

On a New Proof of the Faber-Manteuffel Theorem

Petr Tichý

joint work with

Jörg Liesen and Vance Faber

Institute of Computer Science,
Academy of Sciences of the Czech Republic

June 2, 2008, Zeuthen, Germany
Householder Symposium XVII

Outline

- 1 Introduction
- 2 Formulation of the problem
- 3 The Faber-Manteuffel theorem
- 4 The ideas of a new proof

- 1 Introduction
- 2 Formulation of the problem
- 3 The Faber-Manteuffel theorem
- 4 The ideas of a new proof

Krylov subspace methods

Basis

Methods based on projection onto the Krylov subspaces

$$\mathcal{K}_j(\mathbf{A}, v) \equiv \text{span}(v, \mathbf{A}v, \dots, \mathbf{A}^{j-1}v) \quad j = 1, 2, \dots$$

$$\mathbf{A} \in \mathbb{R}^{n \times n}, v \in \mathbb{R}^n.$$

Krylov subspace methods

Basis

Methods based on projection onto the Krylov subspaces

$$\mathcal{K}_j(\mathbf{A}, v) \equiv \text{span}(v, \mathbf{A}v, \dots, \mathbf{A}^{j-1}v) \quad j = 1, 2, \dots$$

$$\mathbf{A} \in \mathbb{R}^{n \times n}, v \in \mathbb{R}^n.$$

Each method **must generate a basis of $\mathcal{K}_j(\mathbf{A}, v)$** .

- The trivial choice $v, \mathbf{A}v, \dots, \mathbf{A}^{j-1}v$ is computationally infeasible (recall the Power Method).
- For numerical stability: Well conditioned basis.
- For computational efficiency: Short recurrence.

Krylov subspace methods

Basis

Methods based on projection onto the Krylov subspaces

$$\mathcal{K}_j(\mathbf{A}, v) \equiv \text{span}(v, \mathbf{A}v, \dots, \mathbf{A}^{j-1}v) \quad j = 1, 2, \dots$$

$$\mathbf{A} \in \mathbb{R}^{n \times n}, v \in \mathbb{R}^n.$$

Each method **must generate a basis of $\mathcal{K}_j(\mathbf{A}, v)$** .

- The trivial choice $v, \mathbf{A}v, \dots, \mathbf{A}^{j-1}v$ is computationally infeasible (recall the Power Method).
- For numerical stability: Well conditioned basis.
- For computational efficiency: Short recurrence.
- Best of both worlds:
Orthogonal basis computed by short recurrence.

Optimal Krylov subspace methods

with short recurrences

CG (1952), MINRES, SYMMLQ (1975)

- based on three-term recurrences

$$r_{j+1} = \gamma_j \mathbf{A}r_j - \alpha_j r_j - \beta_j r_{j-1},$$

Optimal Krylov subspace methods

with short recurrences

CG (1952), MINRES, SYMMLQ (1975)

- based on three-term recurrences

$$r_{j+1} = \gamma_j \mathbf{A}r_j - \alpha_j r_j - \beta_j r_{j-1},$$

- generate orthogonal (or \mathbf{A} -orthogonal) Krylov subspace basis,

Optimal Krylov subspace methods

with short recurrences

CG (1952), MINRES, SYMMLQ (1975)

- based on three-term recurrences

$$r_{j+1} = \gamma_j \mathbf{A}r_j - \alpha_j r_j - \beta_j r_{j-1},$$

- generate orthogonal (or \mathbf{A} -orthogonal) Krylov subspace basis,
- *optimal* in the sense that they minimize some error norm:

$$\|x - x_j\|_{\mathbf{A}} \text{ in CG,}$$

$$\|x - x_j\|_{\mathbf{A}^T \mathbf{A}} = \|r_j\| \text{ in MINRES,}$$

$$\|x - x_j\| \text{ in SYMMLQ -here } x_j \in x_0 + \mathbf{A}\mathcal{K}_j(\mathbf{A}, r_0).$$

Optimal Krylov subspace methods

with short recurrences

CG (1952), MINRES, SYMMLQ (1975)

