Deadline: Tuesday 27th April 2021

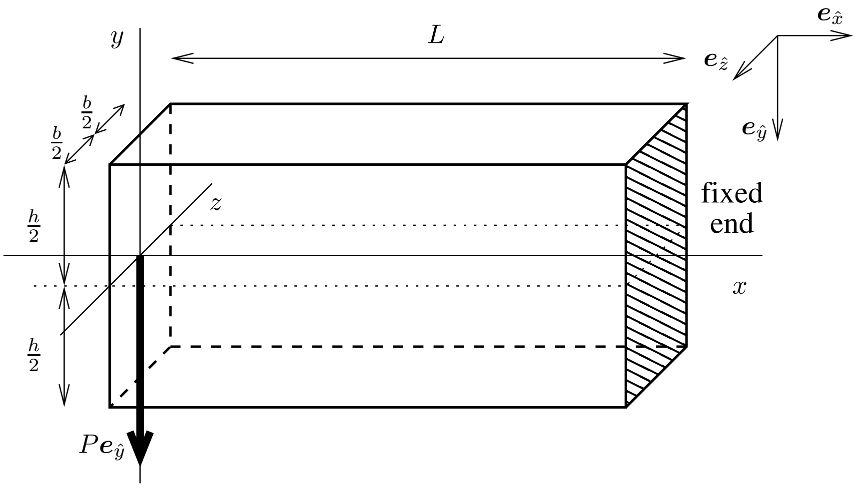

Figure 1: Bending of a narrow rectangular beam by an end load.

1. Consider the problem of bending of a narrow rectangular beam by an end load, see Figure **??**. Find the deflection of the free end.

2.
$$
w\big|_{x=L} = 0, \qquad \left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=L} = 0, \qquad \left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=0} = 0, \qquad wx\big|_{x=0} = \frac{P}{\mathrm{EI}_{\hat{z}\hat{z}}}.
$$

Deadline: Tuesday 20th April 2021

Consider an infinite plate with an elliptic hole, see Figure 1. The plate is uniformly stressed (at infinity), and the boundary of the elliptical hole is traction-free. Find the stress distribution in the plate. Assume that the problem can be solved in the framework of linearised elasticity, and that the material of interest is an isotropic homogeneous elastic solid. Assume that the problem can be solved as a plane strain problem. In solving the problem use the Airy stress function.

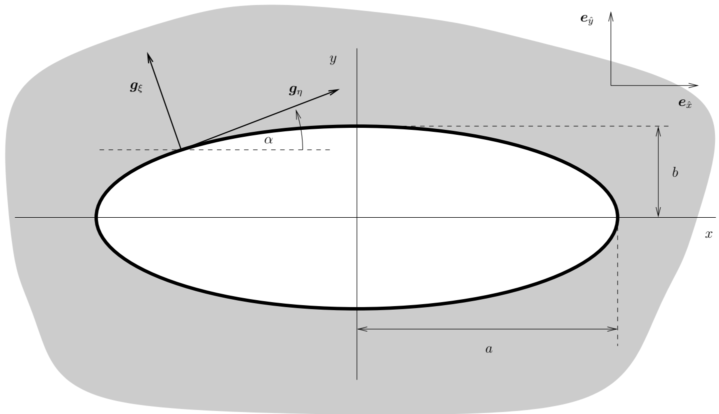

Figure 2: Elliptical hole in an infinite plate.

1. First we need to introduce curvilinear coordinate system well suited to the problem. We use elliptic coordinates defined by the complex function
$$
z = c\cosh\zeta, \tag{1}
$$
where $z = x + \mathrm{i}y$ and $\zeta = \xi + \mathrm{i}\eta$ denote complex numbers and x, y, ξ, η and c denote real numbers. Show that (1) implies that
$$
x = c\cosh\xi\cos\eta, \tag{2a}
$$
$$
y = c\sinh\xi\sin\eta, \tag{2b}
$$

and that

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1, \tag{3a}$$

$$\frac{x^2}{c^2\cos^2\eta} - \frac{y^2}{c^2\sin^2\eta} = 1. \tag{3b}$$

This justifies the nomenclature "elliptic coordinates". (The lines of constant ξ are ellipses.) Finally, show that

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = c\sinh\zeta. \tag{4}$$

The last formula is worthwhile in various chain-rule based manipulations.

