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A tensorial Hencky measure of strain and strain rate for finite deformations

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A proper tensorial logarithmic strain measure and its associated time rate of change is developed using an equivalence between the expressions for an isotropic tensor function and the Sylvester-Lagrange formalism. It is shown that the specific Cauchy stress T/ρ is derivable from a potential with respect to this strain measure. A modified stress T_F is shown to be derivable from a potential with respect to the left polar decomposition of the deformation gradient. Advantages and disadvantages of this formulation are discussed.

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I. INTRODUCTION

In one-dimensional usage, the Hencky strain measure,[1] taken as the logarithm of the extension ratio, has certain well-documented advantages such as additivity of progressive elongation increments. In three dimensions, however, there has heretofore been no readily usable tensorial form available except for the work of Hill.[7]

Dorn and Latter[2] state that the Hencky strain is measured by the logarithms of the principal values and off-diagonal terms which do not form components of any differentiable invariant and, hence, are not admissible for isotropic media as intended.

Truesdell and Toupin[3] discuss the history of the various logarithmic measures and comment that, in effect, using an analytical continuation of the series for logarithms results in off-diagonal terms involving infinite series. Both of the above objections are eliminated in the present development.

II. LOGARITHMIC STRAIN

Where λ represents a one-dimensional stretch ratio, i.e., ratio of deformed to undeformed length, an additively symmetrical measure of strain $f(\lambda)$ must satisfy[3]

$$f(1/\lambda) = -f(\lambda), \tag{1}$$

and one of the infinitely many smooth functions that conform to Eq. (1) is

$$f(\lambda) = \ln\lambda, \tag{2}$$

where ln represents the natural logarithm. The problem at hand is to generalize Eq. (2) to a proper tensorial representation for finite-deformation theory.

With dX and $d\chi$ representing elements of length in the reference configuration and the deformed configuration, respectively, we define the usual deformation gradient two-point tensor F through

$$d\chi_i = \frac{\partial \chi_i}{\partial X_j} dX_j, \quad F_{ij} = \frac{\partial \chi_i}{\partial X_j}, \tag{3}$$

where, without loss of generality, Cartesian coordinates are used.

The polar decomposition of F yields V, the left Cauchy-Green stretch tensor, and R, the proper orthogonal rotation tensor, related through

$$F = VR. \tag{4}$$

The principal values of V are herein defined as the positive square roots of the product FF^T, where the superscript T denotes transpose.

Based upon well-known theorems in matrix analysis,[4] an analytic function of a matrix may be expressed as a second-degree polynomial in the matrix with scalar coefficients which are functions of the invariants of the matrix; since $\ln V$ is analytic except for a singularity at $V = 0$, we can express it as

$$\ln V = \phi_0 1 + \phi_1 \mathrm{V} + \phi_2 \mathrm{V}^2, \tag{5}$$

with the restriction $V > 0$, which means

$$V - 0 \subset \mathscr{S}^{+}ym. \tag{6}$$

Consider V and hence $\ln V$, which is an isotropic tensor function of V) as diagonalized, permitting the interchangeability of the matrix V with its principal values λ_1, λ_2, and λ_3, which are the principal stretches of the deformation, then Eq. (5) yields

$$\begin{aligned} \ln\lambda_1 &= \phi_0 + \phi_1\lambda_1 + \phi_2\lambda_1^2, \\ \ln\lambda_2 &= \phi_0 + \phi_1\lambda_2 + \phi_2\lambda_2^2, \\ \ln\lambda_3 &= \phi_0 + \phi_1\lambda_3 + \phi_3\lambda_3^2, \\ \phi_i &= \phi_i(\lambda_1, \lambda_2, \lambda_3), \quad i = 1,2,3. \end{aligned} \tag{7}$$

By successive substitution in Eq. (7), the solutions for the scalars ϕ_i are obtained as

$$\begin{aligned} \phi_0 &= \frac{\lambda_2\lambda_3 \ln\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 \ln\lambda_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_1\lambda_2 \ln\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, \\ \phi_1 &= \frac{(\lambda_2+\lambda_3) \ln\lambda_1^{-1}}{(\lambda_2-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(\lambda_1+\lambda_3) \ln\lambda_2^{-1}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{(\lambda_1+\lambda_2) \ln\lambda_3^{-1}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, \\ \phi_2 &= \frac{\ln\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\ln\lambda_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\ln\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \end{aligned} \tag{8}$$

