RHEOLOGY SECTION

The Motion of a Viscous Fluid Under a Surface Load. Part II

N. A. HASKELL, *Massachusetts Institute of Technology*
(Received December 14, 1935)

Following up previous work on the subsidence of a circular load on the surface of a highly viscous fluid, the same problem is treated for the case of a load in the form of an infinitely long strip with parallel sides. Simple expressions are derived for the limiting displacements as the system approaches hydrostatic equilibrium and for the form taken by the surface of the fluid at any time during the motion. Curves illustrating these quantities are plotted. It is pointed out that the subsidence of a load on the earth's surface having a span comparable to that of a continental ice sheet must produce appreciable flow at great depth. Using the previously derived figure for the mean viscosity of the earth it is found that with a load 2000 km wide equilibrium would be very nearly reached in about 18,000 years, and that loads of smaller span would require proportionately greater time to reach the same stage.

INTRODUCTION

IN a previous paper[1] the problem of the motion of a highly viscous fluid under a circular load applied on the free surface was treated with the particular object of determining the mean viscosity of the earth from the motion of the surface after the melting of one of the last continental ice sheets. Although it proved possible to derive a figure for the viscosity, the velocity distribution within the fluid could be given only in the form of definite integrals whose evaluation would require numerical integration. Since it is of some geological importance, in connection with the mechanism of isostatic adjustment, to have a roughly quantitative idea of the distribution of velocities, or at least of the final displacements, beneath a subsiding load, it was thought worth while to consider the same problem with a different load distribution which gives more tractable expressions for these quantities. Accordingly in the present paper we shall treat the case of an instantaneously applied load of constant thickness, having parallel sides, and whose length is sufficiently greater than the width so that it may be regarded as effectively infinite. As in I we suppose that the viscosity is so great that we may neglect the acceleration in the equations of motion in comparison with the terms expressing the viscous forces, and that the displacements remain small relative to the width of the load. We also assume the fluid to be incompressible.

1. SOLUTION IN TERMS OF DEFINITE INTEGRALS

We refer the equations of motion to a rectangular coordinate system having the plane $z=0$ in the undisturbed surface of the fluid, the positive z axis directed downward, and the y axis parallel to the length of the load, and let the width of the load be $2l$. Then V_y and $\partial/\partial y$ vanish and the equations of motion become

$$\partial^2 V_x/\partial x^2+\partial^2 V_x/\partial z^2=\partial\bar{p}/\eta\partial x, \quad (1.11)$$

$$\partial^2 V_z/\partial x^2+\partial^2 V_z/\partial z^2=\partial\bar{p}/\eta dz, \quad (1.12)$$

$$\partial V_x/\partial x+\partial V_z/\partial z=0, \quad (1.13)$$

where $\bar{p}=p-\rho gz$.

Following the procedure used in I we let the equation of the free surface be given by $z=\zeta(x, t)$, and let the pressure exerted by the load be $\sigma(x, t)$. The boundary conditions are then

$$\bar{p}(x, o, t)+\rho g\zeta(x, t) -2\eta(\partial V_z/\partial z)(x, o, t)=\sigma(x, t) \quad (1.21)$$

and

$$(\partial V_x/\partial z+\partial V_z/\partial x)_{z=0}=0. \quad (1.22)$$

[1] N. A. Haskell, Physics **6**, 265 (1935). Hereafter referred to as I.

In the case with which we shall be concerned

$$\sigma(x,t)=\begin{cases}0;\ x>l \text{ or } <-l\\ 0;\ t\leq 0\\ \sigma=\text{const.};\ -l<x<l,\ t>0.\end{cases} \tag{1.23}$$

The method of solution is the same as that given in I and need not be repeated in detail. The only difference is that certain trigonometric functions replace the Bessel functions which occur there. The solutions are

$$V_x=(\sigma z/\pi\eta)\int_0^\infty e^{-\rho g t/2\eta\lambda-\lambda z}\sin\lambda l\sin\lambda x d\lambda/\lambda, \tag{1.31}$$

$$V_z=(\sigma/\pi\eta)\int_0^\infty e^{-\rho g t/2\eta\lambda-\lambda z}\sin\lambda l\cos\lambda x((1+\lambda z)/\lambda^2)d\lambda, \tag{1.32}$$

$$\bar{p}=(2\sigma/\pi)\int_0^\infty e^{-\rho g t/2\eta\lambda-\lambda z}\sin\lambda l\cos\lambda x d\lambda/\lambda, \tag{1.33}$$

$$\zeta=(2\sigma/\pi\rho g)\int_0^\infty (1-e^{-\rho g t/2\eta\lambda})\sin\lambda l\cos\lambda x d\lambda/\lambda. \tag{1.34}$$