2. Show that if we consider an ellipse with major axis a and minor axis b, then we can fix

$$c^2 =_{\mathrm{def}} a^2 - b^2, \tag{5}$$

and the ellipse is the curve parametrised by (2) where $\eta \in (0, 2\pi)$ and $\xi =_{\mathrm{def}} \xi_0$. The constant ξ_0 is implicitly given by the equation $\sinh 2\xi_0 = \frac{2ab}{c^2}$ or $\cosh 2\xi_0 = \frac{a^2+b^2}{c^2}$.

3. Show that the tangent vectors to the coordinate lines are given by the formulae

$$\boldsymbol{g}_\xi = J^{\frac{1}{2}}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}, \qquad \boldsymbol{g}_\eta = J^{\frac{1}{2}}\begin{bmatrix}-\sin\alpha\\ \cos\alpha\end{bmatrix}, \tag{6}$$

where

$$J =_{\mathrm{def}} c^2\left(\sinh^2\xi\cos^2\eta + \cosh^2\xi\sin^2\eta\right) \tag{7}$$

and α is the angle defined as a solution to

$$\mathrm{e}^{2\mathrm{i}\alpha} = \frac{\sinh\zeta}{\sinh\overline{\zeta}}. \tag{8}$$

This means that the relation between the canonical basis in the Cartesian coordinate system and the basis in the elliptical coordinate system is

$$\begin{bmatrix}\boldsymbol{g}_{\hat{\xi}}\\ \boldsymbol{g}_{\hat{\eta}}\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_{\hat{x}}\\ \boldsymbol{e}_{\hat{y}}\end{bmatrix}. \tag{9}$$

(Here we slightly abuse the notation.)

4. Show that the relation between Cartesian components of the Cauchy stress tensor $\mathbb{T}$ and the components of the Cauchy stress tensor $\mathbb{T}$ with respect to the elliptical coordinate system reads

$$\tau^{\hat{\xi}\hat{\xi}} = \frac{1}{2}\left(\tau^{\hat{x}\hat{x}} + \tau^{\hat{y}\hat{y}}\right) + \frac{1}{2}\left(\tau^{\hat{x}\hat{x}} - \tau^{\hat{y}\hat{y}}\right)\cos 2\alpha + \tau^{\hat{x}\hat{y}}\sin 2\alpha, \tag{10a}$$

$$\tau^{\hat{\eta}\hat{\eta}} = \frac{1}{2}\left(\tau^{\hat{x}\hat{x}} + \tau^{\hat{y}\hat{y}}\right) - \frac{1}{2}\left(\tau^{\hat{x}\hat{x}} - \tau^{\hat{y}\hat{y}}\right)\cos 2\alpha - \tau^{\hat{x}\hat{y}}\sin 2\alpha, \tag{10b}$$

$$\tau^{\hat{\xi}\hat{\eta}} = -\frac{1}{2}\left(\tau^{\hat{x}\hat{x}} - \tau^{\hat{y}\hat{y}}\right)\sin 2\alpha + \tau^{\hat{x}\hat{y}}\cos 2\alpha, \tag{10c}$$

which yields

$$\tau^{\hat{\xi}\hat{\xi}} + \tau^{\hat{\eta}\hat{\eta}} = \tau^{\hat{x}\hat{x}} + \tau^{\hat{y}\hat{y}}, \tag{11a}$$

$$\tau^{\hat{\eta}\hat{\eta}} - \tau^{\hat{\xi}\hat{\xi}} + 2\mathrm{i}\tau^{\hat{\xi}\hat{\eta}} = \mathrm{e}^{2\mathrm{i}\alpha}\left(\tau^{\hat{y}\hat{y}} - \tau^{\hat{x}\hat{x}} + 2\mathrm{i}\tau^{\hat{x}\hat{y}}\right). \tag{11b}$$

(Equality (11a) is not surprising, the trace of a matrix is invariant with respect to the choice of a basis.)

