mixed-)Integer recourse: definition
- Motivation
- Modification of Recourse Data
- Convex α-approximations for SIR & TU recourse
- Error bounds

Decision making under uncertainty is the real problem we should all be working on *[G.B. Dantzig 2001]*

Stochastic Programming on the Web:

- SP Community Home Page http://stoprog.org
- SP E-Print Series
- SP Bibliography + Books on Stochastic Programming

(Mixed-)Integer recourse models

Coping with m random constraints $T(\omega)x \geq h(\omega)$

Only right-hand side random:

- $h(\omega) = \omega$ with known cdf F_ω
- $T(\omega) = T$ deterministic

Model

$$\min_x \{cx + Q(x) : Ax \geq b,\ x \in \mathbb{R}_+^{n_1}\}$$

where

$$Q(x) = \mathbb{E}_\omega[v(\omega - Tx)]$$

and

$$v(s) = \min_y \{qy : Wy \geq s,\ y \in \mathbb{Z}_+^{n_2}\}$$

Integer recourse actions $y = y(x, \omega)$

Why include integer variables? Modeling power:

- natural integrality of decision variables
  e.g. *The Allocation of Aircraft to Routes* [Ferguson & Dantzig '56]
- yes/no, on/off decisions
- artificial indicator variables for conditional linear constraints (LP formulation of CO problems)

$$0 \leq x \leq Mz, \quad x \in \mathbb{R}, \quad z \in \{0,1\}$$

- satisfy k out of n constraints, e.g. discrete Chance Constraints

$$\Pr\{Tx \geq \omega\} \geq \alpha \in (0,1)$$

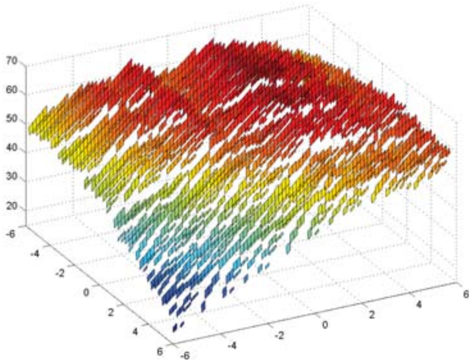
  with $\Pr\{\omega = \omega^s\} = p^s$, $s = 1, \ldots, S$

Why not?

- continuous SLP is already difficult enough
- complexity: 2nd-stage problems NP-hard

[Dyer & Stougie '06] continuous SLP is #P complete
$\longrightarrow$ SMIP not harder (...)

Main issue: integer recourse function Q is non-convex in general



(example from [Schultz et al. '98])

How to solve SMIP?

Borrow from solution approaches for deterministic MILP: e.g.

- (LP + rounding)
- Branch & Bound with LP relaxation
- Benders' decomposition
- Polyhedral theory: valid inequalities
- Lagrangian relaxation

Combine with SLP algorithms $\longrightarrow$ algorithms for SMIP?

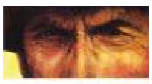
Various authors (+ co-authors):

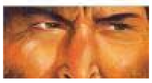
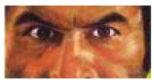
- Schultz
- Louveaux
- Sen
- Ahmed
- ... see e.g. S(I)P Bibliography and SPePS

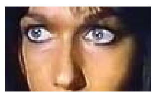
Our perspective: SMIP is a battle between

- randomness: *Good*
- integrality: *Bad*

Usually, result is *Ugly*: non-convex, ...



 $\longrightarrow$ 

However, sometimes result is *Beautiful*: convex! 

Modification of Recourse Data (VdV 2003)

Recourse data (q, W, Y, F_ω)

- structure:
  - (q, W) complete / sufficiently expensive recourse, ...
  - $y \in Y$: simple bounds, integrality
- distribution

Nice properties for recourse models

- Convexity
- Discrete distribution
- Continuous decision variables

Idea of approach

- Modify data $(q, W, Y, F_\omega) \to (\tilde{q}, \tilde{W}, \tilde{Y}, \tilde{F}_\omega)$ so that
  - easy to solve
  - good approximation

MRD for integer recourse

Expected value function $Q(x)$

- $Q(x) = \mathbb{E}_\omega \left[\min_y \{ qy : Wy \geq \omega - Tx,\ y \in \mathbb{Z}_+^{n_2} \} \right]$
  with ω a continuous random vector

Approximation of VdV (2004)

- $\hat{Q}(x) = \mathbb{E}_\xi \left[\min_y \{ qy : Wy \geq \xi - Tx,\ y \in \mathbb{R}_+^{n_2} \} \right]$
  with ξ a discrete random vector

