Finally, similar considerations can also be applied to quantities which characterize the flow but are not functions of the coordinates. Such a quantity is, for instance, the drag force F acting on the body. We can say that the dimensionless ratio of F to some quantity formed from ν, u, l, ρ and having the dimensions of force must be a function of the Reynolds number alone. Such a combination of ν, u, l, ρ can be $\rho u^2 l^2$, for example. Then

$$F = \rho u^2 l^2 f(\mathrm{R}). \tag{19.4}$$

If the force of gravity has an important effect on the flow, then the latter is determined not by three but by four parameters, l, u, ν and the acceleration g due to gravity. From these parameters we can construct not one but two independent dimensionless quantities. These can be, for instance, the Reynolds number and the *Froude number*, which is

$$\mathrm{F} = u^2/lg. \tag{19.5}$$

In formulae (19.2)–(19.4) the function f will now depend on not one but two parameters (R and F), and two flows will be similar only if both these numbers have the same values.

Finally, we may say a little regarding non-steady flows. A non-steady flow of a given type is characterized not only by the quantities ν, u, l but also by some time interval τ characteristic of the flow, which determines the rate of change of the flow. For instance, in oscillations, according to a given law, of a solid body, of a given shape, immersed in a fluid, τ may be the period of oscillation. From the four quantities ν, u, l, τ we can again construct two independent dimensionless quantities, which may be the Reynolds number and the number

$$\mathrm{S} = u\tau/l, \tag{19.6}$$

sometimes called the *Strouhal number*. Similar motion takes place in these cases only if both these numbers have the same values.

If the oscillations of the fluid occur spontaneously (and not under the action of a given external exciting force), then for motion of a given type S will be a definite function of R:

$$\mathrm{S} = f(\mathrm{R}).$$

§20. Flow with small Reynolds numbers

The Navier–Stokes equation is considerably simplified in the case of flow with small Reynolds numbers. For steady flow of an incompressible fluid, this equation is

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\mathbf{grad}\,p + (\eta/\rho)\triangle\,\mathbf{v}.$$

The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ is of the order of magnitude of u^2/l, u and l having the same meaning as in §19. The quantity $(\eta/\rho)\triangle\,\mathbf{v}$ is of the order of magnitude of $\eta u/\rho l^2$. The ratio of the two is just the Reynolds number. Hence the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ may be neglected if the Reynolds number is small, and the equation of motion reduces to a linear equation

$$\eta\triangle\,\mathbf{v} - \mathbf{grad}\,p = 0. \tag{20.1}$$

Together with the equation of continuity

$$\operatorname{div}\mathbf{v} = 0 \tag{20.2}$$

it completely determines the motion. It is useful to note also the equation

$$\triangle\,\mathbf{curl}\,\mathbf{v} = 0, \tag{20.3}$$

which is obtained by taking the curl of equation (20.1).

As an example, let us consider rectilinear and uniform motion of a sphere in a viscous fluid (G. G. Stokes 1851). The problem of the motion of a sphere, it is clear, is exactly equivalent to that of flow past a fixed sphere, the fluid having a given velocity **u** at infinity. The velocity distribution in the first problem is obtained from that in the second problem by simply subtracting the velocity **u**; the fluid is then at rest at infinity, while the sphere moves with velocity $-\mathbf{u}$. If we regard the flow as steady, we must, of course, speak of the flow past a fixed sphere, since, when the sphere moves, the velocity of the fluid at any point in space varies with time.

Since $\operatorname{div}(\mathbf{v}-\mathbf{u}) = \operatorname{div}\mathbf{v} = 0$, $\mathbf{v}-\mathbf{u}$ can be expressed as the curl of some vector **A**:

$$\mathbf{v}-\mathbf{u} = \mathbf{curl}\,\mathbf{A},$$

with **curl A** equal to zero at infinity. The vector **A** must be axial, in order for its curl to be polar, like the velocity. In flow past a sphere, a completely symmetrical body, there is no preferred direction other than that of **u**. This parameter **u** must appear linearly in **A**, because the equation of motion and its boundary conditions are linear. The general form of a vector function $\mathbf{A}(\mathbf{r})$ satisfying all these requirements is $\mathbf{A} = f'(r)\mathbf{n}\times\mathbf{u}$, where **n** is a unit vector parallel to the position vector **r** (the origin being taken at the centre of the sphere), and $f'(r)$ is a scalar function of r. The product $f'(r)\mathbf{n}$ can be represented as the gradient of another function $f(r)$. We shall thus look for the velocity in the form

$$\mathbf{v} = \mathbf{u} + \mathbf{curl}\,(\mathbf{grad}\,f\times\mathbf{u}) = \mathbf{u} + \mathbf{curl}\,\mathbf{curl}\,(f\mathbf{u}); \tag{20.4}$$

the last expression is obtained by noting that **u** is constant.