- based on three-term recurrences

$$r_{j+1} = \gamma_j \mathbf{A}r_j - \alpha_j r_j - \beta_j r_{j-1},$$

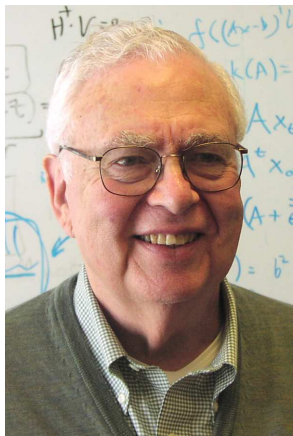
- generate orthogonal (or \mathbf{A} -orthogonal) Krylov subspace basis,
- *optimal* in the sense that they minimize some error norm:

$\|x - x_j\|_{\mathbf{A}}$ in CG,

$\|x - x_j\|_{\mathbf{A}^T \mathbf{A}} = \|r_j\|$ in MINRES,

$\|x - x_j\|$ in SYMMLQ -here $x_j \in x_0 + \mathbf{A}\mathcal{K}_j(\mathbf{A}, r_0)$.

- An important assumption on \mathbf{A} :
 \mathbf{A} is **symmetric** (MINRES, SYMMLQ) & **pos. definite** (CG).



G. H. Golub, 1932–2007

- By the end of the 1970s it was unknown if such methods existed also for general unsymmetric \mathbf{A} .
- Gatlinburg VIII (now Householder VIII) held in Oxford from July 5 to 11, 1981.
- “A prize of \$500 has been offered by Gene Golub for the construction of a 3-term conjugate gradient like descent method for non-symmetric real matrices or a proof that there can be no such method”.

What kind of method Golub had in mind

- We want to solve $\mathbf{A}x = b$ using **CG-like descent method**: error is minimized in some given inner product norm,

$$\|\cdot\|_{\mathbf{B}} = \langle \cdot, \cdot \rangle_{\mathbf{B}}^{1/2}.$$

What kind of method Golub had in mind

- We want to solve $\mathbf{A}x = b$ using **CG-like descent method**: error is minimized in some given inner product norm,
 $\|\cdot\|_{\mathbf{B}} = \langle \cdot, \cdot \rangle_{\mathbf{B}}^{1/2}$.
- Starting from x_0 , compute

$$x_{j+1} = x_j + \alpha_j p_j, \quad j = 0, 1, \dots,$$

p_j is a direction vector, α_j is a scalar (to be determined),

$$\text{span}\{p_0, \dots, p_j\} = \mathcal{K}_{j+1}(\mathbf{A}, r_0), \quad r_0 = b - \mathbf{A}x_0.$$

What kind of method Golub had in mind

- We want to solve $\mathbf{A}x = b$ using **CG-like descent method**: error is minimized in some given inner product norm,

$$\|\cdot\|_{\mathbf{B}} = \langle \cdot, \cdot \rangle_{\mathbf{B}}^{1/2}.$$

- Starting from x_0 , compute

$$x_{j+1} = x_j + \alpha_j p_j, \quad j = 0, 1, \dots,$$

p_j is a direction vector, α_j is a scalar (to be determined),

$$\text{span}\{p_0, \dots, p_j\} = \mathcal{K}_{j+1}(\mathbf{A}, r_0), \quad r_0 = b - \mathbf{A}x_0.$$

- $\|x - x_{j+1}\|_{\mathbf{B}}$ is minimal iff

$$\alpha_j = \frac{\langle x - x_j, p_j \rangle_{\mathbf{B}}}{\langle p_j, p_j \rangle_{\mathbf{B}}} \quad \text{and} \quad \langle p_j, p_i \rangle_{\mathbf{B}} = 0.$$

What kind of method Golub had in mind

- We want to solve $\mathbf{A}x = b$ using **CG-like descent method**: error is minimized in some given inner product norm,

$$\|\cdot\|_{\mathbf{B}} = \langle \cdot, \cdot \rangle_{\mathbf{B}}^{1/2}.$$

- Starting from x_0 , compute

$$x_{j+1} = x_j + \alpha_j p_j, \quad j = 0, 1, \dots,$$

p_j is a direction vector, α_j is a scalar (to be determined),

$$\text{span}\{p_0, \dots, p_j\} = \mathcal{K}_{j+1}(\mathbf{A}, r_0), \quad r_0 = b - \mathbf{A}x_0.$$

- $\|x - x_{j+1}\|_{\mathbf{B}}$ is minimal iff

$$\alpha_j = \frac{\langle x - x_j, p_j \rangle_{\mathbf{B}}}{\langle p_j, p_j \rangle_{\mathbf{B}}} \quad \text{and} \quad \langle p_j, p_i \rangle_{\mathbf{B}} = 0.$$

- p_0, \dots, p_j has to be a **B-orthogonal basis** of $\mathcal{K}_{j+1}(\mathbf{A}, r_0)$.