- Special case: (one-sided) Simple Integer Recourse ($W = I$)

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SR: Modeling linear penalty costs for individual surpluses (shortages):

$$\begin{aligned} v(s) &= \min_y \{ qy : y \geq s,\ y \in \mathbb{Z}_+^m \} \\ &= \sum_{i=1}^m \min_{y_i} \{ q_i y_i : y_i \geq s_i,\ y_i \in \mathbb{Z}_+ \} \\ &= \sum_{i=1}^m q_i \lceil s_i \rceil^+ \qquad \text{(assuming } q \geq 0\text{)} \end{aligned}$$

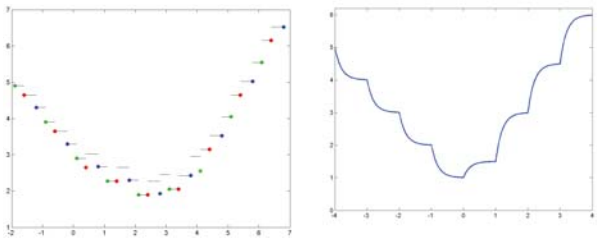
with $\lceil x \rceil^+ := \max\{0, \lceil x \rceil\},\ x \in \mathbb{R}$

SIR value function v is separable
Assume T deterministic $\longrightarrow$ Q separable in tender variables Tx

Special case: Simple integer recourse ($W = I$)

Generic one-dimensional expected value function

- $Q(z) = \mathbb{E}_\omega \left[\lceil \omega - z \rceil^+ \right],\ z \in \mathbb{R}$
- Q is generally non-convex



Examples: ω discrete (left) and exponentially (right) distributed

SIR function Q is non-convex in general. However:

Theorem [Klein Haneveld, Stougie, VdV '06]
SIR function Q is convex if and only if $\omega \sim$ pdf f with

$$f(s) = G(s+1) - G(s), \quad s \in \mathbb{R}$$

where G is an arbitrary cdf with finite mean

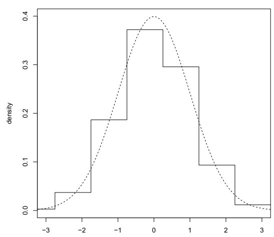
$\longrightarrow$ Idea for MRD:

- Approximate original pdf f_ω with a pdf $\hat{f}$ that is generated by some cdf G

α-approximations

Fix $\alpha \in [0,1)$
Let G be cdf of a discrete r.v. with support in $\alpha + \mathbb{Z}$:

1. $f_\alpha(s) = G(s+1) - G(s)$ is constant on $C_\alpha^l := (\alpha + l - 1, \alpha + l]$, $l \in \mathbb{Z}$
2. For $\omega_\alpha \sim f_\alpha$, $\mathbb{P}\{\omega_\alpha \in C_\alpha^l\} = \mathbb{P}\{\omega \in C_\alpha^l\}$



For later use: analogous for dimension $m \geq 2$

α-approximations (2)

$\longrightarrow$ α-approximation Q_α:

$$Q_\alpha(z) := \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - z \rceil^+\right], \quad z \in \mathbb{R}$$

with $\omega_\alpha \sim f_\alpha$ a continuous random variable.

It turns out that

$$Q_\alpha(z) := \mathbb{E}_{\omega_\alpha}\left[\lceil \omega_\alpha - z \rceil^+\right] = \int_z^\infty (1 - G(x))\, dx$$

$$\Rightarrow Q_\alpha(z) = \mathbb{E}_\xi\left[(\xi - z)^+\right], \quad z \in \mathbb{R}$$

That is: Q_α is a continuous expected surplus function (SR) with discrete random variable ξ having cdf G.

Theorem [Klein Haneveld, Stougie, VdV '93] $\longrightarrow$
Every convex SIR function Q with continuous ω, can be represented as an continuous SR function with discrete ξ

α-approximations (3): MRD for SIR

- Drop integrality in second stage
- Replace $\omega \sim f$ by α-approximation $\xi = \lceil \omega - \alpha \rceil + \alpha$

$\Rightarrow$

1. continuous SR models with discrete $\lceil \omega - \alpha \rceil + \alpha$ can be solved efficiently
2. For SIR models a uniform error bound is available for α-approximations

