Deadline: Monday 16th December 2019

1. We were investigation the deformation of a right circular cylinder as shown in Figure 1. We already know that the stress filed in the cylinder is given (in the cylindrical coordinate system) by the expression

$$\mathbb{T} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{F}{S} \end{bmatrix}.$$

Show that the constitutive relation

$$\mathbb{T} = \lambda \left(\operatorname{Tr} \mathbb{E}\right) \mathbb{I} + 2\mu \mathbb{E},$$

where

$$\mathbb{E} =_{\mathrm{def}} \frac{1}{2} \left(\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^{\top} \right)$$

denotes the linearised strain, can be converted to the form

$$\mathbb{E} = \frac{1}{2\mu} \left(\mathbb{T} - \frac{\lambda}{3\lambda + 2\mu} \left(\operatorname{Tr} \mathbb{T}\right) \mathbb{I} \right). \tag{1}$$

Having the particular formula for $\mathbb{T}$, see above, solve (1) for the *displacement* $\boldsymbol{U}$. *Hint*: It might be again useful to use the cylindrical coordinate system. If we use cylindrical coordinates, and if the displacement $\boldsymbol{U}$ has components

$$\boldsymbol{U} = \begin{bmatrix} \mathrm{U}^{\hat{r}} \\ \mathrm{U}^{\hat{\varphi}} \\ \mathrm{U}^{\hat{z}} \end{bmatrix},$$

then the symmetric gradient of $\boldsymbol{U}$ is given by the formula

$$\frac{1}{2} \left(\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^{\top} \right) = \begin{bmatrix} \frac{\partial \mathrm{U}^{\hat{r}}}{\partial r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial \mathrm{U}^{\hat{r}}}{\partial \varphi} + \frac{\partial \mathrm{U}^{\hat{\varphi}}}{\partial r} - \frac{\mathrm{U}^{\hat{\varphi}}}{r} \right) & \frac{1}{2}\left(\frac{\partial \mathrm{U}^{\hat{r}}}{\partial z} + \frac{\partial \mathrm{U}^{\hat{z}}}{\partial r} \right) \\ \frac{1}{2}\left(\frac{1}{r}\frac{\partial \mathrm{U}^{\hat{r}}}{\partial \varphi} + \frac{\partial \mathrm{U}^{\hat{\varphi}}}{\partial r} - \frac{\mathrm{U}^{\hat{\varphi}}}{r} \right) & \frac{1}{r}\frac{\partial \mathrm{U}^{\hat{\varphi}}}{\partial \varphi} + \frac{\mathrm{U}^{\hat{r}}}{r} & \frac{1}{2}\left(\frac{\partial \mathrm{U}^{\hat{\varphi}}}{\partial z} + \frac{1}{r}\frac{\partial \mathrm{U}^{\hat{z}}}{\partial \varphi} \right) \\ \frac{1}{2}\left(\frac{\partial \mathrm{U}^{\hat{r}}}{\partial z} + \frac{\partial \mathrm{U}^{\hat{z}}}{\partial r} \right) & \frac{1}{2}\left(\frac{\partial \mathrm{U}^{\hat{\varphi}}}{\partial z} + \frac{1}{r}\frac{\partial \mathrm{U}^{\hat{z}}}{\partial \varphi} \right) & \frac{\partial \mathrm{U}^{\hat{z}}}{\partial z} \end{bmatrix}.$$

Once you find the displacement field, show that

$$\frac{F}{S} = \frac{\mu \left(3\lambda + 2\mu\right)}{\lambda + \mu} \frac{\Delta L}{L},$$

$$-\frac{\Delta R}{\Delta L} = \frac{\lambda}{2\left(\lambda + \mu\right)},$$

where ΔL denotes the change of the length of the cylinder in the direction of the z-axis, while ΔR denotes the change of the radius of the cylinder. The proportionality constant $\mathrm{E} =_{\mathrm{def}} \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is referred to as the Young's modulus, while the constant $\nu =_{\mathrm{def}} \frac{\lambda}{2(\lambda+\mu)}$ is referred to as the Poisson's ratio.

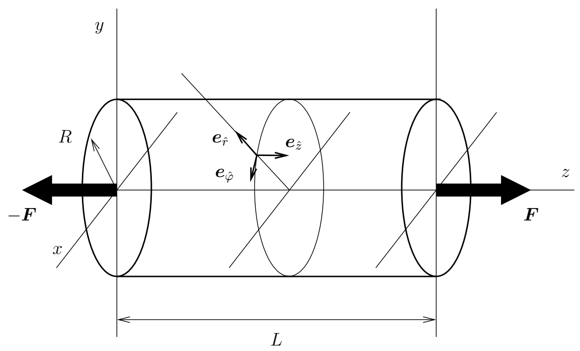

Figure 1: Deformation of a right circular cylinder.