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A CONJUGATE GRADIENT METHOD*

VANCE FABER† AND THOMAS MANTEUFFEL†

Abstract. We characterize the class $CG(s)$ of matrices A for which the linear system $A\mathbf{x}=\mathbf{b}$ can be solved by an s -term conjugate gradient method. We show that, except for a few anomalies, the class $CG(s)$ consists of matrices A for which conjugate gradient methods are already known. These matrices are the Hermitian matrices, $A^*=A$, and the matrices of the form $A=e^{i\theta}(dI+B)$, with $B^*=-B$.

- Faber and Manteuffel gave the answer in 1984:
For a general matrix A there exists *no* short recurrence for generating orthogonal Krylov subspace bases.
- What are the details of this statement?

Outline

- 1 Introduction
- 2 Formulation of the problem**
- 3 The Faber-Manteuffel theorem
- 4 The ideas of a new proof

Formulation of the problem

\mathbf{B} -inner product, Input and Notation

Without loss of generality, $\mathbf{B} = \mathbf{I}$. Otherwise change the basis:

$$\langle x, y \rangle_{\mathbf{B}} = \langle \mathbf{B}^{1/2}x, \mathbf{B}^{1/2}y \rangle, \quad \hat{\mathbf{A}} \equiv \mathbf{B}^{1/2}\mathbf{A}\mathbf{B}^{-1/2}, \quad \hat{v} \equiv \mathbf{B}^{1/2}v.$$

Formulation of the problem

B-inner product, Input and Notation

Without loss of generality, $\mathbf{B} = \mathbf{I}$. Otherwise change the basis:

$$\langle x, y \rangle_{\mathbf{B}} = \langle \mathbf{B}^{1/2}x, \mathbf{B}^{1/2}y \rangle, \quad \hat{\mathbf{A}} \equiv \mathbf{B}^{1/2}\mathbf{A}\mathbf{B}^{-1/2}, \quad \hat{v} \equiv \mathbf{B}^{1/2}v.$$

Input data:

- $\mathbf{A} \in \mathbb{C}^{n \times n}$, a nonsingular matrix.
- $v \in \mathbb{C}^n$, an initial vector.

Formulation of the problem

B-inner product, Input and Notation

Without loss of generality, $\mathbf{B} = \mathbf{I}$. Otherwise change the basis:

$$\langle x, y \rangle_{\mathbf{B}} = \langle \mathbf{B}^{1/2}x, \mathbf{B}^{1/2}y \rangle, \quad \hat{\mathbf{A}} \equiv \mathbf{B}^{1/2}\mathbf{A}\mathbf{B}^{-1/2}, \quad \hat{v} \equiv \mathbf{B}^{1/2}v.$$

Input data:

- $\mathbf{A} \in \mathbb{C}^{n \times n}$, a nonsingular matrix.
- $v \in \mathbb{C}^n$, an initial vector.

Notation:

- $d_{\min}(\mathbf{A})$... the degree of the minimal polynomial of \mathbf{A} .
- $d = d(\mathbf{A}, v)$... the grade of v with respect to \mathbf{A} , the smallest d s.t. $\mathcal{K}_d(\mathbf{A}, v)$ is invariant under mult. with \mathbf{A} .

Formulation of the problem

Our Goal

- Generate a basis v_1, \dots, v_d of $\mathcal{K}_d(\mathbf{A}, v)$ s.t.
 1. $\text{span}\{v_1, \dots, v_j\} = \mathcal{K}_j(A, v)$, for $j = 1, \dots, d$,
 2. $\langle v_i, v_j \rangle = 0$, for $i \neq j$, $i, j = 1, \dots, d$.