5. We already know that the solution to the biharmonic equation $\Delta\Delta\Phi = 0$ can be sought in the form

$$\Phi = \Re\left(\overline{z}\psi(z) + \chi(z)\right), \tag{12}$$

where ψ and χ are complex functions and $\Re$ denotes the real part of the given complex number. The stress components are related to ψ and χ via the formulae

$$\tau^{\hat{x}\hat{x}} + \tau^{\hat{y}\hat{y}} = 4\Re\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right), \tag{13a}$$

$$\tau^{\hat{y}\hat{y}} - \tau^{\hat{x}\hat{x}} + 2\mathrm{i}\tau^{\hat{x}\hat{y}} = 2\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right). \tag{13b}$$

The boundary condition at infinity reads

$$\mathbb{T}|_{x^2+y^2\to+\infty} = S\mathbb{I}, \tag{14}$$

where $S \in \mathbb{R}$ is a given number. (The stress is at infinity a spherical tensor — the material is at infinity in the state of uniform tension.) Show that this boundary condition (14) implies that

$$2\Re\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)\bigg|_{z\to+\infty} = S, \tag{15a}$$

$$\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right)\bigg|_{z\to+\infty} = 0. \tag{15b}$$

The boundary condition on the ellipse is traction-free boundary condition, that is

$$\mathbb{T}\boldsymbol{n}|_{\xi=\xi_0} = \boldsymbol{0}, \tag{16}$$

where n is the unit normal to the ellipse. Show that this boundary condition reduces to

$$\tau^{\hat{\xi}\hat{\xi}}\Big|_{\xi=\xi_0} = 0, \tag{17a}$$

$$\tau^{\hat{\xi}\hat{\eta}}\Big|_{\xi=\xi_0} = 0, \tag{17b}$$

which has to hold for all $\eta \in (0, 2\pi)$. Show that boundary condition (17) implies the following condition for functions ψ and ξ,

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}z} + \overline{\frac{\mathrm{d}\psi}{\mathrm{d}z}}\right)\bigg|_{\xi=\xi_0} = \left(\mathrm{e}^{2\mathrm{i}\alpha}\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right)\right)\bigg|_{\xi=\xi_0}, \tag{18}$$

that has to hold for all $\eta \in (0, 2\pi)$.

6. The problem now reduces to the task to find two complex functions ψ and χ that satisfy the conditions

$$2\Re\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)\bigg|_{z\to+\infty} = S, \tag{19a}$$

$$\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right)\bigg|_{z\to+\infty} = 0, \tag{19b}$$

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}z} + \overline{\frac{\mathrm{d}\psi}{\mathrm{d}z}}\right)\bigg|_{\xi=\xi_0} = \left(\mathrm{e}^{2\mathrm{i}\alpha}\left(z\overline{\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}} + \overline{\frac{\mathrm{d}^2\chi}{\mathrm{d}z^2}}\right)\right)\bigg|_{\xi=\xi_0}. \tag{19c}$$

Show that these conditions are fulfilled provided that we set

$$\psi = Ac\sinh\zeta, \tag{20a}$$

$$\xi = Bc^2\zeta, \tag{20b}$$

where the constants A and B are given by the formulae

$$A = \frac{S}{2}, \tag{20c}$$

$$B = -A\cosh 2\xi_0. \tag{20d}$$

(We are again slightly abusing the notation, since ψ and χ are functions of z, but z and ζ are related via the transformation (1).)