Theorem [Klein Haneveld, Stougie, VdV 2006]
For SIR function Q and for all $\alpha \in [0,1)$,

$$\sup_{z \in \mathbb{R}} |Q(z) - Q_\alpha(z)| \leq \min\left\{\frac{|\Delta| f_\omega}{4}, 1\right\}$$

with $|\Delta| f_\omega :=$ total variation of pdf f_ω

Intuition: for unimodal pdf, $|\Delta| f_\omega$ decreases as σ_ω^2 increases

MRD / α-approximation for more general integer recourse

Expected value function $Q(z)$

- $Q(z) = \mathbb{E}_\omega \left[\min_y \{ qy : Wy \geq \omega - z,\ y \in \mathbb{Z}_+^{n_2} \} \right]$

MRD: (i) drop integrality, (ii) substitute α-approximation of ω

For $\alpha \in \mathbb{R}^m$, α-approximation $Q_\alpha(z)$

- $Q_\alpha(z) = \mathbb{E}_\omega \left[\min_y \{ qy : Wy \geq \lceil \omega - \alpha \rceil + \alpha - z,\ y \in \mathbb{R}_+^{n_2} \} \right]$
- Q_α is a continuous recourse function, convex polyhedral

Consider W complete, Totally Unimodular (TU)

- second-stage problem: IP(s) = LP(s) provided $s \in \mathbb{Z}^m$

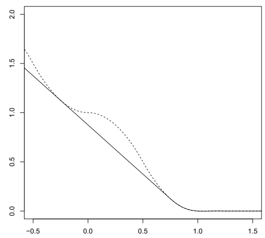
Convex hull

Claim Van der Vlerk (Math Prog, 2004)

- $Q_{\alpha^\star}$ is the convex hull of Q if W is TU
  ($\alpha^\star$ depending on f_ω only)

Counterexample where the convex hull is not polyhedral $\neq Q_{\alpha^\star}$

- $Q(z) = \mathbb{E}_\omega \left[\lceil \omega - z \rceil^+ \right]$
- ω follows a triangular distribution on $[0, 1]$ with mode $1/2$



- Conclusion: Claim needs stronger assumptions

Known results α-approximations of TU integer recourse

- Claim of VdV (2004) holds for uniform distributions [Romeijnders & VdV '12]
- $Q(z) = Q_\alpha(z)$ for $z \in \alpha + \mathbb{Z}^m$

Recall: Error bound only for SIR [KH et al. '06]

- For $\alpha \in \mathbb{R}$, $\sup_{z \in \mathbb{R}} |Q(z) - Q_\alpha(z)| \leq \min \left\{ \frac{|\Delta| f}{4}, 1 \right\}$
- $|\Delta| f$ is the total variation of pdf f

Goal

- Obtain a similar error bound for TU integer recourse

Error bound for α-approximations

New approach for SIR

- Let f_0 be a pdf with $|\Delta| f_0 = B$.
- We are interested in $\sup_{\alpha, z \in \mathbb{R}} |Q(z) - Q_\alpha(z)|$ when $\omega \sim f_0$
- Key insight: Instead, consider

$$\sup_{\alpha \in \mathbb{R}} \sup_{z \in \mathbb{R}} \sup_{f \in \mathcal{F}} \{ |Q(z) - Q_\alpha(z)| : |\Delta| f \leq B \}$$

$\mathcal{F}$ is set of 'nice' density functions f (bounded variation, ...)

Analysis

1. Round-up functions $R(z) := \mathbb{E}_\omega[\lceil \omega - z \rceil],\ z \in \mathbb{R}$
2. SIR $Q(z) = \mathbb{E}_\omega \left[\lceil \omega - z \rceil^+ \right]$
3. TU integer recourse

Property of total variation

Lemma ('flattening')

Let pdf $f \in \mathcal{F}$ be given.
Let I be a bounded interval and define $g \in \mathcal{F}$ as

$$g(x) = \begin{cases} f(x), & x \notin I \\ K_I, & x \in I \end{cases}$$

with $K_I := |I|^{-1} \int_I f(u)du$

Then $|\Delta|g \leq |\Delta|f$

Round-up functions

Expected round-up function and α-approximation

- $R(z) := \mathbb{E}_\omega[\lceil \omega - z \rceil]$
- $R_\alpha(z) := \mathbb{E}_\omega[\lceil \omega - \alpha \rceil + \alpha - z]$