To express these in dimensionless form set

$$\tau=\rho g l t/2\eta,\quad \alpha=x/l,\quad \beta=z/l,\quad \kappa=\lambda l,$$
$$v_x=2\eta V_x/\sigma l,\quad v_z=2\eta V_z/\sigma l,\quad \mu=\rho g\zeta/\sigma.$$

Then

$$v_x=(2\beta/\pi)\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\sin\kappa\sin\kappa\alpha d\kappa/\kappa, \tag{1.41}$$

$$v_z=(2/\pi)\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\sin\kappa \times\cos\kappa\alpha((1+\kappa\beta)/\kappa^2)d\kappa, \tag{1.42}$$

$$(\bar{p}/\sigma)=(2/\pi)\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\sin\kappa\cos\kappa\alpha d\kappa/\kappa, \tag{1.43}$$

$$\mu=(2/\pi)\int_0^\infty (1-e^{-\tau/\kappa})\sin\kappa\cos\kappa\alpha d\kappa/\kappa. \tag{1.44}$$

The components of the displacement are given by

$$U_x=\int_0^t V_x dt=(\sigma/\rho g)\int_0^\tau v_x d\tau,$$

$$U_z=\int_0^t V_z dt=(\sigma/\rho g)\int_0^\tau v_z d\tau.$$

Hence, with $\xi=\rho g U/\sigma$,

$$\xi_x=(2\beta/\pi)\int_0^\tau (1-e^{-\tau/\kappa})e^{-\kappa\beta}\sin\kappa\sin\kappa\alpha d\kappa, \tag{1.51}$$

$$\xi_z=(2/\pi)\int_0^\tau (1-e^{-\tau/\kappa})e^{-\kappa\beta}\sin\kappa \cos\kappa\alpha(1+\kappa\beta)d\kappa/\kappa. \tag{1.52}$$

2. Evaluation of the Integrals

The integrals occurring in these expressions can be evaluated in terms of Bessel functions of the second kind for imaginary argument by the use of the inverted form of integral No. 927.0 in Campbell and Foster's tables.[2] With a slight change in their notation this integral may be written in the form

$$\int_0^\infty \exp(-\tau/\kappa-\kappa\beta-i\gamma\kappa)d\kappa/\kappa^{n+1}=2^{n+1}((\beta+i\gamma)/4\tau)^{n/2}K_n[(4\tau(\beta+i\gamma))^{\frac{1}{2}}]. \tag{2.1}$$

By letting $\gamma=\alpha+1$ this becomes

$$\int_0^\infty e^{-\tau/\kappa-\kappa\beta}(\cos\kappa\alpha\cos\kappa-\sin\kappa\alpha\sin\kappa-i\{\sin\kappa\alpha\cos\kappa+\cos\kappa\alpha\sin\kappa\})d\kappa/\kappa^{n+1}$$
$$=2^{n+1}((\beta+i\alpha+i)/4\tau)^{n/2}\dot{K}_n[(4\tau(\beta+i\alpha+i))^{\frac{1}{2}}]=N_n^{(+)} \tag{2.21}$$

[2] G. A. Campbell and R. M. Foster, "Fourier Integrals for Practical Applications," Bell Sys. Tech. Pub. Monograph B-584 (1931).

and by letting $\gamma=\alpha-1$

$$\int_0^\infty e^{-\tau/\kappa-\kappa\beta}(\cos\kappa\alpha\cos\kappa+\sin\kappa\alpha\sin\kappa-i\{\sin\kappa\alpha\cos\kappa-\cos\kappa\alpha\sin\kappa\})d\kappa/\kappa^{n+1}$$
$$=2^{n+1}((\beta+i\alpha-i)/4\tau)^{n/2}K_n[(4\tau(\beta+i\alpha-i))^{\frac{1}{2}}]=N_n^{(-)}. \quad (2.22)$$

(Note that $N_n^{(-)}$ is not the conjugate of $N_n^{(+)}$.) By adding (2.21) and (2.22) and equating real and imaginary parts we have the two real integrals

$$\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\cos\kappa\alpha\cos\kappa d\kappa/\kappa^{n+1}=\tfrac{1}{2}R[N_n^{(-)}+N_n^{(+)}] \quad (2.31)$$

and

$$\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\sin\kappa\alpha\cos\kappa d\kappa/\kappa^{n+1}=-\tfrac{1}{2}I[N_n^{(+)}+N_n^{(-)}]. \quad (2.32)$$