Formulation of the problem

Our Goal

- Generate a basis v_1, \dots, v_d of $\mathcal{K}_d(\mathbf{A}, v)$ s.t.
 1. $\text{span}\{v_1, \dots, v_j\} = \mathcal{K}_j(A, v)$, for $j = 1, \dots, d$,
 2. $\langle v_i, v_j \rangle = 0$, for $i \neq j$, $i, j = 1, \dots, d$.

Arnoldi's method:

Standard way for generating the orthogonal basis
(no normalization for convenience): $v_1 \equiv v$,

$$v_{j+1} = \mathbf{A}v_j - \sum_{i=1}^j h_{i,j} v_i, \quad h_{i,j} = \frac{\langle \mathbf{A}v_j, v_i \rangle}{\langle v_i, v_i \rangle},$$

$$j = 0, \dots, d-1.$$

Formulation of the problem

Arnoldi's method - matrix formulation

In matrix notation:

$$v_1 = v,$$
$$\mathbf{A} \underbrace{[v_1, \dots, v_{d-1}]}_{\equiv \mathbf{V}_{d-1}} = \underbrace{[v_1, \dots, v_d]}_{\equiv \mathbf{V}_d} \underbrace{\begin{bmatrix} h_{1,1} & \cdots & h_{1,d-1} \\ 1 & \ddots & \vdots \\ & \ddots & h_{d-1,d-1} \\ & & & 1 \end{bmatrix}}_{\equiv \mathbf{H}_{d,d-1}},$$

$\mathbf{V}_d^* \mathbf{V}_d$ is diagonal, $d = \dim \mathcal{K}_n(\mathbf{A}, v)$.

Formulation of the problem

Basic question

What are **sufficient and necessary conditions** for \mathbf{A} to admit an optimal $(s + 2)$ -term recurrence?

Formulation of the problem

Basic question

What are **sufficient and necessary conditions** for \mathbf{A} to admit an optimal $(s + 2)$ -term recurrence?

In other words, **how can we characterize matrices \mathbf{A}** such that for any v , Arnoldi's method applied to \mathbf{A} and v generates an orthogonal basis via a short recurrence of length $s + 2$.

Formulation of the problem

Basic question

What are **sufficient and necessary conditions** for \mathbf{A} to admit an optimal $(s + 2)$ -term recurrence?

In other words, **how can we characterize matrices \mathbf{A}** such that for any v , Arnoldi's method applied to \mathbf{A} and v generates an orthogonal basis via a short recurrence of length $s + 2$.

Example of sufficiency: If $\mathbf{A}^* = \mathbf{A}$, then $s = 1$ and \mathbf{A} admits an optimal 3-term recurrence.

Formulation of the problem

Basic question

What are **sufficient and necessary conditions** for \mathbf{A} to admit an optimal $(s + 2)$ -term recurrence?

In other words, **how can we characterize matrices \mathbf{A}** such that for any v , Arnoldi's method applied to \mathbf{A} and v generates an orthogonal basis via a short recurrence of length $s + 2$.

Example of sufficiency: If $\mathbf{A}^* = \mathbf{A}$, then $s = 1$ and \mathbf{A} admits an optimal 3-term recurrence.

Definition. If

$$\mathbf{A}^* = p_s(\mathbf{A}),$$

where p_s is a polynomial of the smallest possible degree s , \mathbf{A} is called **normal(s)**.

Outline

- 1 Introduction
- 2 Formulation of the problem
- 3 The Faber-Manteuffel theorem**
- 4 The ideas of a new proof

The Faber-Manteuffel theorem

Theorem. [Faber and Manteuffel, 1984], [Liesen and Strakoš, 2008]

Let \mathbf{A} be a nonsingular matrix with minimal polynomial degree $d_{\min}(\mathbf{A})$. Let s be a nonnegative integer, $s + 2 < d_{\min}(\mathbf{A})$:

\mathbf{A} admits an optimal $(s + 2)$ -term recurrence

if and only if

\mathbf{A} is normal(s).

