The Motion of a Viscous Fluid Under a Surface Load

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A formal solution is given for the motion of a highly viscous fluid when a symmetrical pressure is applied at the surface. This is applied to the subsidence of a cylindrical body of constant thickness and to the recovery of the fluid after removal of a load. Applying the latter case to the plastic recoil of the earth after the disappearance of the Pleistocene ice sheets, it is found that the geological data imply a kinematic viscosity of the order of 3×10^{21} c.g.s. units.

INTRODUCTION

OBSERVATIONS on the elevated and tilted shore lines of the glaciated tracts of Europe and North America have shown that during and following the disappearance of the Pleistocene ice sheets the earth returned to a configuration of equilibrium by delayed plastic flow as well as by immediate elastic recovery. The progress of this recovery has been dated by means of varved glacial clays and F. Nansen[1] has constructed curves illustrating the uplift of the Fennoscandian region from 16,000 B.C. to the present time. This suggests the possibility of determining the effective viscosity of the earth's outer shells under forces of long duration.

In the present paper we shall neglect the curvature of the earth and treat the problem of the motion of a semi-infinite, incompressible, viscous fluid under the action of a radially symmetrical pressure applied at the free surface. Since, in the case of the earth, we are dealing with extremely small accelerations and very high viscosity, we may neglect the inertial terms in the equations of motion in comparison with those arising from viscous forces. We shall first treat the case of an arbitrary symmetrical impressed pressure and then apply the results to two special cases, (1) a constant load of radius r_0 applied at $t=0$, and (2) recovery after the removal of an arbitrary symmetrical load.

1. FORMULATION OF THE PROBLEM

Neglecting the terms arising from the acceleration, the equations of motion of a fluid of viscosity η and density ρ in a gravitational field g are;—

$$\eta\nabla^2 V=\operatorname{grad} p-(0,\,0,\,\rho g), \tag{1.1}$$

$$\operatorname{div} V=0, \tag{1.2}$$

where the positive z axis has been taken as directed downward. Transforming to cylindrical coordinates (r, z, ϕ), assuming radial symmetry, and setting $\bar{p}=p-\rho gz$, these become

$$\partial/r\partial r(r\partial V_r/\partial r)-V_r/r^2+\partial^2 V_r/dz^2=\partial\bar{p}/\eta\partial r, \tag{1.11}$$

$$\partial/r\partial r(r\partial V_z/\partial r)+\partial^2 V_z/\partial r^2=\partial\bar{p}/\eta\partial z, \tag{1.12}$$

$$\partial/r\partial r(rV_r)+\partial V_z/\partial z=0. \tag{1.21}$$

The components of stress in which we are interested are

$$p_{zz}=-(\bar{p}+\rho gz)+2\eta\partial V_z/\partial z, \tag{1.31}$$

$$p_{rz}=\eta(\partial V_r/\partial z+\partial V_z/\partial r). \tag{1.32}$$

The boundary conditions are that on the free surface p_{rz} shall vanish and p_{zz} shall equal the applied stress, and at infinity the stresses and velocities shall vanish.

Let the equation of the free surface be $z=\zeta(r, t)$ and take the undisturbed surface as the plane $z=0$. If we assume that ζ remains small in comparison with other distances entering into the problem, such as the radius of the applied load, we may replace the value of $\partial V_z/\partial z$ at $z=\zeta$ by its value at $z=0$, and similarly with all the other quantities appearing in (1.31) and (1.32) except ρgz.[2] Calling the applied pressure $-\sigma(r, t)$, the boundary conditions become

$$\bar{p}(r,0,t)+\rho g\zeta(r,t)-2\eta(\partial V_z/\partial z)(r,0,t)=\sigma(r,t), \tag{1.41}$$

$$(\partial V_r/\partial z+\partial V_z/\partial r)_{z=0}=0 \tag{1.42}$$

and

$$\partial\zeta/\partial t=V_z(r,0,t). \tag{1.43}$$

2. GENERAL SOLUTION FOR ARBITRARY IMPRESSED LOAD

Setting $V_r=R_1(r)Z_1(z)$, $V_z=R_2(r)Z_2(z)$, $\bar{p}=R_3(r)Z_3(z)$ in (1.11), (1.12), and (1.21), and

[1] F. Nansen, *The Earth's Crust, Its Surface Forms and Isostatic Adjustment* (1928), Oslo.