Thus, from Eqs. (5) and (8), the symmetric matrix $\ln V$ is given explicitly by Eq. (9), where the off-diagonal terms contain

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but two entries,

$$\ln V = \begin{pmatrix} \phi_0 + \phi_1 V_{11} + \phi_2 (V^2)_{11} & * & * \\ \phi_1 V_{12} + \phi_2 (V^2)_{12} & \phi_0 + \phi_1 V_{22} + \phi_2 (V^2)_{22} & * \\ \phi_1 V_{13} + \phi_2 (V^2)_{13} & \phi_1 V_{23} + \phi_2 (V^2)_{23} & \phi_0 + \phi_1 V_{33} + \phi_2 (V^2)_{33} \end{pmatrix}. \tag{9}$$

As an alternate approach, one could use the Sylvester-Lagrange interpolation which for $\ln V$ takes the form

$$\ln V = \frac{(V-\lambda_2 1)(V-\lambda_3 1)\ln\lambda_1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(V-\lambda_1 1)(V-\lambda_3 1)\ln\lambda_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{(V-\lambda_1 1)(V-\lambda_2 1)\ln\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, \tag{10}$$

and upon carrying out the indicated multiplication, it is identical to Eq. (5) with coefficients (8).[5]

III. LINEAR STRESS-STRAIN RELATION

The first invariant I_1 of $\ln V$ has the interesting property[3] that

$$\begin{aligned} I_1 &= \mathrm{tr}(\ln V) = \ln\lambda_1 + \ln\lambda_2 + \ln\lambda_3 \\ &= \ln\lambda_1\lambda_2\lambda_3 = \ln(\det V), \end{aligned} \tag{11}$$

where tr and det denote the trace and determinant, respectively. For an example of the above, consider the Cauchy stress T as a linear function of $\ln V$, then

$$T = \alpha_0\,\mathrm{tr}(\ln V)1 + \alpha_1 \ln V. \tag{12}$$

The first term on the right-hand side is a logarithmic measure of volume change by Eq. (11) since $\det V$ represents the ratio of deformed volume to undeformed. Further, Eq. (12) satisfies the rest condition that $T = 0$ for $V = 1$.

IV. HYPERELASTICITY

Assume the existence of a scalar potential function of three independent invariants of $\ln V$. Two different sets will be examined.

$$\begin{aligned} &I_1 = \mathrm{tr}(\ln V), \quad \bar{I}_1 = \mathrm{tr}(\ln V), \\ &I_2 = \tfrac{1}{2}[\mathrm{tr}^2(\ln V) - \mathrm{tr}(\ln V)^2], \quad \bar{I}_2 = \mathrm{tr}(\ln V)^2, \\ &I_3 = \det(\ln V), \quad \bar{I}_3 = \mathrm{tr}(\ln V)^3. \end{aligned} \tag{13}$$

For some stress S to be derivable from a potential function ψ with respect to the strain measure $\ln V$, where

$$\psi = \psi(I_1, I_2, I_3)$$

or

$$\psi = U(\bar{I}_1, \bar{I}_2, \bar{I}_3), \tag{14}$$

we employ the chain rule

$$\begin{aligned} S = \frac{\partial\psi}{\partial(\ln V)} &= \frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial(\ln V)} \\ &+ \frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial(\ln V)} + \frac{\partial\psi}{\partial I_3}\frac{\partial I_3}{\partial \ln V}. \end{aligned} \tag{15}$$

Evaluating the terms $_0I_i/\partial \ln V$ yields

$$\begin{aligned} &\frac{\partial I_1}{\partial \ln V} = 1, \\ &\frac{\partial I_2}{\partial \ln V} = \mathrm{tr}(\ln V)1 - (\ln V)^T = I_1 1 - \ln V, \\ &\frac{\partial I_3}{\partial \ln V} = I_3(\ln V)^{-1}, \end{aligned} \tag{16}$$

[Note: $I_3(\ln V)^{-1} \to 0$ as $V \to 1$]

$$\frac{\partial \bar{I}_1}{\partial \ln V} = 1, \quad \frac{\partial \bar{I}_2}{\partial \ln V} = 2\ln V, \quad \frac{\partial \bar{I}_3}{\partial \ln V} = 3(\ln V)^2. \tag{17}$$

Thus Eqs. (15)–(17) give

$$S = \alpha_0 1 + \alpha_1 \ln V + \alpha_2 (\ln V)^{-1} \tag{18a}$$

and

$$S = \beta_0 1 + \beta_1 \ln V + \beta_2 (\ln V)^2, \tag{18b}$$

with α_i, β_i scalar functions of the invariants of $\ln V$ and the material properties $\partial\psi/\partial I_i$.

