# THERMOMECHANICAL MODELS OF SOILS

# WHAT MAKES A GOOD THEORY?

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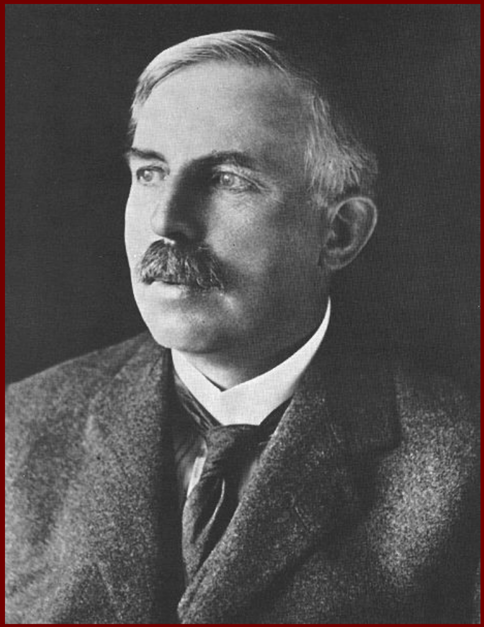
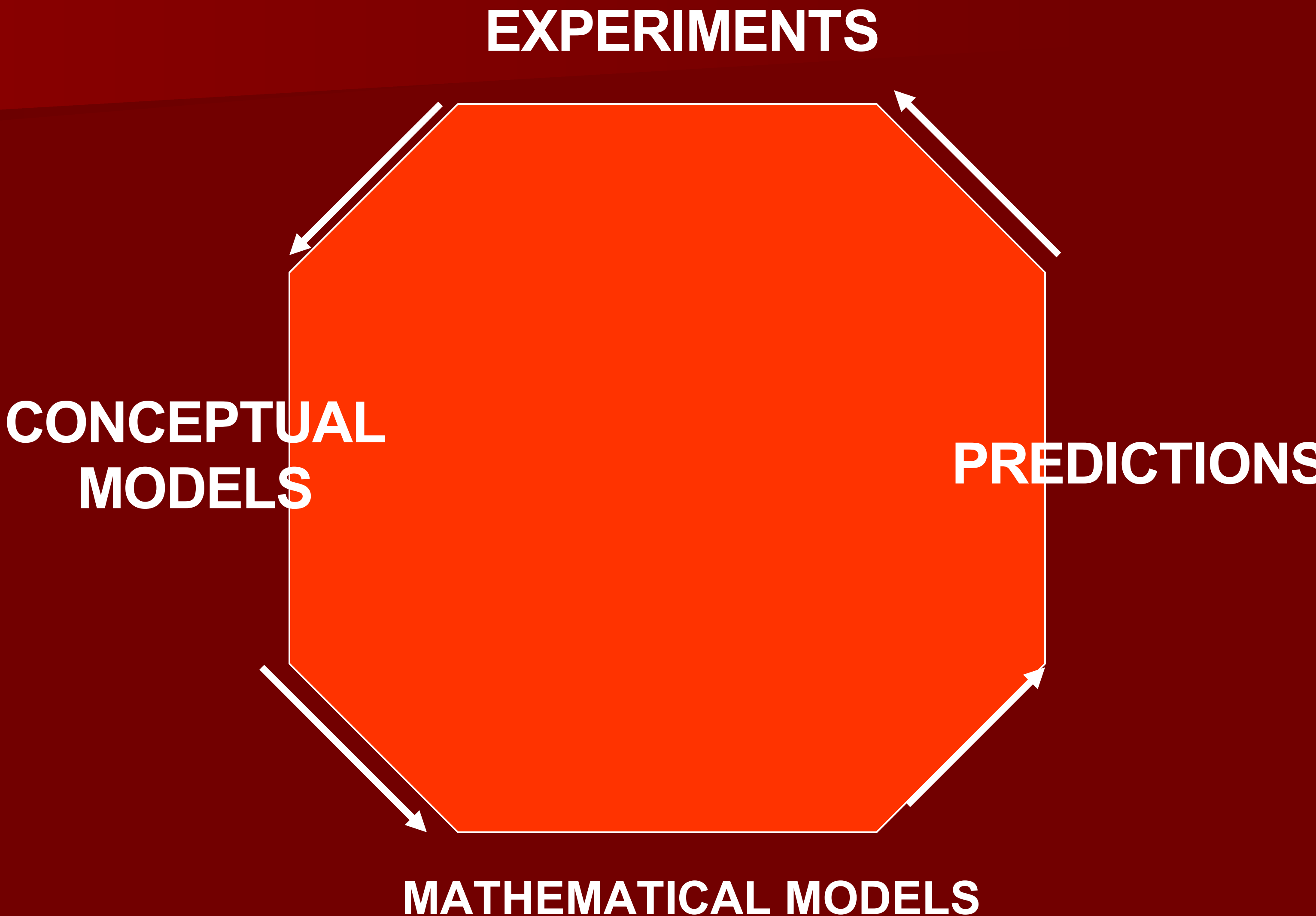


“A GOOD THEORY SHOULD BE EXPLAINABLE TO A BARMAID!!”