[2] This is the sort of approximation commonly made in treating waves on the surface of a fluid. Cf., H. Lamb, *Hydrodynamics*, fifth edition, Chaps. IX. & XI.

separating variables, we have the Bessel equations,

$$d/rdr(rdR_1/dr)-R_1/r^2+\lambda^2R_1=0, \quad (2.11)$$

$$d/rdr(rdR_2/dr)+\lambda^2R_2=0 \quad (2.12)$$

and

$$R_3=\text{const.}\times R_2. \quad (2.13)$$

The appropriate solutions are $R_1=J_1(\lambda r)$, $R_2=R_3=J_0(\lambda r)$, where we have included the constant factor in R_3 in the z factor. The z equations are

$$d^2Z_1/dz^2-\lambda^2Z_1=-\lambda Z_3/\eta, \quad (2.21)$$

$$d^2Z_2/dz^2-\lambda^2Z_2=dZ_3/\eta dz, \quad (2.22)$$

$$\lambda Z_1+dZ_2/dz=0. \quad (2.23)$$

Eliminating Z_1 and Z_3, we have the fourth order equation for Z_2

$$d^4Z_2/dz^4-2\lambda^2d^2Z/dz^2+\lambda^4Z_2=0. \quad (2.3)$$

The solutions of this equation are $e^{\pm\lambda z}$, $ze^{\pm\lambda z}$ of which only those with negative exponents are appropriate to the present problem. The solutions of the set (2.21), (2.22), (2.23) are then,

$$Z_2=e^{-\lambda z}(A+Bz), \quad (2.41)$$

$$Z_1=e^{-\lambda z}(A-B/\lambda+Bz), \quad (2.42)$$

$$Z_3=2\eta Be^{-\lambda z}. \quad (2.43)$$

In order to satisfy the boundary condition (1.42) we have $B=\lambda A$.[3] We must now satisfy (1.41) with functions of the form

$$V_r=z\int_0^\infty A(\lambda)e^{-\lambda z}J_1(\lambda r)\lambda d\lambda, \quad (2.51)$$

$$V_z=\int_0^\infty A(\lambda)e^{-\lambda z}J_0(\lambda r)(1+\lambda z)d\lambda, \quad (2.52)$$

$$\bar{p}=2\eta\int_0^\infty A(\lambda)e^{-\lambda z}J_0(\lambda r)\lambda d\lambda. \quad (2.53)$$

From (2.51) $\partial V_z/\partial z=0$ at $z=0$, hence (1.41) becomes

$$2\eta\int_0^\infty A(\lambda)J_0(\lambda r)\lambda d\lambda+\rho g\zeta=\sigma(r,t). \quad (2.61)$$

A will evidently have to be a function of the time in order to satisfy this equation, hence we may differentiate with respect to t and use (1.43). Then

$$\int_0^\infty J_0(\lambda r)\{2\eta\partial A/\partial t+\rho gA/\lambda\}\lambda d\lambda=\partial\sigma/\partial t. \quad (2.62)$$

[3] Note that this is the same relationship that we would get if instead of the vanishing of p_{rz} at $z=0$ we took as a boundary condition that $V_r=0$ at $z=0$. The latter is the condition we would have if we assumed that on the surface of the fluid there lay a solid crust capable of resisting tangential deformation, but opposing a negligible resistance to vertical deformation.