Error

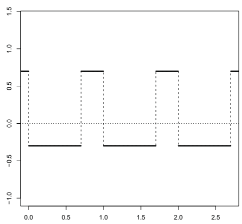
$$\begin{aligned} R(z) - R_\alpha(z) &= \mathbb{E}_\omega\Big[\lceil \omega - z \rceil\Big] - \mathbb{E}_\omega\Big[\lceil \omega - \alpha \rceil + \alpha - z\Big] \\ &= \mathbb{E}_\omega\Big[\phi_{\alpha,z}(\omega)\Big] \end{aligned}$$

with difference function

$$\phi_{\alpha,z}(x) := \Big(\lceil x - z \rceil + z\Big) - \Big(\lceil x - \alpha \rceil + \alpha\Big)$$

The difference function $\phi_{\alpha,z}$

- $\phi_{\alpha,z}(x) := \Big(\lceil x - z \rceil + z\Big) - \Big(\lceil x - \alpha \rceil + \alpha\Big)$
- Solve $\max_{\alpha,z,f}\{|\mathbb{E}_f[\phi_{\alpha,z}(\omega)]| : |\Delta|f \leq B\}$



Properties of $\phi_{\alpha,z}$

- periodic in x, α, z with period 1
- $\phi_{\alpha,z}(x) = -\phi_{z,\alpha}(x)$

Consequences

- Restrict maximization w.r.t. α and z to $[0, 1)$
- Maximize $\mathbb{E}_f[\phi_{\alpha,z}(\omega)]$ instead of $|\mathbb{E}_f[\phi_{\alpha,z}(\omega)]|$

The difference function $\phi_{\alpha,z}$

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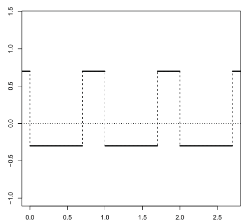


Properties of $\phi_{\alpha,z}$

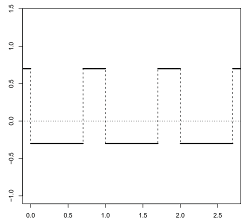
- piecewise constant in x
- jumps of size $+1$ at $z + \mathbb{Z}$
- jumps of size -1 at $\alpha + \mathbb{Z}$

Consequences

- Restrict maximization w.r.t. $f \in \mathcal{F}$ to piecewise constant densities

The difference function $\phi_{\alpha,z}$

- $\phi_{\alpha,z}(x) := \left(\lceil x - z \rceil + z\right) - \left(\lceil x - \alpha \rceil + \alpha\right)$
- Solve $\max\limits_{\alpha,z,f}\{|\mathbb{E}_f[\phi_{\alpha,z}(\omega)]| : |\Delta| f \leq B\}$



Properties of $\phi_{\alpha,z}$

- $\int_I \phi_{\alpha,z}(x)dx = 0$ for any interval I of length $|I| = 1$

Consequences

- Optimal f are (piecewise constant and) alternating (high – low)

The difference function $\phi_{\alpha,z}$

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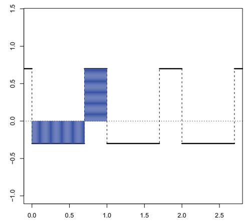
- Solve $\max\limits_{\alpha,z,f}\{|\mathbb{E}_f[\phi_{\alpha,z}(\omega)]| : |\Delta| f \leq B\}$



Properties of $\phi_{\alpha,z}$

- $\int_I \phi_{\alpha,z}(x)dx = 0$ for any interval I of length $|I| = 1$

Consequences

- Optimal f are pc and alternating

The difference function $\phi_{\alpha,z}$

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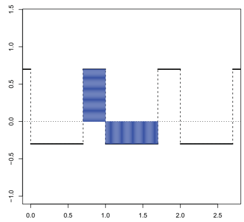

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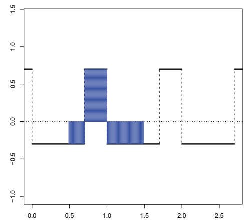


Properties of $\phi_{\alpha,z}$

- $\int_I \phi_{\alpha,z}(x)dx = 0$ for any interval I of length $|I| = 1$

Consequences

- Optimal f are pc and alternating

Round-up functions

This allows to determine, for $\alpha, z \in \mathbb{R}$,

$$\max_{f \in \mathcal{F}} \{ |\mathbb{E}_f[\phi_{\alpha,z}(\omega)]| : |\Delta| f \leq B \} = \min\{\gamma_{\alpha,z}, \gamma_{\alpha,z}(1-\gamma_{\alpha,z})\frac{B}{2}\}$$

where $\gamma_{\alpha,z} := z + 1 - (\lceil z - \alpha \rceil + \alpha) \in [0,1]$