**SIR EARNEST RUTHERFORD**

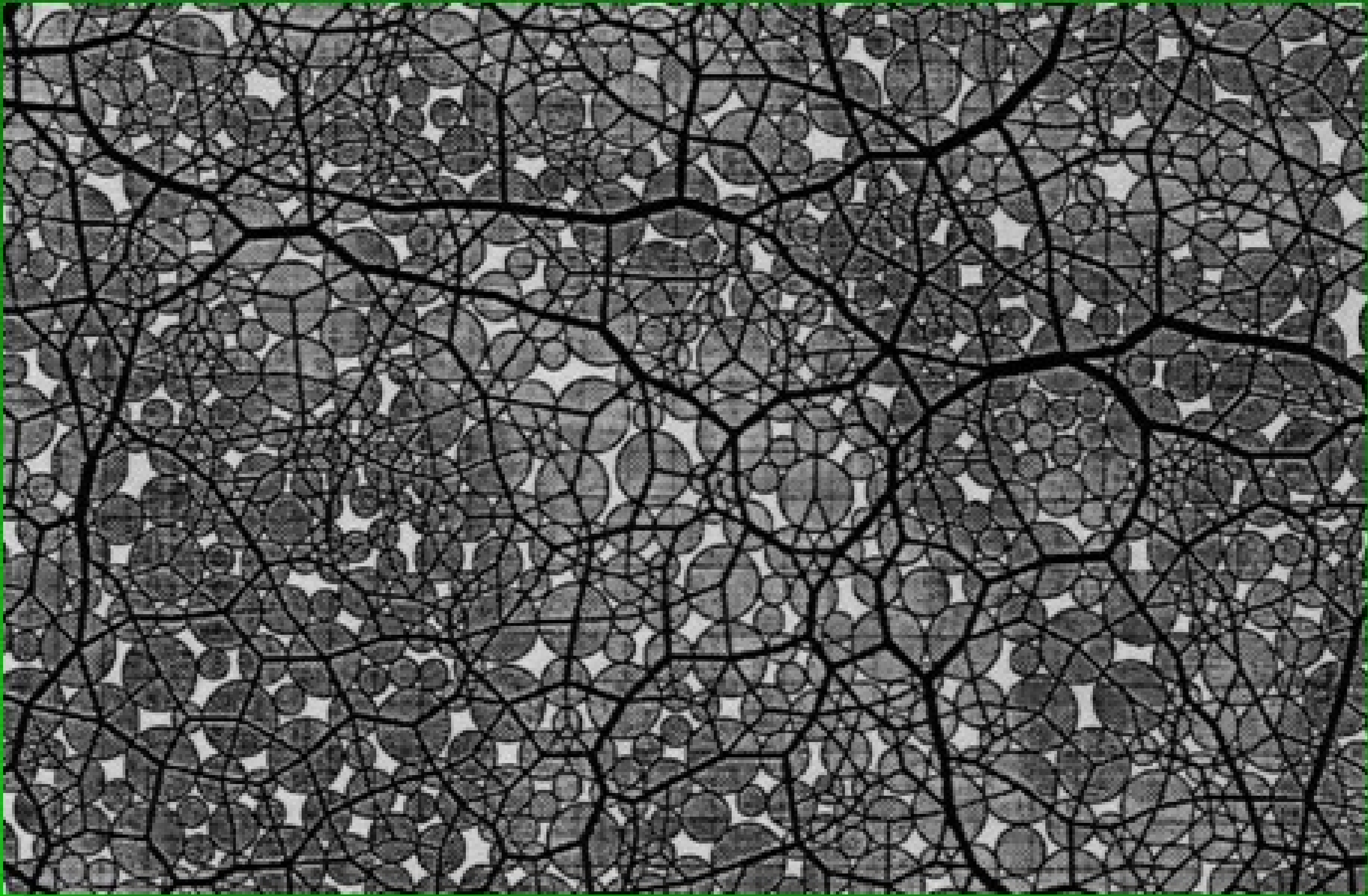
# THEORY CONSTRUCTION



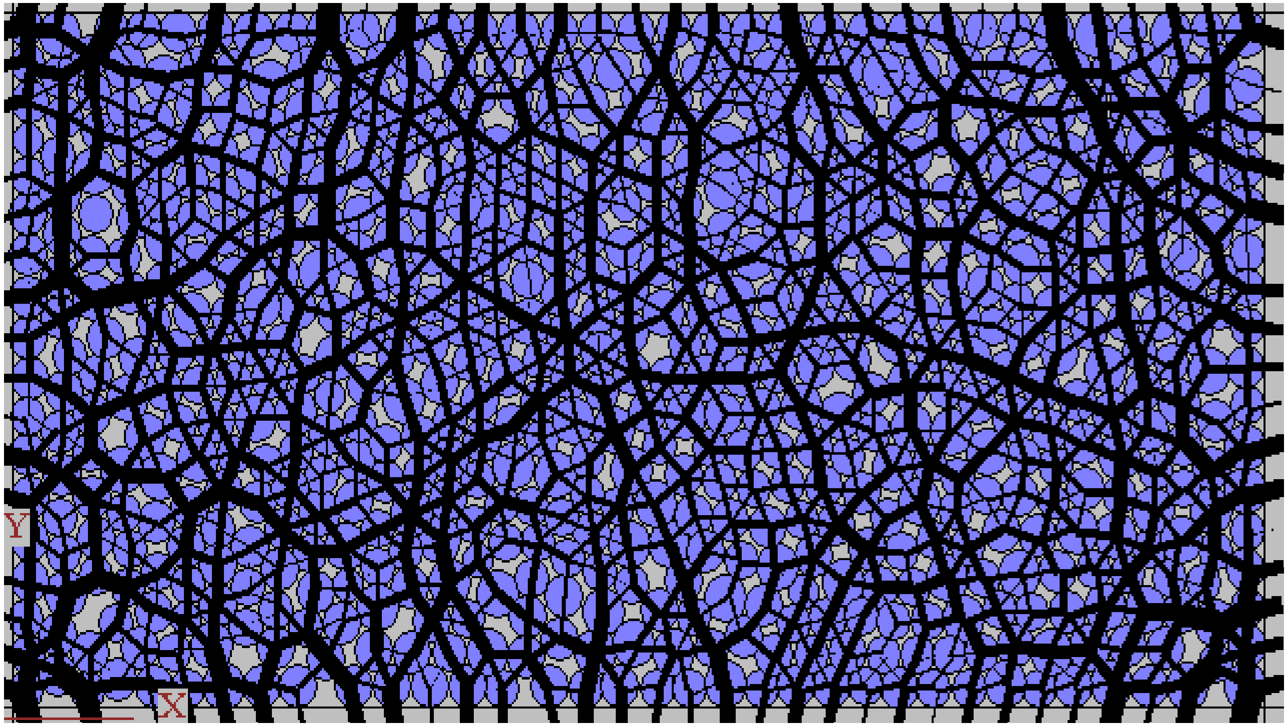
# “STORED PLASTIC WORK”
or
# “FROZEN ELASTIC ENERGY”

Laurits Bjerrum

Discrete element simulation,showing force chains (Radjai et al 1996)



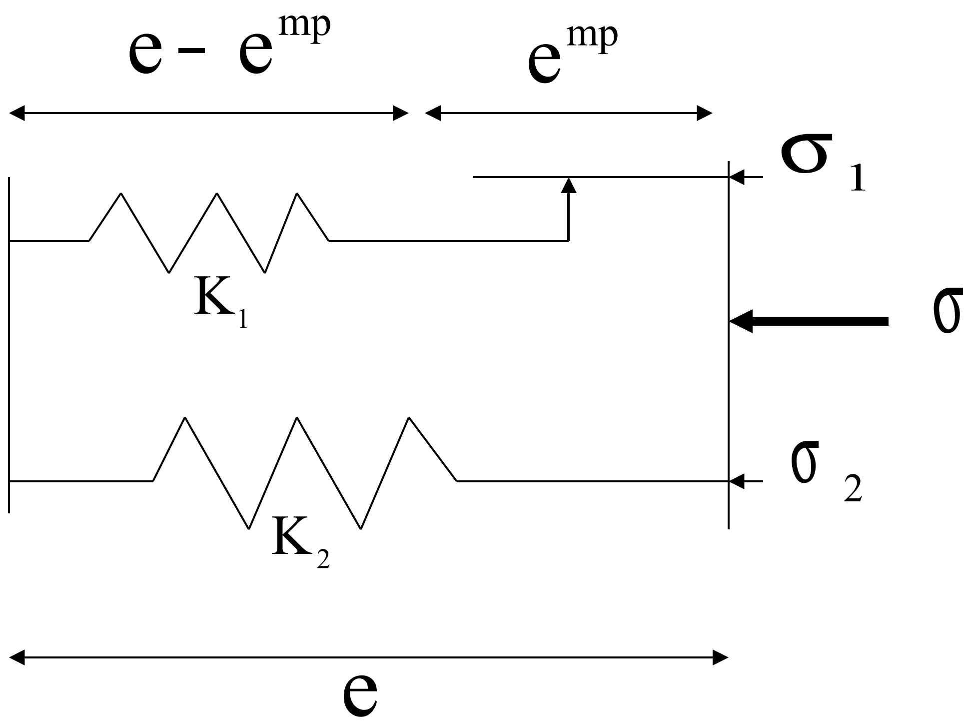
DEM simulation of isotropic compression, showing weak and strong networks (force chains)




The deformation and stress distribution on the micro-scale is highly inhomogeneous