For the stress S, an isotropic function of $\ln V$, Eqs. (18a) and (18b) could have been written directly from relation (5). However, in general, the coefficient α_i, β_i would then not necessarily have the restrictions imposed by Eq. (15), namely,

$$\frac{\partial\alpha_0}{\partial I_2} = \frac{\partial\alpha_1}{\partial I_1}, \quad \text{etc.} \tag{19}$$

Klausner[6] defines a Hencky strain as

$$\underset{\epsilon_{i,(r)}}{H} = \tfrac{1}{2}\ln\bar{a}_{ij}, \tag{20}$$

where his a_{ij} is our V^2. He then defines a strain rate as

$$\underset{\epsilon_{i,(r)}}{\dot{H}} = \tfrac{1}{2}\,\overline{\ln a_{ij}} = \tfrac{1}{2}(\dot{a}_{ij}/a_{ij}). \tag{21}$$

Neither of the above expressions is tensorially proper. The rate expression is also incorrect since

$$\underset{\epsilon_{i,(r)}}{\dot{H}} = \tfrac{1}{2}(\dot{\overline{\ln a}})_{ij} = \frac{1}{2}\frac{\partial(\ln a)_{ij}}{\partial a_{kl}}\dot{a}_{kl}. \tag{22}$$

V. EVALUATION OF $(d/dt)(\ln V)$

From Eq. (5), it is obvious that

$$\ln(V_{\mathrm{diag}}) = (\ln V)_{\mathrm{diag}} = \ln V_d. \tag{23}$$

For the symmetric matrix representation of the tensor $\ln V$, there exists a rotation R such that

$$\ln V = R^{-1}(\ln V_d)R. \tag{24}$$

The chain rule applied to the time derivative of $\ln V$ yields

$$\frac{d}{dt}(\ln V) = R^{-1}\frac{d}{dt}(\ln V_d)R + \dot{R}^T(\ln V_d)R + R^{-1}(\ln V_d)\dot{R} \tag{25a}$$

and

$$\frac{d}{dt}\ln V_d(t) = \lim_{h\to 0}\frac{\ln V_d(t+h) - \ln V_d(t)}{h}, \tag{25b}$$

or alternatively,

$$\overline{\ln V_d}^{\bullet} = \frac{d}{dt}\ln V_d = \frac{\partial(\ln V_d)_{ij}}{\partial V_{d_{kl}}}\dot{V}_{d_{kl}} = V_d^{-1}\dot{V}_d, \tag{26}$$

which is a diagonalized matrix with V_d^{-1} commuting with $\dot{V}_d$ so that Eq. (25a) becomes

$$\frac{d}{dt}(\ln V) = \overline{\ln V}^{\bullet} = R^{-1}V_d^{-1}\dot{V}_dR + \{R^{-1}(\ln V_d)\dot{R} + [R^{-1}(\ln V_d)\dot{R}\,]^T\}, \tag{27}$$

where the term in curly brackets is symmetric and whose diagonal terms vanish, whereas the first term on the right-hand side is diagonalizable.

With Eqs. (24) and (27) multiplied by R on the left and R^{-1} on the right,

$$R(\ln V)R^{-1} = \ln V_d = (\ln V)_d \tag{28a}$$

$$R(\overline{\ln V}^{\bullet})R^{-1} = V_d^{-1}\dot{V}_d + (\ln V_d)\dot{R}R^{-1} + R\dot{R}^T(\ln V_d), \tag{28b}$$

showing that where R diagonalizes $\ln V$ it clearly does not diagonalize $\overline{\ln V}^{\bullet}$ unless, in general, $\dot{R} = 0$.

