Při výpočtu limit není dovoleno používat l'Hospitalovo pravidlo a Taylorovy polynomy zkoumaných funkcí.

Jméno a příjmení: ______________________________

Skupina: ______________________________

Příklad	1	2	3	4	5	6	7	8	9	10	**Celkem bodů**
Bodů	2	2	3	4	4	6	6	8	7	8	50
Získáno											

[2] 1.

$$\lim_{x\to 0+} \frac{\sin\sqrt{2x}}{\sqrt{x}+x^{\frac{3}{2}}}$$

Řešení:

$$\lim_{x\to 0+} \frac{\sin\sqrt{2x}}{\sqrt{x}+x^{\frac{3}{2}}} = \lim_{x\to 0+} \frac{\sin\sqrt{2x}}{\sqrt{x}\,(1+x)} = \lim_{x\to 0+} \frac{\sin\sqrt{2x}}{\sqrt{2x}} \frac{\sqrt{2}}{(1+x)} = \sqrt{2}$$

[2] 2. $\forall i \in \{1, \dots, n\} : a_i \neq 0$, $\forall i \in \{1, \dots, m\} : b_i \neq 0$

$$\lim_{x \to 0} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x}$$

Řešení:

$$\lim_{x \to 0} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_2 x^2 + b_1 x} = \lim_{x \to 0} \frac{x \left(a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_2 x + a_1 \right)}{x \left(b_m x^{m-1} + b_{m-1} x^{m-2} + \cdots + b_2 x + b_1 \right)} = \frac{a_1}{b_1}$$

[3] 3.

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-1}{\sin x}$$

Řešení:

$$\lim_{x\to 0} \frac{\sqrt{1+2x}-1}{\sin x} = \lim_{x\to 0} \frac{\sqrt{1+2x}-1}{x} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\sqrt{1+2x}-1}{x} \frac{\sqrt{1+2x}+1}{\sqrt{1+2x}+1} \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{(1+2x)-1}{x} \frac{1}{\sqrt{1+2x}+1} \frac{x}{\sin x} = 1$$

[4] 4.

$$\lim_{x\to 0} \frac{1-\cos x}{\tan^2 x}$$

Řešení:

$$\lim_{x\to 0} \frac{1-\cos x}{\tan^2 x} = \lim_{x\to 0} \frac{1-\cos x}{x^2}\frac{x^2}{\tan^2 x} = \lim_{x\to 0} \frac{1-\cos x}{x^2}\frac{x^2}{\sin^2 x}\cos^2 x = \lim_{x\to 0} \frac{1-\cos x}{x^2}\left(\frac{1}{\frac{\sin x}{x}}\right)^2 \cos^2 x = \frac{1}{2}.$$

[4] 5.

$$\lim_{x\to 0} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\sqrt{1-\cos x}}$$

Řešení:

$$\lim_{x\to 0\pm} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\sqrt{1-\cos x}} = \lim_{x\to 0\pm} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \frac{\sin^2 x}{x} \frac{1}{\sqrt{1-\cos x}} = \lim_{x\to 0\pm} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \frac{\sin^2 x}{x^2} \frac{x}{\sqrt{1-\cos x}}$$

$$= \lim_{x\to 0\pm} \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \left(\frac{\sin x}{x}\right)^2 \frac{(\operatorname{sgn} x)\,|x|}{\sqrt{1-\cos x}} = \lim_{x\to 0\pm} (\operatorname{sgn} x) \frac{e^{\frac{\sin^2 x}{x}} - 1}{\frac{\sin^2 x}{x}} \left(\frac{\sin x}{x}\right)^2 \sqrt{\frac{1}{\frac{1-\cos x}{x^2}}} = \pm\sqrt{2}$$

Limita tedy neexistuje.

[6] 6. $a \in \mathbb{R}^+$

$$\lim_{x\to 0-} \frac{\ln\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}\right)}{x}$$

Řešení:

$$\lim_{x\to 0-} \frac{\ln\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}\right)}{x} = \lim_{x\to 0-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1\right)+1\right)}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1} \frac{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1}{x} = \lim_{x\to 0-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1\right)+1\right)}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1} \frac{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1}{x}$$

$$= \lim_{x\to 0-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1\right)+1\right)}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1} \frac{\frac{e^{ax}-1+xe^{\frac{a}{x}}}{1+xe^{\frac{a}{x}}}}{x} = \lim_{x\to 0-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1\right)+1\right)}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1} \frac{e^{ax}-1+xe^{\frac{a}{x}}}{x} \frac{1}{1+xe^{\frac{a}{x}}}$$

$$= \lim_{x\to 0-} \frac{\ln\left(\left(\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1\right)+1\right)}{\frac{e^{ax}}{1+xe^{\frac{a}{x}}}-1} \left(a\frac{e^{ax}-1}{ax}+e^{\frac{a}{x}}\right) \frac{1}{1+xe^{\frac{a}{x}}} = a$$

[6] 7.

