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LARGE ELASTIC DEFORMATIONS OF ISOTROPIC MATERIALS

III. SOME SIMPLE PROBLEMS IN CYLINDRICAL POLAR CO-ORDINATES

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Expressions for the components of strain and the incompressibility condition, for large deformations, are obtained in a cylindrical polar co-ordinate system. The stress-strain relations, equations of motion and boundary conditions for an incompressible, neo-Hookean material, in such a co-ordinate system, are also obtained and specialized to the case of cylindrical symmetry. These results are applied to the special cases of the simple torsion of a solid cylinder and of a hollow, cylindrical tube and to their combined simple extension and simple torsion.

In the case of a solid cylinder, it is found that a state of simple torsion can be maintained by surface tractions applied to the ends of the cylinder only, and these consist of a torsional couple together with a compressive force. The necessary torsional couple is proportional to the amount of torsion and the compressive force to the square of the torsion.

In the case of a hollow, cylindrical tube, it is again necessary to exert a torsional couple, proportional to the torsion, and a compressive force, proportional to the square of the torsion, on the plane ends, but it is also necessary to exert a normal surface traction, acting in a positive radial direction, on one or other of the curved surfaces of the tube and proportional to the square of the torsion.

1. Introduction

In Part I of this series (Rivlin 1948) the concept of an incompressible, neo-Hookean material was introduced. Such a material is considered to be isotropic in its undeformed state and to be capable of large, elastic deformation. Also, its stress-strain relationships are defined and it is considered that these form the best possible basis for the development of a mathematical theory of the large, elastic deformation of an incompressible material, isotropic in the undeformed state. In this previous paper, the mathematical theory was developed with reference to a fixed, rectangular Cartesian co-ordinate system. In the present paper the principal results obtained there are expressed with reference to a cylindrical polar co-ordinate system (r, θ, z). The method used is essentially that of moving axes. The simplification of the theory, when the deformation possesses cylindrical symmetry about the z-axis, is considered and the results obtained are applied to calculate the forces necessary to produce a large simple torsion and combined extension and torsion in a solid, right-circular cylinder and in a hollow, cylindrical tube of incompressible, neo-Hookean material. The results obtained are qualitatively different from those obtained from the classical, mathematical theory for the simple torsion of Hookean cylinders and tubes, but reduce to these for the case of vanishingly small torsions and extensions.

PART A. CYLINDRICAL POLAR CO-ORDINATES

2. Components of strain

In a fixed, rectangular Cartesian co-ordinate system (x'', y'', z''), the components of strain $(\epsilon_{x''x''}, \epsilon_{y''y''}, \epsilon_{z''z''}, \epsilon_{y''z''}, \epsilon_{z''x''}, \epsilon_{x''y''})$, at a point of a deformed body which was at (x'', y'', z'') in the undeformed state, are defined by the relation

$$(\delta s')^2 = (1+2\epsilon_{x''x''})(\delta x'')^2 + (1+2\epsilon_{y''y''})(\delta y'')^2 + (1+2\epsilon_{z''z''})(\delta z'')^2 + 2\epsilon_{y''z''}\delta y''\delta z'' + 2\epsilon_{z''x''}\delta z''\delta x'' + 2\epsilon_{x''y''}\delta x''\delta y'', \tag{2·1}$$

where $\delta s'$ is the length of a linear element in the deformed state. This element has components of length $\delta x''$, $\delta y''$ and $\delta z''$, in the undeformed state, parallel to the axes x'', y'' and z'' respectively, and is situated at (x'', y'', z'') in that state.

If u'', v'' and w'' are the components, parallel to the axes x'', y'' and z'', of the displacement undergone by the point in the deformation, then

$$\left.\begin{aligned} 1+2\epsilon_{x''x''} &= (1+u''_{x''})^2 + (v''_{x''})^2 + (w''_{x''})^2, \\ 1+2\epsilon_{y''y''} &= (u''_{y''})^2 + (1+v''_{y''})^2 + (w''_{y''})^2, \\ 1+2\epsilon_{z''z''} &= (u''_{z''})^2 + (v''_{z''})^2 + (1+w''_{z''})^2, \\ \epsilon_{y''z''} &= u''_{y''}u''_{z''} + (1+v''_{y''})\,v''_{z''} + w''_{y''}(1+w''_{z''}), \\ \epsilon_{z''x''} &= u''_{z''}(1+u''_{x''}) + v''_{z''}v''_{x''} + (1+w''_{z''})\,w''_{x''} \\ \text{and} \qquad \epsilon_{x''y''} &= (1+u''_{x''})\,u''_{y''} + v''_{x''}(1+v''_{y''}) + w''_{x''}w''_{y''}. \end{aligned}\right\} \tag{2·2}$$

The position of any point in the undeformed state may be defined by the cylindrical polar co-ordinates (r, θ, z). The cylindrical polar co-ordinate system to which the point is referred is considered to be fixed. Corresponding to each point (r, θ, z), we can choose a system of rectangular, Cartesian co-ordinates (x', y', z'), given by the lines of intersection of the tangent planes to the surfaces $r = \text{const.}$, $\theta = \text{const.}$ and $z = \text{const.}$, which pass through the point, taken in pairs. The x'-axis is the line of intersection of the tangent planes to the surfaces $\theta = \text{const.}$ and $z = \text{const.}$ and so on. If such a co-ordinate system is chosen at each point, then the displacement of any point in a deformation may be expressed by its components u, v and w parallel to the axes x', y' and z' of the co-ordinate system whose origin is at the point.

Now, let us choose a fixed, rectangular Cartesian co-ordinate system (x'', y'', z'') to coincide with the system (x', y', z'), which has its origin at (r, θ, z) and let (u'', v'', w'') be the components of the displacement of any point in this co-ordinate system.

If two neighbouring points are considered, at (r, θ, z) and $(r+\delta r, \theta+\delta\theta, z+\delta z)$ in the cylindrical polar co-ordinate system, in the undeformed state, their co-ordinates in the system (x'', y'', z'') are $(0, 0, 0)$ and $(\delta x'', \delta y'', \delta z'')$ respectively, where

$$\left.\begin{aligned} \delta x'' &= (r+\delta r)\cos\delta\theta - r, \\ \delta y'' &= (r+\delta r)\sin\delta\theta \\ \text{and} \qquad \delta z'' &= \delta z. \end{aligned}\right\} \tag{2·3}$$

Since $\delta\theta$ is infinitesimally small,

$$\cos\delta\theta = 1 \quad \text{and} \quad \sin\delta\theta = \delta\theta. \tag{2·4}$$

Thus (2·3) becomes $\qquad \delta x'' = \delta r, \quad \delta y'' = r\delta\theta \quad \text{and} \quad \delta z'' = \delta z. \qquad (2·5)$

In the deformation, the point (r_1, θ_1, z_1) undergoes a displacement (u_1, v_1, w_1) in the co-ordinate system (x', y', z') whose origin is at (r_1, θ_1, z_1). Let its displacement in the co-ordinate system (x'', y'', z''), whose origin is at (r, θ, z), be (u_1'', v_1'', w_1'').

