

There is no lecture on Thursday 29th November 2018 and on Friday 30th November 2018.

Consider a hollow cylinder of initial inner radius R_{in} and outer radius R_{out} , see Figure 1, and assume that the cylinder is in this configuration in a stress free state. Further, assume that the material of which is the cylinder made is a homogeneous isotropic *incompressible* elastic material specified by constitutive relation

$$\mathbb{T} = -p\mathbb{1} + \mu(\mathbb{B} - \mathbb{1}),$$

where μ is a positive constant and \mathbb{B} denotes the left Cauchy–Green tensor with respect to the initial configuration with the inner radius R_{in} and the outer radius R_{out} . Let us now apply a pressure P_{in} inside the cylinder and a pressure P_{out} outside the cylinder. If the inner pressure is higher than the outer pressure, then the cylinder inflates, and its inner radius changes to r_{in} , while its outer radius changes to r_{out} . The task is to find a relation between the relative change in the void area

$$c =_{\text{def}} \frac{r_{in}^2 - R_{in}^2}{R_{in}^2}$$

and the pressure difference

$$P_{in} - P_{out}.$$

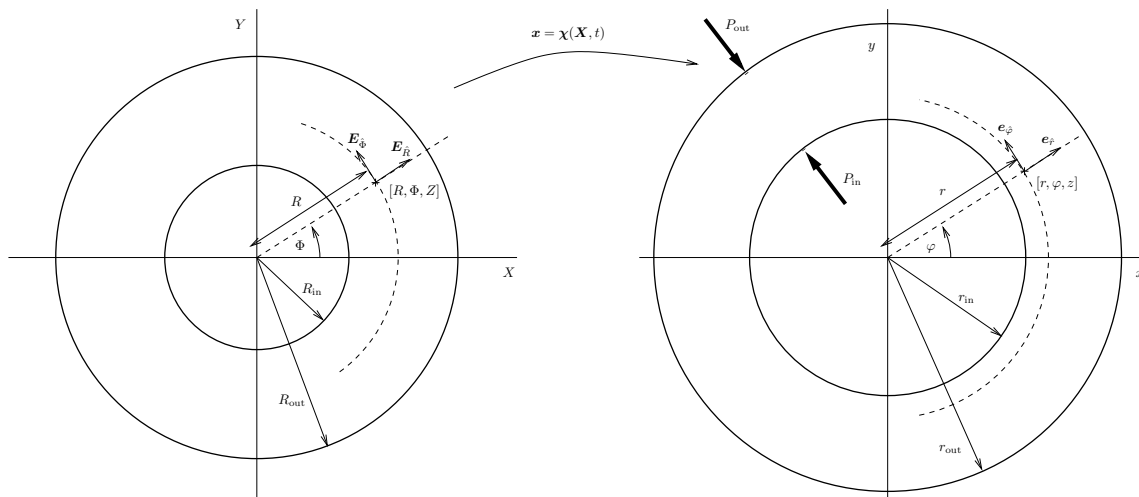


Figure 1: Inflation of a hollow cylinder made of an incompressible elastic material.

1. Find the answer using *finite elasticity* theory, that is use the governing equations in the form

$$\begin{aligned} \operatorname{div} \mathbb{T} &= \mathbf{0}, \\ \det \mathbb{F} &= 1, \end{aligned}$$

with boundary conditions

$$\begin{aligned} \mathbb{T} \mathbf{e}_{\hat{r}}|_{r=r_{in}} &= -P_{in} \mathbf{e}_{\hat{r}}|_{r=r_{in}}, \\ \mathbb{T} \mathbf{e}_{\hat{r}}|_{r=r_{out}} &= -P_{out} \mathbf{e}_{\hat{r}}|_{r=r_{out}}. \end{aligned}$$

Let me give you few hints concerning the solution procedure. We know that the deformation gradient for the deformation shown in Figure 1 is given by the formula

$$\mathbb{F} = \begin{bmatrix} \frac{df}{dR} & 0 & 0 \\ 0 & \frac{f}{R} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The solution can be found as follows.

- Use the incompressibility constraint $\det \mathbb{F} = 1$ and find a formula for f . You should obtain

$$f = \sqrt{R^2 + cR_{in}^2},$$

- Assume that the Cauchy stress tensor has the structure

$$\mathbb{T} = -p\mathbb{1} + \begin{bmatrix} S^{\hat{r}\hat{r}} & 0 & 0 \\ 0 & S^{\hat{\varphi}\hat{\varphi}} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the pressure p and the components $S^{\hat{r}\hat{r}}$ and $S^{\hat{\varphi}\hat{\varphi}}$ are functions of r . Show that under these assumptions the equation $\text{div } \mathbb{T} = \mathbf{0}$ reduces to a single nontrivial equation

$$-\frac{dp}{dr} + \frac{dS^{\hat{r}\hat{r}}}{dr} + \frac{S^{\hat{r}\hat{r}} - S^{\hat{\varphi}\hat{\varphi}}}{r} = 0.$$