# SIMPLE SPRING MODELS

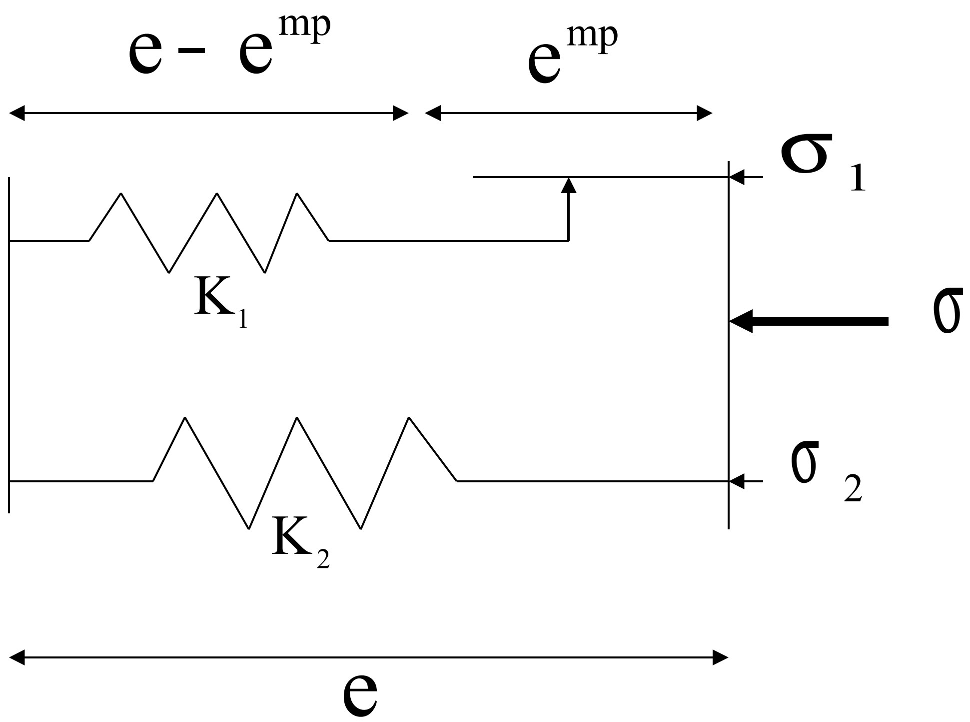
# A SIMPLE SCHEMATIC MODEL -1




$$\sigma_1 = K_1(e - e^{mp}), \qquad \sigma_2 = K_2 e, \qquad \sigma = \sigma_1 + \sigma_2$$

so: $\sigma = (K_1 + K_2)e - K_1 e^{mp}.$

# A SIMPLE SCHEMATIC MODEL -2



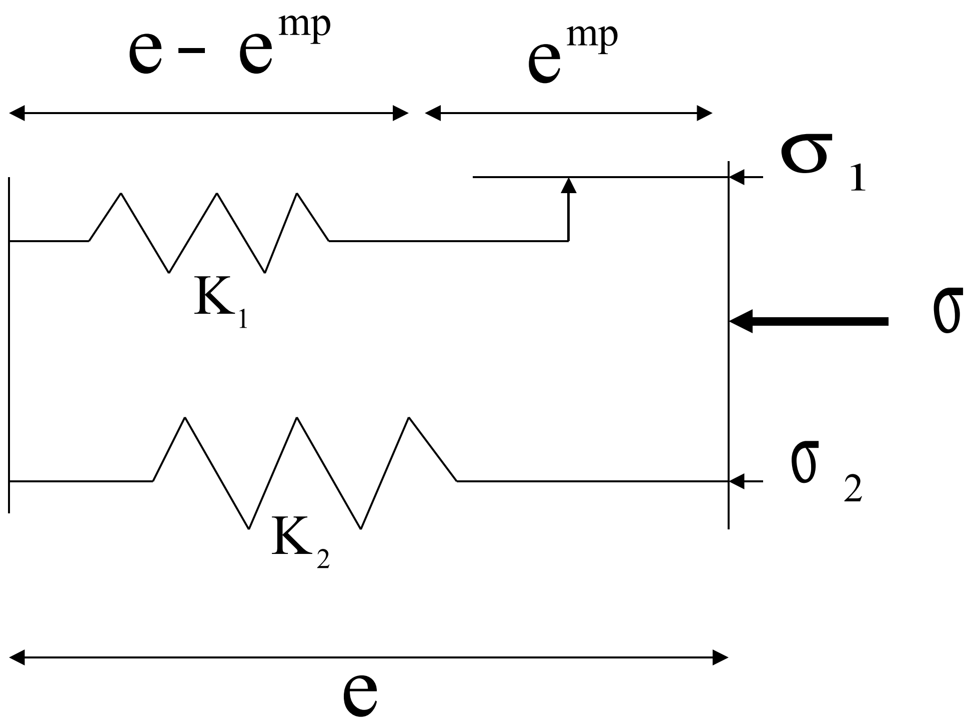

$$\sigma = (K_1 + K_2)e - K_1 e^{mp}.$$

$e^E = \sigma/(K_1 + K_2)$, $e^P = K_1 e^{mp}/(K_1 + K_2)$ and $e = e^E + e^P$

continuum elastic strain
(no micro-plastic strain)

continuum plastic strain
(at zero stress)

# A SIMPLE SCHEMATIC MODEL -3




The micro - elastic energy in springs is:

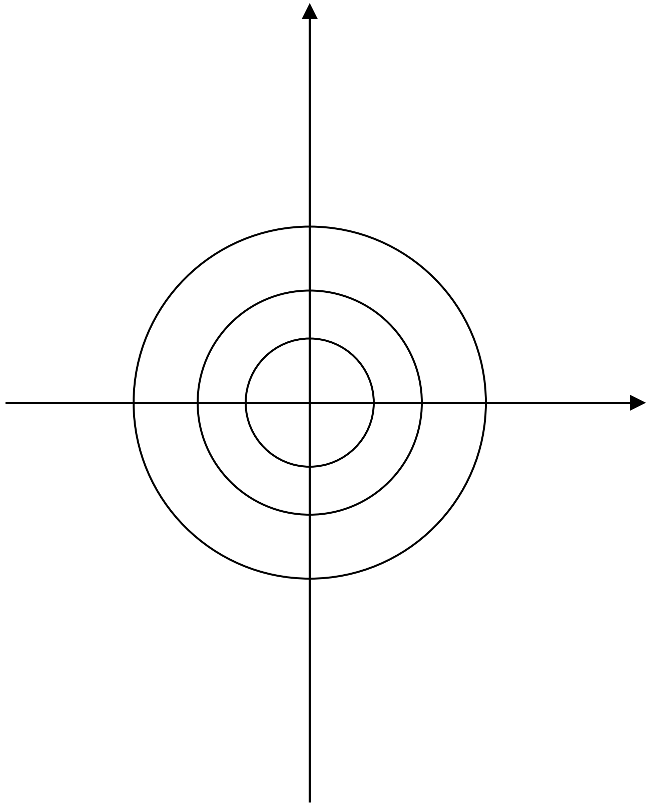
$$\Psi = \tfrac{1}{2}K_1(e - e^{mp})^2 + \tfrac{1}{2}K_2 e^2$$

$$= \tfrac{1}{2}(K_1 + K_2)e^{E^2} + \tfrac{1}{2}(K_2 / K_1)(K_1 + K_2)e^{P^2} = \Psi^E(e^E) + \Psi^P(e^P)$$

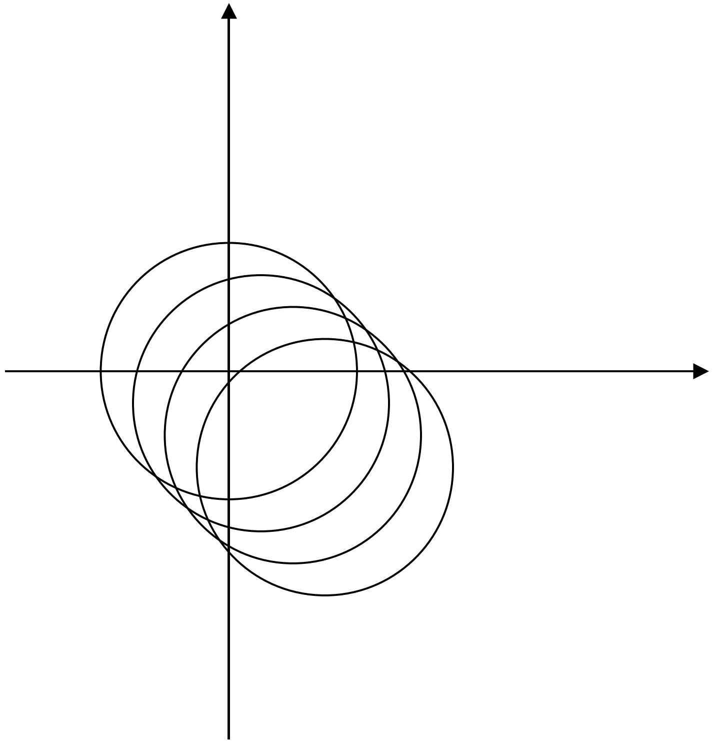
**RECOVERABLE ELASTIC ENERGY**

**FROZEN ELASTIC ENERGY**

# HARDENING LAWS



ISOTROPIC HARDENING



KINEMATIC HARDENING

# SOURCE OF KINEMATIC HARDENING

$$\Psi = \tfrac{1}{2}(K_1 + K_2){e^E}^2 + \tfrac{1}{2}(K_2/K_1)(K_1 + K_2){e^P}^2 = \Psi^E(e^E) + \Psi^P(e^P)$$