Then,

$$\left.\begin{aligned} u_1'' &= u_1 \cos(\theta_1 - \theta) - v_1 \sin(\theta_1 - \theta), \\ v_1'' &= u_1 \sin(\theta_1 - \theta) + v_1 \cos(\theta_1 - \theta) \\ \text{and} \quad w_1'' &= w_1. \end{aligned}\right\} \tag{2·6}$$

Whence,

$$\left.\begin{aligned} \delta u_1'' &= \delta u_1 \cos(\theta_1 - \theta) - \delta v_1 \sin(\theta_1 - \theta) - u_1 \sin(\theta_1 - \theta)\, \delta\theta_1 - v_1 \cos(\theta_1 - \theta)\, \delta\theta_1, \\ \delta v_1'' &= \delta u_1 \sin(\theta_1 - \theta) + \delta v_1 \cos(\theta_1 - \theta) + u_1 \cos(\theta_1 - \theta)\, \delta\theta_1 - v_1 \sin(\theta_1 - \theta)\, \delta\theta_1, \\ \text{and} \quad \delta w_1'' &= \delta w_1. \end{aligned}\right\} \tag{2·7}$$

In the limit as $\theta_1 \to \theta$ and $r_1 \to r$, the displacement (u_1'', v_1'', w_1'') becomes (u'', v'', w''), the displacement of the point (r, θ, z) parallel to the axes of the co-ordinate system (x'', y'', z''), and the displacement (u_1, v_1, w_1) becomes (u, v, w), the displacement of the point (r, θ, z) parallel to the axes of the co-ordinate system (x', y', z') whose origin is at (r, θ, z). Thus, letting $\theta_1 \to \theta$ in (2·7), we obtain

$$\left.\begin{aligned} \delta u'' &= \delta u - v\,\delta\theta, \\ \delta v'' &= \delta v + u\,\delta\theta \\ \text{and} \quad \delta w'' &= \delta w. \end{aligned}\right\} \tag{2·8}$$

From (2·5) and (2·8), we obtain

$$\left.\begin{aligned} &\frac{\partial u''}{\partial x''} = \frac{\partial u}{\partial r}, \quad \frac{\partial u''}{\partial y''} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}, \quad \frac{\partial u''}{\partial z''} = \frac{\partial u}{\partial z}, \\ &\frac{\partial v''}{\partial x''} = \frac{\partial v}{\partial r}, \quad \frac{\partial v''}{\partial y''} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad \frac{\partial v''}{\partial z''} = \frac{\partial v}{\partial z}, \\ &\frac{\partial w''}{\partial x''} = \frac{\partial w}{\partial r}, \quad \frac{\partial w''}{\partial y''} = \frac{1}{r}\frac{\partial w}{\partial \theta} \quad \text{and} \quad \frac{\partial w''}{\partial z''} = \frac{\partial w}{\partial z}. \end{aligned}\right\} \tag{2·9}$$

It is noted from (2·5) that if x'' is constant, $\delta r = 0$, if y'' is constant, $\delta\theta = 0$ and if z'' is constant, $\delta z = 0$.

Introducing these relations, together with (2·5) and (2·2), into (2·1), we obtain

$$\begin{aligned} (\delta s')^2 = (1 + 2\epsilon_{rr})(\delta r)^2 + (1 + 2\epsilon_{\theta\theta})(r\,\delta\theta)^2 + (1 + 2\epsilon_{zz})(\delta z)^2 \\ + 2\epsilon_{\theta z}\, r\,\delta\theta\,\delta z + 2\epsilon_{zr}\,\delta z\,\delta r + 2\epsilon_{r\theta}\,\delta r\, r\,\delta\theta, \end{aligned} \tag{2·10}$$

where

$$\left.\begin{aligned} 1 + 2\epsilon_{rr} &= (1 + u_r)^2 + v_r^2 + w_r^2, \\ 1 + 2\epsilon_{\theta\theta} &= \frac{1}{r^2}(u_\theta - v)^2 + \left(1 + \frac{u}{r} + \frac{1}{r} v_\theta\right)^2 + \frac{1}{r^2} w_\theta^2, \\ 1 + 2\epsilon_{zz} &= u_z^2 + v_z^2 + (1 + w_z)^2, \\ \epsilon_{\theta z} &= \frac{1}{r}(u_\theta - v)\, u_z + \left(1 + \frac{u}{r} + \frac{1}{r} v_\theta\right) v_z + \frac{1}{r} w_\theta (1 + w_z), \\ \epsilon_{zr} &= u_z(1 + u_r) + v_z v_r + (1 + w_z)\, w_r \\ \text{and} \quad \epsilon_{r\theta} &= (1 + u_r)\frac{1}{r}(u_\theta - v) + v_r\left(1 + \frac{u}{r} + \frac{1}{r} v_\theta\right) + w_r \frac{1}{r} w_\theta. \end{aligned}\right\} \tag{2·11}$$

ϵ_{rr}, $\epsilon_{\theta\theta}$, ϵ_{zz}, $\epsilon_{\theta z}$, ϵ_{zr} and $\epsilon_{r\theta}$ are defined as the components of strain, at the point which is at (r, θ, z) in the undeformed state, in the cylindrical polar co-ordinate system (r, θ, z). It is readily seen that these components of strain are defined from (2·10) in a manner analogous to that in which the components of strain (2·2) are defined from (2·1).

3. The incompressibility condition

Using the notation of § 2, the incompressibility condition is given, in the co-ordinate system (x'', y'', z''), by

$$\tau = \begin{vmatrix} 1+u''_{x''} & u''_{y''} & u''_{z''} \\ v''_{x''} & 1+v''_{y''} & v''_{z''} \\ w''_{x''} & w''_{y''} & 1+w''_{z''} \end{vmatrix} = 1.$$

Making use of the relations (2·9), this becomes

$$\tau = \begin{vmatrix} 1+u_r & \frac{1}{r}(u_\theta - v) & u_z \\ v_r & 1+\frac{u}{r}+\frac{1}{r}v_\theta & v_z \\ w_r & \frac{1}{r}w_\theta & 1+w_z \end{vmatrix} = 1. \tag{3·1}$$

4. The stress-strain relationships

In the co-ordinate system (x'', y'', z''), the stress-strain relationships for an incompressible, neo-Hookean material are given (Rivlin 1948, I, § 3) by

$$\begin{aligned} t_{x''x''} &= \tfrac{1}{3}E[(1+u''_{x''})^2 + u''^2_{y''} + u''^2_{z''}] + p, \\ t_{y''y''} &= \tfrac{1}{3}E[v''^2_{x''} + (1+v''_{y''})^2 + v''^2_{z''}] + p, \\ t_{z''z''} &= \tfrac{1}{3}E[w''^2_{x''} + w''^2_{y''} + (1+w''_{z''})^2] + p, \\ t_{y''z''} &= \tfrac{1}{3}E[v''_{x''}w''_{x''} + (1+v''_{y''})\, w''_{y''} + v''_{z''}(1+w''_{z''})], \\ t_{z''x''} &= \tfrac{1}{3}E[w''_{x''}(1+u''_{x''}) + w''_{y''}u''_{y''} + (1+w''_{z''})\, u''_{z''}] \end{aligned}$$

and
$$t_{x''y''} = \tfrac{1}{3}E[(1+u''_{x''})\, v''_{x''} + u''_{y''}(1+v''_{y''}) + u''_{z''}v''_{z''}],$$

at the point which, in the undeformed state of the material, lies at (x'', y'', z''). E is a physical constant characterizing the material.

Choosing the co-ordinate system (x'', y'', z'') with its origin at the point (r, θ, z) in the cylindrical polar co-ordinate system and with its axes directed along the intersections of the tangent planes to $r =$ const., $\theta =$ const. and $z =$ const., as before, we have, employing the relations (2·9),

$$\left.\begin{aligned}
t_{x''x''} &= \tfrac{1}{3}E\Big[(1+u_r)^2+\frac{1}{r^2}(u_\theta-v)^2+u_z^2\Big]+p,\\
t_{y''y''} &= \tfrac{1}{3}E\Big[v_r^2+\Big(1+\frac{u}{r}+\frac{1}{r}v_\theta\Big)^2+v_z^2\Big]+p,\\
t_{z''z''} &= \tfrac{1}{3}E\Big[w_r^2+\frac{1}{r^2}w_\theta^2+(1+w_z)^2\Big]+p,\\
t_{y''z''} &= \tfrac{1}{3}E\Big[v_r w_r+\Big(1+\frac{u}{r}+\frac{1}{r}v_\theta\Big)\frac{1}{r}w_\theta+v_z(1+w_z)\Big],\\
t_{z''x''} &= \tfrac{1}{3}E\Big[w_r(1+u_r)+\frac{1}{r^2}w_\theta(u_\theta-v)+(1+w_z)\,u_z\Big]\\
\text{and}\quad t_{x''y''} &= \tfrac{1}{3}E\Big[(1+u_r)\,v_r+\frac{1}{r}(u_\theta-v)\Big(1+\frac{u}{r}+\frac{1}{r}v_\theta\Big)+u_z v_z\Big].
\end{aligned}\right\} \qquad (4\cdot1)$$

In the deformation, the point whose cylindrical polar co-ordinates in the undeformed state are (r, θ, z) moves to a point whose cylindrical polar co-ordinates are $(r+s, \theta+\phi, z+w)$. Let t_{rr}, $t_{\theta\theta}$, t_{zz}, ... be the components of stress in a rectangular, Cartesian co-ordinate system whose origin is at $(r+s, \theta+\phi, z+w)$ and whose axes are directed along the intersections of the tangent planes to the surfaces $r = \text{const.}$, $\theta = \text{const.}$ and $z = \text{const.}$, which pass through this point, taken in pairs.

