Stochastic Programming and Convexity

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March 1, 2012

Background

We consider

- Sets in a finite-dimensional Euclidian space $\mathbb{R}^n$.
- Functions defined on a finite-dimensional Euclidian space $\mathbb{R}^n$ with values in the generalized Euclidian space $\mathbb{R}^* = [-\infty, +\infty]$.

Having function $f : \mathbb{R}^n \to \mathbb{R}^*$ we define its graph, epigraph, hypograph, domain

$$\begin{aligned}
\text{graph}\,(f) &= \{(x, f(x)) : x \in \mathbb{R}^n\} \\
&= \{(x, \eta) : f(x) = \eta,\ x \in \mathbb{R}^n,\ \eta \in \mathbb{R}^*\}, \\
\text{epi}\,(f) &= \{(x, \eta) : f(x) \le \eta,\ x \in \mathbb{R}^n,\ \eta \in \mathbb{R}\}, \\
\text{hypo}\,(f) &= \{(x, \eta) : f(x) \ge \eta,\ x \in \mathbb{R}^n,\ \eta \in \mathbb{R}\}, \\
\text{Dom}\,(f) &= \{x : f(x) < +\infty,\ x \in \mathbb{R}^n\}.
\end{aligned}$$

Convex sets

A set $A \subset \mathbb{R}^n$ is called convex whenever for each couple of points $x, y \in A$ and $0 < \lambda < 1$ we have $\lambda x + (1 - \lambda) y \in A$.

Equivalently,
A is convex
iff
for each couple of points $x, y \in A$ we have

$$[x, y] \quad = \quad \{tx + (1 - t)y \,:\, 0 \le t \le 1\} \subset A.$$

Convex functions

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^*$ is called convex if

- Dom (f) is a convex set.
- For all $x, y \in \text{Dom}\,(f)$ and $0 < \lambda < 1$

$$f\,(\lambda x + (1-\lambda)y) \quad \leq \quad \lambda f\,(x) + (1-\lambda) f\,(y)\,.$$

Equivalently, f is convex iff epi (f) is convex.

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^*$ is called concave iff $-f$ is convex .

Properties of convex functions

- Each level set of a convex function is a convex set, i.e. for each $\alpha \in \mathbb{R}$

$$\begin{aligned} \mathrm{lev}_{[\leq \alpha]}(f) &= \{x : f(x) \leq \alpha,\ x \in \mathbb{R}^n\}, \\ \mathrm{lev}_{[< \alpha]}(f) &= \{x : f(x) < \alpha,\ x \in \mathbb{R}^n\}. \end{aligned}$$

- Each convex function f is continuous on $\mathrm{int}(\mathrm{Dom}(f))$.

Stronger cases

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called strictly convex if

- Dom (f) is a convex set.
- For all $x, y \in \text{Dom}(f)$, $x \neq y$ and $0 < \lambda < 1$

$$f(\lambda x + (1-\lambda)y) \quad < \quad \lambda f(x) + (1-\lambda) f(y) .$$

Equivalently, f is strictly convex iff epi (f) is convex and each tangent hyperplane to epi (f) which is not perpendicular to horizontal hyperplane contains at most one point of graph (f).

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called strictly concave iff $-f$ is strictly convex .

Stronger cases

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^*$ is called strongly convex if

- Dom (f) is a convex set.
- For all $x, y \in$ Dom (f), $x \neq y$ and $0 < \lambda < 1$

$$f(\lambda x + (1-\lambda)y) \quad \leq \quad \lambda f(x) + (1-\lambda)f(y) - \frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 .$$

for an appropriate constant $C > 0$.

Equivalently,
$f : \mathbb{R}^n \longrightarrow \mathbb{R}^*$ is strongly convex
iff
$g(x) = f(x) - \frac{C}{2}\|x\|^2$ is a convex function.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^*$ is called strongly concave iff $-f$ is strongly convex .

Strongly convex functions - verification

We verify that $g(x) = f(x) - \frac{C}{2}\|x\|^2$ is a convex function:

$$\begin{aligned}
&\lambda g(x) + (1-\lambda) g(y) - g(\lambda x + (1-\lambda)y) = \\
&= \lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda)y) - \\
&\quad - \frac{C}{2}\left(\lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \|\lambda x + (1-\lambda)y\|^2\right) \\
&\geq \frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 - \frac{C}{2}\left(\lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \right. \\
&\quad \left. - \lambda^2\|x\|^2 - (1-\lambda)^2\|y\|^2 - 2\lambda(1-\lambda)x^\top y\right) \\
&= \frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 - \frac{C}{2}\lambda(1-\lambda)\left(\|x\|^2 + \|y\|^2 - 2x^\top y\right) \\
&= \frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 - \frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 = 0.
\end{aligned}$$

Semiconvex functions

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called semiconvex function with linear modulus if

- $\mathrm{Dom}\,(f)$ is an open set.
- f is continuous on $\mathrm{Dom}\,(f)$.
- For all $x, h \in \mathbb{R}^n$, $[x - h, x + h] \subset \mathrm{Dom}\,(f)$

$$2f(x) \leq f(x-h) + f(x+h) + \mathrm{C}\,\|h\|^2 .$$

for an appropriate constant $\mathrm{C} \geq 0$.