Deadline: Monday 9th December 2019

1. Let $\mathbb{T}_{\mathrm{R}}$ and $\mathbb{T}$ denote the first Piola–Kirchhoff tensor and the Cauchy stress tensor respectively. Show that

$$\operatorname{Div} \mathbb{T}_{\mathrm{R}} = \left(\det \mathbb{F}\right) \operatorname{div} \mathbb{T},$$

or, in detail, that

$$\operatorname{Div}_{\boldsymbol{X}} \mathbb{T}_{\mathrm{R}}(\boldsymbol{X}, T) = \left(\det \mathbb{F}(\boldsymbol{X}, t)\right) \left.\left(\operatorname{div}_{\boldsymbol{x}} \mathbb{T}(\boldsymbol{x}, t)\right)\right|_{\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)},$$

where $\mathbb{F}$ denotes the deformation gradient and $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is the deformation. (Direct differentiation is not a good idea.)

Deadline: Monday 2nd December 2019

1. Consider the energetic equation of state for the calorically perfect ideal gas $e(\eta, \rho)$, and show that the specific Helmholtz free energy for the calorically perfect ideal gas is given by the formula

$$\psi(\theta, \rho) = -c_{\mathrm{V}} \theta \left(\ln \left(\frac{\theta}{\theta_{\mathrm{ref}}} \right) - 1 \right) + c_{\mathrm{V}} \theta \left(\gamma - 1 \right) \ln \left(\frac{\rho}{\rho_{\mathrm{ref}}} \right).$$

2. Consider a homogeneous isotropic elastic solid with the Helmholtz free energy in the form

$$\psi = \psi(\theta, \mathbb{B}), \tag{2}$$

and show that the corresponding evolution equation for the entropy reads

$$\rho \frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{\theta} \left[\left(\mathbb{T} - 2\rho \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B} \right) : \mathbb{D} - \operatorname{div} \boldsymbol{j}_e \right].$$

Use this equation and argue that the material does not produce entropy in mechanical processes (our definition of elastic material) provided that the Cauchy stress tensor is related to the derivative of the Helmholtz free energy via the formula

$$\mathbb{T} = 2\rho \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B}. \tag{3}$$

Since the material is isotropic, the Helmholtz free energy must be in fact a function of the invariants of $\mathbb{B}$,

$$\psi = \psi(\theta, \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3), \tag{4}$$

where the invariants are given by the formulae

$$\mathrm{I}_1 =_{\mathrm{def}} \operatorname{Tr} \mathbb{B}, \qquad \mathrm{I}_2 =_{\mathrm{def}} \frac{1}{2} \left((\operatorname{Tr} \mathbb{B})^2 - \operatorname{Tr} \mathbb{B}^2 \right), \qquad \mathrm{I}_3 =_{\mathrm{def}} \det \mathbb{B}.$$

(Recall that we have the *representation theorem for scalar isotropic functions.*) Show that if we use (4) in (3), then the formula for the Cauchy stress tensor reads

$$\mathbb{T} = \beta_0 \mathbb{I} + \beta_1 \mathbb{B} + \beta_{-1} \mathbb{B}^{-1}, \tag{5a}$$

where

$$\beta_0 =_{\mathrm{def}} 2\rho \left(\mathrm{I}_2 \frac{\partial \psi}{\partial \mathrm{I}_2} + \mathrm{I}_3 \frac{\partial \psi}{\partial \mathrm{I}_3} \right), \tag{5b}$$

$$\beta_1 =_{\mathrm{def}} 2\rho \frac{\partial \psi}{\partial \mathrm{I}_1}, \tag{5c}$$

$$\beta_{-1} =_{\mathrm{def}} -2\rho \mathrm{I}_3 \frac{\partial \psi}{\partial \mathrm{I}_2}. \tag{5d}$$

Note that the corresponding formulae are in the most textbooks on solid mechanics written in a slightly different form. The reason is that the Helmholtz free energy is frequently defined as the Helmholtz free energy per unit *volume*, while we have defined the Helmholtz free energy as the energy per unit *mass*.

