Solve ONE of the following problem sets.

1. Let φ, ψ, $\boldsymbol{u}$, $\boldsymbol{v}$ and $\mathbb{A}$ be smooth scalar, vector and tensor fields in $\mathbb{R}^3$. Show that[1]

$$
\begin{aligned}
\operatorname{div}(\varphi \boldsymbol{v}) &= \boldsymbol{v} \bullet (\nabla \varphi) + \varphi \operatorname{div} \boldsymbol{v}, \\
\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) &= \boldsymbol{v} \bullet \operatorname{rot} \boldsymbol{u} - \boldsymbol{u} \bullet \operatorname{rot} \boldsymbol{v}, \\
\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{v}) &= [\nabla \boldsymbol{u}]\, \boldsymbol{v} + \boldsymbol{u} \operatorname{div} \boldsymbol{v}, \\
\operatorname{div}(\varphi \mathbb{A}) &= \mathbb{A} (\nabla \varphi) + \varphi \operatorname{div} \mathbb{A}.
\end{aligned}
$$

Further, show that

$$
\begin{aligned}
\nabla (\varphi \psi) &= \psi \nabla \varphi + \varphi \nabla \psi, \\
\nabla (\varphi \boldsymbol{v}) &= \boldsymbol{v} \otimes \nabla \varphi + \varphi \nabla \boldsymbol{v}, \\
\nabla (\boldsymbol{u} \bullet \boldsymbol{v}) &= (\nabla \boldsymbol{u})^{\top} \boldsymbol{v} + (\nabla \boldsymbol{v})^{\top} \boldsymbol{u}, \\
\operatorname{rot}(\varphi \boldsymbol{v}) &= \varphi \operatorname{rot} \boldsymbol{v} - \boldsymbol{v} \times \nabla \varphi.
\end{aligned}
$$

2. Let $\mathbb{U} \in \mathbb{R}^{3\times 3}$ be a symmetric positive definite matrix, and let $\mathbb{A} \in \mathbb{R}^{3\times 3}$ be a symmetric matrix. Show that the solution $\mathbb{X}$ of the matrix equation

$$\mathbb{X}\mathbb{U} + \mathbb{U}\mathbb{X} = \mathbb{A},$$

is given by the formula

$$\mathbb{X} = \int_{u=0}^{+\infty} \mathrm{e}^{-u\mathbb{U}} \mathbb{A} \mathrm{e}^{-u\mathbb{U}} \, \mathrm{d}u.$$

(The matrix equation $\mathbb{X}\mathbb{U} + \mathbb{U}\mathbb{X} = \mathbb{A}$ is usually called the Lyapunov equation.) Using the Lypunov equation find a formula for the derivative of the square root of a symmetric positive definite matrix. Show that

$$\frac{\partial \sqrt{\mathbb{K}}}{\partial \mathbb{K}} [\mathbb{H}] = \int_{\tau=0}^{+\infty} \mathrm{e}^{-\tau\sqrt{\mathbb{K}}} \mathbb{H} \mathrm{e}^{-\tau\sqrt{\mathbb{K}}} \, \mathrm{d}\tau,$$

where $\mathbb{H}$ is a symmetric matrix. Moreover, show that if $\{\lambda_i\}_{i=1}^{3}$ are the eigenvalues of $\mathbb{K}$, and H_{ij} denote the components of matrix $\mathbb{H}$, then the components of the derivative are given by the formula

$$\left[\frac{\partial \sqrt{\mathbb{K}}}{\partial \mathbb{K}} [\mathbb{H}]\right]_{ij} = \frac{\mathrm{H}_{ij}}{\sqrt{\lambda_i} + \sqrt{\lambda_j}}.$$

[1] We use the notation $([\nabla \boldsymbol{u}]\, \boldsymbol{v})_i = \frac{\partial \mathrm{u}_i}{\partial x_j} \mathrm{v}_j$.