The Field of Values Bounds on Ideal GMRES

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Solving linear systems

using the generalized minimal residual (GMRES) method

- A nonsingular, $\|b\| = 1$, solve

$$Ax = b\,.$$

- GMRES $\;\rightarrow\;$ $x_0 = 0$, find $x_k \in \mathcal{K}_k(A, b)$:

$$\| \overbrace{b - Ax_k}^{r_k} \| \;=\; \min_{p \in \pi_k} \| p(A) b \| \;\leq\; \min_{p \in \pi_k} \| p(A) \|$$

π_k ... polynomials $\deg(p) \leq k$, $p(0) = 1$

bound ... ideal GMRES

Toh's example

$$\| r_k \| = \min_{p \in \pi_k} \| p(A)b \| \le \min_{p \in \pi_k} \| p(A) \|$$

$$A = \begin{bmatrix} 1 & \epsilon & & \\ & -1 & \epsilon^{-1} & \\ & & 1 & \epsilon \\ & & & -1 \end{bmatrix}, \qquad \epsilon > 0\,.$$

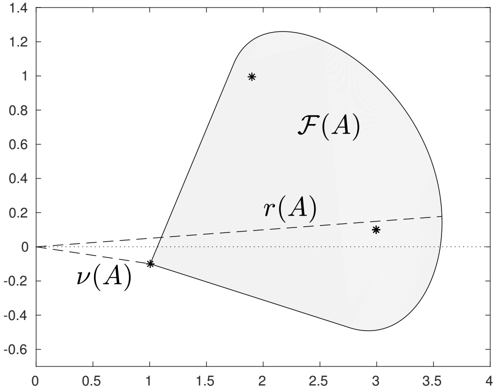
For $k = 3$ and each b,

$$0 \overset{\epsilon \to 0}{\longleftarrow} \|r_k\| < \min_{p \in \pi_k} \| p(A) \| = \frac{4}{5}\,.$$

[Toh '97]

Where in the complex plane does a matrix live?

A question of Nick Trefethen



Field of values of A

$$\mathcal{F}(A) \equiv \{\langle Av, v\rangle : v \in \mathbb{C}^n,\ \|v\| = 1\}$$

The field of values bounds

on the GMRES residual norm

- Elman's bound

$$\| r_k \| \;\leq\; \left(1 - \frac{\nu(A)^2}{\|A\|^2}\right)^{k/2}$$

[Elman 1982, Beckermann et al 2005]

- Starke's bound

$$\| r_k \| \;\leq\; \left(1 - \nu(A)\nu(A^{-1})\right)^{k/2}$$

[Starke 1997, Eiermann, Ernst 2001]

Doest it hold that

$$\| r_k \| \;\leq\; \min_{p \in \pi_k} \| p(A) \| \;\leq\; \left(1 - \nu(A)\nu(A^{-1})\right)^{k/2} \;?$$

Proof

Preliminaries

GMRES in the first iteration, and the minmax equality

- GMRES $k = 1$

$$\min_{\alpha \in \mathbb{R}} \| b - \alpha A b \|^2 \;=\; 1 - \frac{\langle Ab, b \rangle}{\langle b, b \rangle} \frac{\langle b, Ab \rangle}{\langle Ab, Ab \rangle}$$

- Minmax equality

$$\min_{\alpha \in \mathbb{R}} \max_{\substack{b \in \mathbb{R}^n \\ \|b\|=1}} \| b - \alpha A b \| \;=\; \max_{\substack{b \in \mathbb{R}^n \\ \|b\|=1}} \min_{\alpha \in \mathbb{R}} \| b - \alpha A b \|$$

[Joubert 1994, Greenbaum, Gurvits 1994]

see also [Gustafson 1968, Asplund, Pták 1971]