7. Having obtained the solution in terms of ψ and ξ,

$$\psi = \frac{1}{2}Sc\sinh\zeta, \tag{21a}$$

$$\chi = -\frac{1}{2}Sc^2\left(\cosh 2\xi_0\right)\zeta, \tag{21b}$$

we are ready to investigate the stress field. Show that

$$\tau^{\hat{\eta}\hat{\eta}}\Big|_{\xi=\xi_0} = \frac{2S\sinh 2\xi}{\cosh 2\xi - \cos 2\eta}\bigg|_{\xi=\xi_0}, \tag{22}$$

and that

$$\max_{\eta\in(0,2\pi)} \tau^{\hat{\eta}\hat{\eta}}\Big|_{\xi=\xi_0} = 2S\frac{a}{b}, \tag{23a}$$

$$\max_{\eta\in(0,2\pi)} \tau^{\hat{\eta}\hat{\eta}}\Big|_{\xi=\xi_0} = 2S\frac{b}{a}, \tag{23b}$$

while the maximum is attained at $\eta = 0$ and $\eta = \pi$, and the minimum is attained at $\eta = \frac{\pi}{2}$ and $\eta = \frac{3}{2}\pi$. This means that the maximum value of $\tau^{\hat{\eta}\hat{\eta}}$ is attained at the tip of the ellipse with the largest curvature, while the minimum is attained at points where the curvature is the smallest.

Deadline: Tuesday 6th April 2021

Consider a right circular cylinder of radius R_{in} and height L, see Figure 2, and assume that the cylinder is in this configuration in a stress free state. Further, assume that the material of which is the cylinder made is a homogeneous isotropic *incompressible* elastic material where the constitutive relation for the Cauchy stress tensor reads

$$\mathbb{T} = -p\mathbb{I} + \mu\left(\mathbb{B} - \mathbb{I}\right), \tag{24}$$

where μ is a positive constant and $\mathbb{B}$ denotes the left Cauchy–Green tensor with respect to the initial configuration.

Assume that the deformation of the cylinder takes the form

$$r = R, \tag{25a}$$
$$\varphi = \Phi + \psi Z, \tag{25b}$$
$$z = Z, \tag{25c}$$

where $[r, \varphi, z]$ denote the cylindrical coordinates in the current configuration, while $[R, \Phi, Z]$ denote the cylindrical coordinates in the reference configuration, and ψ is a constant. The deformation prescribed in (25) is referred to as *torsion*. The planes parallel to the XY plane are transformed to planes parallel to the xy plane, and the top base of the cylinder is rotated with respect to the bottom base by the angle ψL.

The task is to find the relation between the constant ψ and the *torque* that is necessary to maintain such a deformation.

1. Show that the deformation gradient $\mathbb{F}$ is given by the formula

$$\mathbb{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r\psi \\ 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

   or, in other words, $\mathbb{F} = \boldsymbol{e}_{\hat{r}} \otimes \boldsymbol{E}^{\hat{R}} + \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{\Phi}} + \boldsymbol{e}_{\hat{z}} \otimes \boldsymbol{E}^{\hat{Z}} + r\psi \boldsymbol{e}_{\hat{\varphi}} \otimes \boldsymbol{E}^{\hat{Z}}$. (From now on we are physicists, hence we work with *physical components* of the given tensors.)

2. Use the formula for the deformation gradient and show that the corresponding formula for the Cauchy stress tensor reads

$$\mathbb{T} = -p\mathbb{I} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & (r\psi)^2 & r\psi \\ 0 & r\psi & 0 \end{bmatrix}. \tag{27}$$

   (We are again working in cylindrical coordinates.)