Differentiate with respect to time:

$$\dot{\Psi} = (K_1 + K_2)e^E\dot{e}^E + (K_2/K_1)(K_1 + K_2)e^P\dot{e}^P = \dot{\Psi}^E(e^E) + \dot{\Psi}^P(e^P)$$

ie $$\dot{\Psi} = \sigma\,\dot{e}^E + \sigma^S\dot{e}^P = \dot{\Psi}^E(e^E) + \dot{\Psi}^P(e^P)$$

where $\sigma^S \equiv (K_2/K_1)(K_1 + K_2)e^P$ is the shift stress.

If the "micro-yield condition" is: $-Y < \sigma_1 < Y$

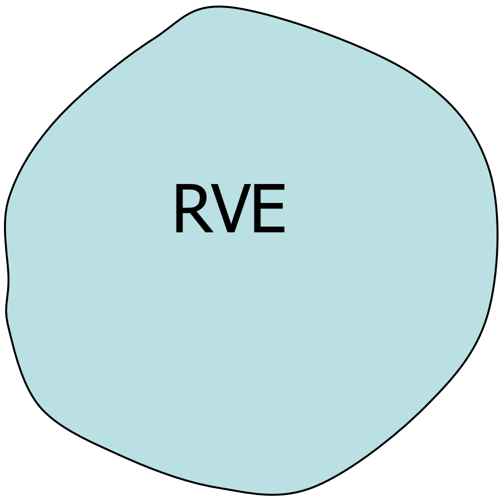
The "macro or continuum yield condition" is $-Y + \sigma^S < \sigma < Y + \sigma^S$

**THUS KINEMATIC HARDENING/SOFTENING IS DUE TO GENERATION/RECOVERY OF FROZEN ELASTIC ENERGY**

# HOMOGENIZATION

# BASIC HOMOGENIZATION THEORY

$$\langle * \rangle = \frac{1}{V} \int_{RVE} (*) dV$$




$$\langle \sigma^m \rangle = \sigma, \quad \langle e^m \rangle = e$$

$$\langle \sigma^m : e^m \rangle = \sigma : e$$

PROVIDED THE MICRO-STRESS FIELD IS STATICALLY ADMISSIBLE, AND MICRO-STRAIN FIELD IS KINEMATICALLY ADMISSIBLE

# ELASTIC-PLASTIC MATERIALS - 1

THE ELASTIC AND PLASTIC PARTS OF THE STRAIN TENSOR ARE NOT KINEMATICALLY ADMISSIBLE

$$e^{m} = e^{me} + e^{mp}$$

$$e^{e} \neq \left\langle e^{me} \right\rangle, \text{ and } e^{p} \neq \left\langle e^{mp} \right\rangle$$

THE CONTINUUM ELASTIC AND PLASTIC STRAINS ARE NOT THE AVERAGES OF THE MICRO-ELASTIC AND PLASTIC STRAINS.

# ELASTIC-PLASTIC MATERIALS - 2

## IN UNLOADED STATE:

$$\sigma = 0$$

$$e^{me} = e^{mr}, \text{ so } e^{m} \equiv e^{mR} = e^{mr} + e^{mp}$$

$$e^{P} \equiv \langle e^{mR} \rangle$$

$$\sigma^{mr} \equiv K : e^{mr}, \text{ where } \langle \sigma^{mr} \rangle = 0$$

$$\text{Define: } \sigma^{mE} = \sigma^{m} - \sigma^{mr} \Rightarrow \langle \sigma^{mE} \rangle = \langle \sigma^{m} \rangle = \sigma$$

# ELASTIC-PLASTIC MATERIALS - 3

## IN LOADED STATE:

$$\sigma^{mE} = \sigma^{m} - \sigma^{mr} \Rightarrow K^{-1} : \sigma^{mE} = K^{-1} : \sigma^{m} - K^{-1} : \sigma^{mr}$$

$$\Rightarrow e^{mE} = e^{me} - e^{mr}$$

$$\Rightarrow e^{m} = e^{me} + e^{mp} = e^{mE} + e^{mr} + e^{mp} = e^{mE} + e^{mR}$$

$$\Rightarrow e = \langle e^{m} \rangle = \langle e^{mE} \rangle + \langle e^{mR} \rangle = e^{e} + e^{p}$$

$$\text{ie} : \langle e^{mE} \rangle = e^{e}, \quad \langle e^{mR} \rangle = e^{p}$$

# ELASTIC-PLASTIC MATERIALS - SUMMARY

(1) The continuum elastic strain is the average of the “fictitious” micro-elastic strain, which would pertain in the RVE, if there were no yielding.

(2) The continuum plastic strain is the average of the sum of the micro-plastic and micro-elastic residual strain

# STORED PLASTIC WORK

THE INCREMENT OF WORK IS $dW = \sigma : de = \langle \sigma^{m} : de^{m} \rangle \Rightarrow$

..............

$\Rightarrow \; dW = dW^{e} + dW^{s} + dW^{d}$, where

$dW^{e} = \langle \sigma^{mE} : de^{mE} \rangle$ "ELASTIC WORK"

$dW^{s} = \langle \sigma^{mr} : de^{mr} \rangle$ "STORED PLASTIC WORK"

$dW^{d} = \langle \sigma \; : de^{mp} \rangle$ "DISSIPATED WORK"

# THERMOMECHANICAL FORMULATION

- The first and second laws of thermodynamics for isothermal deformations:

$$\hat{W} = \dot{\Psi} + \hat{\Phi}, \quad \text{where } \hat{\Phi} \geq 0$$

Rate of working of applied stresses

Rate of change of free energy

Rate of dissipation

Note $\hat{W}$ and $\hat{\Phi}$ are not "proper" time derivatives

We will assume that the elastic and plastic strains can be taken as state variables

# UNIMODAL, DECOUPLED MODELS

$$\hat{W} = \sigma_{ij}\dot{e}_{ij} = \sigma_{ij}\dot{e}^{E}_{ij} + \sigma_{ij}\dot{e}^{P}_{ij}$$

("Unimodal" means only one dissipation mechanism, so only one plastic strain)

$$\Psi(e^{E}_{ij}, e^{P}_{ij}) = \Psi^{E}(e^{E}_{ij}) + \Psi^{P}(e^{P}_{ij})$$

("Decoupled")

$$\hat{\Phi}(\sigma_{ij}, e^{P}_{ij}; \dot{e}^{P}_{ij}) = \frac{\partial \hat{\Phi}}{\partial \dot{e}^{p}_{ij}} \dot{e}^{p}_{ij}$$

(By Euler's Theorem for a homogeneous function of degree 1, since material is rate independent.)