The Faber-Manteuffel theorem

Theorem. [Faber and Manteuffel, 1984], [Liesen and Strakoš, 2008]

Let \mathbf{A} be a nonsingular matrix with minimal polynomial degree $d_{\min}(\mathbf{A})$. Let s be a nonnegative integer, $s + 2 < d_{\min}(\mathbf{A})$:

\mathbf{A} admits an optimal $(s + 2)$ -term recurrence

if and only if

\mathbf{A} is normal(s).

- Sufficiency is rather straightforward, necessity *is not*. Key words from the proof of necessity in (Faber and Manteuffel, 1984) include: “continuous function” (analysis), “closed set of smaller dimension” (topology), “wedge product” (multilinear algebra).

The Faber-Manteuffel theorem

Why is necessity so hard?

Optimal $(s + 2)$ -term recurrence:

$$\mathbf{A} \mathbf{V}_{d-1} = \mathbf{V}_d$$

Prove something about the linear operator \mathbf{A} , without complete knowledge of the structure of its matrix representation.

- 1 Introduction
- 2 Formulation of the problem
- 3 The Faber-Manteuffel theorem
- 4 The ideas of a new proof**

- Motivated by the paper [J. Liesen and Z. Strakoš, 2008] which contains a completely reworked theory of short recurrences for generating orthogonal Krylov subspace bases.

“It is unknown if a simpler proof of the necessity part can be found. In view of the fundamental nature of the Faber-Manteuffel Theorem, such proof would be a welcome addition to the existing literature. It would lead to a better understanding of the theorem by enlightening some (possibly unexpected) relationships, and it would also be more suitable for classroom teaching.”

- Motivated by the paper [J. Liesen and Z. Strakoš, 2008] which contains a completely reworked theory of short recurrences for generating orthogonal Krylov subspace bases.

“It is unknown if a simpler proof of the necessity part can be found. In view of the fundamental nature of the Faber-Manteuffel Theorem, such proof would be a welcome addition to the existing literature. It would lead to a better understanding of the theorem by enlightening some (possibly unexpected) relationships, and it would also be more suitable for classroom teaching.”

- We give two new proofs of the Faber-Manteuffel theorem that use more elementary tools,
- first proof - improved version of the Faber-Manteuffel proof,
- second proof - completely new proof based on orthogonal transformations of upper Hessenberg matrices.

Idea of the second proof (1)

V. Faber, J. Liesen and P. Tichý, 2008

(for simplicity, we omit indices by \mathbf{V}_d and $\mathbf{H}_{d,d}$)

Let \mathbf{A} admit an optimal $(s + 2)$ -term recurrence

$$\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{H}, \quad \mathbf{V}^*\mathbf{V} = \mathbf{I}.$$

Up to the last column, \mathbf{H} is $(s + 2)$ -band Hessenberg.

Idea of the second proof (1)

V. Faber, J. Liesen and P. Tichý, 2008

(for simplicity, we omit indices by \mathbf{V}_d and $\mathbf{H}_{d,d}$)

Let \mathbf{A} admit an optimal $(s + 2)$ -term recurrence

$$\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{H}, \quad \mathbf{V}^*\mathbf{V} = \mathbf{I}.$$

Up to the last column, \mathbf{H} is $(s + 2)$ -band Hessenberg.

Let \mathbf{G} be a $d \times d$ unitary matrix, $\mathbf{G}^*\mathbf{G} = \mathbf{I}$. Then

$$\mathbf{A} \underbrace{(\mathbf{V}\mathbf{G})}_{\mathbf{W}} = \underbrace{(\mathbf{V}\mathbf{G})}_{\mathbf{W}} \underbrace{(\mathbf{G}^*\mathbf{H}\mathbf{G})}_{\tilde{\mathbf{H}}}.$$

\mathbf{W} is unitary.

Idea of the second proof (1)

V. Faber, J. Liesen and P. Tichý, 2008

(for simplicity, we omit indices by \mathbf{V}_d and $\mathbf{H}_{d,d}$)

Let \mathbf{A} admit an optimal $(s + 2)$ -term recurrence

$$\mathbf{A} \mathbf{V} = \mathbf{V} \mathbf{H}, \quad \mathbf{V}^* \mathbf{V} = \mathbf{I}.$$

Up to the last column, \mathbf{H} is $(s + 2)$ -band Hessenberg.