Application of the Fourier-Bessel inversion formula yields a first order differential equation for A as a function of t.

$$2\eta\partial A/\partial t+\rho gA/\lambda=\int_0^\infty(\partial\sigma/\partial t)J_0(\lambda r)rdr. \quad (2.7)$$

The solution is[4]

$$A=K(\lambda)e^{-\rho gt/2\eta\lambda}+\frac{1}{2\eta}e^{-\rho gt/2\eta\lambda}\int_0^t\int_0^\infty\frac{\partial\sigma}{\partial t}e^{\rho gt/2\eta\lambda}J_0(\lambda r)rdrdt, \quad (2.81)$$

where $K(\lambda)$ must be determined from the initial conditions. Integrating the second term by parts with respect to t

$$\int_0^t\frac{\partial\sigma}{\partial t}e^{\rho gt/2\eta\lambda}dt=\sigma e^{\rho gt/2\eta\lambda}-\sigma_0-\frac{\rho g}{2\eta\lambda}\int_0^t\sigma e^{\rho gt/2\eta\lambda}dt.$$

Hence

$$A=K(\lambda)e^{-\rho gt/2\eta\lambda}+\frac{1}{2\eta}\int_0^\infty\sigma J_0(\lambda r)rdr-\frac{1}{2\eta}e^{-\rho gt/2\eta\lambda}\int_0^\infty\sigma_0J_0(\lambda r)rdr-\frac{\rho g}{4\eta^2\lambda}e^{-\rho gt/2\eta\lambda}\int_0^t\int_0^\infty\sigma e^{\rho gt/2\eta\lambda}J_0(\lambda r)rdrdt. \quad (2.82)$$

This expression, with Eqs. (2.51), (2.52), (2.53) is the formal solution of the problem.

3. Subsidence of a Cylindrical Body

If we suppose that the fluid is initially at rest and that at $t=0$ a uniform circular load of radius r_0 is placed on the surface,

$$\sigma(r,t)=\begin{cases}0; & r>r_0\\ 0; & t\leq 0\\ \sigma=\text{const}; & r<r_0,\ t>0.\end{cases} \quad (3.1)$$

Then $K(\lambda)=0$, and by using

$$\int_0^{r_0}J_0(\lambda r)rdr=(r_0/\lambda)J_1(\lambda r)$$

$$A=(\sigma r_0/2\eta\lambda)e^{-\rho gt/2\eta\lambda}J_1(\lambda r_0). \quad (3.2)$$

By substituting in (2.51), (2.52) and (2.53),

$$V_r=(z\sigma r_0/2\eta)\int_0^\infty e^{-\rho gt/2\eta\lambda-\lambda z}\times J_1(\lambda r_0)J_1(\lambda r)d\lambda, \quad (3.31)$$

[4] E. L. Ince, *Ordinary Differential Equations*, p. 21.

$$V_z=(\sigma r_0/2\eta)\int_0^\infty e^{-\rho g t/2\eta\lambda-\lambda z}\times J_1(\lambda r_0)J_0(\lambda r)((1+\lambda z)/\lambda)d\lambda, \quad (3.32)$$

$$\bar{p}=\sigma r_0\int_0^\infty e^{-\rho g t/2\eta\lambda-\lambda z}J_1(\lambda r_0)J_0(\lambda r)d\lambda, \quad (3.33)$$

$$\zeta=\int_0^t V_z(r,0,t)dt=(\sigma r_0/\rho g)\int_0^\infty(1-e^{-\rho g t/2\eta\lambda})\times J_1(\lambda r_0)J_0(\lambda r)d\lambda. \quad (3.34)$$

We may note as a check that as $t\rightarrow\infty$

$$\zeta\rightarrow(\sigma r_0/\rho g)\int_0^\infty J_1(\lambda r_0)J_0(\lambda r)d\lambda = \begin{cases} 0; & r>r_0 \\ \sigma/\rho g; & r<r_0, \end{cases} \quad (3.35)$$

so that the system approaches the configuration of hydrostatic equilibrium asymptotically as it should.

It does not appear to be possible to express the above integrals in closed form. They may be put in dimensionless form for numerical or mechanical integration by setting

$$\tau=\rho g r_0 t/2\eta, \quad x=r/r_0, \quad y=z/r_0,$$

$$K=\lambda r_0, \quad v_r=2\eta V_r/\sigma r_0, \quad v_z=2\eta V_z/\sigma r_0,$$

$$\pi=\bar{p}/\sigma, \quad \mu=\rho g\zeta/\sigma.$$

Then

$$v_r=y\int_0^\infty e^{-\tau/K-Ky}J_1(K)J_1(Kx)dK, \quad (3.41)$$

$$v_z=\int_0^\infty e^{-\tau/K-Ky}J_1(K)\times J_0(Kx)((1+Ky)/K)dK, \quad (3.42)$$