The $V_d^{-1}\dot{V}_d$ term is essentially the rate of change of the principal stretch ratios λ_i with respect to the current value of λ_i, i.e.,

$$V_d^{-1}\dot{V}_d = \begin{pmatrix} \dot{\lambda}/\lambda_1 & 0 & 0 \\ 0 & \dot{\lambda}_2/\lambda_2 & 0 \\ 0 & 0 & \dot{\lambda}_3/\lambda_3 \end{pmatrix}. \tag{29}$$

The $\dot{R}R^{-1}$ term and its transpose $R\dot{R}^T$ are not, however, the rate of change of rotation with respect to the current rotation R that diagonalizes V. Rather, $\dot{R}R^{-1}$ represents the instantaneous value of the rate of change of the Eulerian angles of the principal directions of $\ln V$. For example, for a rotation about the 3 $-$3 axis of $\ln V$ of ω,

$$\dot{R}R^{-1} = \dot{\omega}\begin{pmatrix} 0, & 1, & 0 \\ -1, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}, \tag{30}$$

showing that it is a function only of the current rate of change of ω.

VI. STRESS POWER

In this section we shall formally form a stress power for the Cauchy stress T and $\ln V$, with which it commutes. Justification for calling the above formed scalar product a stress power will be a given in Sec. VII.

Take the scalar product of T and $\overline{\ln V}^{\bullet}$ [Eq. (25a)] with which T does not in general commute

$$T\cdot\overline{\ln V}^{\bullet} = \mathrm{tr}[TR^{-1}\overline{\ln V_d}^{\bullet}R + T\dot{R}^T(\ln V_d)R + TR^{-1}(\ln V_d)\dot{R}\,]. \tag{31}$$

Consider the second term on the right-hand side and insert $RR^{-1} = 1$ before $\ln V_d$, and it becomes

$$\mathrm{tr}T\dot{R}^T(\ln V_d)R, \tag{32}$$

which by Eq. (24) yields

$$\mathrm{tr}T\dot{R}^TR\ln V, \tag{33}$$

and with the cycle property of trace becomes

$$\mathrm{tr}\dot{R}^TR(\ln V)T = 0, \tag{34}$$

since $\dot{R}^TR$ is antisymmetric and T commutes with $\ln V$ and is symmetric.

Similarly the last term vanishes.

$$\mathrm{tr}TR^{-1}(\ln V_d)\dot{R} = 0. \tag{35}$$

Thus Eq. (31) with Eq. (26) yields

$$T\cdot\overline{\ln V}^{\bullet} = \mathrm{tr}TR^{-1}V_d^{-1}\dot{V}_dR = \mathrm{tr}TV^{-1}R^{-1}\dot{V}_dR. \tag{36}$$

Now similarly to Eqs. (24) and (25),

$$V = R^{-1}V_dR,$$

hence
$$\dot{V} = R^{-1}\dot{V}_dR + \dot{R}^TV_dR + R^{-1}V_d\dot{R}, \tag{37}$$

so that

$$\dot{V}_d = R\dot{V}R^{-1} - R\dot{R}^TV_d - V_d\dot{R}R^{-1}, \tag{38}$$

where the last two terms contain the antisymmetric matrice $R\dot{R}^T$ and $\dot{R}R^{-1}$ so that Eq. (36) with Eq. (38) becomes

$$T\cdot\overline{\ln V}^{\bullet} = \mathrm{tr}TV^{-1}\dot{V} = TV^{-1}\cdot\dot{V} = T\cdot\dot{V}V^{-1}. \tag{39}$$

VII. STRESS DERIVABLE FROM A POTENTIAL

Assume the existence of a hyperelastic material wherein the stress is derivable from a potential ψ which is itself a function of the invariants of its associated strain measure. Many such forms are in use.

The usual form for the Cauchy stress uses a specific strain energy potential ψ (which is the isothermal free-energy potential) and the left Cauchy–Green stretch tensor B for isotropic hyperelastic materials with ρ being the density in the deformed configuration to give

$$\frac{T}{\rho} = 2B\frac{\partial\psi}{\partial B} = V\frac{\partial\psi}{\partial V}, \quad B = V^2. \tag{40}$$

The above may be written in terms of rates to yield the stress power as

$$(T/\rho)V^{-1}\cdot\dot{V} = \dot{\psi}. \tag{41}$$

From Eq. (39) we can then write

$$\frac{T}{\rho}\cdot\overline{\ln V}^{\bullet} = \dot{\psi} \quad \text{or} \quad T = \rho\frac{\partial\psi}{\partial\ln V}. \tag{42}$$

Thus, the Cauchy specific stress T/ρ is derivable from a potential with respect to $\ln V$ but not T itself (see Ref. 7).