To determine the function f, we use equation (20.3). Since

$$\mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{curl}(f\mathbf{u}) = (\mathbf{grad}\,\operatorname{div} - \triangle)\,\mathbf{curl}(f\mathbf{u})$$
$$= -\triangle\,\mathbf{curl}(f\mathbf{u}),$$

(20.3) takes the form $\triangle^2\,\mathbf{curl}(f\mathbf{u}) = \triangle^2(\mathbf{grad}\,f\times\mathbf{u}) = (\triangle^2\mathbf{grad}\,f)\times\mathbf{u} = 0$. It follows from this that

$$\triangle^2\mathbf{grad}\,f = 0. \tag{20.5}$$

A first integration gives

$$\triangle^2 f = \text{constant}.$$

It is easy to see that the constant must be zero, since the velocity difference $\mathbf{v}-\mathbf{u}$ must vanish at infinity, and so must its derivatives. The expression $\triangle^2 f$ contains fourth derivatives of f, whilst the velocity is given in terms of the second derivatives of f. Thus we have

$$\triangle^2 f \equiv \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)\triangle f = 0.$$

Hence

$$\triangle f = 2a/r + c.$$

The constant c must be zero if the velocity $v-u$ is to vanish at infinity. From $\triangle f = 2a/r$ we obtain

$$f = ar + b/r. \tag{20.6}$$

The additive constant is omitted, since it is immaterial (the velocity being given by derivatives of f).

Substituting in (20.4), we have after a simple calculation

$$\mathbf{v} = \mathbf{u} - a\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + b\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})-\mathbf{u}}{r^3}. \tag{20.7}$$

The constants a and b have to be determined from the boundary conditions: at the surface of the sphere ($r = R$), $\mathbf{v} = 0$, i.e.

$$-\mathbf{u}\left(\frac{a}{R}+\frac{b}{R^3}-1\right)+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})\left(-\frac{a}{R}+\frac{3b}{R^3}\right)=0.$$

Since this equation must hold for all $\mathbf{n}$, the coefficients of $\mathbf{u}$ and $\mathbf{n}(\mathbf{u}\cdot\mathbf{n})$ must each vanish. Hence $a = \frac{3}{4}R$, $b = \frac{1}{4}R^3$. Thus we have finally

$$f = \tfrac{3}{4}Rr + \tfrac{1}{4}R^3/r, \tag{20.8}$$

$$\mathbf{v} = -\tfrac{3}{4}R\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} - \tfrac{1}{4}R^3\frac{\mathbf{u}-3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r^3} + \mathbf{u}, \tag{20.9}$$

or, in spherical polar components with the axis parallel to $\mathbf{u}$,

$$\left.\begin{aligned} v_r &= u\cos\theta\left[1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right],\\ v_\theta &= -u\sin\theta\left[1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right].\end{aligned}\right\} \tag{20.10}$$

This gives the velocity distribution about the moving sphere. To determine the pressure, we substitute (20.4) in (20.1):

$$\begin{aligned}\mathbf{grad}\,p &= \eta\triangle\mathbf{v} = \eta\triangle\,\mathbf{curl}\,\mathbf{curl}\,(f\mathbf{u})\\ &= \eta\triangle(\mathbf{grad}\,\mathrm{div}\,(f\mathbf{u}) - \mathbf{u}\triangle f).\end{aligned}$$

But $\triangle^2 f = 0$, and so

$$\mathbf{grad}\,p = \mathbf{grad}[\eta\triangle\mathrm{div}(f\mathbf{u})] = \mathbf{grad}(\eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f).$$

Hence

$$p = \eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f + p_0, \tag{20.11}$$

where p_0 is the fluid pressure at infinity. Substitution for f leads to the final expression

$$p = p_0 - \tfrac{3}{2}\eta\frac{\mathbf{u}\cdot\mathbf{n}}{r^2}R. \tag{20.12}$$