Let \mathbf{G} be a $d \times d$ unitary matrix, $\mathbf{G}^* \mathbf{G} = \mathbf{I}$. Then

$$\mathbf{A} \underbrace{(\mathbf{V}\mathbf{G})}_{\mathbf{W}} = \underbrace{(\mathbf{V}\mathbf{G})}_{\mathbf{W}} \underbrace{(\mathbf{G}^* \mathbf{H} \mathbf{G})}_{\tilde{\mathbf{H}}}.$$

\mathbf{W} is unitary. If \mathbf{G} is chosen such that $\tilde{\mathbf{H}}$ is again unreduced upper Hessenberg matrix, then

$$\mathbf{A} \mathbf{W} = \mathbf{W} \tilde{\mathbf{H}}.$$

represents the result of Arnoldi's method applied to \mathbf{A} and w_1 .

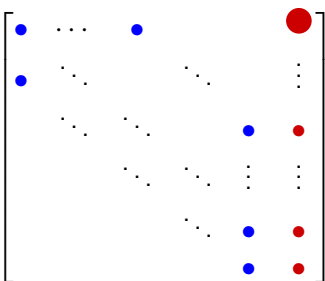
Up to the last column, $\tilde{\mathbf{H}}$ has to be $(s + 2)$ -band Hessenberg.

Idea of the second proof (2)

V. Faber, J. Liesen and P. Tichý, 2008

Proof by contradiction. Let \mathbf{A} admit an optimal $(s + 2)$ -term recurrence and \mathbf{A} not be normal(s).

Then there exists a starting vector v such that $h_{1,d} \neq 0$.

$$\mathbf{A} \mathbf{V} = \mathbf{V}$$


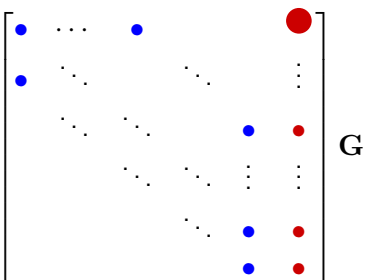
The diagram illustrates the matrix equation $\mathbf{A} \mathbf{V} = \mathbf{V}$. The matrix \mathbf{A} is represented by a large square bracket containing a grid of dots. The top row has a blue dot, followed by three dots, then a blue dot, and a large red dot in the top right corner. The second row has a blue dot, followed by two dots, then a diagonal of three dots, and a vertical ellipsis. The third row has two dots, then a diagonal of two dots, then a blue dot, and a red dot. The fourth row has a diagonal of two dots, then a vertical ellipsis, and a red dot. The fifth row has a diagonal of one dot, then a blue dot, and a red dot. The sixth row has a blue dot and a red dot. The vector \mathbf{V} is represented by a large square bracket containing a column of dots. The top dot is large and red. The bottom three dots are blue, and the other dots are small and black.

Idea of the second proof (2)

V. Faber, J. Liesen and P. Tichý, 2008

Proof by contradiction. Let \mathbf{A} admit an optimal $(s + 2)$ -term recurrence and \mathbf{A} not be normal(s).

Then there exists a starting vector v such that $h_{1,d} \neq 0$.

$$\mathbf{A}(\mathbf{V}\mathbf{G}) = (\mathbf{V}\mathbf{G})\mathbf{G}^*$$


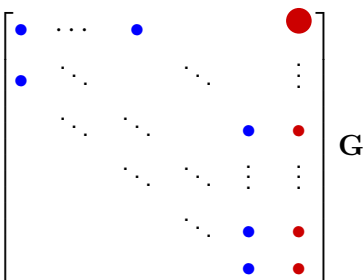
The diagram shows a matrix structure enclosed in large square brackets. The matrix is partitioned into several blocks. The top-left block contains a blue dot, followed by horizontal ellipses, and then another blue dot. The top-right block contains a large red dot. The middle-left block contains a blue dot. The middle-right block contains a vertical ellipsis, followed by a blue dot, and then a red dot. The bottom-left block contains a vertical ellipsis, followed by a blue dot, and then a red dot. The bottom-right block contains a vertical ellipsis, followed by a blue dot, and then a red dot. The entire matrix structure is labeled with \mathbf{G} to the right of the closing bracket.