3. Consider a generic tensor $\mathbb{A}$ and show that the explicit formula for $\operatorname{div} \mathbb{A}$ in cylindrical coordinates reads

$$\operatorname{div}\mathbb{A} = \begin{bmatrix} \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{r}}}{\partial r} + \frac{1}{r^2}\left(r\frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}}}{\partial \varphi} - r\mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}} + r\mathrm{A}^{\hat{r}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{z}}}{\partial z} \\ r\left(\frac{\partial}{\partial r}\left(\frac{\mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}}{r}\right) + \frac{\mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}}{r^2} + \frac{1}{r^2}\left(\frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}\right) + \frac{1}{r}\frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{z}}}{\partial z}\right) \\ \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{r}}}{\partial r} + \frac{1}{r^2}\left(r\frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{\varphi}}}{\partial \varphi} + r\mathrm{A}^{\hat{z}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{z}}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}}}{\partial \varphi} - \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{r}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{r}}{}_{\hat{z}}}{\partial z} \\ \frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{r}}{}_{\hat{\varphi}} + \mathrm{A}^{\hat{\varphi}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{\varphi}}{}_{\hat{z}}}{\partial z} \\ \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{r}}}{\partial r} + \frac{1}{r}\left(\frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{\varphi}}}{\partial \varphi} + \mathrm{A}^{\hat{z}}{}_{\hat{r}}\right) + \frac{\partial \mathrm{A}^{\hat{z}}{}_{\hat{z}}}{\partial z} \end{bmatrix}. \tag{28}$$

4. Solve

$$\operatorname{div}\mathbb{T} = \mathbf{0} \tag{29}$$

   subject to boundary condition $\left.\mathbb{T}\boldsymbol{e}_{\hat{r}}\right|_{r=R_{\mathrm{in}}} = \mathbf{0}$.

5. The z component of the torque $\boldsymbol{r} \times \boldsymbol{F}$ acting on the top surface of the cylinder is given by the formula

$$\tau^{\hat{z}} = -\boldsymbol{e}_{\hat{z}} \bullet \int_{\varphi=0}^{2\pi} \int_{r=0}^{R_{\mathrm{in}}} r\boldsymbol{e}_{\hat{r}} \times \left.\mathbb{T}\boldsymbol{e}_{\hat{z}}\right|_{z=L} r\,\mathrm{d}r\,\mathrm{d}\varphi. \tag{30}$$

   Show that

$$\tau^{\hat{z}} = -\frac{\pi\mu\psi R_{\mathrm{in}}^4}{2}. \tag{31}$$

6. The z component of the total force $\boldsymbol{F}$ acting on the top base of the cylinder is given by the formula

$$\mathrm{F}^{\hat{z}} = -\boldsymbol{e}_{\hat{z}} \bullet \int_{\varphi=0}^{2\pi} \int_{r=0}^{R_{\mathrm{in}}} \left.\mathbb{T}\boldsymbol{e}_{\hat{z}}\right|_{z=L} r\,\mathrm{d}r\,\mathrm{d}\varphi. \tag{32}$$