Using the above formulae, we can calculate the force $\mathbf{F}$ exerted on the sphere by the moving fluid (or, what is the same thing, the drag on the sphere as it moves through the fluid). To do so, we take spherical polar coordinates with the axis parallel to $\mathbf{u}$; by symmetry, all quantities are functions only of r and of the polar angle θ. The force $\mathbf{F}$ is evidently parallel to the velocity $\mathbf{u}$. The magnitude of this force can be determined from (15.14). Taking from this formula the components, normal and tangential to the surface, of the force on an element of the surface of the sphere, and projecting these components on the direction of $\mathbf{u}$, we find

$$F = \oint(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\mathrm{d}f, \tag{20.13}$$

where the integration is taken over the whole surface of the sphere.

Substituting the expressions (20.10) in the formulae

$$\sigma'_{rr} = 2\eta\frac{\partial v_r}{\partial r}, \qquad \sigma'_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

(see (15.20)), we find that at the surface of the sphere

$$\sigma'_{rr} = 0, \qquad \sigma'_{r\theta} = -(3\eta/2R)u\sin\theta,$$

while the pressure (20.12) is $p = p_0 - (3\eta/2R)u\cos\theta$. Hence the integral (20.13) reduces to $F = (3\eta u/2R)\oint \mathrm{d}f$. In this way we finally arrive at *Stokes' formula* for the drag on a sphere moving slowly in a fluid:†

$$F = 6\pi\eta Ru. \tag{20.14}$$

The drag is proportional to the velocity and linear size of the body. This could have been foreseen from dimensional arguments: the fluid density ρ does not appear in the approximate equations (20.1), (20.2), and so the force F which they give must be expressed only in terms of η, u and R; from these, only one combination with the dimensions of force can be formed, namely the product ηRu.

A similar dependence occurs for slowly moving bodies with other shapes. The direction of the drag on a body of arbitrary shape is not the same as that of the velocity; the general form of the dependence of **F** on **u** can be written

$$F_i = \eta a_{ik} u_k, \tag{20.15}$$

where a_{ik} is a tensor of rank two, independent of the velocity. It is important to note that this tensor is symmetrical, a result which holds in the linear approximation with respect to the velocity, and is a particular case of a general law valid for slow motion accompanied by dissipative processes (see *SP*1, §121).

Refinement of Stokes' formula

The above solution of the problem of flow past a sphere is not valid at large distances, even if the Reynolds number is small. To see this, let us estimate the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ neglected in (20.1). At large distances, $\mathbf{v} \cong \mathbf{u}$; the velocity derivatives there are of the order of uR/r^2, as is seen from (20.9). Hence $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim u^2R/r^2$. The terms retained in (20.1) are of the order of $\eta Ru/\rho r^3$, as can be seen from the same expression (20.9) for the velocity or (20.12) for the pressure. The condition $\eta Ru/\rho r^3 \gg u^2R/r^2$ is satisfied only for distances such that

$$r \ll \nu/u. \tag{20.16}$$

At greater distances, the terms neglected are not negligible, and the velocity distribution so found is incorrect.

† With a view to later applications, it may be mentioned that calculations with (20.7) and the constants a and b undetermined give

$$F = 8\pi\eta au. \tag{20.14a}$$

The drag can also be calculated for a slowly moving ellipsoid with any shape. The relevant formulae are given by H. Lamb, *Hydrodynamics*, 6th ed., §339, Cambridge 1932. Here we shall give the limiting expressions for a plane circular disk with radius R moving perpendicular to its plane:

$$F = 16\eta Ru,$$

and for a similar disk moving in its plane:

$$F = 32\eta Ru/3.$$

The moment of the frictional forces on both sides of the disk is

$$M = 2\int_0^R r\cdot 2\pi r\eta(\partial v/\partial z)_{z=0}\,\mathrm{d}r = \omega\theta_0\pi\sqrt{(\omega\rho\eta)}R^4\cos(\omega t-\tfrac{1}{4}\pi).$$

PROBLEM 4. Determine the flow between two parallel planes when there is a pressure gradient which varies harmonically with time.