- Using the boundary conditions and the last differential equation show that

$$P_{\text{out}} - P_{\text{in}} = \int_{r=R_{\text{in}}}^{r_{\text{out}}} \frac{S^{\hat{r}\hat{r}} - S^{\hat{\varphi}\hat{\varphi}}}{r} dr.$$

- Use the fact that $S = \mu (\mathbb{B} - \mathbb{1})$ and show that

$$P_{\text{out}} - P_{\text{in}} = \mu \int_{r=\sqrt{R_{\text{in}}^2+cR_{\text{in}}^2}}^{\sqrt{R_{\text{out}}^2+cR_{\text{in}}^2}} \frac{(r^2 - cR_{\text{in}}^2)^2 - r^4}{r^3 (r^2 - cR_{\text{in}}^2)} dr. \tag{1}$$

This is the sought formula that provides us a relation between the change in the void area c and the pressure difference $P_{\text{out}} - P_{\text{in}}$.

2. Find the answer using *linearised elasticity* theory, that is use the governing equations in the form

$$\begin{aligned} \text{div } \boldsymbol{\tau} &= \mathbf{0}, \\ \text{Tr} (\nabla \mathbf{U}) &= 0, \end{aligned}$$

where $\boldsymbol{\tau} \stackrel{\text{def}}{=} -p\mathbb{1} + 2\mu\boldsymbol{\epsilon}$ is the linearised version of the constitutive relation and $\boldsymbol{\epsilon} \stackrel{\text{def}}{=} \frac{1}{2} (\nabla \mathbf{U} + (\nabla \mathbf{U})^\top)$ denotes the linearised strain tensor. The boundary conditions read

$$\begin{aligned} \boldsymbol{\tau} \mathbf{E}_{\hat{z}} \Big|_{R=R_{\text{in}}} &= -P_{\text{in}} \mathbf{E}_{\hat{z}} \Big|_{R=R_{\text{in}}}, \\ \boldsymbol{\tau} \mathbf{E}_{\hat{z}} \Big|_{R=R_{\text{out}}} &= -P_{\text{out}} \mathbf{E}_{\hat{z}} \Big|_{R=R_{\text{out}}}, \end{aligned}$$

while the displacement \mathbf{U} is assumed to take the form

$$\begin{aligned} U^{\hat{r}} &= g(R), \\ U^{\hat{\varphi}} &= 0, \\ U^{\hat{z}} &= 0. \end{aligned}$$

The solution can be found by following the same procedure as the procedure outlined above for the fully nonlinear setting. (Provided that you modify it in an appropriate manner.)

In the linearised case you should obtain formula

$$P_{\text{out}} - P_{\text{in}} = \mu \int_{r=R_{\text{in}}}^{R_{\text{out}}} \frac{2cR_{\text{in}}^2}{r^3} dr,$$

which is a formula that we could have obtained via the linearisation of the solution to the complete system of nonlinear governing equations with respect to c . (Previous question, equation (1).) Check this! *In this sense we see that the result obtained by solving the linearised governing equations is the same as the result obtained by linearising the solution to the fully nonlinear governing equations.*

For your convenience, I recall that the formula for the divergence of a tensorial quantity \mathbb{A} in the cylindrical coordinate system reads

$$\text{div } \mathbb{A} = \begin{bmatrix} \frac{\partial A^{\hat{r}}}{\partial r} + \frac{1}{r} \left(\frac{\partial A^{\hat{r}}}{\partial \varphi} - A^{\hat{\varphi}}_{\hat{\varphi}} + A^{\hat{r}}_{\hat{r}} \right) + \frac{\partial A^{\hat{r}}}{\partial z} \\ \frac{\partial A^{\hat{\varphi}}}{\partial r} + \frac{1}{r} \left(\frac{\partial A^{\hat{\varphi}}}{\partial \varphi} + A^{\hat{r}}_{\hat{\varphi}} + A^{\hat{\varphi}}_{\hat{r}} \right) + \frac{\partial A^{\hat{\varphi}}}{\partial z} \\ \frac{\partial A^{\hat{z}}}{\partial r} + \frac{1}{r} \left(\frac{\partial A^{\hat{z}}}{\partial \varphi} + A^{\hat{z}}_{\hat{r}} \right) + \frac{\partial A^{\hat{z}}}{\partial z} \end{bmatrix},$$

while the formula for the gradient of a vector field \mathbf{v} reads

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial v^{\hat{r}}}{\partial r} & \frac{1}{r} \frac{\partial v^{\hat{r}}}{\partial \varphi} - \frac{v^{\hat{\varphi}}}{r} & \frac{\partial v^{\hat{r}}}{\partial z} \\ r \frac{\partial}{\partial r} \left(\frac{v^{\hat{\varphi}}}{r} \right) + \frac{v^{\hat{\varphi}}}{r} & \frac{1}{r} \frac{\partial v^{\hat{\varphi}}}{\partial \varphi} + \frac{v^{\hat{r}}}{r} & \frac{\partial v^{\hat{\varphi}}}{\partial z} \\ \frac{\partial v^{\hat{z}}}{\partial r} & \frac{1}{r} \frac{\partial v^{\hat{z}}}{\partial \varphi} & \frac{\partial v^{\hat{z}}}{\partial z} \end{bmatrix}.$$