   Show that

$$\mathrm{F}^{\hat{z}} = -\frac{\pi\mu\psi^2 R_{\mathrm{in}}^4}{4}. \tag{33}$$

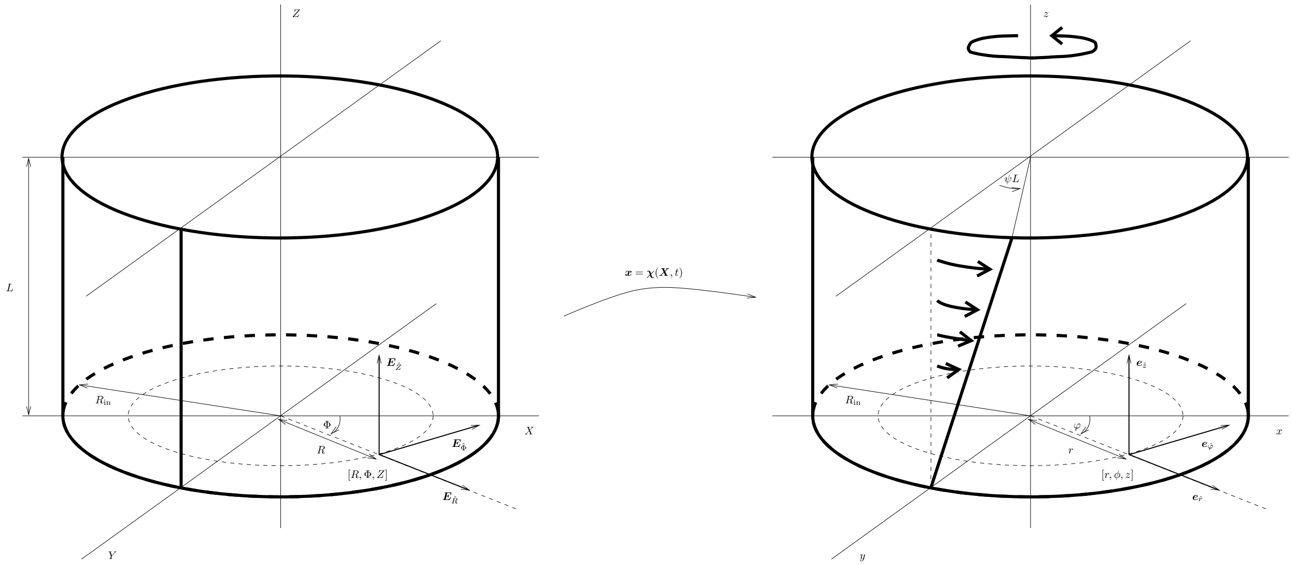

Figure 3: Torsion of a right circular cylinder.

Deadline: Tuesday 23rd March 2021

We already know that

$$\frac{\partial \boldsymbol{t}_\alpha}{\partial u^\beta} = \Gamma^\gamma{}_{\alpha\beta} \boldsymbol{t}_\gamma + \mathrm{b}_{\alpha\beta} \boldsymbol{n},$$
$$\frac{\partial \boldsymbol{n}}{\partial u^\alpha} = -\mathrm{g}^{\gamma\beta} \mathrm{b}_{\beta\alpha} \boldsymbol{t}_\gamma.$$

1. Show that the equality

$$\frac{\partial^2 \boldsymbol{t}_\alpha}{\partial u^\delta \partial u^\beta} - \frac{\partial^2 \boldsymbol{t}_\alpha}{\partial u^\beta \partial u^\delta} = 0$$

implies

$$\left[\mathrm{R}^\gamma{}_{\beta\delta\alpha} + \left(\mathrm{b}_{\alpha\delta}\mathrm{b}_{\pi\beta} - \mathrm{b}_{\alpha\beta}\mathrm{b}_{\pi\delta}\right)\mathrm{g}^{\gamma\pi}\right]\boldsymbol{t}_\gamma + \left[\Gamma^\gamma{}_{\alpha\beta}\mathrm{b}_{\gamma\delta} + \frac{\partial \mathrm{b}_{\alpha\beta}}{\partial u^\delta} - \Gamma^\gamma{}_{\alpha\delta}\mathrm{b}_{\gamma\beta} - \frac{\partial \mathrm{b}_{\alpha\delta}}{\partial u^\beta}\right]\boldsymbol{n} = 0, \tag{34}$$

where

$$\mathrm{R}^\gamma{}_{\alpha\delta\beta} =_{\text{def}} \frac{\partial \Gamma^\gamma{}_{\alpha\beta}}{\partial u^\delta} - \frac{\partial \Gamma^\gamma{}_{\delta\beta}}{\partial u^\alpha} + \Gamma^\gamma{}_{\rho\delta}\Gamma^\rho{}_{\alpha\beta} - \Gamma^\gamma{}_{\rho\alpha}\Gamma^\rho{}_{\delta\beta}.$$