Then,

$$\left.\begin{aligned}
t_{rr} &= t_{x''x''}\cos^2\phi+t_{y''y''}\sin^2\phi+2t_{x''y''}\sin\phi\,\cos\phi,\\
t_{\theta\theta} &= t_{x''x''}\sin^2\phi+t_{y''y''}\cos^2\phi-2t_{x''y''}\sin\phi\,\cos\phi,\\
t_{zz} &= t_{z''z''},\\
t_{\theta z} &= t_{y''z''}\cos\phi-t_{z''x''}\sin\phi,\\
t_{zr} &= t_{y''z''}\sin\phi+t_{z''x''}\cos\phi\\
\text{and}\quad t_{r\theta} &= (t_{y''y''}-t_{x''x''})\sin\phi\,\cos\phi+t_{x''y''}(\cos^2\phi-\sin^2\phi).
\end{aligned}\right\} \qquad (4\cdot2)$$

Since, in the deformation, the point (r, θ, z) moves to $(r+s, \theta+\phi, z+w)$,

$$u = (r+s)\cos\phi-r \quad \text{and} \quad v = (r+s)\sin\phi. \qquad (4\cdot3)$$

Substituting from (4·1) and (4·3) in (4·2), we have

$$\left.\begin{aligned}
t_{rr} &= \tfrac{1}{3}E\Big[(1+s_r)^2+\frac{1}{r^2}s_\theta^2+s_z^2\Big]+p,\\
t_{\theta\theta} &= \tfrac{1}{3}E(r+s)^2\Big[\phi_r^2+\frac{1}{r^2}(1+\phi_\theta)^2+\phi_z^2\Big]+p,\\
t_{zz} &= \tfrac{1}{3}E\Big[w_r^2+\frac{1}{r^2}w_\theta^2+(1+w_z)^2\Big]+p,\\
t_{\theta z} &= \tfrac{1}{3}E(r+s)\Big[\phi_r w_r+\frac{1}{r^2}w_\theta(1+\phi_\theta)+(1+w_z)\,\phi_z\Big],\\
t_{zr} &= \tfrac{1}{3}E\Big[w_r(1+s_r)+\frac{1}{r^2}w_\theta s_\theta+(1+w_z)\,s_z\Big]\\
\text{and}\quad t_{r\theta} &= \tfrac{1}{3}E(r+s)\Big[(1+s_r)\,\phi_r+\frac{1}{r^2}s_\theta(1+\phi_\theta)+s_z\phi_z\Big].
\end{aligned}\right\} \qquad (4\cdot4)$$

5. The equations of motion

The equations of motion, in the co-ordinate system (x'', y'', z''), at the point (r, θ, z) in the cylindrical polar co-ordinate system, take the form (Rivlin 1948, I, § 20)

$$\left.\begin{aligned}\rho\frac{\partial^2 u''}{\partial t^2} &= \rho X'' + \tfrac{1}{3}E\nabla^2 u'' + \frac{\partial\tau}{\partial u''_{x''}}\frac{\partial p}{\partial x''} + \frac{\partial\tau}{\partial u''_{y''}}\frac{\partial p}{\partial y''} + \frac{\partial\tau}{\partial u''_{z''}}\frac{\partial p}{\partial z''},\\ \rho\frac{\partial^2 v''}{\partial t^2} &= \rho Y'' + \tfrac{1}{3}E\nabla^2 v'' + \frac{\partial\tau}{\partial v''_{x''}}\frac{\partial p}{\partial x''} + \frac{\partial\tau}{\partial v''_{y''}}\frac{\partial p}{\partial y''} + \frac{\partial\tau}{\partial v''_{z''}}\frac{\partial p}{\partial z''}\\ \text{and}\qquad \rho\frac{\partial^2 w''}{\partial t^2} &= \rho Z'' + \tfrac{1}{3}E\nabla^2 w'' + \frac{\partial\tau}{\partial w''_{x''}}\frac{\partial p}{\partial x''} + \frac{\partial\tau}{\partial w''_{y''}}\frac{\partial p}{\partial y''} + \frac{\partial\tau}{\partial w''_{z''}}\frac{\partial p}{\partial z''}.\end{aligned}\right\} \tag{5·1}$$

X'', Y'' and Z'' are the components, in the directions of the axes x'', y'' and z'' respectively, of the body forces, per unit mass, acting on the material at the point (r, θ, z). ρ is the density of the material.

If R, Θ, Z are the components of the body forces, per unit mass, parallel to the axes (x', y', z') of the rectangular, Cartesian co-ordinate system whose origin is at (r, θ, z), then

$$R = X'', \quad \Theta = Y'' \quad \text{and} \quad Z = Z'', \tag{5·2}$$

for these axes (x', y', z') and the axes (x'', y'', z'') are coincident at the point (r, θ, z).

Using the relations (2·5), we have

$$\frac{\partial p}{\partial x''} = \frac{\partial p}{\partial r}, \quad \frac{\partial p}{\partial y''} = \frac{1}{r}\frac{\partial p}{\partial \theta} \quad \text{and} \quad \frac{\partial p}{\partial z''} = \frac{\partial p}{\partial z}. \tag{5·3}$$

Also, for the point (r, θ, z)

$$u = u'', \quad v = v'' \quad \text{and} \quad w = w'',$$

so that

$$\frac{\partial^2 u''}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial^2 v''}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 w''}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}. \tag{5·4}$$

$\partial\tau/\partial u''_{x''}$, $\partial\tau/\partial u''_{y''}$, etc., can be expressed in terms of u_r, v_r, etc., by means of the relations (2·9). Thus, for example, from (3·1) and (2·9),

$$\begin{aligned}\frac{\partial\tau}{\partial u''_{x''}} &= \begin{vmatrix} 1+v''_{y''} & v''_{z''} \\ w''_{y''} & 1+w''_{z''} \end{vmatrix}\\ &= \begin{vmatrix} 1+\dfrac{u}{r}+\dfrac{1}{r}v_\theta & v_z \\ \dfrac{1}{r}w_\theta & 1+w_z \end{vmatrix}.\end{aligned} \tag{5·5}$$

Now, in the fixed, rectangular Cartesian co-ordinate system (x'', y'', z''), the operator ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial x''^2} + \frac{\partial^2}{\partial y''^2} + \frac{\partial^2}{\partial z''^2}.$$

In the cylindrical polar co-ordinate system, it is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{5·6}$$

Now, from (2·8),

$$\left.\begin{aligned}&\frac{\partial u''}{\partial r}=\frac{\partial u}{\partial r}, \quad \frac{\partial u''}{\partial \theta}=\frac{\partial u}{\partial \theta}-v, \quad \frac{\partial u''}{\partial z}=\frac{\partial u}{\partial z},\\ &\frac{\partial v''}{\partial r}=\frac{\partial v}{\partial r}, \quad \frac{\partial v''}{\partial \theta}=\frac{\partial v}{\partial \theta}+u, \quad \frac{\partial v''}{\partial z}=\frac{\partial v}{\partial z},\\ &\frac{\partial w''}{\partial r}=\frac{\partial w}{\partial r}, \quad \frac{\partial w''}{\partial \theta}=\frac{\partial w}{\partial \theta}, \quad \frac{\partial w''}{\partial z}=\frac{\partial w}{\partial z}.\end{aligned}\right\} \tag{5·7}$$