Proof

$$
\begin{aligned}
\min_{p\in\pi_k} \| p(A) \| &\le \min_{\alpha\in\mathbb{R}} \| (I-\alpha A)^k \| \le \min_{\alpha\in\mathbb{R}} \| I-\alpha A \|^k \\
&= \min_{\alpha\in\mathbb{R}} \max_{\substack{b\in\mathbb{R}^n \\ \|b\|=1}} \| b-\alpha Ab \|^k = \max_{\substack{b\in\mathbb{R}^n \\ \|b\|=1}} \min_{\alpha\in\mathbb{R}} \| b-\alpha Ab \|^k \\
&= \max_{\substack{b\in\mathbb{R}^n \\ \|b\|=1}} \left(1 - \frac{\langle Ab, b\rangle}{\langle b, b\rangle} \frac{\langle b, Ab\rangle}{\langle Ab, Ab\rangle}\right)^{k/2} \\
&\le \left(1 - \min_{\substack{v\in\mathbb{C}^n \\ \|v\|=1}} \left| \frac{\langle Av, v\rangle}{\langle v, v\rangle} \right| \min_{w\in\mathbb{C}^n\setminus\{0\}} \left| \frac{\langle A^{-1}w, w\rangle}{\langle w, w\rangle} \right| \right)^{k/2} \\
&= \left(1 - \nu(A)\nu(A^{-1})\right)^{k/2}
\end{aligned}
$$

Summary

$$\begin{aligned} \| r_k \| &\leq \min_{p \in \pi_k} \| p(A) \| \\ &\leq \Big(1 - \nu(A)\nu(A^{-1})\Big)^{k/2} \\ &\leq \left(1 - \frac{\nu(A)^2}{\|A\|^2}\right)^{k/2} \end{aligned}$$

Improvements of bounds?

Improvement of Starke's bound?

It can be shown that

$$1 - \frac{\nu(A)}{r(A)} \ \leq \ 1 - \nu(A)\nu(A^{-1})\,.$$

Does it hold that

$$\|\, r_k \,\| \ \leq \ \left(1 - \frac{\nu(A)}{r(A)}\right)^{k/2} ?$$

No, in general.

$$A = \left[\begin{array}{cc} \lambda & 1 \\ 0 & \lambda \end{array}\right] \quad \text{with} \quad \lambda > \frac{1}{2}.$$

$$\left(1 - \frac{\nu(A)}{r(A)}\right)^{1/2} = \frac{1}{\sqrt{\lambda + \frac{1}{2}}}\,, \qquad \min_{\mu\in\mathbb{R}} \|I - \mu A\| = \frac{1}{\lambda + \frac{1}{4\lambda}}\,.$$

$$\lambda = \frac{2}{3} \quad \to \quad \left(\frac{6}{7}\right)^{1/2} \ < \ 0.96$$

Improvement of Elman's bound?

From the previous

$$\min_{p \in \pi_k} \| p(A) \| \;\le\; \left(1 - \min_{\substack{b \in \mathbb{R}^n \\ \|b\|=1}} \frac{|\langle b, Ab \rangle|^2}{\|Ab\|^2} \right)^{k/2} .$$

- We **conjecture** that

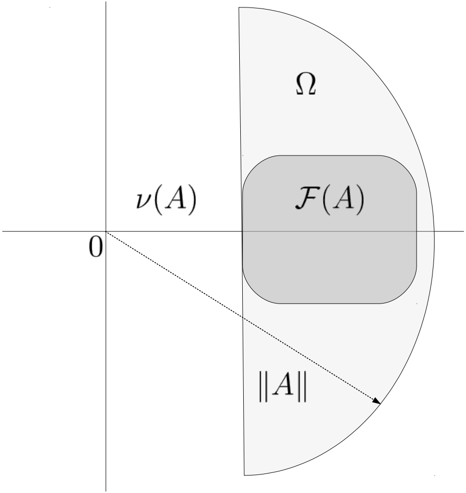
$$\min_{\|b\|=1} \cos \angle(b, Ab) \;\ge\; \frac{\nu(A)}{r(A)} .$$

- If this is true, then

$$\min_{p \in \pi_k} \| p(A) \| \;\le\; \left(1 - \frac{\nu(A)^2}{r(A)^2} \right)^{k/2} \;\le\; \left(1 - \frac{\nu(A)^2}{\|A\|^2} \right)^{k/2} .$$

Bound by Beckermann, Goreinov, Tyrtyshnikov

inclusion domain $\Omega \to$ circular segment



Bound by Beckermann, Goreinov, Tyrtyshnikov

Ideas

- **Circular segment** Ω, radius $\|A\|$, distance $\nu(A)$

$$\mathcal{F}(A) \subseteq \Omega\,.$$

- Determine a **constant** C such that

$$\min_{p\in\pi_k} \| \, p(A) \, \| \;\leq\; C \min_{p\in\pi_k} \max_{z\in\Omega} |p(z)|$$

- and **estimate**

$$\min_{p\in\pi_k} \max_{z\in\Omega} |p(z)|.$$

[Beckermann, Goreinov, Tyrtyshnikov, 2005]

Find a smaller Ω, use results of [Crouzeix, Palencia, 2017].