SOLUTION. We take the xz-plane half-way between the two planes, with the x-axis parallel to the pressure gradient, which we write in the form

$$-(1/\rho)\partial p/\partial x = ae^{-i\omega t}.$$

The velocity is everywhere in the x-direction, and is determined by the equation

$$\partial v/\partial t = ae^{-i\omega t} + \nu\partial^2 v/\partial y^2.$$

The solution of this equation which satisfies the conditions $v = 0$ for $y = \pm\frac{1}{2}h$ is

$$v = \frac{ia}{\omega}e^{-i\omega t}\left[1-\frac{\cos ky}{\cos\frac{1}{2}kh}\right].$$

The mean value of the velocity over a cross-section is

$$\bar{v} = \frac{ia}{\omega}e^{-i\omega t}\left(1-\frac{2}{kh}\tan\tfrac{1}{2}kh\right).$$

For $h/\delta \ll 1$ this becomes

$$\bar{v} \cong ae^{-i\omega t}h^2/12\nu,$$

in agreement with (17.5), while for $h/\delta \gg 1$ we have

$$\bar{v} \cong (ia/\omega)e^{-i\omega t},$$

in accordance with the fact that in this case the velocity must be almost constant over the cross-section, varying only in a thin surface layer.

PROBLEM 5. Determine the drag on a sphere with radius R which executes translatory oscillations in a fluid.

SOLUTION. We write the velocity of the sphere in the form $\mathbf{u} = \mathbf{u}_0 e^{-i\omega t}$. As in §20, we seek the fluid velocity in the form $\mathbf{v} = e^{-i\omega t}\,\mathbf{curl}\,\mathbf{curl}\,f\mathbf{u}_0$, where f is a function of r only (the origin is taken at the instantaneous position of the centre of the sphere). Substituting in (24.9) and effecting transformations similar to those in §20, we obtain the equation

$$\triangle^2 f + (i\omega/\nu)\triangle f = 0$$

(instead of the equation $\triangle^2 f = 0$ in §20). Hence we have

$$\triangle f = \text{constant}\times e^{ikr}/r,$$

the solution being chosen which decreases exponentially with r. Integrating, we have

$$\mathrm{d}f/\mathrm{d}r = [ae^{ikr}(r-1/ik)+b]/r^2; \tag{1}$$

the function f itself is not needed, since only the derivatives f' and f'' appear in the velocity. The constants a and b are determined from the condition that $\mathbf{v} = \mathbf{u}$ for $r = R$, and are found to be

$$a = -\frac{3R}{2ik}e^{-ikR}, \qquad b = -\tfrac{1}{2}R^3\left(1-\frac{3}{ikR}-\frac{3}{k^2R^2}\right). \tag{2}$$

It may be pointed out that, at high frequencies ($R \gg \delta$), $a \to 0$ and $b \to -\frac{1}{2}R^3$, the values for potential flow obtained in §10, Problem 2; this is in accordance with what was said in §24.

The drag is calculated from formula (20.13), in which the integration is over the surface of the sphere. The result is

$$F = 6\pi\eta R\left(1+\frac{R}{\delta}\right)u + 3\pi R^2\sqrt{(2\eta\rho/\omega)}\left(1+\frac{2R}{9\delta}\right)\frac{\mathrm{d}u}{\mathrm{d}t}. \tag{3}$$

For $\omega = 0$ this becomes Stokes' formula, while for large frequencies we have

$$F = \tfrac{2}{3}\pi\rho R^3\frac{\mathrm{d}u}{\mathrm{d}t} + 3\pi R^2\sqrt{(2\eta\rho\omega)}u.$$

The first term in this expression corresponds to the inertial force in potential flow past a sphere (see §11, Problem 1), while the second gives the limit of the dissipative force. This second term could also have been found by calculating the energy dissipation according to (24.14); see Problem 6.

PROBLEM 6. Find the expression, in the limit of high frequencies ($\delta \ll R$), for the dissipative drag on an infinite cylinder with radius R oscillating at right angles to its axis.