From the first of equations (2·7), we have

$$\begin{aligned}\delta^2 u_1'' = \delta^2 u_1 \cos(\theta_1-\theta) - \delta^2 v_1 \sin(\theta_1-\theta) - \delta u_1 \sin(\theta_1-\theta)\,\delta\theta_1\\ -\delta v_1 \cos(\theta_1-\theta)\,\delta\theta_1 - \delta u_1 \sin(\theta_1-\theta)\,\delta\theta_1 - \delta v_1 \cos(\theta_1-\theta)\,\delta\theta_1\\ -u_1 \cos(\theta_1-\theta)\,(\delta\theta_1)^2 + v_1 \sin(\theta_1-\theta)\,(\delta\theta_1)^2.\end{aligned}$$

In the limit as $\theta_1 \to \theta$, this yields

$$\delta^2 u'' = \delta^2 u - 2\delta v\,\delta\theta - u(\delta\theta)^2.$$

Similarly, $\delta^2 v''$ and $\delta^2 w''$ can be obtained from the remaining two equations of (2·7). We then have

$$\delta^2 u'' = \delta^2 u - 2\delta v\,\delta\theta - u(\delta\theta)^2, \quad \delta^2 v'' = \delta^2 v + 2\delta u\,\delta\theta - v(\delta\theta)^2 \quad \text{and} \quad \delta^2 w'' = \delta^2 w.$$

Whence,

$$\left.\begin{aligned}&\frac{\partial^2 u''}{\partial r^2}=\frac{\partial^2 u}{\partial r^2}, \quad \frac{\partial^2 u''}{\partial \theta^2}=\frac{\partial^2 u}{\partial \theta^2}-2\frac{\partial v}{\partial \theta}-u, \quad \frac{\partial^2 u''}{\partial z^2}=\frac{\partial^2 u}{\partial z^2},\\ &\frac{\partial^2 v''}{\partial r^2}=\frac{\partial^2 v}{\partial r^2}, \quad \frac{\partial^2 v''}{\partial \theta^2}=\frac{\partial^2 v}{\partial \theta^2}+2\frac{\partial u}{\partial \theta}-v, \quad \frac{\partial^2 v''}{\partial z^2}=\frac{\partial^2 v}{\partial z^2}\\ \text{and}\quad &\frac{\partial^2 w''}{\partial r^2}=\frac{\partial^2 w}{\partial r^2}, \quad \frac{\partial^2 w''}{\partial \theta^2}=\frac{\partial^2 w}{\partial \theta^2}, \quad \frac{\partial^2 w''}{\partial z^2}=\frac{\partial^2 w}{\partial z^2}.\end{aligned}\right\} \tag{5·8}$$

Making use of the relationships (5·6), (5·7) and (5·8), we have

$$\left.\begin{aligned}&\nabla^2 u'' = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}-\frac{2}{r^2}v_\theta-\frac{u}{r^2}+u_{zz},\\ &\nabla^2 v'' = v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}+\frac{2}{r^2}u_\theta-\frac{v}{r^2}+v_{zz}\\ \text{and}\quad &\nabla^2 w'' = w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}+w_{zz}.\end{aligned}\right\} \tag{5·9}$$

Introducing the results (5·2), (5·3), (5·4), (5·5) and (5·9) into equations (5·1), the equations of motion become, in cylindrical polar co-ordinates,

$$\left.\begin{aligned}\alpha &= \left[\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)(1+w_z)-\frac{1}{r}w_\theta v_z\right]\frac{\partial p}{\partial r}+\left[v_z w_r - v_r(1+w_z)\right]\frac{1}{r}\frac{\partial p}{\partial \theta}\\ &\qquad +\left[\frac{1}{r}w_\theta v_r - w_r\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)\right]\frac{\partial p}{\partial z},\\ \beta &= \left[\frac{1}{r}w_\theta u_z - \frac{1}{r}(u_\theta - v)(1+w_z)\right]\frac{\partial p}{\partial r}+\left[(1+u_r)(1+w_z)-u_z w_r\right]\frac{1}{r}\frac{\partial p}{\partial \theta}\\ &\qquad +\left[\frac{1}{r}(u_\theta - v)\,w_r - \frac{1}{r}w_\theta(1+u_r)\right]\frac{\partial p}{\partial z}\\ \text{and}\quad \gamma &= \left[\frac{1}{r}(u_\theta - v)\,v_z - \left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)u_z\right]\frac{\partial p}{\partial r}+\left[u_z v_r - (1+u_r)\,v_z\right]\frac{1}{r}\frac{\partial p}{\partial \theta}\\ &\qquad +\left[(1+u_r)\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)-\frac{1}{r}v_r\,(u_\theta - v)\right]\frac{\partial p}{\partial z},\end{aligned}\right\} \tag{5·10}$$

where
$$\left.\begin{aligned} \alpha &= \rho\frac{\partial^2 u}{\partial t^2}-\rho R-\tfrac{1}{3}E\left(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}-\frac{2v_\theta}{r^2}-\frac{u}{r^2}+u_{zz}\right),\\ \beta &= \rho\frac{\partial^2 v}{\partial t^2}-\rho\Theta-\tfrac{1}{3}E\left(v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}+\frac{2u_\theta}{r^2}-\frac{v}{r^2}+v_{zz}\right)\\ \text{and}\quad \gamma &= \rho\frac{\partial^2 w}{\partial t^2}-\rho Z-\tfrac{1}{3}E\left(w_{rr}+\frac{1}{r}w_r+\frac{1}{r^2}w_{\theta\theta}+w_{zz}\right). \end{aligned}\right\} \tag{5·11}$$

6. The boundary conditions

In the rectangular, Cartesian co-ordinate system (x'', y'', z''), the boundary conditions take the form (Rivlin 1948, I, § 20)

$$\left.\begin{aligned} &\tfrac{1}{3}E[(1+u''_{x''})\cos(x'',\nu)+u''_{y''}\cos(y'',\nu)+u''_{z''}\cos(z'',\nu)]-X''_\nu\\ &\quad=-p\left[\frac{\partial\tau}{\partial u''_{x''}}\cos(x'',\nu)+\frac{\partial\tau}{\partial u''_{y''}}\cos(y'',\nu)+\frac{\partial\tau}{\partial u''_{z''}}\cos(z'',\nu)\right],\\ &\tfrac{1}{3}E[v''_{x''}\cos(x'',\nu)+(1+v''_{y''})\cos(y'',\nu)+v''_{z''}\cos(z'',\nu)]-Y''_\nu\\ &\quad=-p\left[\frac{\partial\tau}{\partial v''_{x''}}\cos(x'',\nu)+\frac{\partial\tau}{\partial v''_{y''}}\cos(y'',\nu)+\frac{\partial\tau}{\partial v''_{z''}}\cos(z'',\nu)\right]\\ \text{and}\quad &\tfrac{1}{3}E[w''_{x''}\cos(x'',\nu)+w''_{y''}\cos(y'',\nu)+(1+w''_{z''})\cos(z'',\nu)]-Z''_\nu\\ &\quad=-p\left[\frac{\partial\tau}{\partial w''_{x''}}\cos(x'',\nu)+\frac{\partial\tau}{\partial w''_{y''}}\cos(y'',\nu)+\frac{\partial\tau}{\partial w''_{z''}}\cos(z'',\nu)\right]. \end{aligned}\right\} \tag{6·1}$$

ν is the direction of the normal to the surface in the undeformed state of the body at the point considered.