3. [Optional] Some people prefer to write (2) as

$$\psi = \psi(\theta, J, \overline{\mathbb{B}}),$$

where $J =_{\mathrm{def}} \det \mathbb{F}$ and

$$\overline{\mathbb{B}} =_{\mathrm{def}} \frac{\mathbb{B}}{J^{\frac{2}{3}}}.$$

This decomposition is motivated by the fact that J is related to the volume-changing part of the deformation, while $\overline{\mathbb{B}}$ characterises the volume-preserving part of the deformation. (Check that $\det \overline{\mathbb{B}} = 1$.) Show that in this case the counterpart of (3) is

$$\mathbb{T} = \rho J \frac{\partial \psi}{\partial J} \mathbb{I} + 2\rho \left(\frac{\partial \psi}{\partial \overline{\mathbb{B}}} \overline{\mathbb{B}} \right)_{\delta},$$

where $\mathbb{A}_{\delta} =_{\mathrm{def}} \mathbb{A} - \frac{1}{3} (\operatorname{Tr} \mathbb{A}) \mathbb{I}$ denotes the traceless part of the corresponding tensorial quantity.

There is no lecture on Monday 25th November 2019 and on Wednesday 27th November 2019.

Deadline: Monday 4th November 2019

1. Using the assumption that the internal energy e is a function of the density ρ and the yet unknown quantity referred to as the *entropy* η, that is $e = e(\eta, \rho)$, we were able to show that the quantity η has all the desired properties provided that $e = e(\eta, \rho)$ satisfies partial differential equations

$$\frac{\partial e}{\partial \eta}(\eta, \rho) = \theta, \tag{6a}$$

$$\rho^2 \frac{\partial e}{\partial \rho}(\eta, \rho) = p. \tag{6b}$$

Let us assume that our substance of interest is the *calorically perfect ideal gas*. We know that the equation of state for this substance reads

$$p = c_{\mathrm{V}} (\gamma - 1) \rho\theta, \tag{7a}$$

where c_{V} is a positive constant (specific heat at constant volume) and γ is a positive constant greater that one (adiabatic exponent). Symbol θ denotes the absolute temperature. (This is the way how one writes the equation of state $PV = nRT$ in the continuum mechanics setting. We pretend that we do not know anything about atoms, hence we can hardly use concepts such as "molar mass" or "molar gas constant" and so forth.) Further, we also know that the internal energy of the substance is proportional to the temperature

$$e = c_{\mathrm{V}}\theta. \tag{7b}$$

Note that in the last equation we write e as a *function of the density and the temperature*, that is it should read $e(\rho, \theta) = c_{\mathrm{V}}\theta$. As you probably know, the relations (7) are outcomes of experiments, see Gay-Lussac, Boyle, Charles and Joule expansion.

Now use characterisations (7) and solve the partial differential equations (6) for the internal energy e as a *function of the entropy and the density.* Once you find the function $e(\eta, \rho)$, find also explicit formula for the entropy as a function of the temperature and the density, $\eta(\theta, \rho)$. You should obtain something like

$$\eta(\theta, \rho) = c_{\mathrm{V}} \ln \left[\frac{\theta}{\theta_{\mathrm{ref}}} \left(\frac{\rho}{\rho_{\mathrm{ref}}} \right)^{1-\gamma} \right].$$

Deadline: Monday 4th November 2019

1. Let $\boldsymbol{v}$ be a smooth vector field in $\Omega \subset \mathbb{R}^3$, and let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Show that there exist a scalar field φ and a vector field $\boldsymbol{A}$ such that

$$\boldsymbol{v} = -\nabla\varphi + \operatorname{rot} \boldsymbol{A},$$

where

$$\varphi(\boldsymbol{r}) =_{\mathrm{def}} -\frac{1}{4\pi} \int_{\Omega} \frac{\operatorname{div}_{\boldsymbol{r}'} \boldsymbol{v}(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{r}|} \,\mathrm{d}v' + \frac{1}{4\pi} \int_{\partial\Omega} \frac{\boldsymbol{v}(\boldsymbol{r}') \bullet \boldsymbol{n}}{|\boldsymbol{r}' - \boldsymbol{r}|} \mathrm{d}s',$$

$$\boldsymbol{A}(\boldsymbol{r}) =_{\mathrm{def}} -\frac{1}{4\pi} \int_{\Omega} \frac{\operatorname{rot}_{\boldsymbol{r}'} \boldsymbol{v}(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{r}|} \,\mathrm{d}v' - \frac{1}{4\pi} \int_{\partial\Omega} \frac{\boldsymbol{v}(\boldsymbol{r}') \times \boldsymbol{n}}{|\boldsymbol{r}' - \boldsymbol{r}|} \,\mathrm{d}s'.$$

Function φ is called the *scalar potential* and function $\boldsymbol{A}$ is called the *vector potential* of the vector field $\boldsymbol{v}$. This theorem is referred to as Helmholtz decomposition, hence if you need a hint regarding the proof, you can search the Internet for keyword "Helmholtz decomposition".

There is no lecture on Monday 28th October 2019, Monday is a public holiday in the Czech republic. We are celebrating Czechoslovak Independence Day.