SOLUTION. The velocity distribution round a cylinder at rest in a transverse flow is

$$\mathbf{v} = (R^2/r^2)[2\mathbf{n}(\mathbf{u}\cdot\mathbf{n}) - \mathbf{u}] - \mathbf{u};$$

see §10, Problem 3. From this, we find as the tangential velocity at the surface of the cylinder

$$v_\phi = -2u \sin\phi,$$

where r and ϕ are polar coordinates in the transverse plane, with ϕ measured from the direction of $\mathbf{u}$. From (24.14) we find the energy dissipated per unit length of the cylinder:

$$\dot{E}_{\text{kin}} = \pi u^2 R\sqrt{(2\eta\rho\omega)}.$$

Comparison with (24.15) and (24.16) gives the result

$$F_{\text{dis}} = 2\pi u R\sqrt{(2\eta\rho\omega)}.$$

PROBLEM 7. Determine the drag on a sphere moving in an arbitrary manner, the velocity being given by a function $u(t)$.

SOLUTION. We represent $u(t)$ as a Fourier integral:

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_\omega e^{-i\omega t}\,d\omega, \qquad u_\omega = \int_{-\infty}^{\infty} u(\tau)e^{i\omega\tau}\,d\tau.$$

Since the equations are linear, the total drag may be written as the integral of the drag forces for velocities which are the separate Fourier components $u_\omega e^{-i\omega t}$; these forces are given by (3) of Problem 5, and are

$$\pi\rho R^3 u_\omega e^{-i\omega t}\left\{\frac{6\nu}{R^2} - \frac{2i\omega}{3} + \frac{3\sqrt{(2\nu)}}{R}(1-i)\sqrt{\omega}\right\}.$$

Noticing that $(du/dt)_\omega = -i\omega u_\omega$, we can rewrite this as

$$\pi\rho R^3 e^{-i\omega t}\left\{\frac{6\nu}{R^2}u_\omega + \tfrac{2}{3}(\dot{u})_\omega + \frac{3\sqrt{(2\nu)}}{R}(\dot{u})_\omega\frac{1+i}{\sqrt{\omega}}\right\}.$$

On integration over $\omega/2\pi$, the first and second terms give respectively $u(t)$ and $\dot{u}(t)$. To integrate the third term, we notice first of all that for negative ω this term must be written in the complex conjugate form, $(1+i)\sqrt{\omega}$ being replaced by $(1-i)/\sqrt{|\omega|}$; this is because formula (3) of Problem 5 was derived for a velocity $u = u_0 e^{-i\omega t}$ with $\omega > 0$, and for a velocity $u_0 e^{i\omega t}$ we should obtain the complex conjugate. Instead of an integral over ω from $-\infty$ to $+\infty$, we can therefore take twice the real part of the integral from 0 to ∞. We write

$$\frac{1}{\pi}\operatorname{re}\left\{(1+i)\int_0^\infty \frac{(\dot{u})_\omega e^{-i\omega t}}{\sqrt{\omega}}\,d\omega\right\} = \frac{1}{\pi}\operatorname{re}\left\{(1+i)\int_{-\infty}^{\infty}\int_0^\infty \frac{\dot{u}(\tau)e^{i\omega(\tau-t)}}{\sqrt{\omega}}\,d\omega\,d\tau\right\}$$

$$= \frac{1}{\pi}\operatorname{re}\left\{(1+i)\int_{-\infty}^{t}\int_0^\infty \frac{\dot{u}(\tau)e^{-i\omega(t-\tau)}}{\sqrt{\omega}}\,d\omega\,d\tau + (1+i)\int_t^\infty\int_0^\infty \frac{\dot{u}(\tau)e^{i\omega(\tau-t)}}{\sqrt{\omega}}\,d\omega\,d\tau\right\}$$

$$= \sqrt{\frac{2}{\pi}}\operatorname{re}\left\{\int_{-\infty}^{t}\frac{\dot{u}(\tau)}{\sqrt{(t-\tau)}}\,d\tau + i\int_t^\infty \frac{\dot{u}(\tau)}{\sqrt{(\tau-t)}}\,d\tau\right\}$$

$$= \sqrt{\frac{2}{\pi}}\int_{-\infty}^{t}\frac{\dot{u}(\tau)}{\sqrt{(t-\tau)}}\,d\tau.$$

Thus we have finally for the drag

$$F = 2\pi\rho R^3 \left\{ \frac{1}{3}\frac{du}{dt} + \frac{3\nu u}{R^2} + \frac{3}{R}\sqrt{\frac{\nu}{\pi}} \int_{-\infty}^{t} \frac{du}{d\tau}\frac{d\tau}{\sqrt{(t-\tau)}} \right\}. \tag{4}$$

PROBLEM 8. Determine the drag on a sphere which at time $t = 0$ begins to move with a uniform acceleration, $u = \alpha t$.