By subtracting we have

$$\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\sin\kappa\alpha\sin\kappa d\kappa/\kappa^{n+1}=\tfrac{1}{2}R[N_n^{(-)}-N_n^{(+)}], \quad (2.33)$$

$$\int_0^\infty e^{-\tau/\kappa-\kappa\beta}\cos\kappa\alpha\sin\kappa d\kappa/\kappa^{n+1}=\tfrac{1}{2}I[N_n^{(-)}-N_n^{(+)}]. \quad (2.34)$$

Eqs. (1.41) to (1.44) may then be written in the form

$$v_x=(\beta/\pi)R[N_0^{(-)}-N_0^{(+)}], \quad (2.41)$$

$$v_z=(1/\pi)I[N_1^{(-)}-N_1^{(+)}]+(\beta/\pi)I[N_0^{(-)}-N_0^{(+)}], \quad (2.42)$$

$$\bar{p}/\sigma=(1/\pi)I[N_0^{(-)}-N_0^{(+)}], \quad (2.43)$$

$$\mu=\mu_\infty-(\bar{p}/\sigma)_{\beta=0}=\mu_\infty-(2/\pi)I[K_0\{(4i\tau(\alpha-1))^{\frac{1}{2}}\}-K_0\{4i\tau(\alpha+1))^{\frac{1}{2}}\}], \quad (2.44)$$

where $\mu_\infty=\begin{cases}0; & x<-l \text{ or } x>l\\ 1; & -l<x<l.\end{cases}$

The displacements are

$$\xi_x=\xi_{x\infty}-(\beta/\pi)R[N_{-1}^{(-)}-N_{-1}^{(+)}], \quad (2.51)$$

$$\xi_z=\xi_{z\infty}-(1/\pi)I[N_0^{(-)}-N_0^{(+)}]-(\beta/\pi)I[N_{-1}^{(-)}-N_{-1}^{(+)}], \quad (2.52)$$

where

$$\xi_{x\infty}=(2\beta/\pi)\int_0^\infty e^{-\kappa\beta}\sin\kappa\sin\kappa\alpha d\kappa, \quad (2.53)$$

$$\xi_{z\infty}=(2/\pi)\int_0^\infty e^{-\kappa\beta}\sin\kappa\cos\kappa\alpha(1+\kappa\beta)d\kappa/\kappa. \quad (2.54)$$

The integrals (2.53) and (2.54) cannot be evaluated directly by setting $\tau=0$ in (2.33) and (2.34), but may be treated separately as follows.

Integral No. 632.2 in Campbell and Foster's tables is

$$\int_{-\infty}^{+\infty} e^{-2\pi(\beta|f|+ixf-igf)}df=\beta/\pi[\beta^2+(g-x)^2]. \quad (2.61)$$

Let $2\pi f=\kappa$, $g=\alpha$, $x=-1$. Then

$$\int_{-\infty}^{+\infty} e^{-|\kappa|\beta+i\kappa\alpha+i\kappa}d\kappa=2\beta/[\beta^2+(\alpha+1)^2].$$

By letting $x=+1$

$$\int_{-\infty}^{+\infty} e^{-|\kappa|\beta+i\kappa\alpha-i\kappa}d\kappa=2\beta/[\beta^2+(\alpha-1)^2].$$

By subtracting,

$$\int_{-\infty}^{+\infty} e^{-|\kappa|\beta+i\kappa\alpha}\sin\kappa d\kappa=(\beta/i)[\{\beta^2+(\alpha+1)^2\}^{-1}-\{\beta^2+(\alpha-1)^2\}^{-1}]$$

or

$$\int_0^\infty e^{-\kappa\beta}\sin\kappa\sin\kappa\alpha d\kappa=(\beta/2)[\{\beta^2+(\alpha-1)^2\}^{-1}-\{\beta^2+(\alpha+1)^2\}^{-1}]. \quad (2.62)$$

Hence

$$\xi_{x\infty}=4\alpha\beta^2/\pi\{\beta^2+(\alpha-1)^2\}\{\beta^2+(\alpha+1)^2\}. \quad (2.71)$$