$$\lim_{x\to 1-} \frac{1-x^2}{(\arccos x)^2}$$

Řešení:

Označme si $y = \arccos x$, aneb $\cos y = x$, pak $x \to 1-$ znamená $y \to 0+$. Je tedy

$$\lim_{x\to 1-} \frac{1-x^2}{(\arccos x)^2} = \lim_{y\to 0+} \frac{1-\cos^2 y}{y^2} = \lim_{y\to 0+} \frac{1-\cos y}{y^2}(1+\cos y) = 1.$$

[8] 8.

$$\lim_{x\to\frac{\pi}{4}-}(\tan x)^{3\tan 2x}$$

Řešení:

$$\lim_{x\to\frac{\pi}{4}-}(\tan x)^{3\tan 2x} = \lim_{x\to\frac{\pi}{4}-} e^{3\tan 2x\ln(\tan x)} = e^{\lim_{x\to\frac{\pi}{4}-} 3\frac{\sin 2x}{\cos 2x}\ln((\tan x-1)+1)}$$

$$= e^{\lim_{x\to\frac{\pi}{4}-} 3\frac{2\sin x\cos x}{\cos^2 x-\sin^2 x}\frac{\ln((\tan x-1)+1)}{(\tan x-1)}(\tan x-1)} = e^{\lim_{x\to\frac{\pi}{4}-} 3\frac{2\tan x}{1-\tan^2 x}\frac{\ln((\tan x-1)+1)}{(\tan x-1)}(\tan x-1)}$$

$$= e^{\lim_{x\to\frac{\pi}{4}-} -3\frac{2\tan x}{1+\tan x}\frac{\ln((\tan x-1)+1)}{(\tan x-1)}} = e^{-3}$$

[7] 9.

$$\lim_{x\to 1}\left(1+\cos\left(\frac{\pi}{2}x\right)\right)^{\frac{1}{x-1}}$$

Řešení:

$$\lim_{x\to 1}\left(1+\cos\left(\frac{\pi}{2}x\right)\right)^{\frac{1}{x-1}} = \lim_{y\to 0}\left(1+\cos\left(\frac{\pi}{2}(y+1)\right)\right)^{\frac{1}{y}} = \lim_{y\to 0} e^{\frac{\ln\left(1+\cos\left(\frac{\pi}{2}(y+1)\right)\right)}{y}} = e^{\lim_{y\to 0}\frac{\ln\left(1+\cos\left(\frac{\pi}{2}(y+1)\right)\right)}{\cos\left(\frac{\pi}{2}(y+1)\right)}\frac{\cos\left(\frac{\pi}{2}(y+1)\right)}{y}}$$

$$= e^{\lim_{y\to 0}\frac{\ln\left(1+\cos\left(\frac{\pi}{2}(y+1)\right)\right)}{\cos\left(\frac{\pi}{2}(y+1)\right)}\frac{\cos\left(\frac{\pi}{2}y\right)\cos\frac{\pi}{2}-\sin\left(\frac{\pi}{2}y\right)\sin\frac{\pi}{2}}{y}} = e^{\lim_{y\to 0}\frac{\ln\left(1+\cos\left(\frac{\pi}{2}(y+1)\right)\right)}{\cos\left(\frac{\pi}{2}(y+1)\right)}\frac{-\sin\left(\frac{\pi}{2}y\right)}{y}}$$

$$= e^{\lim_{y\to 0}\frac{\ln\left(1+\cos\left(\frac{\pi}{2}(y+1)\right)\right)}{\cos\left(\frac{\pi}{2}(y+1)\right)}\left(-\frac{\sin\left(\frac{\pi}{2}y\right)}{\frac{\pi}{2}y}\right)\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$$

[8] 10.

$$\lim_{x\to 0+} \frac{1}{\ln(x+1)} \ln\left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}\right)$$

Řešení:

$$\lim_{x\to 0+} \frac{1}{\ln(x+1)} \ln\left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}\right) = \lim_{x\to 0+} \frac{1}{\ln(x+1)} \frac{\ln\left(\left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1\right)+1\right)}{\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1} \left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1\right)$$

$$= \lim_{x\to 0+} \frac{1}{x} \frac{x}{\ln(x+1)} \frac{\ln\left(\left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1\right)+1\right)}{\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1} \left(\frac{\left(\frac{1}{x}+a\right)-\left(\frac{1}{x}-a\right)}{\frac{1}{x}-a}\right)$$

$$= \lim_{x\to 0+} \frac{x}{\ln(x+1)} \frac{\ln\left(\left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1\right)+1\right)}{\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1} \frac{1}{x}\left(\frac{2a}{\frac{1}{x}-a}\right) = \lim_{x\to 0+} \frac{x}{\ln(x+1)} \frac{\ln\left(\left(\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1\right)+1\right)}{\frac{\frac{1}{x}+a}{\frac{1}{x}-a}-1} \frac{2a}{1-ax} = 2a$$