ELASTIC and PLASTIC STRAINS taken as STATE VARIABLES

# STRESS DECOMPOSITION

Since $\hat{W} = \dot{\Psi} + \hat{\Phi}$, where $\hat{\Phi} \geq 0$

$$\hat{W} = \sigma_{ij}\dot{e}_{ij} = \sigma_{ij}\dot{e}^{E}_{ij} + \sigma_{ij}\dot{e}^{P}_{ij} = \frac{\partial \Psi^{E}}{\partial e^{E}_{ij}}\dot{e}^{E}_{ij} + \frac{\partial \Psi^{P}}{\partial e^{P}_{ij}}\dot{e}^{P}_{ij} + \frac{\partial \hat{\Phi}}{\partial \dot{e}^{P}_{ij}}\dot{e}^{P}_{ij}$$

Hence $\sigma_{ij} = \dfrac{\partial \Psi^{E}}{\partial e^{E}_{ij}}$ and $\sigma_{ij} = + \dfrac{\partial \Psi^{P}}{\partial e^{P}_{ij}} + \dfrac{\partial \hat{\Phi}}{\partial \dot{e}^{P}_{ij}}$

Ziegler's hypothesis

So $\sigma_{ij} = \sigma^{S}_{ij} + \sigma^{D}_{ij}$

$\sigma^{S}_{ij}$ $\sigma^{D}_{ij}$

Total stress is sum of shift and dissipative stress .

**{ Principal axes of stress and plastic strain rate no NOT coincide}**

# A NOTE ON PLASTIC WORK

FROM THE PREVIOUS SLIDE, THE RATE OF PLASTIC WORK IS:

$$\hat{W}^{P} \equiv \sigma_{ij}\dot{e}_{ij}^{P} = \sigma_{ij}^{S}\dot{e}_{ij}^{P} + \sigma_{ij}^{D}\dot{e}_{ij}^{P}$$

RATE OF PLASTIC WORK

RATE AT WHICH PLASTIC WORK IS BEING STORED OR RECOVERED

RATE OF ENERGY DISSIPATION

RATE AT WHICH MICRO-ELASTIC ENERGY IS BEING FROZEN OR RELEASED

# VOLUMETRIC HARDENING (TRIAXIAL)

$$\hat{W}^{P} = p\dot{e}_{V}^{P} + q\dot{e}_{\gamma}^{P} = \dot{\psi}^{P}(e_{V}^{P}) + \hat{\Phi}(p, p_{S}, e_{V}^{P}; \dot{e}_{V}^{P}, \dot{e}_{\gamma}^{P})$$

$$\dot{\psi}^{P} = p_{S}\dot{e}_{V}^{P}, \text{where } p_{S} \equiv \frac{\partial \psi^{P}}{\partial e_{V}^{P}} \text{ is shift pressure.}$$

$$\hat{\Phi} = p_{D}\dot{e}_{V}^{P} + q_{D}\dot{e}_{\gamma}^{p}, \text{where } p_{D} \equiv \frac{\partial \hat{\Phi}}{\partial \dot{e}_{V}^{P}}, q_{D} \equiv \frac{\partial \hat{\Phi}}{\partial \dot{e}_{\gamma}^{P}},$$

are dissipative stresses.

$$p = p_{S} + p_{D}, \text{ and } q = q_{D}$$

# MODIFIED CAM CLAY

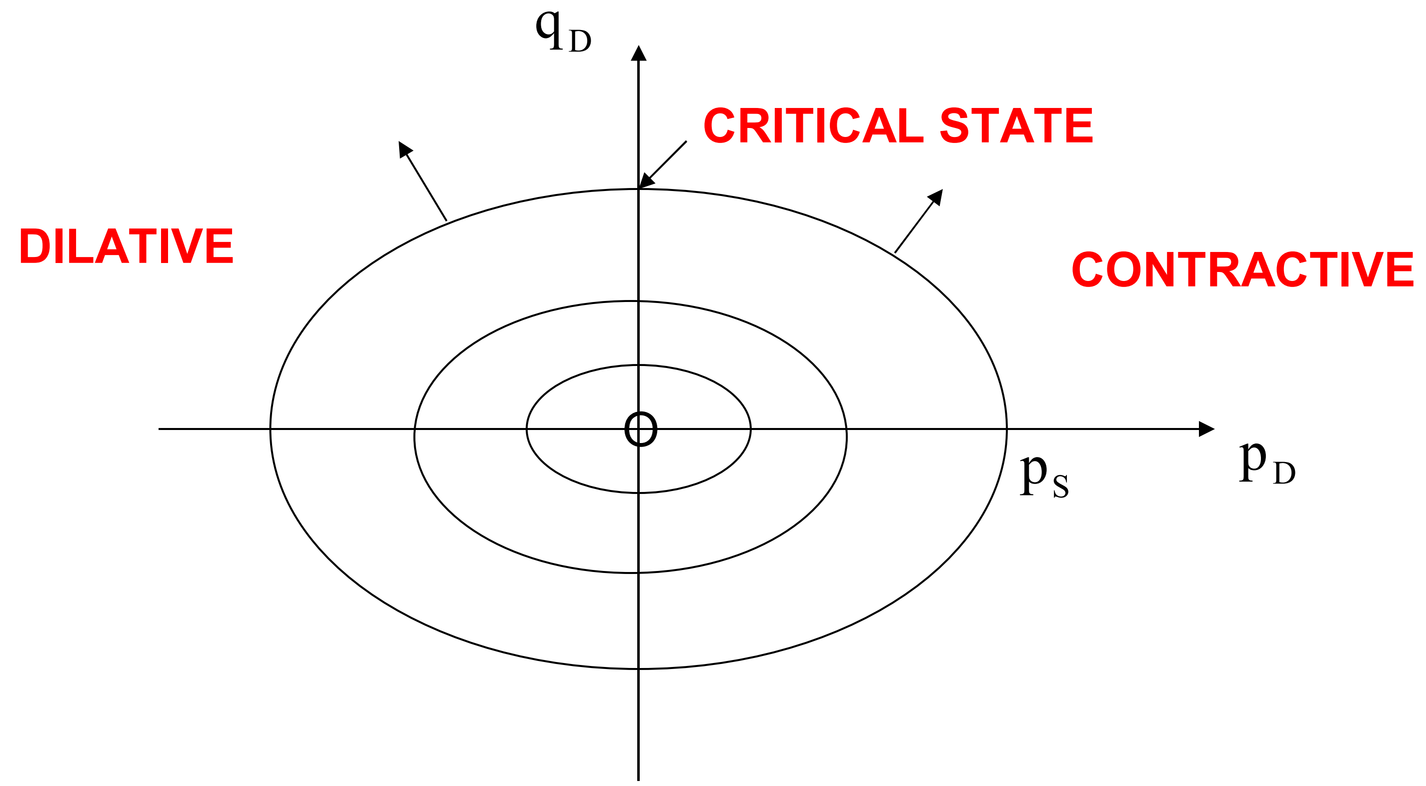
$$\hat{\Phi} = p_S \sqrt{\dot{e}_v^{P\,2} + M^2 \dot{e}_\gamma^{P\,2}}$$

$$p_D \equiv \frac{\partial \hat{\Phi}}{\partial \dot{e}_v^P} = \frac{p_S{}^2 \dot{e}_v^P}{\hat{\Phi}} \qquad q_D \equiv \frac{\partial \hat{\Phi}}{\partial \dot{e}_\gamma^P} = \frac{p_S{}^2 M^2 \dot{e}_\gamma^P}{\hat{\Phi}}$$

$$\frac{p_D{}^2}{p_S{}^2} + \frac{q_D{}^2}{M^2 p_S{}^2} = 1 \qquad \tan\psi \equiv -\frac{\dot{e}_v^P}{\dot{e}_\gamma^P} = \frac{M^2 p_D}{q_D}$$