Integrating (2.62) with respect to α from 0 to α gives

$$\int_0^\infty e^{-\kappa\rho}\sin\kappa\cos\kappa\alpha d\kappa/\kappa=\int_0^\infty e^{-\kappa\beta}\sin\kappa d\kappa/\kappa$$

$$-(\beta/2)\left[\int_0^\alpha\{\beta^2+(\alpha-1)^2\}^{-1}d\alpha-\int_0^\alpha\{\beta^2+(\alpha+1)^2\}^{-1}d\alpha\right].$$

The first integral on the right is

$$\int_0^\infty e^{-\kappa\beta}\sin\kappa d\kappa/\kappa=\int_0^\infty\int_\beta^\infty e^{-\kappa\beta}\sin\kappa d\beta d\kappa=\int_\beta^\infty\{\beta^2+1\}^{-1}d\beta=\tan^{-1}(1/\beta).$$

The second and third are

$$\int_0^\alpha\{\beta^2+(\alpha-1)^2\}^{-1}d\alpha=(1/\beta)[\tan^{-1}((\alpha-1)/\beta)+\tan^{-1}(1/\beta)]$$

and

$$\int_0^\alpha\{\beta^2+(\alpha+1)^2\}^{-1}d\alpha=(1/\beta)[\tan^{-1}((\alpha+1)/\beta)-\tan^{-1}(1/\beta)].$$

Hence

$$\int_0^\infty e^{-\kappa\beta}\sin\kappa\cos\kappa\alpha d\kappa/\kappa$$

$$=\tfrac{1}{2}[\tan^{-1}((\alpha+1)/\beta)-\tan^{-1}((\alpha-1)/\beta)]$$
$$=\tfrac{1}{2}\tan^{-1}(2\beta/(\beta^2+\alpha^2-1)). \quad (2.63)$$

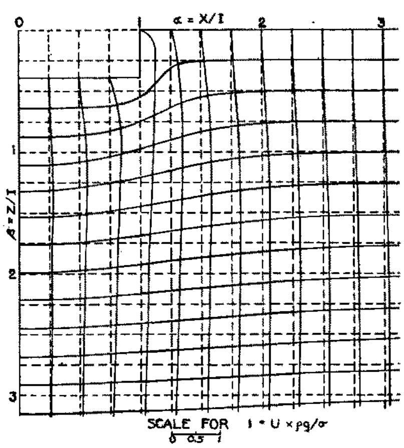

FIG. 1. Asymptotic displacements beneath a constant load.

For the second part of $\xi_{z\infty}$ the use of integral No. 632.11 of the tables gives

$$\int_0^\infty e^{-\kappa\rho}\sin\kappa\cos\kappa\alpha d\kappa$$

$$=(\beta^2-\alpha^2+1)/(\beta^2+(\alpha+1)^2)(\beta^2+(\alpha-1)^2). \quad (2.64)$$

Hence

$$\xi_{z\infty}=(1/\pi)[\tan^{-1}(2\beta/(\beta^2+\alpha^2-1))$$
$$+2\beta(\beta^2-\alpha^2+1)/(\beta^2+(\alpha+1)^2)(\beta^2+(\alpha-1)^2)]. \quad (2.72)$$

It is easily shown that if $r=(\alpha^2+\beta^2)^{\frac{1}{2}}$ and $\theta=\tan^{-1}\alpha/\beta$, then for large values of r

$$\xi_{x\infty}\to\xi\sin\theta,\quad \xi_{z\infty}\to\xi\cos\theta,$$

where $\xi=(4/\pi)(\cos^2\theta)/r$.

3. Numerical Values

In Table I values of $\pi\xi_{x\infty}$ and $\pi\xi_{z\infty}$ computed from (2.71) and (2.72) are given for various values of α and β. In Fig. 1 these values are represented graphically, the solid lines giving the final positions of a set of rectangular lines whose initial positions are given by the dashed lines. Since the system is symmetrical in the zy plane, only one-half the field is represented.

To get a roughly quantitative idea of what this means in the case of a continental ice sheet on the earth let us suppose $l=10^8$ cm, take $\rho=5$ (i.e., a little less than the mean density of the earth), $g=10^3$, and let $\sigma=1.5\times10^8$ dynes cm^{-2} (approximately the pressure exerted by a sheet of ice 1500 m thick). Then $(x, z)=10^8(\alpha, \beta)$ and $U=0.955\times10^4(\pi\xi)$, hence with the scales used in Fig. 1 the displacement as shown is exaggerated by a factor of 1310. The great persistence with depth is noteworthy. In spite of the obviously great simplification of the problem we have treated as compared with the actual case, it seems clear that the isostatic compensation of such a load must be regarded as a distortion of the whole earth, as Daly[3] has contended, rather

[3] R. A. Daly, "Pleistocene Changes of Level," Am. J. Sci. **10**, 281 (1925).