SOLUTION. Putting, in formula (4) of Problem 7, $u = 0$ for $t < 0$ and $u = \alpha t$ for $t > 0$ we have for $t > 0$

$$F = 2\pi\rho R^3\alpha\left[\frac{1}{3} + \frac{3\nu t}{R^2} + \frac{6}{R}\sqrt{\frac{t\nu}{\pi}}\right].$$

PROBLEM 9. The same as Problem 8, but for a sphere brought instantaneously into uniform motion.

SOLUTION. We have $u = 0$ for $t < 0$ and $u = u_0$ for $t > 0$. The derivative du/dt is zero except at the instant $t = 0$, when it is infinite, but the time integral of du/dt is finite, and equals u_0. As a result, we have for all $t > 0$

$$F = 6\pi\rho\nu R u_0\left[1 + \frac{R}{\sqrt{(\pi\nu t)}}\right] + \tfrac{2}{3}\pi\rho R^3 u_0 \delta(t),$$

where $\delta(t)$ is the delta function. For $t \to \infty$ this expression tends asymptotically to the value given by Stokes' formula. The impulsive drag on the sphere at $t = 0$ is obtained by integrating the last term and is $\frac{2}{3}\pi\rho R^3 u_0$.

PROBLEM 10. Determine the moment of the forces on a sphere executing rotary oscillations about a diameter in a viscous fluid.

SOLUTION. For the same reasons as in §20, Problem 1, the pressure-gradient term can be omitted from the equation of motion, so that we have $\partial \mathbf{v}/\partial t = \nu \triangle \mathbf{v}$. We seek a solution in the form $\mathbf{v} = \mathbf{curl}\, f \boldsymbol{\Omega}_0 e^{-i\omega t}$, where $\boldsymbol{\Omega} = \boldsymbol{\Omega}_0 e^{-i\omega t}$ is the angular velocity of rotation of the sphere. We then obtain for f, instead of the equation $\triangle f = \text{constant}$,

$$\triangle f + k^2 f = \text{constant}.$$

Omitting an unimportant constant term in the solution of this equation, we find $f = ae^{ikr}/r$, taking the solution which vanishes at infinity. The constant a is determined from the boundary condition that $\mathbf{v} = \boldsymbol{\Omega}\times\mathbf{r}$ at the surface of the sphere. The result is

$$f = \frac{R^3}{r(1-ikR)}e^{ik(r-R)}, \qquad \mathbf{v} = (\boldsymbol{\Omega}\times\mathbf{r})\left(\frac{R}{r}\right)^3 \frac{1-ikr}{1-ikR}e^{ik(r-R)},$$

where R is the radius of the sphere. A calculation like that in §20, Problem 1, gives the following expression for the moment of the forces exerted on the sphere by the fluid:

$$M = -\frac{8\pi}{3}\eta R^3 \Omega \frac{3 + 6R/\delta + 6(R/\delta)^2 + 2(R/\delta)^3 - 2i(R/\delta)^2(1 + R/\delta)}{1 + 2R/\delta + 2(R/\delta)^2}.$$

For $\omega \to 0$ (i.e. $\delta \to \infty$), we obtain $M = -8\pi\eta R^3\Omega$, corresponding to uniform rotation of the sphere (see §20, Problem 1). In the opposite limiting case $R/\delta \gg 1$, we find

$$M = \frac{4\sqrt{2}}{3}\pi R^4 \sqrt{(\eta\rho\omega)}\,(i-1)\Omega.$$

This expression can also be obtained directly: for $\delta \ll R$ each element of the surface of the sphere may be regarded as plane, and the frictional force acting on it is found by substituting $u = \Omega R \sin\theta$ in formula (24.6).

PROBLEM 11. Determine the moment of the forces on a hollow sphere filled with viscous fluid and executing rotary oscillations about a diameter.

SOLUTION. We seek the velocity in the same form as in Problem 10. For f we take the solution $(a/r)\sin kr$, which is finite everywhere within the sphere, including the centre. Determining a from the boundary condition, we have

$$\mathbf{v} = (\boldsymbol{\Omega}\times\mathbf{r})\left(\frac{R}{r}\right)^3 \frac{kr\cos kr - \sin kr}{kR\cos kR - \sin kR}.$$