(x'', ν) is the angle between the direction of x'' and ν. (y'', ν) and (z'', ν) are similarly defined.

X''_ν, Y''_ν and Z''_ν are the components parallel to x'', y'' and z'' respectively, of the surface force, per unit area of the surface measured in the undeformed state. At the point (r, θ, z),

$$(x'',\nu)=(r,\nu),\quad (y'',\nu)=(\theta,\nu)\quad\text{and}\quad (z'',\nu)=(z,\nu), \tag{6·2}$$

where (r,ν), (θ,ν) and (z,ν) are the angles between ν and the directions of the axes x', y' and z' respectively, which pass through (r,θ,z).

Also, since at the point (r,θ,z) the axes (x'',y'',z'') and (x',y',z') coincide,

$$R_\nu = X''_\nu,\quad \Theta_\nu = Y''_\nu\quad\text{and}\quad Z_\nu = Z''_\nu, \tag{6·3}$$

where R_ν, Θ_ν and Z_ν are the components of the surface force per unit area parallel to x', y' and z' respectively.

Introducing into (6·1) the relations (2·9), (6·2), (6·3) and such relations as (5·5), the boundary conditions become, in the cylindrical co-ordinate system,

$$\left.\begin{aligned}
&\tfrac{1}{3}E\left[(1+u_r)\cos(r,\nu)+\frac{1}{r}(u_\theta-v)\cos(\theta,\nu)+u_z\cos(z,\nu)\right]-R_\nu\\
&\quad=-p\left[\left\{\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)(1+w_z)-\frac{1}{r}w_\theta v_z\right\}\cos(r,\nu)\right.\\
&\qquad+\{v_z w_r-v_r(1+w_z)\}\cos(\theta,\nu)\\
&\qquad\left.+\left\{\frac{1}{r}w_\theta v_r-w_r\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)\right\}\cos(z,\nu)\right],\\
&\tfrac{1}{3}E\left[v_r\cos(r,\nu)+\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)\cos(\theta,\nu)+v_z\cos(z,\nu)\right]-\Theta_\nu\\
&\quad=-p\left[\left\{\frac{1}{r}w_\theta u_z-\frac{1}{r}(u_\theta-v)(1+w_z)\right\}\cos(r,\nu)\right.\\
&\qquad+\{(1+u_r)(1+w_z)-u_z w_r\}\cos(\theta,\nu)\\
&\qquad\left.+\left\{\frac{1}{r}(u_\theta-v)w_r-\frac{1}{r}w_\theta(1+u_r)\right\}\cos(z,\nu)\right]\\
\text{and}\quad&\tfrac{1}{3}E\left[w_r\cos(r,\nu)+\frac{1}{r}w_\theta\cos(\theta,\nu)+(1+w_z)\cos(z,\nu)\right]-Z_\nu\\
&\quad=-p\left[\left\{\frac{1}{r}(u_\theta-v)v_z-\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)u_z\right\}\cos(r,\nu)\right.\\
&\qquad+\{u_z v_r-(1+u_r)v_z\}\cos(\theta,\nu)\\
&\qquad\left.+\left\{(1+u_r)\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)-\frac{1}{r}(u_\theta-v)v_r\right\}\cos(z,\nu)\right].
\end{aligned}\right\}\qquad(6\cdot4)$$

If part of the surface of the body, whose deformation is considered, has, in the undeformed state, the form of a cylinder with the z-axis as axis,

$$\cos(r,\nu)=1\quad\text{and}\quad\cos(\theta,\nu)=\cos(z,\nu)=0.$$

The boundary conditions (6·4) then become, over this surface,

$$\left.\begin{aligned}
&\tfrac{1}{3}E(1+u_r)-R_\nu=-p\left[\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)(1+w_z)-\frac{1}{r}w_\theta v_z\right],\\
&\tfrac{1}{3}Ev_r-\Theta_\nu=-p\left[\frac{1}{r}w_\theta u_z-\frac{1}{r}(u_\theta-v)(1+w_z)\right]\\
\text{and}\quad&\tfrac{1}{3}Ew_r-Z_\nu=-p\left[\frac{1}{r}(u_\theta-v)v_z-\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)u_z\right].
\end{aligned}\right\}\qquad(6\cdot5)$$

Again, if part of the surface consists of a plane normal to the z-axis,

$$\cos(r,\nu)=\cos(\theta,\nu)=0\quad\text{and}\quad\cos(z,\nu)=1.$$

Over this plane, the boundary conditions (6·4) become

$$\left.\begin{aligned}
&\tfrac{1}{3}Eu_z-R_\nu=-p\left[\frac{1}{r}w_\theta v_r-w_r\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)\right],\\
&\tfrac{1}{3}Ev_z-\Theta_\nu=-p\left[\frac{1}{r}(u_\theta-v)w_r-\frac{1}{r}w_\theta(1+u_r)\right]\\
\text{and}\quad&\tfrac{1}{3}E(1+w_z)-Z_\nu=-p\left[(1+u_r)\left(1+\frac{u}{r}+\frac{1}{r}v_\theta\right)-\frac{1}{r}(u_\theta-v)v_r\right].
\end{aligned}\right\}\qquad(6\cdot6)$$

7. Deformation possessing cylindrical symmetry

If the deformation possesses cylindrical symmetry about the z-axis, then u, v, w and p are independent of θ; i.e.

$$u_\theta = v_\theta = w_\theta = 0 \quad \text{and} \quad \partial p/\partial\theta = 0.$$

Introducing this into (3·1), the incompressibility condition becomes

$$\tau = \begin{vmatrix} 1+u_r & -v/r & u_z \\ v_r & 1+(u/r) & v_z \\ w_r & 0 & 1+w_z \end{vmatrix} = 1. \tag{7·1}$$

The equations of motion (5·10) become

$$\left.\begin{aligned} \alpha &= \left(1+\frac{u}{r}\right)(1+w_z)\frac{\partial p}{\partial r} - w_r\left(1+\frac{u}{r}\right)\frac{\partial p}{\partial z}, \\ \beta &= \frac{v}{r}(1+w_z)\frac{\partial p}{\partial r} - \frac{v}{r}w_r\frac{\partial p}{\partial z} \\ \text{and} \qquad \gamma &= \left[-\frac{v}{r}v_z - \left(1+\frac{u}{r}\right)u_z\right]\frac{\partial p}{\partial r} + \left[(1+u_r)\left(1+\frac{u}{r}\right)+\frac{v}{r}v_r\right]\frac{\partial p}{\partial z}, \end{aligned}\right\} \tag{7·2}$$

$$\left.\begin{aligned} \text{where} \qquad \alpha &= \rho\frac{\partial^2 u}{\partial t^2} - \rho R - \tfrac{1}{3}E\left(u_{rr}+\frac{1}{r}u_r - \frac{u}{r^2}+u_{zz}\right), \\ \beta &= \rho\frac{\partial^2 v}{\partial t^2} - \rho\Theta - \tfrac{1}{3}E\left(v_{rr}+\frac{1}{r}v_r - \frac{v}{r^2}+v_{zz}\right) \\ \text{and} \qquad \gamma &= \rho\frac{\partial^2 w}{\partial t^2} - \rho Z - \tfrac{1}{3}E\left(w_{rr}+\frac{1}{r}w_r + w_{zz}\right). \end{aligned}\right\} \tag{7·3}$$