VIII. MODIFIED STRESS T_F

Let us define a modified stress T_F through

$$T_F = JTV^{-1}, \quad J = \rho_0/\rho, \tag{43}$$

where ρ_0 is the density in the underformed configuration. Then by using Eqs. (40) and (41), we obtain

$$T_F \cdot \dot{V} = \rho_0 \dot{\psi}, \quad T_F = \rho_0 \frac{\partial \psi}{\partial V}, \tag{44}$$

so that the modified stress is derivable from a potential with respect to the left Cauchy–Green stretch tensor V.

The modified stress is essentially a symmetric tensor whose principal values are the surface tractions per unit area of undeformed configuration.

Where the first Piola–Kirchhoff stress S is defined through

$$S = \rho_0 \frac{\partial \bar{\psi}}{\partial F}, \quad S \cdot \dot{F} = \rho_0 \dot{\bar{\psi}}, \tag{45}$$

it is well known that the relation of S to T is

$$S = JTV^{-1}R, \tag{46}$$

so that the modified stress T_F is the symmetric left polar decomposition of S,

$$S = T_F R. \tag{47}$$

IX. EQUILIBRIUM EQUATIONS

The equilibrium equations for T and S [3] are

$$\text{div}T + \rho \text{b} = \rho \ddot{\chi}, \tag{48a}$$

$$\text{Div}S + \rho_0 b = \rho_0 \ddot{\chi}, \tag{48b}$$

where b is the body force, $\ddot{\chi}$ is the acceleration term, and div and Div are the divergence operators with respect to χ and X, respectively.

Both of the above equations for elastostatic zero-body force problems become

$$\text{div}T = 0, \quad \text{Div}S = 0, \tag{49}$$

which are "nice" in the sense that a wealth of mathematical literature exists for the existence, uniqueness, and solutions of the above.

The modified stress T_F does not possess such a "nice" equilibrium equation;

$$\text{div}(J^{-1}T_F V) = 0. \tag{50}$$

(It should again be stressed that V is the positive square root of B).

X. APPLICATION OF THE LOGARITHMIC STRAIN MEASURE

Consider a deformation gradient mapping from the reference configuratin to a deformed configuration at time t, i.e., $F(t)$. Now consider a mapping from the reference configuration to a later time $\tau \geqslant t$, i.e., $F(\tau)$. The relative deformation gradient from t to τ will be denoted $F_{(t)}(\tau)$ and the composition of mapping yields

$$F(\tau) = F_{(t)}(\tau) F(t) \tag{51a}$$

or

$$F_{(t)}(\tau) = F_{(\tau)} F^{-1}(t). \tag{51b}$$

Since $F(t)$ and $F^{-1}(\tau)$ exist and are unique and invertible, then $F_{(t)}^{-1}(\tau)$ exists and is unique and the polar decomposition applies,

$$F_{(t)}(\tau) = R_{(t)}(\tau) U_{(t)}(\tau) = V_{(t)}(\tau) R_{(t)}(\rho). \tag{52}$$

Now let $F(\tau) = V(\tau) R(\tau)$ and $F^{-1}(t) = R^{-1}(t) V^{-1}(t)$ so that Eq. (51b) is

$$F_{(t)}(\tau) = V(\tau) R(\tau) R^{-1}(t) V^{-1}(t), \tag{53}$$

which, when $t = \tau$, yields

$$F_{(t)}(t) = 1. \tag{54}$$

Assume that the rotations $R(\tau)$ and $R(t)$ are equal, that is, no change in R occurs from t to τ,

$$F_{(t)}(\tau) = V(\tau) V^{-1}(\tau). \tag{55}$$

Unfortunately, $V(\tau)$ and $V^{-1}(t)$ do not, in general, commute. For example, in simple shear, the principal directions of V are a function of the amount of shear, so that neither $R(\tau) = R(t)$ nor $V(\tau)$ and $V(t)$ commute.

The only case where $V(\tau)$ commutes with $V(t)$ is where the deformation gradient at t and τ consists of nonrotating principal directions.

$$F(t) = V_d(t) \quad \text{and} \quad F(\tau) = V_d(\tau). \tag{56}$$

It is thus only in this highly restrictive instance that the advantage of using a logarithmic strain measure exists.