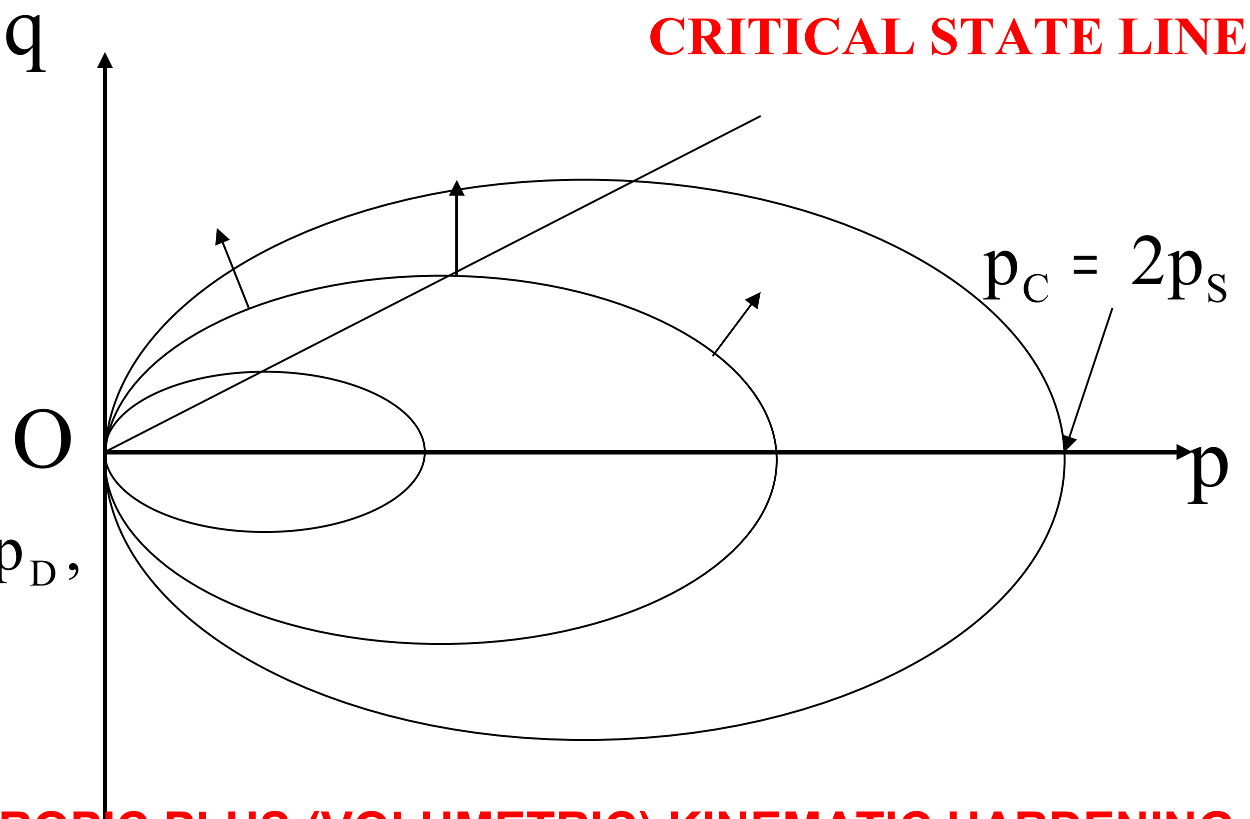
$$\frac{(p - p_S)^2}{p_S{}^2} + \frac{q^2}{M^2 p_S{}^2} = 1 \qquad q = 0 \Rightarrow p_S = \tfrac{1}{2} p_C$$

# DISSIPATIVE YIELD LOCI




**NB: ISOTROPIC HARDENING and NORMAL FLOW RULE**

# SHIFT TO GIVE MODIFIED CAM CLAY




$p = p_S + p_D$,

$q = q_D$

**ISOTROPIC PLUS (VOLUMETRIC) KINEMATIC HARDENING.**

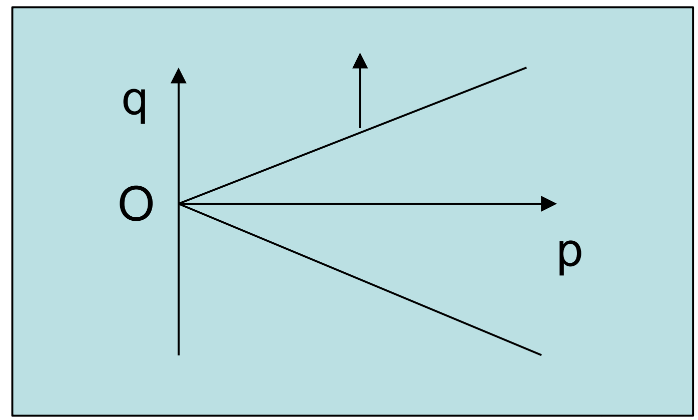
**HALF PLASTIC WORK IS STORED**

**FLOW RULE IS STILL NORMAL, SINCE DISSIPATION DOES NOT DEPEND ON p**

# FRICTION IMPLIES NON-ASSOCIATED FLOW RULES (Collins and Houlsby 1997).

$$\hat{\Phi} = Mp\left|\dot{e}^{p}_{\gamma}\right| \Rightarrow$$

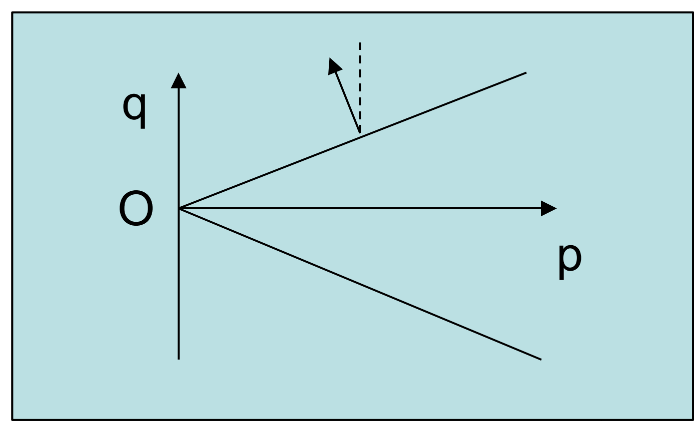
Schofield and Wroth
"Original Cam Clay"




"Incompressible Coulomb"

$$\hat{\Phi} = p\sqrt{\dot{e}^{p\,2}_{v} + M^{2}\dot{e}^{p\,2}_{\gamma}} \Rightarrow$$

Roscoe and Burland
"Modified Cam Clay




"Dilatational Coulomb"

# THE ALPHA-GAMMA FAMILY OF MODELS (Collins and Kelly,Collins and Hilder (2002))

$$\Phi = \sqrt{[A^2 \dot{e}_v^{p\,2} + B^2 \dot{e}_\gamma^{p\,2}]} \Rightarrow \frac{p_d^{\,2}}{A^2} + \frac{q_d^{\,2}}{B^2} = 1$$