TABLE I. *Asymptotic values of the displacements* ($\times\pi$).

	$\beta \backslash \alpha$	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
	0.00	3.142	3.142	3.142	3.142	1.571	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	3.121	3.115	3.040	2.854	1.570	.285	.063	.022	.009	.005	.003	.002	.001
	0.50	3.016	2.980	2.836	2.421	1.562	.700	.280	.123	.060	.024	.020	.014	.008
	0.75	2.815	2.768	2.557	2.149	1.541	.927	.504	.273	.154	.091	.058	.041	.026
	1.00	2.517	2.507	2.187	1.963	1.507	1.044	.672	.420	.264	.169	.112	.083	.054
	1.25	2.325	2.267	2.094	1.813	1.462	1.099	.782	.539	.368	.253	.177	.135	.091
$\pi\xi_{z\infty}$	1.50	2.100	2.050	1.908	1.683	1.409	1.117	.850	.628	.457	.333	.243	.191	.133
	1.75	1.900	1.860	1.746	1.567	1.348	1.107	.887	.690	.538	.401	.304	.247	.178
	2.00	1.729	1.696	1.604	1.462	1.285	1.093	.903	.730	.581	.457	.359	.298	.222
	2.25	1.579	1.553	1.481	1.366	1.223	1.065	.905	.753	.618	.502	.404	.343	.263
	2.50	1.451	1.431	1.371	1.279	1.162	1.031	.896	.763	.642	.535	.442	.382	.299
	2.75	1.340	1.323	1.275	1.201	1.104	.995	.880	.765	.657	.558	.471	.413	.331
	3.00	1.244	1.230	1.191	1.129	1.050	.957	.859	.760	.664	.574	.492	.438	.358
	0.00	0.000	0.000	0.000	0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	0.000	.062	.173	.480	.984	.488	.190	.092	.052	.033	.022	.016	.011
	0.50	0.000	.170	.400	.725	.941	.754	.461	.276	.173	.115	.080	.058	.043
	0.75	0.000	.235	.492	.745	.876	.800	.610	.431	.301	.214	.156	.117	.089
	1.00	0.000	.250	.492	.695	.800	.776	.662	.523	.400	.304	.232	.180	.141
	1.25	0.000	.235	.452	.624	.719	.726	.662	.564	.462	.371	.297	.238	.192
$\pi\xi_{x\infty}$	1.50	0.000	.210	.400	.550	.640	.665	.635	.571	.492	.415	.345	.286	.236
	1.75	0.000	.183	.348	.480	.566	.603	.596	.556	.500	.437	.376	.321	.273
	2.00	0.000	.165	.301	.418	.500	.544	.551	.531	.492	.445	.394	.345	.300
	2.25	0.000	.136	.261	.365	.441	.488	.505	.499	.475	.440	.400	.358	.318
	2.50	0.000	.117	.226	.319	.390	.438	.461	.465	.452	.429	.397	.363	.329
	2.75	0.000	.102	.197	.280	.346	.393	.420	.431	.427	.412	.389	.362	.333
	3.00	0.000	.089	.173	.247	.308	.353	.382	.398	.400	.392	.376	.356	.332

than as a shallow horizontal flow beneath the crust. In this connection it may be of interest to compute the depth, z_0, below which one-half the total lateral displacement across a given vertical plane, $x=\text{const.}$, takes place. The quantity of matter transported across a given vertical plane above an arbitrary depth z, per unit length of the load, is

$$\rho\int_0^z U_x dz=(\sigma l/g)\int_0^\beta \xi_x d\beta=\begin{cases}(\sigma l/\pi g)\{(\alpha+1)\tan^{-1}(\beta/(\alpha+1))-(\alpha-1)\tan^{-1}(\beta/(\alpha-1))\}; & \alpha>1\\(\sigma l/\pi g)\{(\alpha+1)\tan^{-1}(\beta/(\alpha+1))-(1-\alpha)\tan^{-1}(\beta/(1-\alpha))\}; & \alpha<1.\end{cases}\tag{3.1}$$

The total quantity is

$$\rho\int_0^\infty U_x dz=\begin{cases}(\sigma l/g); & \alpha>1\\(\sigma l\alpha/g); & \alpha<1.\end{cases}\tag{3.2}$$