Constant C is called a semiconvex constant for f in $\mathrm{Dom}\,(f)$.

This definition is frequently used as a definition of "semiconvex functions" in literature. Here we accept a more general concept. Therefore, we added the second prefix "with linear modulus". Latter, we will see why.

Semiconvex functions

Let $\mathrm{Dom}\,(f)$ be an open set. Hence,

$f : \mathbb{R}^n \to \mathbb{R}^*$ is semiconvex function with linear modulus and a semiconvex constant $C \geq 0$
iff
For all $x, y \in \mathrm{Dom}\,(f)$, $[x, y] \subset \mathrm{Dom}\,(f)$ and $0 < \lambda < 1$

$$f\left(\lambda x + (1-\lambda)y\right) \quad \leq \quad \lambda f\left(x\right) + (1-\lambda) f\left(y\right) + \frac{C}{2}\lambda(1-\lambda)\left\|x-y\right\|^2 .$$

Semiconvex functions - equivalences

Let $\mathrm{Dom}\,(f)$ be an open convex set. Hence,

$f : \mathbb{R}^n \to \mathbb{R}^*$ is semiconvex function with linear modulus and a semiconvex constant $C \geq 0$

iff

Function $x \mapsto f(x) + \frac{C}{2}\|x\|^2$ is convex.

iff

There are $u, v : \mathrm{Dom}\,(f) \to \mathbb{R}$ such that $f = u + v$, u is convex, $v \in C^2(\mathrm{Dom}\,(f))$ and $\forall\, x \in \mathrm{Dom}\,(f) : \left\|\nabla^2_{x,x} v(x)\right\|_\infty \leq C$.

Semiconvex functions - verification

We verify that $g(x) = f(x) + \frac{C}{2}\|x\|^2$ is a convex function:

$$
\begin{aligned}
&\lambda g(x) + (1-\lambda)g(y) - g(\lambda x + (1-\lambda)y) = \\
&= \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) + \\
&\quad + \frac{C}{2}\left(\lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \|\lambda x + (1-\lambda)y\|^2\right) \\
&\geq -\frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 + \frac{C}{2}\Big(\lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \\
&\quad -\lambda^2\|x\|^2 - (1-\lambda)^2\|y\|^2 - 2\lambda(1-\lambda)x^\top y\Big) \\
&= -\frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 + \frac{C}{2}\lambda(1-\lambda)\left(\|x\|^2 + \|y\|^2 - 2x^\top y\right) \\
&= -\frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 + \frac{C}{2}\lambda(1-\lambda)\|x-y\|^2 = 0.
\end{aligned}
$$

Semiconvex functions - general definition

General definition is taken from the book

Cannarsa, Piermarco; Sinestrari, Carlo:
Semiconcave Functions, HamiltonJacobi Equations, and Optimal Control.
Birkhuser, Boston, 2004.

Semiconvex functions - general definition

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called semiconvex function with modulus ω if

- $\omega : \mathbb{R}_{+,0} \to \mathbb{R}_{+,0}$ is a nondecreasing upper semicontinuous function with $\omega(0) = 0$.
- For all $x, y \in \mathrm{Dom}(f)$, $[x, y] \subset \mathrm{Dom}(f)$ and $0 < \lambda < 1$

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\le \lambda f(x) + (1-\lambda) f(y) + \\ &\quad + \lambda(1-\lambda)\|x-y\|\,\omega(\|x-y\|). \end{aligned}$$

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called semiconcave iff $-f$ is semiconvex .

Semiconvex functions - general definition

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^*$ is called locally semiconvex function
if
it is semiconvex function on every compact subset of $\mathbb{R}^n$.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^*$ is called locally semiconcave iff $-f$ is locally semiconvex .