$$\pi=\int_0^\infty e^{-\tau/K-Ky}J_1(K)J_0(Kx)dK, \quad (3.43)$$

$$\mu=\int_0^\infty(1-e^{-\tau/K})J_1(K)J_0(Kx)dK. \quad (3.44)$$

4. Recovery after Removal of Load

Setting the applied pressure, σ, equal to zero over the whole surface in (2.82) we have

$$A=K(\lambda)e^{-\rho g t/2\eta\lambda}.$$

$K(\lambda)$ can then be determined either from the initial velocity at the surface or from the initial configuration of the surface. From (2.52)

$$V_z(r,0,0)=\int_0^\infty K(\lambda)J_0(\lambda r)d\lambda. \quad (4.11)$$

Hence, by inversion,

$$K(\lambda)=\lambda\int_0^\infty V_z(r,0,0)J_0(\lambda r)rdr. \quad (4.12)$$

We also have

$$\zeta=\zeta(r,0)+\int_0^t V_z(r,0,0)dt.$$
$$=\zeta(r,0)+(2\eta/\rho g)\int_0^\infty K(\lambda)(1-e^{-\rho g t/2\eta\lambda})\times J_0(\lambda r)\lambda d\lambda. \quad (4.21)$$

As t becomes infinite, ζ must approach zero, therefore

$$\zeta(r,0)=-(2\eta/\rho g)\int_0^\infty K(\lambda)J_0(\lambda r)\lambda d\lambda \quad (4.22)$$

or, inverting,

$$K(\lambda)=-(\rho g/2\eta)\int_0^\infty \zeta(r,0)J_0(\lambda r)rdr. \quad (4.23)$$

Thus the subsequent motion is completely determined if we know either $\zeta(r,0)$ or $V_z(r,0,0)$.

5. Application to Post-Glacial Uplift of Fennoscandia

It would, of course, be impossible to calculate the motion of the earth throughout the history of the retreat of a continental ice sheet, since both the radius and the thickness varied with time in an irregular and, in the case of the thickness, an unknown manner. Moreover, an important part of the uplift (estimated as one-sixth to one-quarter of the total) was due not to plastic flow but to elastic recovery. However, the results of the last section show that we may treat the part of the uplift which took place after the complete disappearance of the ice without any knowledge of how fast it retreated or how its thickness varied provided we know the rate of uplift at all distances from the center, or the configuration of the surface, at some time subsequent to the completion of the melting. During this part of the motion we do not need to consider the elastic component, since that would keep pace with the decrease in the load and would be complete when the ice had disappeared.

Nansen's curves (Figs. 1 and 2) give the amount and rate of uplift at a point in Ångermanland near the center of the Fennoscandian glaciated region and at Oslo, about 500 km distant. In order to use this data it will be necessary to assume a reasonable form for either $\zeta(r,0)$, the initial configuration of the surface, or $V_2(r,0,0)$, the initial rate of uplift, and fit the assumed function to the values given for these two points. It is reasonable to suppose that $V_z(r,0,0)$ will

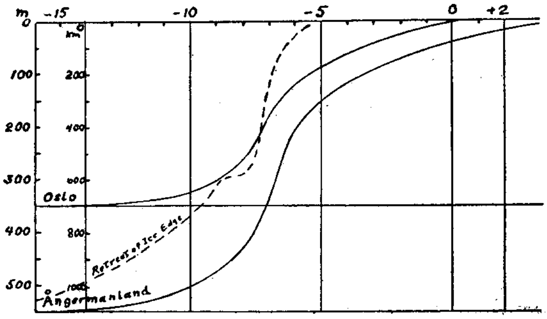

FIG. 1. Uplift of the Fennoscandian region. Abscissas are times in units of 1000 years B.C. and A.D. Ordinates (solid curves) are depths of the surface in meters below equilibrium level. Ordinates of the dotted curve give the radius of the ice cap in kilometers.

