

Při výpočtu limit není dovoleno používat l'Hospitalovo pravidlo a Taylorovy polynomy zkoumaných funkcí.

Jméno a příjmení: \_\_\_\_\_

Skupina: \_\_\_\_\_

Příklad	1	2	3	4	5	6	7	8	9	10	Celkem bodů
Bodů	2	2	3	4	4	6	6	8	7	8	50
Získáno											

[2] 1.

$$\lim_{x \rightarrow 0^+} \frac{\sin \sqrt{2x}}{\sqrt{x} + x^{\frac{3}{2}}}$$

**Řešení:**

$$\lim_{x \rightarrow 0^+} \frac{\sin \sqrt{2x}}{\sqrt{x} + x^{\frac{3}{2}}} = \lim_{x \rightarrow 0^+} \frac{\sin \sqrt{2x}}{\sqrt{x}(1+x)} = \lim_{x \rightarrow 0^+} \frac{\sin \sqrt{2x}}{\sqrt{2x}} \frac{\sqrt{2}}{(1+x)} = \sqrt{2}$$

[2] 2.  $\forall i \in \{1, \dots, n\} : a_i \neq 0, \forall i \in \{1, \dots, m\} : b_i \neq 0$

$$\lim_{x \rightarrow 0} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_2 x^2 + b_1 x}$$

**Řešení:**

$$\lim_{x \rightarrow 0} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_2 x^2 + b_1 x} = \lim_{x \rightarrow 0} \frac{x (a_n x^{n-1} + a_{n-1} x^{n-2} + \dots + a_2 x + a_1)}{x (b_m x^{m-1} + b_{m-1} x^{m-2} + \dots + b_2 x + b_1)} = \frac{a_1}{b_1}$$

[3] 3.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - 1}{\sin x}$$

**Řešení:**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - 1}{\sin x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - 1}{x} \frac{x}{\sin x} = \lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - 1}{x} \frac{\sqrt{1+2x} + 1}{\sqrt{1+2x} + 1} \frac{x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{(1+2x) - 1}{x} \frac{1}{\sqrt{1+2x} + 1} \frac{x}{\sin x} = 1 \end{aligned}$$

[4] 4.

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\tan^2 x}$$

**Řešení:**

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\tan^2 x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \frac{x^2}{\tan^2 x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \frac{x^2}{\sin^2 x} \cos^2 x = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \left( \frac{1}{\frac{\sin x}{x}} \right)^2 \cos^2 x = \frac{1}{2}.$$

[4] 5.

$$\lim_{x \rightarrow 0} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\sqrt{1 - \cos x}}$$

**Řešení:**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\sqrt{1 - \cos x}} &= \lim_{x \rightarrow 0} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \frac{\sin^2 x}{x} \frac{1}{\sqrt{1 - \cos x}} = \lim_{x \rightarrow 0} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \frac{\sin^2 x}{x^2} \frac{x}{\sqrt{1 - \cos x}} \\ &= \lim_{x \rightarrow 0} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \left( \frac{\sin x}{x} \right)^2 \sqrt{\frac{1}{\frac{1 - \cos x}{x^2}}} = \sqrt{2} \end{aligned}$$

[6] 6.  $a \in \mathbb{R}^+$ 

$$\lim_{x \rightarrow 0^-} \frac{\ln\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}\right)}{x}$$

**Řešení:**

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{\ln\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}\right)}{x} &= \lim_{x \rightarrow 0^-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1\right) + 1\right) \frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1} \frac{1}{x}} = \lim_{x \rightarrow 0^-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1\right) + 1\right) \frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1} \frac{1}{x}} \\ &= \lim_{x \rightarrow 0^-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1\right) + 1\right) \frac{e^{ax} - 1 + xe^{\frac{a}{x}}}{1+xe^{\frac{a}{x}}}}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1} \frac{1}{x}} = \lim_{x \rightarrow 0^-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1\right) + 1\right) e^{ax} - 1 + xe^{\frac{a}{x}}}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1} \frac{1}{x} \frac{1}{1+xe^{\frac{a}{x}}}} \\ &= \lim_{x \rightarrow 0^-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1\right) + 1\right)}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}} - 1} \left(a \frac{e^{ax} - 1}{ax} + e^{\frac{a}{x}}\right) \frac{1}{1+xe^{\frac{a}{x}}} = a \end{aligned}$$

[6] 7.

$$\lim_{x \rightarrow 1^-} \frac{1 - x^2}{(\arccos x)^2}$$

**Řešení:**

Označme si  $y = \arccos x$ , aneb  $\cos y = x$ , pak  $x \rightarrow 1^-$  znamená  $y \rightarrow 0^+$ . Je tedy

$$\lim_{x \rightarrow 1^-} \frac{1 - x^2}{(\arccos x)^2} = \lim_{y \rightarrow 0^+} \frac{1 - \cos^2 y}{y^2} = \lim_{y \rightarrow 0^+} \frac{1 - \cos y}{y^2} (1 + \cos y) = 1.$$