Deadline: Monday 21st October 2019

1. Plot the following velocity fields

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix},$$

and calculate $\operatorname{rot} \boldsymbol{v}_i$, $i = 1, \dots, 3$. Think of the following vague statement: "If the fluid is rotating, then $\operatorname{rot} \boldsymbol{v} \neq \boldsymbol{0}$." In which sense is this statement true or false?

2. Show that

$$\nabla(\varphi \boldsymbol{w}) = \boldsymbol{w} \otimes \nabla\varphi + \varphi\nabla\boldsymbol{w},$$
$$\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{w}) = [\nabla\boldsymbol{u}]\,\boldsymbol{w} + \boldsymbol{u} \operatorname{div} \boldsymbol{w},$$
$$\operatorname{div}(\varphi\mathbb{A}) = \mathbb{A}(\nabla\varphi) + \varphi \operatorname{div} \mathbb{A},$$

where φ, $\boldsymbol{v}$ and $\mathbb{A}$ denote a smooth scalar, vector and tensor field respectively.

3. [Optional] If the previous problem is too boring, you can try to prove that

$$\int_{\Omega} \operatorname{rot} \boldsymbol{v} \, \mathrm{dv} = - \int_{\partial\Omega} \boldsymbol{v} \times \boldsymbol{n} \, \mathrm{ds},$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, and $\boldsymbol{v}$ is a smooth vector field. (Yes, the integral on the left-hand side is a volume integral. There is no misprint in the formula.)

Deadline: Monday 14th October 2019

1. Let $\boldsymbol{a}$ be a unit vector and let $\boldsymbol{u}$ be an arbitrary vector. Show that vectors $(\mathbb{I} - \boldsymbol{a} \otimes \boldsymbol{a}) \boldsymbol{u}$ and $\boldsymbol{a} \times \boldsymbol{u}$ have the same length, that is

$$|(\mathbb{I} - \boldsymbol{a} \otimes \boldsymbol{a}) \boldsymbol{u}| = |\boldsymbol{a} \times \boldsymbol{u}| .$$

In solving this problem, you should showcase your mastery in the manipulations with Levi–Civita symbol.

2. [Optional] If the previous problem is too boring, you can also try to prove that

$$\operatorname{Tr} \mathbb{A} = \frac{\mathbb{A}\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u} \bullet (\mathbb{A}\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{u} \bullet (\boldsymbol{v} \times \mathbb{A}\boldsymbol{w})}{\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})},$$

where $\mathbb{A} \in \mathbb{R}^{3\times3}$ is a given matrix and $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are arbitrary non-coplanar vectors.

3. Show that if $\boldsymbol{a}$ is a given unit vector and φ is a given angle, then the matrix $\mathbb{Q}$ representing the rotation along the axis $\boldsymbol{a}$ by the angle φ is given by the formula

$$\mathbb{Q} = \mathbb{I} + (\sin\varphi) \mathbb{A} + (1 - \cos\varphi) \mathbb{A}^2,$$

where matrix $\mathbb{A}$ is the matrix associated to the axial vector $\boldsymbol{a}$. The solution of this problem is a little bit tricky, if you are in trouble in solving this problem, you can try to search the Internet for keywords "Rodrigues formula" or "axis-angle representation".

Deadline: Monday 7th October 2019

1. Let $\mathbb{A} \in \mathbb{R}^{3\times3}$ be an invertible matrix. Show that

$$\mathrm{I}_2(\mathbb{A}) = \operatorname{Tr}(\operatorname{cof} \mathbb{A}),$$

where $\operatorname{cof} \mathbb{A} =_{\mathrm{def}} (\det \mathbb{A}) \mathbb{A}^{-\top}$ denotes the cofactor matrix of matrix $\mathbb{A}$, and $\mathrm{I}_2(\mathbb{A}) =_{\mathrm{def}} \frac{1}{2}\left((\operatorname{Tr}\mathbb{A})^2 - \operatorname{Tr}\left(\mathbb{A}^2\right)\right)$ is the second invariant of matrix $\mathbb{A}$.

2. Let $\mathbb{A} \in \mathbb{R}^{3\times3}$ a $\mathbb{B} \in \mathbb{R}^{3\times3}$ be invertible matrices. Show that

$$\det(\mathbb{A} + \mathbb{B}) = \det\mathbb{A} + \operatorname{Tr}\left(\mathbb{A}^\top \operatorname{cof}\mathbb{B}\right) + \operatorname{Tr}\left(\mathbb{B}^\top \operatorname{cof}\mathbb{A}\right) + \det\mathbb{B},$$

where $\operatorname{cof} \mathbb{C} =_{\mathrm{def}} (\det \mathbb{C}) \mathbb{C}^{-\top}$ denotes the cofactor matrix of matrix $\mathbb{C}$.