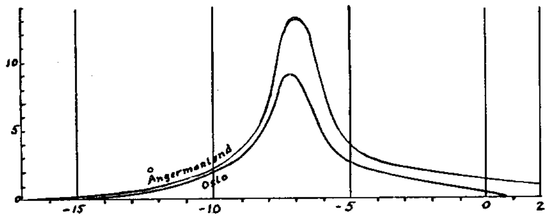

FIG. 2. Rate of uplift in meters per century. Abscissas in units of 1000 years.

have a maximum at the center and will decrease outward in a way that can be represented with sufficient accuracy by an exponential function. We shall suppose then that

$$V_z(r, 0, 0) = -ae^{-b^2r^2}, \qquad (5.1)$$

using the square of r in the exponent rather than the first power in order to avoid having a cusp at the center. By substituting in (4.12),

$$K(\lambda) = -a\lambda \int_0^\infty e^{-b^2r^2} J_0(\lambda r) r dr = -(a\lambda/2b^2)e^{-\lambda^2/4b^2}. \qquad (5.2)$$

By using (4.21) and (4.22),

$$\zeta = (a\eta/\rho g b^2)\int_0^\infty e^{-\lambda^2/4b^2 - \rho g t/2\eta\lambda} J_0(\lambda r)\lambda^2 d\lambda. \qquad (5.3)$$

At $r=0$, $t=0$

$$\zeta(0, 0) = (a\eta/\rho g b^2)\int_0^\infty e^{-\lambda^2/4b^2}\lambda^2 d\lambda = 2\sqrt{\pi}(\eta ab/\rho g).$$

The kinematic viscosity, $\nu = \eta/\rho$, is then given by

$$\nu = g\zeta(0, 0)/2\sqrt{\pi} ab. \qquad (5.41)$$

Since we are at liberty, in applying the above theory, to take any time after the disappearance of the ice as the initial instant $t=0$, we may check roughly the validity of the assumption that $V_z(r, 0, 0)$ does not depart very widely from the form given in (5.1) by calculating ν from (5.41) for various initial instants and seeing that it remains of the same order of magnitude. If we express $\zeta(0, 0)$ in meters, a in meters per century, and b in reciprocal kilometers, ν is given in c.g.s. units by

$$\nu = 8.73\times 10^{16}\zeta(0, 0)/ab. \qquad (5.42)$$

$\zeta(0, 0)$ may then be read from the curve for Ångermanland in Fig. 1, a from that in Fig. 2, and b may be calculated from the curve for Oslo in Fig. 2 by setting $r=500$ in (5.1). In Table I the values of these quantities are given for various times taken as the initial instant.

The initial configuration of the surface, corresponding to the assumed initial velocity (5.1), is given by

$$\zeta(r, 0) = (a\eta/\rho g b^2)\int_0^\infty e^{-\lambda^2/4b^2} J_0(\lambda r)\lambda^2 d\lambda \qquad (5.51)$$

or

$$\zeta(r, 0)/\zeta(0, 0) = (1/2\sqrt{\pi})\int_0^\infty e^{-\lambda^2/4} J_0(\lambda x)\lambda^2 d\lambda, \qquad (5.52)$$

where $x = br$. Values of this quantity as a function of x are given in Table II. Taking the initial instant at 5000 B.C., and using $b = 1.27\times 10^{-3}$ km^{-1}, the initial profile of the surface and rate of uplift are plotted in Fig. 3. A complete observed profile is not available to compare with the calculated curve in order to determine how closely the assumptions made fit the actual case, but it may be noted that Fig. 3 indicates a depression of the surface at 500 km radius of about 76 meters. This is to be compared with 85 meters at Oslo at 5000 B.C. as read from the curve of Fig. 1.