[8] 8.

$$\lim_{x \rightarrow \frac{\pi}{4}^-} (\tan x)^{3 \tan 2x}$$

**Řešení:**

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{4}^-} (\tan x)^{3 \tan 2x} &= \lim_{x \rightarrow \frac{\pi}{4}^-} e^{3 \tan 2x \ln(\tan x)} = e^{\lim_{x \rightarrow \frac{\pi}{4}^-} 3 \frac{\sin 2x}{\cos 2x} \ln((\tan x - 1) + 1)} \\ &= e^{\lim_{x \rightarrow \frac{\pi}{4}^-} 3 \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x} \frac{\ln((\tan x - 1) + 1)}{(\tan x - 1)} (\tan x - 1)} = e^{\lim_{x \rightarrow \frac{\pi}{4}^-} 3 \frac{2 \tan x}{1 - \tan^2 x} \frac{\ln((\tan x - 1) + 1)}{(\tan x - 1)} (\tan x - 1)} \\ &= e^{\lim_{x \rightarrow \frac{\pi}{4}^-} -3 \frac{2 \tan x}{1 + \tan x} \frac{\ln((\tan x - 1) + 1)}{(\tan x - 1)}} = e^{-3} \end{aligned}$$

[7] 9.

$$\lim_{x \rightarrow 1} \left( 1 + \cos \left( \frac{\pi}{2} x \right) \right)^{\frac{1}{x-1}}$$

**Řešení:**

$$\begin{aligned} \lim_{x \rightarrow 1} \left( 1 + \cos \left( \frac{\pi}{2} x \right) \right)^{\frac{1}{x-1}} &= \lim_{y \rightarrow 0} \left( 1 + \cos \left( \frac{\pi}{2} (y+1) \right) \right)^{\frac{1}{y}} = \lim_{y \rightarrow 0} e^{\frac{\ln \left( 1 + \cos \left( \frac{\pi}{2} (y+1) \right) \right)}{y}} = e^{\lim_{y \rightarrow 0} \frac{\ln \left( 1 + \cos \left( \frac{\pi}{2} (y+1) \right) \right) \cos \left( \frac{\pi}{2} (y+1) \right)}{\cos \left( \frac{\pi}{2} (y+1) \right) y}} \\ &= e^{\lim_{y \rightarrow 0} \frac{\ln \left( 1 + \cos \left( \frac{\pi}{2} (y+1) \right) \right) \cos \left( \frac{\pi}{2} y \right) \cos \frac{\pi}{2} - \sin \left( \frac{\pi}{2} y \right) \sin \frac{\pi}{2}}{\cos \left( \frac{\pi}{2} (y+1) \right) y}} = e^{\lim_{y \rightarrow 0} \frac{\ln \left( 1 + \cos \left( \frac{\pi}{2} (y+1) \right) \right) - \sin \left( \frac{\pi}{2} y \right)}{\cos \left( \frac{\pi}{2} (y+1) \right) y}} \\ &= e^{\lim_{y \rightarrow 0} \frac{\ln \left( 1 + \cos \left( \frac{\pi}{2} (y+1) \right) \right)}{\cos \left( \frac{\pi}{2} (y+1) \right)} \left( -\frac{\sin \left( \frac{\pi}{2} y \right)}{\frac{\pi}{2} y} \right)^{\frac{\pi}{2}}} = e^{-\frac{\pi}{2}} \end{aligned}$$

[8] 10.

$$\lim_{x \rightarrow 0^+} \frac{1}{\ln(x+1)} \ln \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} \right)$$

**Řešení:**

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{1}{\ln(x+1)} \ln \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} \right) &= \lim_{x \rightarrow 0^+} \frac{1}{\ln(x+1)} \frac{\ln \left( \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1 \right) + 1 \right)}{\frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1} \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1 \right) \\ &= \lim_{x \rightarrow 0^+} \frac{1}{x \ln(x+1)} \frac{\ln \left( \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1 \right) + 1 \right)}{\frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1} \left( \frac{(\frac{1}{x} + a) - (\frac{1}{x} - a)}{\frac{1}{x} - a} \right) \\ &= \lim_{x \rightarrow 0^+} \frac{x}{\ln(x+1)} \frac{\ln \left( \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1 \right) + 1 \right)}{\frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1} \frac{1}{x} \left( \frac{2a}{\frac{1}{x} - a} \right) = \lim_{x \rightarrow 0^+} \frac{x}{\ln(x+1)} \frac{\ln \left( \left( \frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1 \right) + 1 \right)}{\frac{\frac{1}{x} + a}{\frac{1}{x} - a} - 1} \frac{2a}{1 - ax} = 2a \end{aligned}$$