2. Use the definition of the covariant derivative and show that

$$\Gamma^\gamma{}_{\alpha\beta}\mathrm{b}_{\gamma\delta} + \frac{\partial \mathrm{b}_{\alpha\beta}}{\partial u^\delta} - \Gamma^\gamma{}_{\alpha\delta}\mathrm{b}_{\gamma\beta} - \frac{\partial \mathrm{b}_{\alpha\delta}}{\partial u^\beta} = \mathrm{b}_{\alpha\beta}\big|_\delta - \mathrm{b}_{\alpha\delta}\big|_\beta,$$

which means that (34) can be rewritten as

$$\left[\mathrm{R}^\gamma{}_{\beta\delta\alpha} + \left(\mathrm{b}_{\alpha\delta}\mathrm{b}_{\pi\beta} - \mathrm{b}_{\alpha\beta}\mathrm{b}_{\pi\delta}\right)\mathrm{g}^{\gamma\pi}\right]\boldsymbol{t}_\gamma + \left[\mathrm{b}_{\alpha\beta}\big|_\delta - \mathrm{b}_{\alpha\delta}\big|_\beta\right]\boldsymbol{n} = 0.$$

Deadline: Tuesday 16th March 2021

Consider again bipolar coordinate system introduced a week ago.

1. Show that the domain Ω described in bipolar coordinates as $\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^2 \middle| \; \alpha \in (\alpha_{\text{outer}}, \alpha_{\text{inner}}),\; \beta \in (0, 2\pi)\right\}$ corresponds to the domain Ω shown in Figure 3. (The domain is bounded by two eccentrically placed circles of radii R_1 and R_2, and the distance between the centres of the circles is e. This "complex" domain is however described as a *rectangular* domain in the bipolar coordinate system — meaning that the coordinates α and β are taken from the Cartesian product of two intervals.) Show that the value of parameter a in (39) is in this case given as

$$a = \frac{1}{2e}\sqrt{((R_1^2 + R_2^2) - e^2) - 4R_1^2R_2^2}, \tag{35}$$

and that the upper and lower bounds of the interval $\alpha \in (\alpha_{\text{outer}}, \alpha_{\text{inner}})$ are in terms of R_1, R_2 and e given as

$$\sinh \alpha_{\text{inner}} = \frac{a}{R_1}, \tag{36}$$
$$\sinh \alpha_{\text{outer}} = \frac{a}{R_2}. \tag{37}$$

2. The Laplace operator acting on a scalar function is defined as

$$\Delta\varphi =_{\mathrm{def}} \operatorname{div}\nabla\varphi = \left.\left(\mathrm{g}^{ij}\frac{\partial\varphi}{\partial\xi^j}\right)\right|_i .$$

(This is the definition we used in the lecture.) However, in the textbooks you can find the following formula

$$\Delta\varphi =_{\mathrm{def}} \frac{1}{\sqrt{\det\mathbb{g}}}\frac{\partial}{\partial\xi^i}\left(\sqrt{\det\mathbb{g}}\,\mathrm{g}^{ij}\frac{\partial\varphi}{\partial\xi^j}\right). \tag{38}$$

Show that both definitions are equivalent. (The symbol $\mathbb{g}$ denotes the metric tensor. You might want to exploit the already known formula for the derivative of the determinant, $\frac{\partial}{\partial\mathbb{A}}\det\mathbb{A} = (\det\mathbb{A})\,\mathbb{A}^{-\top}$.) Note that the definition (38) does not force you to find explicit formulae for the Christoffel symbols.

3. Show that the formula for the Laplace operator acting on a scalar field $\varphi(\alpha,\beta)$, where α and β are bipolar coordinates, reads

$$\Delta\varphi = \left(\frac{\cosh\alpha - \cos\beta}{a}\right)^2\left(\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{\partial^2\varphi}{\partial\beta^2}\right).$$

(Formula (38) derived in the previous question might be useful. Recall that a week ago you also derived a formula for the metric tensor in bipolar coordinate system.)

4. Write down the formula for the analytical solution of the problem of steady heat conduction between eccentrically placed circles of radii R_1 and R_2 that are kept at (constant) temperature φ_{inner} and φ_{outer} respectively. (The distance between the centres of the circles is e.) In other words solve

$$\Delta\varphi = 0,$$
$$\varphi|_{\text{inner circle}} = \varphi_{\mathrm{inner}},$$
$$\varphi|_{\text{outer circle}} = \varphi_{\mathrm{outer}},$$

in the domain Ω shown in Figure 3. (Write down the solution as a function of α and β, and as a function of x and y. In order to do that, you might want to exploit the relation $\alpha = \ln\frac{d_1}{d_2}$, see Figure 3.) If you want, you can plot the solution. (Heat map, isolines, force lines or whatever form of visualisation you prefer.)