In comparing the figure given above for the earth's kinematic viscosity with experimentally determined viscosities of solid substances it should be noted that two quite distinct properties

TABLE I. *Values of the quantities entering in Eq. (5.42) for various times taken as the initial instant.*

$t=0$	a	$b\times 10^3$	$\zeta(0, 0)$	$\nu\times 10^{-21}$
5,000 B.C.	3.9	1.27±.07	147	2.6±0.2
4,000 "	2.7	1.19±.09	118	3.2±0.3
3,000 "	2.2	1.24±.11	94	3.0±0.4
2,000 "	1.8	1.25±.15	74	2.9±0.5
Mean value: $\nu = 2.9\times 10^{21}$ cm^2 sec.$^{-1}$				

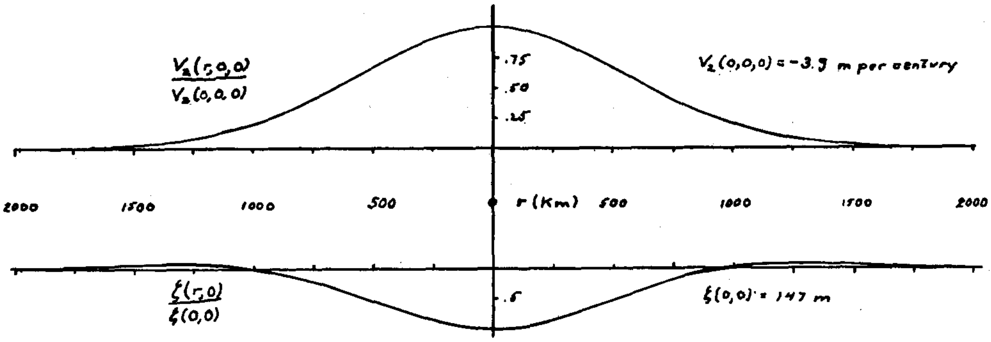

FIG. 3. Rate of uplift and surface profile at 5000 B.C. Vertical exaggeration of profile 1.7×10^3 times.

TABLE II.

x	$\zeta(r, 0)/\zeta(0, 0)$	x	$\zeta(r, 0)/\zeta(0, 0)$
0.0	+1.000	1.0	+0.155
0.2	+0.922	1.5	−0.076
0.4	+0.780	2.0	−0.068
0.6	+0.562	2.5	−0.033
0.8	+0.328		

TABLE III. *Values taken for the coefficient of viscosity of various substances.*

MATERIAL	TEMPERATURE	η (g. cm^{-1} sec.$^{-1}$)
Shoemakers wax	15°C	$2\ \times10^{8}$
Asphalt	15	$3\ \times10^{10}$
Ice	0	$5\ \times10^{12}$
Glass	575	$1\ \times10^{13}$
Lead	20	$1\ \times10^{16}$
Calcite	18	1.5×10^{16}
Rocksalt	18	$2\ \times10^{18}$

of matter are often included in the term.[5] One, which is better designated as inner friction, relates to the damping of elastic vibrations, the other relates to plastic flow. The coefficient of inner friction is of the order of 10^8 or 10^9 c.g.s. units for most metals and is higher for soft than for hard substances. The "true" coefficient of viscosity is more difficult to measure experimentally, different observers giving widely varying values,[6] and, in general, increases rapidly with the hardness of the substance. The figures given in Table III are taken from Gutenberg and Schlechtweg's paper. In view of the great increase of viscosity with the hardness of the material and the pressure, it is not surprising to find that for silicates under the enormous pressures within the earth the viscosity is as high as we have found here, even when the decrease of viscosity at high temperatures is taken into consideration.

R. W. van Bemmelen and H. P. Berlage[7] have recently made an estimate of the viscosity of the earth on the basis of post-glacial uplift using a method quite different from that followed hree. They find $\eta=1.3\times10^{20}$ which, since they assume a density of 3, is equivalent to $\nu=0.4\times10^{20}$. Their method makes use of special hypotheses concerning the structure and dynamics of the earth's crust and is only indirectly based on hydrodynamical theory. They also assume that the flow is confined to a layer 100 km thick and it is therefore to be expected that their value should be lower than that found by the present method. According to their theory ν varies as the cube of the assumed thickness of the layer, hence if it were taken to be 400 km, their figure would be comparable with ours.

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[5] B. Gutenberg and H. S. Schlechtweg, *Viscosity and Inner Friction of Solid Bodies*, Physik. Zeits. **31**, 745 (1930).
[6] W. D. Kusnezow, Zeits. f. Physik. **51**, 239 (1928).
[7] R. W. van Bemmelen and H. P. Berlage, Gerl. Beit. z. Geophys. **43**, 19 (1934).