Improved bound

The bound by [Beckermann, Goreinov, Tyrtyshnikov, 2005]

$$\min_{p \in \pi_k} \| \, p(A) \, \| \quad \leq \quad C \, (2 + \rho_\beta) \, \rho_\beta^k \, ,$$

where $C = 2 + 2/\sqrt{3}$, and

$$\rho_\beta \, \equiv \, 2 \sin \left(\frac{\beta}{4 - 2\beta/\pi} \right) \, < \, \sin \beta, \quad \cos \beta \, = \, \frac{\nu(A)}{\|A\|}$$

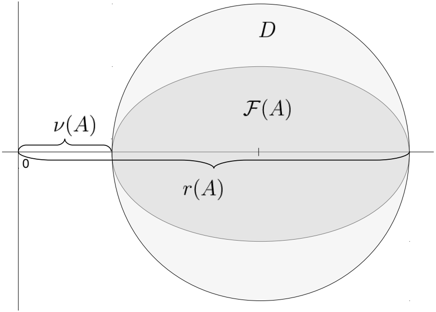
can be improved by taking $C = 1 + \sqrt{2}$ and β such that

$$\cos \beta \, = \, \frac{\nu(A)}{r(A)} \, .$$

[Liesen, Tichy, 2018]

Disk inclusion domain

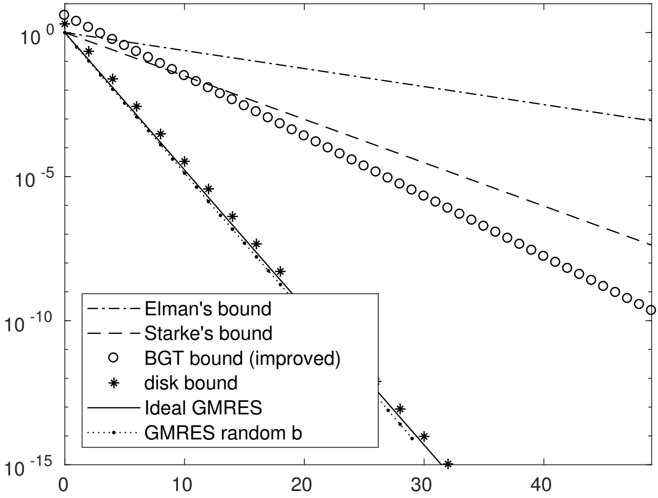
Sometimes, $\mathcal{F}(A) \subseteq D$, D is the following disk



Then

$$\min_{p \in \pi_k} \| p(A) \| \;\leq\; 2 \left(\frac{\frac{r(A)}{\nu(A)} - 1}{\frac{r(A)}{\nu(A)} + 1} \right)^k$$

is **better** than the other FOV bounds. [Liesen, Tichy, 2018]



Jordan block $J_\lambda,\ \lambda = 3,\ n = 100$

Conclusions

- Elman's and Starke's bounds hold also for **ideal GMRES**.
- **Conjecture**

$$\| r_k \| \leq \left(1 - \frac{\nu(A)^2}{r(A)^2}\right)^{k/2} .$$

- **Improvement** of the bound by Beckermann et al.
- Bound for a **disk** inclusion domain.

Related papers

- B. Beckermann, S. A. Goreinov, and E. E. Tyrtyshnikov, [Some remarks on the Elman estimate for GMRES, SIMAX 27 (2005), pp. 772–778]
- M. Crouzeix and C. Palencia, [The numerical range is a $(1+\sqrt{2})$-spectral set, SIMAX 38 (2017), pp. 649–655]
- H. C. Elman, [terative methods for large sparse nonsymmetric systems of linear equations., PhD thesis, Yale University, New Haven, 1982]
- J. Liesen and P. Tichý, [The field of values bounds on ideal GMRES, in preparation, 2018]
- G. Starke, [Field-of-values analysis of preconditioned iterative methods for nonsymmetric elliptic problems, Numer. Math. 78 (1997), pp. 103–117.]

Thank you for your attention!