For example, in the one-dimensional case, it is generally shown that with l_0 the original length and $l_1 a$ deformed length that

$$\lambda_{(1)} = l_1/l_0, \quad \ln\lambda_{(1)} = \ln(l_1/l_0). \tag{57}$$

For a subsequent increase in the length to l_3, $\ln(l_2/l_1)$ defines $\lambda_{(2)}$, and so forth up to the nth extension to length l_n, where we define

$$\ln\lambda = \ln(l_n/l_0), \tag{58}$$

which can also be written

$$\ln\lambda = \frac{\ln l_n l_{n-1} \cdots l_2 l_1}{l_{n-1} l_{n-2} \cdots l_1 l_0} = \ln\frac{l_n}{l_{n-1}} + \ln\frac{l_{n-1}}{l_{n-2}} + \cdots + \ln\frac{l_2}{l_1} + \ln\frac{l_1}{l_0} \tag{59}$$

or

$$\ln\lambda = \ln\lambda_{(n)} + \ln\lambda_{(n-1)} + \cdots + \ln\lambda_{(2)} + \ln\lambda_{(1)},$$

so that a progressive straining in logarithmic terms forms an additive group.

It is the above attribute that is claimed as an advantage for Hencky strain when applying a strain measure to plastic deformations, materials with damage and permanent set, and materials with slow viscous flow effects.

For a proper tensorial representation of $\ln V$ as given herein, the relative stretch tensor defined similarly to Eq. (51) is

$$V(\tau) = V_{(t)}(\tau) V(t), \tag{60}$$

so that

$$\ln V_{(t)}(\tau) = \ln V(\tau) V^{-1}(t). \tag{61}$$

Expression (55) may be written

$$\ln V_{(t)}(\tau) = \ln V(\tau) - \ln V(t) \tag{62}$$

analogously to Eq. (53) *if and only if* $V(\tau)$ and $V(t)$ commute for all times t and τ, respectively.

For all practical purposes, the above means that $V(\tau)$ and $V(t)$ diagonalize together for all times, so that one might as well define $\ln V$ as

$$\ln V = \ln V_{\text{diag}}, \tag{63}$$

where V_{diag} is known and its directions do not change with time. Then and only then does the additivity of Eq. (59) exist. Tensorial transformations are then lost.

The off-diagonal terms with the above definition then consist of functions of the λ_i and the rotation angles.

The rate of change of $\ln V$ is given as before in Eq. (27) or Eq. (26).

XI. CONCLUSION

A proper tensorial measure of logarithmic strain H is presented, where

$$H_{ij} = (\ln V)_{ij} = \phi_0\delta_{ij} + \phi_1 V_{ij} + \phi_2(V^2)_{ij}, \tag{64}$$

with the scalar coefficients ϕ_i given by Eq. (8). The time derivative of H is derived in Eq. (27).

The stress power relation (39) yields

$$T\cdot\dot{H} = \operatorname{tr} TV^{-1}\dot{V} = T\cdot\dot{V}V^{-1}. \tag{39}$$

It is shown that the specific Cauchy stress T/ρ is derivable from a potential with respect to H, i.e.,

$$\frac{T}{\rho} = \frac{\partial\psi}{\partial H}. \tag{42}$$

A modified stress T_F is developed wherein

$$T_F = \rho_0 \frac{\partial\psi}{\partial V}. \tag{44}$$

The main conclusion then is that the tensorial logarithmic strain measure presented is not of particular usefulness except when $V(t)$ is diagonalized with respect to a fixed set of coordinates for all $t \in (0, \infty)$. This conclusion holds for any tensorial logarithmic strain measure because the logarithmic tensor forms a non-Abelian group under addition. Thus, the desirable logarithmic additivity property of scalars holds for tensors if and only if each relative deformation from one time to the next commutes.

For example, a simple stretching $V(t)$ followed by a simple shear relative to the stretch $V_{(t)}(\tau)$ produces

$$\ln V(\tau) = \ln V_{(t)}(\tau)\mathrm{V}(\mathrm{t}), \tag{64}$$

and the right-hand side cannot be decomposed into

$$\ln V_{(t)}(\tau) + \ln V(t).$$

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