Consequences

- $\omega(0+) = \lim_{t \to 0+} \omega(t) = 0$.
- If $\omega(t) = \frac{C}{2} t$ then f is semiconvex function with linear modulus and semiconvexity constant C.
- If $v \in C^1$ and $\mathrm{Dom}(v)$ is an open convex set then both v, $-v$ are semiconvex with modulus $\omega(t) = \max\{\|\nabla_x v(x) - \nabla_x v(y)\| : \|x - y\| \leq t\}$.
- If $\mathrm{Dom}(f)$ is an open convex set, $f = u + v$, u is convex, $v \in C^1(\mathrm{Dom}(f))$ then f is locally semiconvex.

Consequences

- For each $0 < \alpha < 1$ there exists a function $f_\alpha : [0,1] \to \mathbb{R}$ which is semiconvex with modulus $\omega(t) = Ct^\alpha$, $C > 0$ and cannot be written as $f_\alpha = u + v$, where u is convex, $v \in C^1([0,1])$.
- If f_λ, $\lambda \in \Lambda$ is a family of semiconvex functions with the same modulus ω then $\sup_{\lambda \in \Lambda} f_\lambda$ is a semiconvex function with the modulus ω.
- Each semiconvex function f is locally Lipschitz continuous in $\mathrm{int}(\mathrm{Dom}(f))$.
- Each locally semiconvex function f is locally Lipschitz continuous in $\mathrm{int}(\mathrm{Dom}(f))$.

Quasi-convex functions

Here we refer from the paper

Prékopa, András; Yoda, Kunikazu; Subasi, Munevver Mine:
Uniform Quasi-Concavity in Probabilistic Constrained Stochastic Programming.
Operations Research Letters, 39,1(2011), 188-192.

Quasi-convex functions

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called quasi-convex if all its level sets are convex sets, i.e. for each $\alpha \in \mathbb{R}$

$$\begin{aligned} \mathrm{lev}_{[\leq\alpha]}(f) &= \{x : f(x) \leq \alpha,\ x \in \mathbb{R}^n\}, \\ \mathrm{lev}_{[<\alpha]}(f) &= \{x : f(x) < \alpha,\ x \in \mathbb{R}^n\}. \end{aligned}$$

A function $f : \mathbb{R}^n \to \mathbb{R}^*$ is called quasi-concave iff $-f$ is quasi-convex .

Examples

$$
\begin{aligned}
f(x) &= \log(x) \quad \forall x > 0, \\
&= +\infty \quad \forall x \leq 0, \\
f(x) &= \arctan(x) \quad \forall x \in \mathbb{R}, \\
f(x) &= \sqrt{|x|} \quad \forall x \in \mathbb{R}.
\end{aligned}
$$

Properties

If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^*$ is quasi-convex and $\varphi : \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is non-decreasing then $\varphi \circ f$ is also quasi-convex.

Example (Kataoka 1963, de Panne & Popp 1963)

Let $b \in \mathbb{R}$ be deterministic and row $T \in \mathbb{R}^{1\times n}$ be random and normally distributed.
Then the function $h(x) = \mathrm{P}(Tx \leq b)$ is a quasi-concave function on $\left\{y \in \mathbb{R}^n : h(y) \geq \frac{1}{2}\right\}$.

Proof 1)

If $x^\top \mathrm{var}\,(T)\,x = 0$ then $Tx = \mathrm{E}\,[T]\,x$ a.s. and

$$
\begin{aligned}
h\,(x) = \mathrm{P}\,(Tx \le b) = \mathrm{P}\,(\mathrm{E}\,[T]\,x \le b) \quad &= \quad 1 \quad \text{whenever } \mathrm{E}\,[T]\,x \le b, \\
&= \quad 0 \quad \text{whenever } \mathrm{E}\,[T]\,x \not\le b.
\end{aligned}
$$

Proof 2)

If $x^{\top}\mathrm{var}\,(T)\,x \neq 0$ then

$$h\,(x) = \mathrm{P}\,(Tx \leq b) = \Phi\left(\frac{b - \mathrm{E}\,[T]\,x}{\sqrt{x^{\top}\mathrm{var}\,(T)\,x}}\right).$$

Therefore,

$$h\,(x) \geq \Delta$$

$$\Updownarrow$$

$$\frac{b - \mathrm{E}\,[T]\,x}{\sqrt{x^{\top}\mathrm{var}\,(T)\,x}} \geq \Phi^{-1}\,(\Delta)$$

$$\Updownarrow$$

$$\Phi^{-1}\,(\Delta)\,\sqrt{x^{\top}\mathrm{var}\,(T)\,x} + \mathrm{E}\,[T]\,x \leq b.$$

Proof

The left-hand side of the inequality is a convex function in x iff $\Phi^{-1}(\Delta) \geq 0$.
Consequently, if $\Delta \geq \frac{1}{2}$ then

$$\{x \in \mathbb{R}^n : h(x) \geq \Delta\} =$$
$$= \left\{x \in \mathbb{R}^n : \Phi^{-1}(\Delta) \sqrt{x^\top \mathrm{var}(T)\, x} + \mathrm{E}[T]\, x \leq b\right\}$$

is a convex set.

