CLASSICAL PROBLEMS IN CONTINUUM MECHANICS

VÍT PRŮŠA

Problem 1. *Consider the following system of ordinary differential equations, $t \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mathrm{Re}} & 1 \\ 0 & -\frac{1}{\mathrm{Re}} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + u\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

Show that $\boldsymbol{u} = \mathbf{0}$ is a steady solution to the problem. Investigate linear stability *of the steady solution. In particular, show that*

(1) the eigenvalues of the linearized operator are negative,
(2) the disturbances decay for $t \to +\infty$,
(3) the disturbances can experience a strong transient growth,

Now consider the full problem. Show that

(1) there exist Reynolds number Re *(small), such that* any *disturbance to the steady solution decays for $t \to +\infty$,*
(2) there exist Reynolds number Re *(high) and an initial condition $\boldsymbol{u}_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$, such that the disturbance to the steady solution starting form this initial condition does not decay for $t \to +\infty$. (You can solve the problem numerically.)*

Problem 2. *Consider differential equation*

$$\epsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + (x+1)\frac{\mathrm{d}f}{\mathrm{d}x} + f = 2x,$$

with boundary conditions

$$f(0) = 1,$$
$$f(1) = 2.$$

Use the boundary layer technique and show that the exact solution can be, for small ϵ, approximated by

$$f_{\text{approximation}} = \frac{x^2+3}{x+1} - 2\mathrm{e}^{-\frac{x}{\epsilon}}.$$

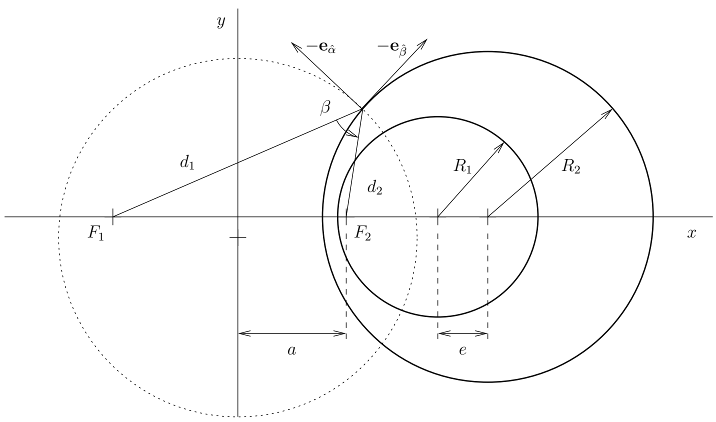

FIGURE 1. Bipolar coordinates.

Problem 3. *Consider bipolar coordinate system, see Figure 1. The relation between the Cartesian coordinates x, y and the bipolar coordinates α, β reads*

$$x = \frac{a \sinh\alpha}{\cosh\alpha - \cos\beta},$$
$$y = \frac{a \sin\beta}{\cosh\alpha - \cos\beta}.$$

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where $\beta \in [0, 2\pi]$, $\alpha \in [\alpha_1, \alpha_2]$ *and* a *is a parameter. Show that the coordinate curves, that is curves* $\beta = \text{const.}$ *and* $\alpha = \text{const.}$ *respectively, are given by implicit equations*

$$(x - a\coth\alpha)^2 + y^2 = \frac{a^2}{\sinh^2\alpha},$$
$$x^2 + (y - a\cot\beta)^2 = \frac{a^2}{\sin^2\beta}.$$

Find tangent vectors to these curves, and denote them $\boldsymbol{e}_\alpha$ *a* $\boldsymbol{e}_\beta$. *Find the components of the metric tensor* g_{ij} *and show that the Christoffel symbols* ${\Gamma^\alpha}_{\alpha\alpha}$ *a* ${\Gamma^\beta}_{\beta\beta}$ *are given by the following formulae,*

$${\Gamma^\alpha}_{\alpha\alpha} = -\frac{\sinh\alpha}{\cosh\alpha - \cos\beta},$$
$${\Gamma^\beta}_{\beta\beta} = -\frac{\sin\beta}{\cosh\alpha - \cos\beta},$$

and that the remaining Christoffel symbols are ${\Gamma^\beta}_{\alpha\beta} = {\Gamma^\alpha}_{\alpha\alpha}$, ${\Gamma^\alpha}_{\beta\beta} = -{\Gamma^\alpha}_{\alpha\alpha}$, ${\Gamma^\beta}_{\alpha\alpha} = -{\Gamma^\beta}_{\beta\beta}$, ${\Gamma^\alpha}_{\alpha\beta} = {\Gamma^\beta}_{\beta\beta}$. *Find a formula for* $\Delta\phi$. *(Laplace operator acting on a scalar function.)*

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