Finally, maximization w.r.t. α and z yields
desired uniform error bound on $|R(z) - R_\alpha(z)|$

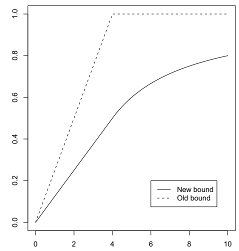
Round-up functions

Uniform error bound: for $\alpha \in \mathbb{R}$

- $\sup_{z \in \mathbb{R}} |R(z) - R_\alpha(z)| \leq \frac{|\Delta| f}{8}$
- In fact, for $|\Delta| f \geq 4$, $\sup_{z \in \mathbb{R}} |R(z) - R_\alpha(z)| \leq 1 - \frac{2}{|\Delta| f}$

Simple integer recourse

- The same error bound holds for SIR: $Q(z) = \mathbb{E}_\omega[\lceil \omega - z \rceil^+]$
- $\sup_{z \in \mathbb{R}} |Q(z) - Q_\alpha(z)| \leq \frac{|\Delta| f}{8}$
- Bound is sharp; improvement of [KH et al. '06] by factor 2



Similar approach for recourse with TU matrix W

Second-stage value function v: for $s \in \mathbb{R}^m$

$$\begin{aligned} v(s) &:= \min_y \{qy : Wy \geq s,\ y \in \mathbb{Z}_+^{n_2}\} \\ &= \min_y \{qy : Wy \geq \lceil s \rceil,\ y \in \mathbb{Z}_+^{n_2}\} \\ &= \min_y \{qy : Wy \geq \lceil s \rceil,\ y \in \mathbb{R}_+^{n_2}\} \quad \text{(because } W \text{ is TU)} \\ &= \max_\lambda \{\lambda \lceil s \rceil : \lambda W \leq q,\ \lambda \in \mathbb{R}_+^m\} \quad \text{(by strong LP duality)} \end{aligned}$$

- Dual feasible region: $\Lambda := \{\lambda \in \mathbb{R}_+^m : \lambda W \leq q\}$
- $v(s)$ is finite for all $s \in \mathbb{R}^m \Rightarrow \Lambda$ is non-empty and bounded
- Extreme points of Λ: $\lambda_k, k = 1, \ldots, K$

$$v(s) = \max_{k=1,\ldots,K} \lambda^k \lceil s \rceil$$

TU recourse matrix W

Expected recourse function Q: for $z \in \mathbb{R}^m$

$$Q(z) = \mathbb{E}_\omega[v(\omega - z)] = \mathbb{E}_\omega \left[\max_{k=1,\ldots,K} \lambda^k \lceil \omega - z \rceil \right]$$

MRD: (i) drop integrality, (ii) substitute α-approximation of ω
For $\alpha \in [0,1)^m$

$$Q_\alpha(z) = \mathbb{E}_\omega \left[\max_{k=1,\ldots,K} \lambda^k (\lceil \omega - \alpha \rceil + \alpha - z) \right]$$

$Q(z)$ similar to expected round-up functions $\lambda^k \mathbb{E}_\omega[\lceil \omega - z \rceil]$

$\longrightarrow$ 'same' analysis (...) yields error bound for $|Q(z) - Q_\alpha(z)|$

First error bound for TU integer recourse

Error bound when components of ω are independent

- $\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sum_{i=1}^{m} \lambda_i^* \frac{|\Delta| f_i}{8}$
- with $\lambda_i^* := \max_{k=1,\dots,K} \lambda_i^k$
- Special case: simple integer recourse
- Approximation is good when all total variations are small

First error bound for TU integer recourse

Notation: $x_{(i)} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$

Error bound when components of ω are dependent

- $\sup_{z \in \mathbb{R}^m} |Q(z) - Q_\alpha(z)| \leq \sum_{i=1}^{m} \lambda_i^* \mathbb{E}_{\omega_{(i)}} \left[\frac{|\Delta| f_i(\cdot|\omega_{(i)})}{8} \right]$
- with $f_i(\cdot|\omega_{(i)})$ a conditional density function
- Special case: independent random vectors

Example:

Bivariate normal distribution with correlation ρ.
If variances are sufficiently large and $|\rho| \leq .4$
then EB(dep.) $\leq$ 1.1 EB(indep.)

Future research

Construct convex hull of integer recourse models

Extending MRD / α-approximations to

- General integer recourse (non TU)
- Mixed-integer recourse
- Multi-stage recourse
- Binary recourse variables
- ...

Approximation by piecewise constant densities
also promising in other context