Dissipative yield loci are again concentric ellipses, but no longer self similar

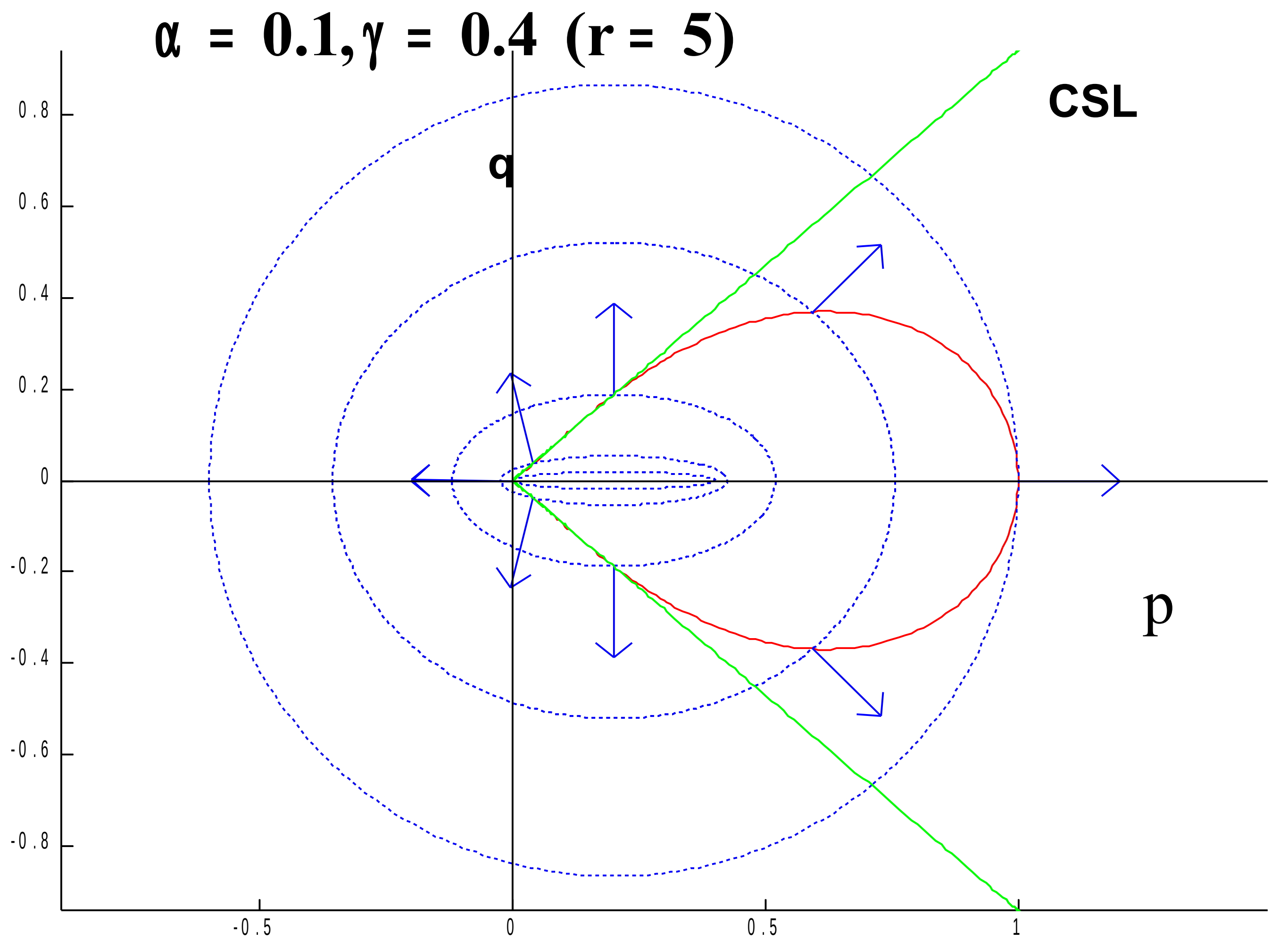
Assume a linear dependence on p and $p_s$

