1. Let us assume that the material of interest is an isotropic elastic solid with the constitutive relation

$$\mathbb{T} = \lambda \left(\operatorname{Tr} \mathbb{\varepsilon}\right) \mathbb{I} + 2\mu \mathbb{\varepsilon},$$

and let us assume that we are interested in the static problem in which the balance of linear momentum reduces to

$$\mathbf{0} = \operatorname{div} \mathbb{T} + \boldsymbol{f},$$

where $\boldsymbol{f} =_{\mathrm{def}} \rho_{\mathrm{R}} \boldsymbol{b}$. Show that the compatibility conditions for the linearised strain

$$\operatorname{rot} \left(\operatorname{rot} \mathbb{\varepsilon}\right)^{\top} = \mathbb{O},$$

the constitutive relation and the balance of linear momentum imply that $\mathbb{T}$ solves the system

$$\Delta \mathbb{T} + \frac{1}{1+\nu} \nabla \left(\nabla \left(\operatorname{Tr} \mathbb{T}\right)\right) = -\left(\nabla \boldsymbol{f} + \left(\nabla \boldsymbol{f}\right)^{\top}\right) - \frac{\nu}{1-\nu} \left(\operatorname{div} \boldsymbol{f}\right) \mathbb{I}, \tag{1}$$

This equation is referred to as Beltrami–Michell equation. (The symbol ν denotes the Poisson's ratio.) You might find that it is more convenient to use the compatibility conditions rewritten in the form

$$\frac{\partial^2 \varepsilon_{ik}}{\partial x_l \partial x_j} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_l \partial x_i} = \frac{\partial^2 \varepsilon_{il}}{\partial x_k \partial x_j} - \frac{\partial^2 \varepsilon_{jl}}{\partial x_k \partial x_i}.$$

There is no need to prove the equivalence between $\operatorname{rot} \left(\operatorname{rot} \mathbb{\varepsilon}\right)^{\top} = \mathbb{O}$ and this formula.