The boundary conditions (6·4) become

$$\left.\begin{aligned} &\tfrac{1}{3}E\left[(1+u_r)\cos(r,\nu) - \frac{v}{r}\cos(\theta,\nu) + u_z\cos(z,\nu)\right] - R_\nu \\ &\quad = -p\left[\left(1+\frac{u}{r}\right)(1+w_z)\cos(r,\nu) + \{v_z w_r - v_r(1+w_z)\}\cos(\theta,\nu) - w_r\left(1+\frac{u}{r}\right)\cos(z,\nu)\right], \\ &\tfrac{1}{3}E\left[v_r\cos(r,\nu) + \left(1+\frac{u}{r}\right)\cos(\theta,\nu) + v_z\cos(z,\nu)\right] - \Theta_\nu \\ &\quad = -p\left[\frac{v}{r}(1+w_z)\cos(r,\nu) + \{(1+u_r)(1+w_z) - u_z w_r\}\cos(\theta,\nu) - \frac{v}{r}w_r\cos(z,\nu)\right] \\ &\text{and} \\ &\tfrac{1}{3}E[w_r\cos(r,\nu) + (1+w_z)\cos(z,\nu)] - Z_\nu \\ &\quad = -p\left[\left\{-\frac{v}{r}v_z - \left(1+\frac{u}{r}\right)u_z\right\}\cos(r,\nu) + \{u_z v_r - (1+u_r)v_z\}\cos(\theta,\nu)\right. \\ &\qquad\qquad \left. + \left\{(1+u_r)\left(1+\frac{u}{r}\right)+\frac{v}{r}v_r\right\}\cos(z,\nu)\right]. \end{aligned}\right\} \tag{7·4}$$

If, further, a part of the surface of the body is cylindrical in the undeformed state and has its generators parallel to the z-axis, then, for this surface, equations (7·4) become

$$\left.\begin{aligned} \tfrac{1}{3}E(1+u_r)-R_\nu &= -p\left(1+\frac{u}{r}\right)(1+w_z), \\ \tfrac{1}{3}Ev_r-\Theta_\nu &= -p\frac{v}{r}(1+w_z) \\ \text{and}\qquad \tfrac{1}{3}Ew_r-Z_\nu &= -p\left[-\frac{v}{r}v_z-\left(1+\frac{u}{r}\right)u_z\right]. \end{aligned}\right\} \tag{7·5}$$

Again, if part of the surface is plane in the undeformed state and normal to the z-axis, the boundary conditions for this surface become

$$\left.\begin{aligned} \tfrac{1}{3}Eu_z-R_\nu &= -p\left[-w_r\left(1+\frac{u}{r}\right)\right], \\ \tfrac{1}{3}Ev_z-\Theta_\nu &= -p\left[-\frac{v}{r}w_r\right] \\ \text{and}\qquad \tfrac{1}{3}E(1+w_z)-Z_\nu &= -p\left[(1+u_r)\left(1+\frac{u}{r}\right)+\frac{v}{r}v_r\right]. \end{aligned}\right\} \tag{7·6}$$

PART B. SOLUTIONS OF SOME SIMPLE PROBLEMS

8. The torsion of a right-circular cylinder

As an example of the application of the formulae deduced in the foregoing pages, the problem of the torsion of a right-circular solid cylinder of incompressible, neo-Hookean material, by forces applied to its plane ends, will be considered. Let us assume that in its undeformed state, i.e. under the action of no external forces, the cylinder has a length l and radius a. We choose our cylindrical polar reference system (r, θ, z) in such a way that its z-axis coincides with the axis of the cylinder and its origin is at the centre of one of the ends of the cylinder. The curved surface of the cylinder is then part of the surface $r = a$ and the plane ends form parts of the surfaces $z = l$ and $z = 0$.

Now, consider the torsional deformation of the cylinder, in which planes normal to the z-axis remain plane and suffer only a pure rotation about the z-axis through an angle ϕ proportional to their distance from $z = 0$, i.e.

$$\phi = \psi z, \tag{8·1}$$

where ψ is a constant. It is noted that the body will then remain cylindrical in form.

It can readily be shown that such a deformation can be supported by surface tractions applied to the ends of the cylinder only.

In the deformation, the displacement (u, v, w), defined as in § 2, of a point which is at (r, θ, z) in the undeformed state, can readily be seen to be given by

$$\left.\begin{aligned} u &= -r(1-\cos\phi) = -r(1-\cos\psi z), \\ v &= r\sin\phi \qquad\;\; = r\sin\psi z \\ \text{and}\qquad w &= 0. \end{aligned}\right\} \tag{8·2}$$

Since the deformation is cylindrically symmetrical about the z-axis, the incompressibility condition is given by (7·1). Introducing into the expression for τ the values of u, v and w given by (8·2),

$$\tau = \begin{vmatrix} \cos\psi z & -\sin\psi z & -\psi r\sin\psi z \\ \sin\psi z & \cos\psi z & \psi r\cos\psi z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos^2\psi z + \sin^2\psi z = 1.$$

So, the incompressibility condition is automatically satisfied by a deformation given by (8·2).

Introducing (8·2) into equations (7·2) and (7·3), and restricting the argument to the case of equilibrium, where $\partial^2 u/\partial t^2 = \partial^2 v/\partial t^2 = \partial^2 w/\partial t^2 = 0$, it is seen that the equations of motion for this case become

$$\alpha = \frac{\partial p}{\partial r}\cos\psi z, \quad \beta = \frac{\partial p}{\partial r}\sin\psi z \quad \text{and} \quad \gamma = \frac{\partial p}{\partial z}, \tag{8·3}$$

where

$$\left.\begin{aligned} \alpha &= -\rho R + \tfrac{1}{3}E\psi^2 r\cos\psi z, \\ \beta &= -\rho\Theta + \tfrac{1}{3}E\psi^2 r\sin\psi z \\ \gamma &= -\rho Z. \end{aligned}\right\} \tag{8·4}$$

and

If we make $Z = 0$, then $\partial p/\partial z = 0$ (from the last of equations (8·3) and the last of equations (8·4)), i.e. p is a function of r only. The first two of equations (8·4) are then simultaneously satisfied if $R = \Theta = 0$.

Thus, if the body forces R, Θ, Z are zero throughout the volume of the material, the equations of motion are satisfied, provided p is a function of r only. The form of p, as a function of r, can be found from the first two of equations (8·4). These become, putting $R = \Theta = 0$,

$$\tfrac{1}{3}E\psi^2 r = \partial p/\partial r.$$

This yields

$$p = \tfrac{1}{6}E\psi^2 r^2 + \text{const.} \tag{8·5}$$

The boundary conditions (7·5) over the curved surface of the cylinder become

$$\left.\begin{aligned} \tfrac{1}{3}E\cos\psi z - R_\nu &= -p\cos\psi z, \\ \tfrac{1}{3}E\sin\psi z - \Theta_\nu &= -p\sin\psi z \\ -Z_\nu &= 0. \end{aligned}\right\} \tag{8·6}$$

and

If the surface tractions on this cylindrical boundary are zero, i.e. $R_\nu = \Theta_\nu = Z_\nu = 0$, these three conditions can be simultaneously satisfied if $p = -\frac{1}{3}E$.

Introducing this value of p on the cylindrical boundary $r = a$ into equation (8·5), we find that the constant there has the value $(-\frac{1}{3}E - \frac{1}{6}E\psi^2 a^2)$.

Thus

$$p = -\tfrac{1}{6}E\psi^2(a^2 - r^2) - \tfrac{1}{3}E. \tag{8·7}$$

The boundary conditions (7·6) over the plane end $z = l$ of the cylinder become

$$\left.\begin{aligned} -\tfrac{1}{3}E\psi r\sin\psi l - R_\nu &= 0, \\ \tfrac{1}{3}E\psi r\cos\psi l - \Theta_\nu &= 0 \\ \tfrac{1}{3}E - Z_\nu &= -p. \end{aligned}\right\} \tag{8·8}$$

and

Thus, introducing the expression for p given in (8·7), we have

$$\left.\begin{aligned} R_\nu &= -\tfrac{1}{3}E\psi r\sin\psi l, \\ \Theta_\nu &= \tfrac{1}{3}E\psi r\cos\psi l \\ \text{and}\qquad Z_\nu &= -\tfrac{1}{6}E\psi^2(a^2-r^2). \end{aligned}\right\} \tag{8·9}$$

It should be borne in mind that the surface tractions R_ν, Θ_ν and Z_ν at a point of the surface are defined as being in the radial, azimuthal and axial directions respectively, at that point, in the undeformed state of the material. It is of advantage for the practical interpretation of the results to replace R_ν, Θ_ν and Z_ν by the components R'_ν, Θ'_ν and Z'_ν, parallel to the radial, azimuthal and axial directions respectively, at the corresponding point of the deformed body.