Idea of the second proof (2)

V. Faber, J. Liesen and P. Tichý, 2008

Proof by contradiction. Let \mathbf{A} admit an optimal $(s + 2)$ -term recurrence and \mathbf{A} not be normal(s).

Then there exists a starting vector v such that $h_{1,d} \neq 0$.

$$\mathbf{A}(\mathbf{V}\mathbf{G}) = (\mathbf{V}\mathbf{G})\mathbf{G}^*$$


Find unitary \mathbf{G} (a product of Givens rotations) such that $\tilde{\mathbf{H}}$ is unreduced upper Hessenberg, but $\tilde{\mathbf{H}}$ is not $(s + 2)$ -band (up to the last column) - **contradiction**.

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$\begin{bmatrix} \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & \bullet \\ & \bullet & \bullet & \bullet & \bullet & 0 & 0 & \bullet \\ & & \bullet & \bullet & \bullet & 0 & & \bullet \\ & & & \bullet & \bullet & \bullet & & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix} G_{7,8}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$G_{7,8}^* \begin{bmatrix} \bullet & \bullet & \bullet & 0 & 0 & 0 & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & 0 & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$G_{6,7}^* \begin{bmatrix} \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \\ & & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$\begin{bmatrix} \bullet & \bullet & \bullet & 0 & 0 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \\ & & & & & & & \bullet \\ & & & & & & & \bullet \end{bmatrix} G_{5,6}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$G_{5,6}^* \begin{bmatrix} \bullet & \bullet & \bullet & 0 & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$\begin{bmatrix} \bullet & \bullet & \bullet & 0 & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix} G_{4,5}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$G_{4,5}^* \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet & \bullet \\ & & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix} G_{3,4}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$G_{3,4}^* \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

The diagram shows an 8x8 matrix enclosed in large square brackets. The matrix contains dots representing non-zero entries. The first two columns each have two blue dots in the first and second rows. The remaining six columns (columns 3 through 8) each have a diagonal sequence of green dots. Specifically, column 3 has a red dot in the third row and green dots in rows 4, 5, 6, 7, and 8. Column 4 has green dots in rows 4, 5, 6, 7, and 8. Column 5 has green dots in rows 5, 6, 7, and 8. Column 6 has green dots in rows 6, 7, and 8. Column 7 has green dots in rows 7 and 8. Column 8 has green dots in rows 8 and 9. To the right of the matrix is the label $G_{2,3}$.

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

$$G_{2,3}^* \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

The diagram shows an 8x8 matrix $G_{1,2}$ enclosed in large square brackets. The matrix is represented by a grid of dots. The first column contains a blue dot at the top, a green dot in the second row, and a red dot in the third row. All other dots in the matrix are green. The dots are arranged in a pattern that suggests a Givens rotation in the first two rows, with the rest of the matrix being identity. Specifically, the first two rows have dots in all eight columns, while the remaining rows have dots only in the columns corresponding to their row index.

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix} G_{1,2}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $G_{7,8}$.

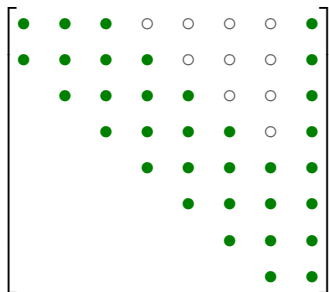
$$G_{1,2}^* \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & \bullet & \bullet & \bullet \\ & & & & & \bullet & \bullet & \bullet \\ & & & & & & \bullet & \bullet \end{bmatrix}$$

Idea of the second proof (3)

V. Faber, J. Liesen and P. Tichý, 2008

Let v be a starting vector such that $h_{1,8} \neq 0$.

Choose Givens rotation $\mathbf{G}_{7,8}$.



$$\mathbf{G} \equiv \mathbf{G}_{7,8} \mathbf{G}_{6,7} \dots \mathbf{G}_{1,2}, \quad \tilde{\mathbf{H}} \equiv \mathbf{G}^* \mathbf{H} \mathbf{G}.$$

We proved: It is possible to choose $\mathbf{G}_{7,8}$ such that

$$h_{1,8} \neq 0 \implies \tilde{h}_{1,7} \neq 0 \text{ or } \tilde{h}_{2,7} \neq 0.$$

Summary

Generating of orthogonal basis of $\mathcal{K}_d(\mathbf{A}, v)$ via short recurrences

Arnoldi-type recurrence
($s + 2$)-term



\mathbf{A} is normal(s)
 $\mathbf{A}^* = p(\mathbf{A})$

- When is \mathbf{A} normal(s)?