The equation which determines $\beta_0=z_0/l$ is then

$$\begin{aligned}&(\alpha+1)\tan^{-1}(\beta_0/(\alpha+1))\\&-(\alpha-1)\tan^{-1}(\beta_0/(\alpha-1))=(\pi/2);\quad \alpha>1;\\&(\alpha+1)\tan^{-1}(\beta_0/(\alpha+1))\\&-(1-\alpha)\tan^{-1}(\beta_0/(1-\alpha))=(\pi\alpha/2);\quad \alpha<1.\end{aligned}\tag{3.3}$$

TABLE II. *Depths below which one-half the total lateral displacement takes place.*

α	β_0	α	β_0
0.25	2.25	1.00	2.00
0.50	2.20	1.50	3.23
0.75	2.12	2.00	4.40

Table II gives the values of β_0 corresponding to various values of α.

The configuration of the surface at any time, given by (2.44), may be readily computed by expressing it in terms of the Hankel function $H_0^{(1)}(x\sqrt{i})$, where x is real, for which tables are available.[4] Let

$$x_1=\begin{cases}[4\tau(\alpha-1)]^{\frac{1}{2}}; & \alpha>1.\\[4\tau(1-\alpha)]^{\frac{1}{2}}; & \alpha<1.\end{cases}\qquad x_2=[4\tau(\alpha+1)]^{\frac{1}{2}}.$$

Then

$$\mu=\begin{cases}-(2/\pi)I[K_0(x_1\sqrt{i})-K_0(x_2\sqrt{i})]; & \alpha>1\\1-(2/\pi)I[K_0(x_1\sqrt{-i})-K_0(x_2\sqrt{i})]; & \alpha<1.\end{cases}\tag{3.41}$$

By using the relationship $K_0(z)=(\pi i/2)H_0^{(1)}(iz)$, where z is any complex number, and taking care

[4] E. Jahnke and F. Emde, *Function Tables*, second edition (1933).

TABLE III. *Surface configuration at various times.*

α \ τ	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	3	10
0.00	0.145	0.236	0.370	0.545	0.742	0.977	1.007
0.25	.144	.234	.366	.538	.732	.967	1.008
0.50	.140	.227	.352	.515	.697	.929	1.007
0.75	.130	.213	.327	.471	.627	.837	.976
1.00	.118	.185	.273	.371	.457	.507	.500
1.25	.106	.155	.217	.269	.283	.177	.024
1.50	.094	.139	.186	.217	.204	.082	−.006
2.00	.082	.116	.145	.152	.118	.016	−.004
3.00	.067	.088	.099	.086	.045	−.005	+.002
4.00	.057	.070	.071	.052	.017	−.004	.000
5.00	.049	.058	.054	.032	.006	−.002	.000

to choose the proper root of $\sqrt{i}$ and $\sqrt{-i}$,[5] we have $I[K_0(x\sqrt{i})] = -(\pi/2)R[H_0^{(1)}(x\sqrt{i})]$ and $I[K_0(x\sqrt{-i})] = (\pi/2)R[H_0^{(1)}(x\sqrt{i})]$. Hence

$$\mu = \begin{cases} R[H_0^{(1)}[x_1\sqrt{i}) - H_0^{(1)}(x_2\sqrt{i})]; & \alpha > 1 \\ 1 - R[H_0^{(1)}(x_1\sqrt{i}) + H_0^{(1)}(x_2\sqrt{i})]; & \alpha < 1. \end{cases} \tag{3.42}$$

[5] $H_0^{(1)}$ vanishes at infinity if the imaginary part of the argument is positive, but becomes infinite if it is negative.

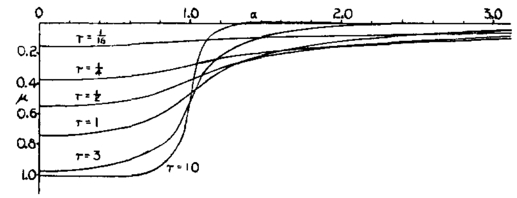

FIG. 2. Surface configuration at various times.

This function is tabulated in Table III and plotted in Fig. 2. If we use the value of $\eta/\rho = 2.9 \times 10^{21}$ found in I and the l assumed above, then the unit of τ is approximately 1800 years. The figure for $\tau = 10$ then shows that equilibrium would be very nearly reached in about 18,000 years. A load of smaller span would take a correspondingly greater time, e.g., one such as a river delta having $l = 100$ km would require 180,000 years.