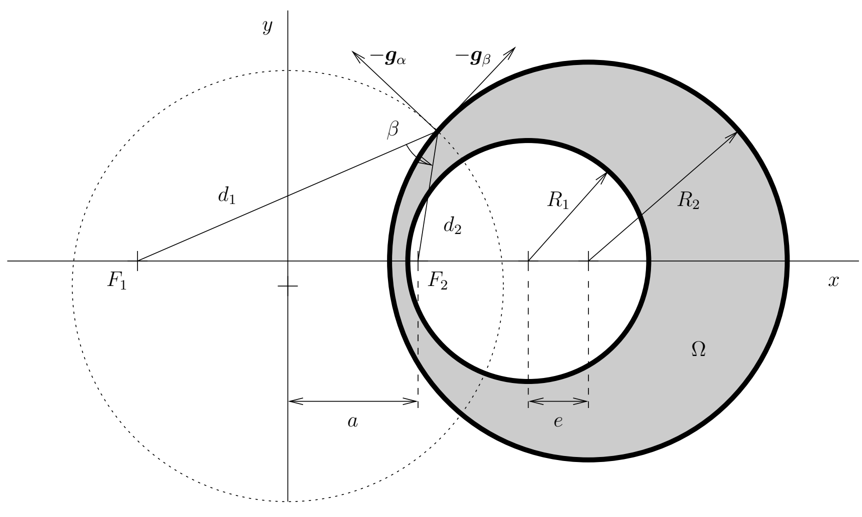

Figure 4: Bipolar coordinates.

Note: The calculation shown above might be also interpreted as an application of the theory of conformal mappings in the context of solution of (two-dimensional) Laplace equation. Especially people working in the field of electrical engineering are obsessed by applications of the outlined method, see, for example, Bewley (1948).

Deadline: Tuesday 9th March 2021

Bipolar coordinates (α,β) are related to the Cartesian coordinates (x,y) by the formula

$$\alpha + \mathrm{i}\beta = \ln\frac{y + \mathrm{i}(x+a)}{y + \mathrm{i}(x-a)}, \tag{39}$$

where i denotes the imaginary unit. (We work in $\mathbb{R}^2$, the complex numbers are used for the sake of convenience.)

1. Show that (39) implies that
$$x = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta}, \quad y = \frac{a \sin \beta}{\cosh \alpha - \cos \beta}. \tag{40}$$
and that
$$(x - a \coth \alpha)^2 + y^2 = \frac{a^2}{\sinh^2 \alpha}, \quad x^2 + (y - a \cot \beta)^2 = \frac{a^2}{\sin^2 \beta}. \tag{41}$$
(This could help you to visualise the coordinate curves.)

2. Show that the tangent vectors to the coordinate curves are given by the formulae
$$\boldsymbol{g}_\alpha = \frac{a}{(\cosh \alpha - \cos \beta)^2} \begin{bmatrix} 1 - \cosh \alpha \cos \beta \\ - \sinh \alpha \sin \beta \end{bmatrix}, \quad \boldsymbol{g}_\beta = \frac{a}{(\cosh \alpha - \cos \beta)^2} \begin{bmatrix} - \sinh \alpha \sin \beta \\ \cosh \alpha \cos \beta - 1 \end{bmatrix}.$$
Check that the tangent vectors are perpendicular to each other. (Is that a coincidence? Check your notes from a lecture on complex analysis.)

3. Show that the metric tensor is given by the formula
$$[g_{ij}] = \frac{a^2}{(\cosh \alpha - \cos \beta)^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

References

Bewley, L. V. (1948). *Two-dimensional fields in electrical engineering.* New York: Macmillan.