We have proved that h is quasi-concave on the set $\left\{x \in \mathbb{R}^n : h(x) \geq \frac{1}{2}\right\}$.

Example (Prékopa 1974)

Let $b \in \mathbb{R}^m$ be deterministic and matrix $T \in \mathbb{R}^{m \times n}$ be random.
Rows $T_{1,\bullet}, \ldots, T_{m,\bullet}$ are independent normally distributed and
$\mathrm{var}\left(T_{1,\bullet}\right) = \sigma_1^2 D, \ldots, \mathrm{var}\left(T_{m,\bullet}\right) = \sigma_m^2 D$.
Then the function $h(x) = \mathrm{P}\left(Tx \leq b\right)$ is a quasi-concave function on $\left\{y \in \mathbb{R}^n \,:\, h(y) \geq \frac{1}{2}\right\}$.

Let $b \in \mathbb{R}^m$ be deterministic and matrix $T \in \mathbb{R}^{m \times n}$ be random.
Columns $T_{\bullet,1}, \ldots, T_{\bullet,n}$ are independent normally distributed and
$\mathrm{var}\left(T_{\bullet,1}\right) = \sigma_1^2 D, \ldots, \mathrm{var}\left(T_{\bullet,n}\right) = \sigma_n^2 D$.
Then the function $h(x) = \mathrm{P}\left(Tx \leq b\right)$ is a quasi-concave function on $\left\{y \in \mathbb{R}^n \,:\, h(y) \geq \frac{1}{2}\right\}$.

Uniformly quasi-convex functions

Let $E \subset \mathbb{R}^n$ and $f_i : E \to \mathbb{R}^*$, $i = 1, 2, \ldots, m$ be given.
We say $f_i : E \to \mathbb{R}^*$, $i = 1, 2, \ldots, m$ are uniformly quasi-concave if

1. E is convex.
2. For each $i = 1, 2, \ldots, m$ the function f_i is quasi-concave on E.
3. For each $x, y \in E$ either

$$\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(x)$$

or

$$\forall i = 1, 2, \ldots, m \quad \min\{f_i(x), f_i(y)\} = f_i(y).$$

Properties

Sum of uniformly quasi-concave functions is quasi-concave.

Product of uniformly quasi-concave functions which are nonnegative is quasi-concave.

Example (Prékopa 2010)

Let for each $i = 1, 2, \ldots, m$ be given $b_i > 0$ deterministic and row $T_i \in \mathbb{R}^{1\times n}$ be random with normal probability distribution.
We set functions $h_i(x) = \mathrm{P}\left(T_i x \le b_i\right)$ for all $i = 1, 2, \ldots, m$.
We suppose to have a given set E with properties

1. E is convex.
2. $0 \in \mathrm{int}\,(E)$.
3. For each $i = 1, 2, \ldots, m$ the function h_i is quasi-concave on E.

Example (Prékopa 2010)

Then
The family of functions h_i, $i = 1, 2, \dots, m$ is uniformly quasi-concave on E.
iff
There are constants $\gamma \in \mathbb{R}$, $c_1, c_2, \dots, c_m \in \mathbb{R}_{+,0}$ and positive semidefinite matrix Γ such that

$$\begin{aligned} &\mathsf{E}\left[T_1\right] = b_1\gamma,\ \mathsf{E}\left[T_2\right] = b_2\gamma,\ \dots,\ \mathsf{E}\left[T_m\right] = b_m\gamma, \\ &\mathrm{var}\left(T_1\right) = c_1\Gamma,\ \mathrm{var}\left(T_2\right) = c_2\Gamma,\ \dots,\ \mathrm{var}\left(T_m\right) = c_m\Gamma. \end{aligned}$$

For example $E = \bigcap_{i=1}^{m} \left\{ y \in \mathbb{R}^n \ :\ h_i(y) \geq \frac{1}{2} \right\}$.

Cannarsa, Piermarco; Sinestrari, Carlo: Semiconcave Functions, HamiltonJacobi Equations, and Optimal Control. Birkhuser, Boston, 2004.

Prékopa, András; Yoda, Kunikazu; Subasi, Munevver Mine: Uniform Quasi-Concavity in Probabilistic Constrained Stochastic Programming.. Operations Research Letters **39,1**(2011), 188-192.

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