Now, the axial directions are not altered by the deformation and the displacement of each point lies in a plane at right angles to this axial direction. Therefore

$$Z'_\nu = Z_\nu. \tag{8·10}$$

Since the plane end of the cylinder considered turns through an angle ψl in the deformation,

$$R'_\nu = R_\nu\cos\psi l + \Theta_\nu\sin\psi l \quad\text{and}\quad \Theta'_\nu = \Theta_\nu\cos\psi l - R_\nu\sin\psi l. \tag{8·11}$$

Substituting in (8·10) and (8·11) from (8·9), we obtain

$$R'_\nu = 0, \quad \Theta'_\nu = \tfrac{1}{3}E\psi r \quad\text{and}\quad Z'_\nu = -\tfrac{1}{6}E\psi^2(a^2-r^2). \tag{8·12}$$

Thus the deformation described by equations (8·2) can be supported by the following set of surface tractions applied to the plane end of the cylinder:

(i) an azimuthal tangential traction increasing linearly from zero at the centre to $\frac{1}{3}E\psi a$ at the periphery; and

(ii) a normal compressive traction increasing from zero at the periphery to $\frac{1}{6}E\psi^2a^2$ at the centre.

These tractions are of course measured per unit area of the surface to which they are applied measured in the deformed state. However, in the deformation we are considering the area of an element of the end surface of the cylinder does not change.

The azimuthal forces have the nature of a couple. The total moment of this couple which must be applied to produce a torsion ψ is

$$\begin{aligned} \int_0^a r\Theta'_\nu\,2\pi r\,dr &= \int_0^a \tfrac{1}{3}E\psi r^2\,2\pi r\,dr \\ &= \tfrac{1}{6}\pi E\psi a^4. \end{aligned} \tag{8·13}$$

We note that it is proportional to the torsion produced.

The total compressive force which is exerted is

$$\begin{aligned} \int_0^a -Z'_\nu\,2\pi r\,dr &= \int_0^a \tfrac{1}{6}E\psi^2(a^2-r^2)\,2\pi r\,dr \\ &= \tfrac{1}{12}\pi E\psi^2a^4. \end{aligned} \tag{8·14}$$

We note that this force is proportional to the square of the torsion produced and therefore is negligibly small for very small torsions. Thus, vanishingly small torsions can be produced by a torsional couple proportional to the torsion, in accordance with the results of the classical theory of the torsion of a cylinder of incompressible, Hookean material.

If ϕ_l is the angle, in radians, through which one end of the cylinder is turned relative to the other,

$$\phi_l = \psi l.$$

Introducing this relation into the results (8·13) and (8·14), we see that the total couple which must be applied has moment

$$= \tfrac{1}{6}\pi E \frac{\phi_l}{l} a^4 \tag{8·15}$$

and the total compressive force

$$= \tfrac{1}{12}\pi E \left(\frac{\phi_l}{l}\right)^2 a^4. \tag{8·16}$$

The stress at any point of the deformed cylinder can be obtained from equations (4·4) by putting

$$s = 0, \quad \phi = \psi z \quad \text{and} \quad w = 0.$$

Also, since the value of p throughout the cylinder is given by (8·7), this expression can also be introduced into equations (4·4).

We obtain

$$\left.\begin{aligned} t_{rr} = -\tfrac{1}{6}E\psi^2(a^2-r^2), \quad t_{\theta\theta} = \tfrac{1}{3}E\psi^2 r^2 - \tfrac{1}{6}E\psi^2(a^2-r^2), \quad t_{zz} = -\tfrac{1}{6}E\psi^2(a^2-r^2), \\ t_{\theta z} = \tfrac{1}{3}E\psi r, \quad t_{zr} = 0 \quad \text{and} \quad t_{r\theta} = 0. \end{aligned}\right\} \tag{8·17}$$

The stress system is seen to be equivalent to a shearing stress $t_{\theta z}$ $(= \tfrac{1}{3}E\psi r)$ together with a normal stress $t_{\theta\theta}$ $(= \tfrac{1}{3}E\psi^2 r^2)$, acting azimuthally, and a superposed negative hydrostatic pressure of amount $-\tfrac{1}{6}E\psi^2(a^2-r^2)$.

The forces which must be applied to the ends of the cylinder to produce the simple torsion, given by (8·1) or (8·2), have been calculated in detail as regards their distribution over these ends. However, we can invoke Saint Venant's Principle to generalize this result in the case of a right-circular cylinder whose diameter is small compared with its length. Thus, provided the external forces are applied over, or close to, the ends of the cylinder and the total torsional couple and compressive force are given by (8·13) and (8·14) respectively, the torsion produced in the cylinder, at distances from the ends large compared with the diameter of the cylinder, will be given by (8·1) or (8·2). It must be borne in mind, however, that it has only been proved that this torsion represents a possible equilibrium state under the deduced system of forces and in view of the non-linearity of the equations of motion and boundary conditions for an incompressible, neo-Hookean material and the results obtained in Part II of this series, the possibility of alternative equilibrium states should be borne in mind. Analogous remarks will apply to the examples discussed in §§ 9 and 10 of this paper, but will not be repeated there.

It has already been pointed out (Rivlin 1948, I, § 11) that vulcanized rubbers behave approximately as incompressible, neo-Hookean materials. The system of forces (8·12) could be applied to a rubber cylinder by bonding on to its plane ends two metal plates and rotating these relative to each other through an angle ψl, while restraining their motion so that they remain parallel and a distance l apart. To do this the torsional couple (8·13) and compressive force (8·14) will have to be applied to the cylinder. If the compressive force is not applied and we consider for the moment that the metal plates are flexible, the rubber cylinder will tend to take up a form such as that shown in axial section by the full line in figure 1. The dotted line in the figure represents an axial section of the cylinder before deformation. The volumes before and after deformation must be equal.

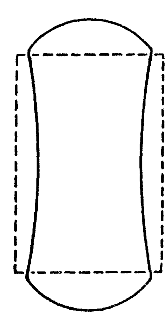

FIGURE 1

9. Simultaneous extension and torsion of a right-circular cylinder

In this section, we shall investigate the forces necessary to produce a combined uniform extension and torsion of a right-circular cylinder. Let us assume, as in § 8, that planes normal to the z-axis remain plane and suffer a translation w parallel to the z-axis, given by

$$w = (\lambda - 1)\, z, \tag{9·1}$$

where λ is a constant.

Suppose now that the cylinder is subjected to a uniform torsion, i.e. planes normal to the z-axis are rotated in their own plane through an angle ϕ given by

$$\phi = \psi\lambda z. \tag{9·2}$$

As a result of the deformation described by (9·1), the cylinder undergoes a radial contraction s, which, since the material is incompressible, is given by

$$r^2 z = (r+s)^2\, \lambda z;$$

i.e.

$$r + s = r/\surd\lambda.$$

The displacements u, v, w in the axial system x', y', z' (defined as in § 2) are now given by

$$u = -r\left(1 - \frac{1}{\surd\lambda}\cos\psi\lambda z\right), \quad v = \frac{r}{\surd\lambda}\sin\psi\lambda z \quad \text{and} \quad w = (\lambda - 1)\, z. \tag{9·3}$$

It can readily be seen that the incompressibility condition (7·1) is satisfied for the displacement (9·3), by direct substitution.

We now proceed to find the system of body and surface forces which is necessary to produce the deformation (9·3), as in § 8.