Summary

Generating of orthogonal basis of $\mathcal{K}_d(\mathbf{A}, v)$ via short recurrences

Arnoldi-type recurrence
($s + 2$)-term



\mathbf{A} is normal(s)
 $\mathbf{A}^* = p(\mathbf{A})$

- When is \mathbf{A} normal(s)?
- \mathbf{A} is normal and
 - [Faber and Manteuffel, 1984],
[Khavinson and Świątek, 2003]
[Liesen and Strakoš, 2008]
 - 1. $s = 1$ if and only if the eigenvalues of \mathbf{A} lie on a line in \mathbb{C} .
 - 2. If the eigenvalues of \mathbf{A} are *not* on a line, then $d_{\min}(\mathbf{A}) \leq 3s - 2$.

Summary

Generating of orthogonal basis of $\mathcal{K}_d(\mathbf{A}, v)$ via short recurrences

Arnoldi-type recurrence
($s + 2$)-term



\mathbf{A} is normal(s)
 $\mathbf{A}^* = p(\mathbf{A})$



the only interesting case
is $s = 1$,
collinear eigenvalues

- When is \mathbf{A} normal(s)?
- \mathbf{A} is normal and
 - [Faber and Manteuffel, 1984],
[Khavinson and Świątek, 2003]
[Liesen and Strakoš, 2008]
 - 1. $s = 1$ if and only if the eigenvalues of \mathbf{A} lie on a line in \mathbb{C} .
 - 2. If the eigenvalues of \mathbf{A} are *not* on a line, then $d_{\min}(\mathbf{A}) \leq 3s - 2$.

Summary

Generating of orthogonal basis of $\mathcal{K}_d(\mathbf{A}, v)$ via short recurrences

Arnoldi-type recurrence
($s + 2$)-term



\mathbf{A} is normal(s)
 $\mathbf{A}^* = p(\mathbf{A})$



the only interesting case
is $s = 1$,
collinear eigenvalues

- When is \mathbf{A} normal(s)?
- \mathbf{A} is normal and
 - [Faber and Manteuffel, 1984],
[Khavinson and Świątek, 2003]
[Liesen and Strakoš, 2008]
 - 1. $s = 1$ if and only if the eigenvalues of \mathbf{A} lie on a line in \mathbb{C} .
 - 2. If the eigenvalues of \mathbf{A} are *not* on a line, then $d_{\min}(\mathbf{A}) \leq 3s - 2$.
- All classes of “interesting” matrices are known.

Related papers

- J. Liesen and Z. Strakoš, [On optimal short recurrences for generating orthogonal Krylov subspace bases, to appear in SIAM Review, 2008].
Completely reworked theory of short recurrences for generating orthogonal Krylov subspace bases
- V. Faber, J. Liesen and P. Tichý, [The Faber-Manteuffel Theorem for Linear Operators, SIAM J. Numer. Anal., 2008, 46, 1323-1337].
New proofs of the fundamental theorem of Faber and Manteuffel

More details can be found at

<http://www.cs.cas.cz/tichy>
<http://www.math.tu-berlin.de/~liesen>
<http://www.cs.cas.cz/strakos>

Related papers

- J. Liesen and Z. Strakoš, [On optimal short recurrences for generating orthogonal Krylov subspace bases, to appear in SIAM Review, 2008].
Completely reworked theory of short recurrences for generating orthogonal Krylov subspace bases
- V. Faber, J. Liesen and P. Tichý, [The Faber-Manteuffel Theorem for Linear Operators, SIAM J. Numer. Anal., 2008, 46, 1323-1337].
New proofs of the fundamental theorem of Faber and Manteuffel

More details can be found at

<http://www.cs.cas.cz/tichy>
<http://www.math.tu-berlin.de/~liesen>
<http://www.cs.cas.cz/strakos>

Thank you for your attention!