The equations of motion (7·2) become

$$\alpha = \surd\lambda\cos\psi\lambda z\frac{\partial p}{\partial r}, \quad \beta = \surd\lambda\sin\psi\lambda z\frac{\partial p}{\partial r} \quad \text{and} \quad \gamma = \frac{1}{\lambda}\frac{\partial p}{\partial z}, \tag{9·4}$$

where $\alpha = -\rho R + \frac{1}{3}E\psi^2\lambda^{\frac{3}{2}} r\cos\psi\lambda z$, $\beta = -\rho\Theta + \frac{1}{3}E\psi^2\lambda^{\frac{3}{2}} r\sin\psi\lambda z$ and $\gamma = -\rho Z$.

As in § 8, we see that the equations of motion (9·4) can be simultaneously satisfied by taking $\partial p/\partial z = 0$ and $R = \Theta = Z = 0$, i.e. no body forces are acting, provided that

$$\partial p/\partial r = \tfrac{1}{3}E\psi^2\lambda r;$$

i.e.

$$p = \tfrac{1}{6}E\psi^2\lambda r^2 + \text{const.} \tag{9·5}$$

The boundary conditions (7·5) over the cylindrical portion of the surface yield

$$R_\nu = \Theta_\nu = Z_\nu = 0$$

over this surface, taking $p = -\frac{1}{3}E/\lambda$ on the surface.

Introducing $p = -\frac{1}{3}E/\lambda$, when $r = a$, into (9·5), we have

$$p = -\tfrac{1}{6}E\psi^2\lambda(a^2 - r^2) - \frac{1}{3}\frac{E}{\lambda}. \tag{9·6}$$

The boundary conditions (7·6) for the plane end of the cylinder, which in the undeformed state forms part of $z = l$, become

$$R_\nu = -\tfrac{1}{3}E\psi\lambda^{\frac{1}{2}} r\sin\psi\lambda l, \quad \Theta_\nu = \tfrac{1}{3}E\psi\lambda^{\frac{1}{2}} r\cos\psi\lambda l \quad \text{and} \quad Z_\nu = \tfrac{1}{3}E\lambda + \frac{p}{\lambda}. \tag{9·7}$$

Introducing the value of p given in (9·6), and employing relations (8·11), in which ψl is replaced by $\psi\lambda l$, we obtain

$$R'_\nu = 0, \quad \Theta'_\nu = \tfrac{1}{3}E\psi\lambda^{\frac{1}{2}} r \quad \text{and} \quad Z'_\nu = \tfrac{1}{3}E\left(\lambda - \frac{1}{\lambda^2}\right) - \tfrac{1}{6}E\psi^2(a^2 - r^2). \tag{9·8}$$

It is thus seen that the combined simple extension and torsion of the cylinder can be produced by the combination of a couple and normal force applied to the ends of the cylinder.

10. Torsion of a hollow cylindrical tube

We shall now consider the torsion of a hollow, cylindrical tube of length l and internal and external radii b and a respectively, in the undeformed state. The axis of the tube is considered to coincide with the z-axis. Suppose this tube is subjected to a simple torsion defined as in § 8 by

$$\phi = \psi z.$$

Then, as before, u, v and w are given by equations (8·2). The incompressibility condition is automatically satisfied and the equations of motion (8·3) are satisfied for zero body forces, i.e. $R = \Theta = Z = 0$, if

$$\frac{\partial p}{\partial z} = 0 \quad \text{and} \quad \frac{\partial p}{\partial r} = \tfrac{1}{3}E\psi^2 r;$$

i.e.

$$p = \tfrac{1}{6}E\psi^2 r^2 + \text{const.} \tag{10·1}$$

The boundary conditions (7·5) over the curved surface $r = b$ become

$$\left.\begin{aligned} \tfrac{1}{3}E\cos\psi z + R_\nu &= -p_b\cos\psi z, \\ \tfrac{1}{3}E\sin\psi z + \Theta_\nu &= -p_b\sin\psi z \\ \text{and} \qquad Z_\nu &= 0, \end{aligned}\right\} \tag{10·2}$$

where p_b is the value of p on $r = b$.

The boundary conditions (7·5) over the curved surface $r = a$ become

$$\left.\begin{aligned} \tfrac{1}{3}E\cos\psi z - R_\nu &= -p_a\cos\psi z, \\ \tfrac{1}{3}E\sin\psi z - \Theta_\nu &= -p_a\sin\psi z \\ \text{and} \qquad Z_\nu &= 0, \end{aligned}\right\} \tag{10·3}$$

where p_a is the value of p on $r = a$.

We shall now consider two separate cases.

Case 1. $p_a = -\tfrac{1}{3}E$.

Then, from (10·3), $R_\nu = \Theta_\nu = Z_\nu = 0$, over the surface $r = a$ and, from (10·1),

$$p = -\tfrac{1}{6}E\psi^2(a^2 - r^2) - \tfrac{1}{3}E. \tag{10·4}$$

Then,

$$p_b = -\tfrac{1}{6}E\psi^2(a^2 - b^2) - \tfrac{1}{3}E.$$

Substituting this value in (10·2), we have

$$R_\nu = \tfrac{1}{6}E\psi^2(a^2 - b^2)\cos\psi z, \quad \Theta_\nu = \tfrac{1}{6}E\psi^2(a^2 - b^2)\sin\psi z \quad \text{and} \quad Z_\nu = 0. \tag{10·5}$$

Now, with the notation of § 8, we have

$$R'_\nu = \tfrac{1}{6}E\psi^2(a^2-b^2), \quad \Theta'_\nu = 0 \quad \text{and} \quad Z'_\nu = 0, \tag{10·6}$$

over the surface $r = b$.

It can readily be seen, as in § 8, that the surface tractions acting on the plane ends of the cylindrical shell are again given by (8·12).

Case 2. $p_b = -\tfrac{1}{3}E$.

In this case, we see, from (10·2), that $R_\nu = \Theta_\nu = Z_\nu = 0$ over the surface $r = b$. Also, from (10·1),

$$p = -\tfrac{1}{6}E\psi^2(b^2-r^2) - \tfrac{1}{3}E. \tag{10·7}$$

Then,

$$p_a = -\tfrac{1}{6}E\psi^2(b^2-a^2) - \tfrac{1}{3}E.$$

Substituting this value in (10·3), and proceeding as in Case 1, we have

$$R'_\nu = \tfrac{1}{6}E\psi^2(a^2-b^2), \quad \Theta'_\nu = 0 \quad \text{and} \quad Z'_\nu = 0, \tag{10·8}$$

over the surface $r = a$.

Again, it can readily be seen, as in § 8, that, in Case 2, the surface tractions acting on the plane ends of the cylindrical tube are given by expressions similar to (8·12) in which a is replaced by b.

In Case 1 the force system which must be exerted on the inner surface of the tube is a constant normal force acting outwards and could be produced by creating an appropriate positive pressure of gas inside the tube as compared with that outside the tube. If the gas pressures inside and outside the tube are equal, then the tube will tend to collapse under torsion.

It is of interest to note that if one end of a rubber tube is forced over the end of a glass tube and the other end of the rubber tube is turned in its own plane, so that the rubber is in torsion, it will tend to slip further over the glass tube owing to the absence of the normal traction Z'_ν, given by (8·12), on its end.

In a similar manner, it can be shown that if the tube is subjected to a combined simple extension and torsion, similar to that of the cylinder in § 9, so that the deformation is described by (9·3), the forces exerted on the ends of the tube are given by (9·8), or a similar expression in which a is replaced by b, and normal surface tractions of magnitude $\tfrac{1}{6}E\psi^2\lambda^{\frac{3}{2}}(a^2-b^2)$ per unit area of surface, measured in the undeformed state, acting along the positive direction of r, must be exerted on either the inner or outer curved surface of the tube. The area of either of these curved surfaces changes by the factor $\lambda^{\frac{1}{2}}$ in the extension and not at all in the torsion and consequently the normal surface traction which must be exerted has the magnitude $\tfrac{1}{6}E\psi^2\lambda(a^2-b^2)$ per unit area, measured in the deformed state.

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Reference

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