$$A \equiv [(1-\gamma)p + p_s] \qquad B \equiv M[(1-\alpha)p + \alpha p_s]$$

$$p_c = \frac{2}{\gamma} p_s = r p_s$$

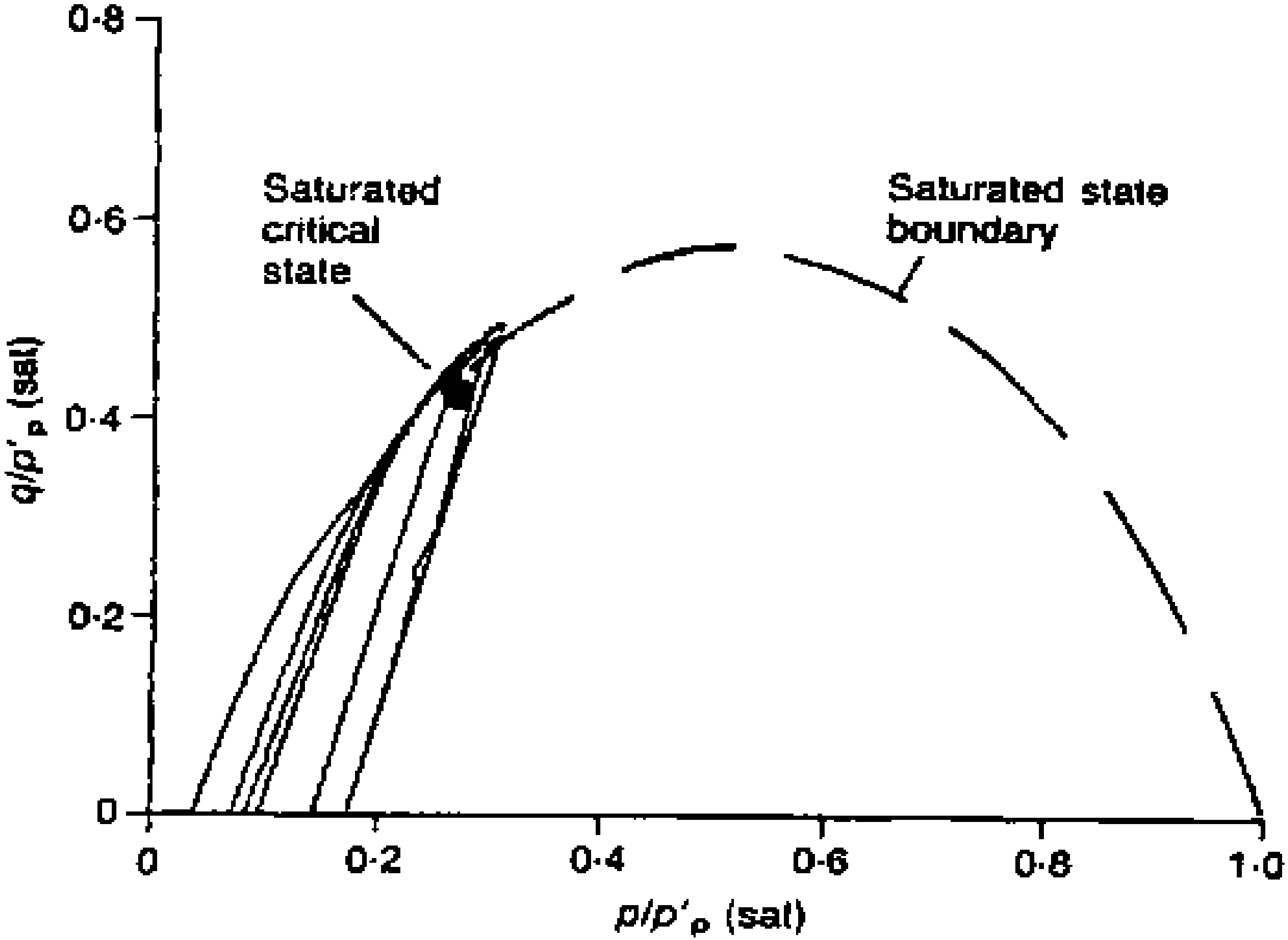
Where r is the SPACING RATIO

$$\alpha = \gamma = 1 \Rightarrow \textbf{Modified Cam Clay}$$

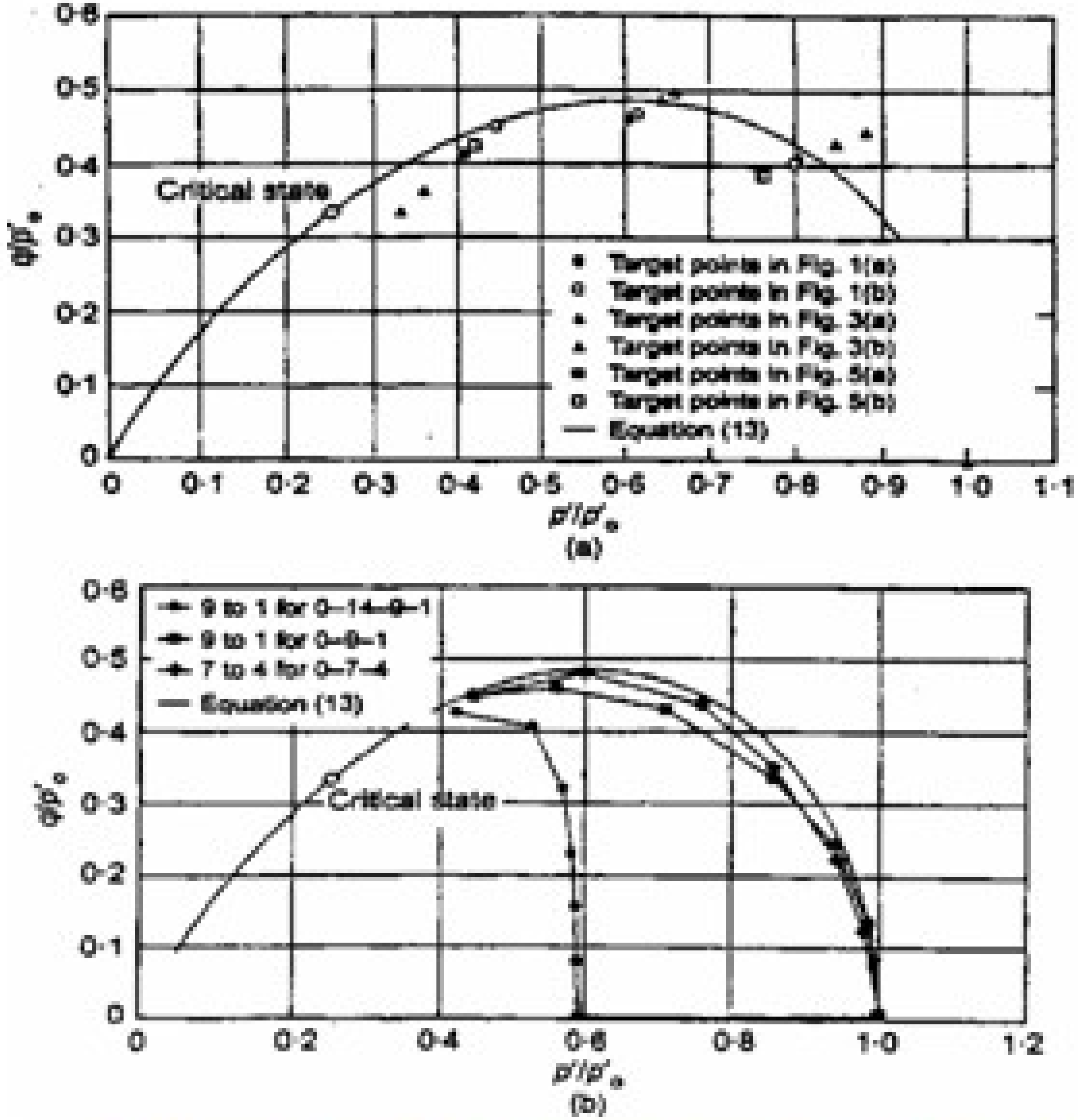



YIELD LOCUS and DISSIPATIVE YIELD LOCI (COLLINS AND HILDER 2002)

TRANSFORMATION DISTORTS YIELD LOCI, AND FLOW RULE IS NON-ASSOCIATED

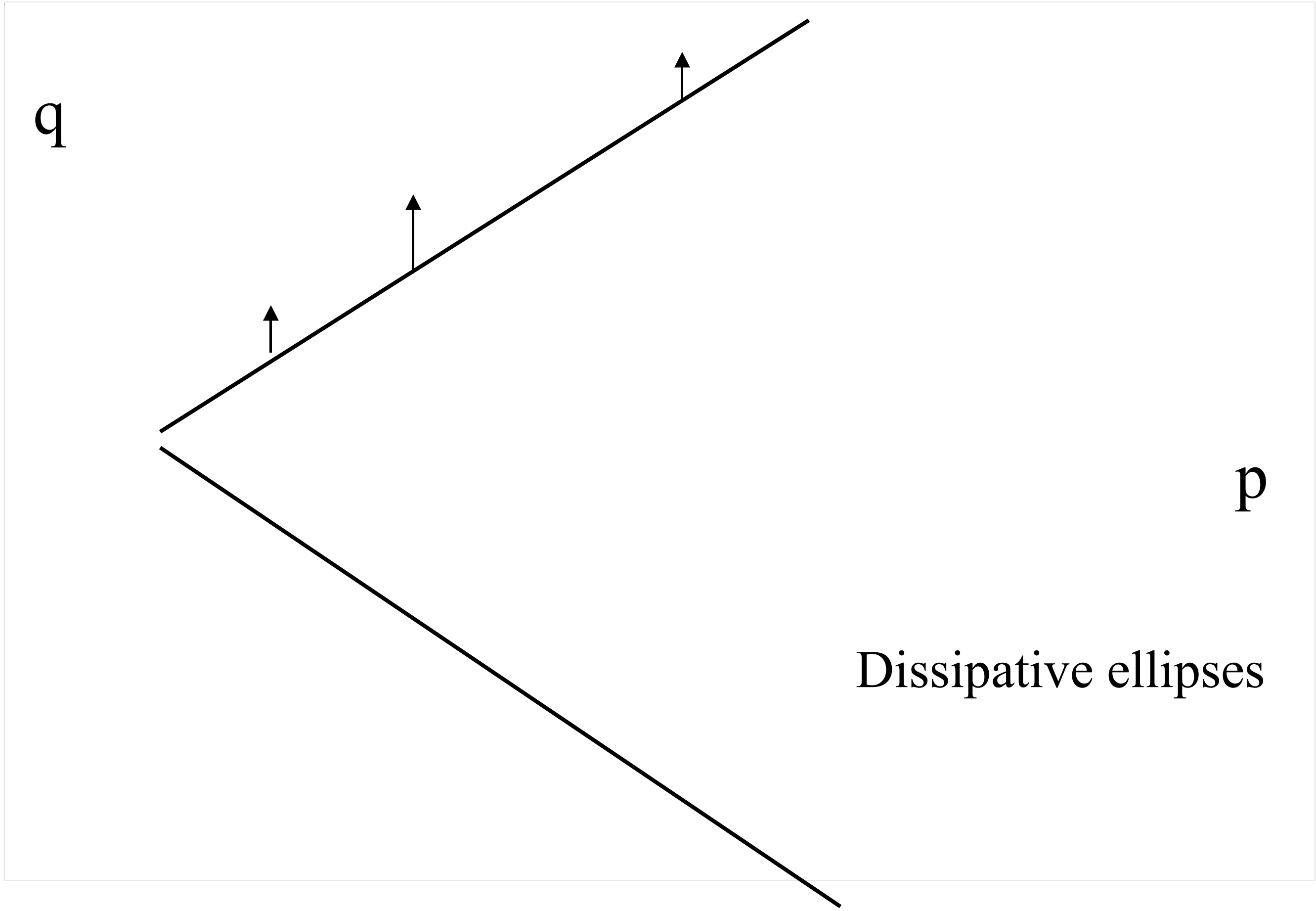



Crushed granite (Lee and Coop (1995))




Silica Sand: McDowell et al (2002)

Coulomb failure line

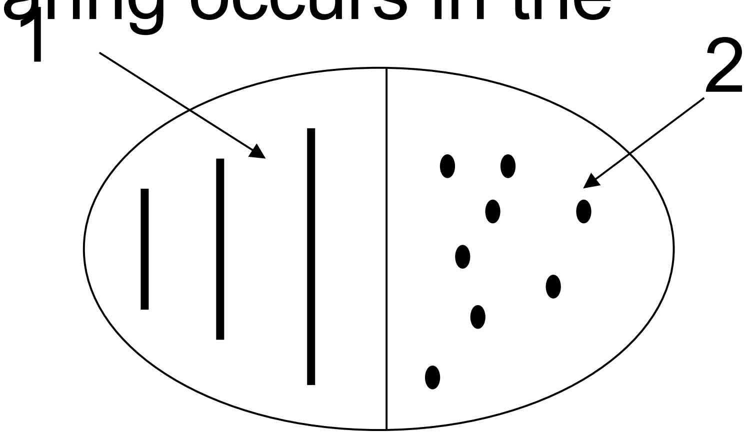



$\alpha = \gamma = 0$

COULOMB MODEL

# A BI-MODAL MODEL (Radjai, Roux, et al,Thornton)

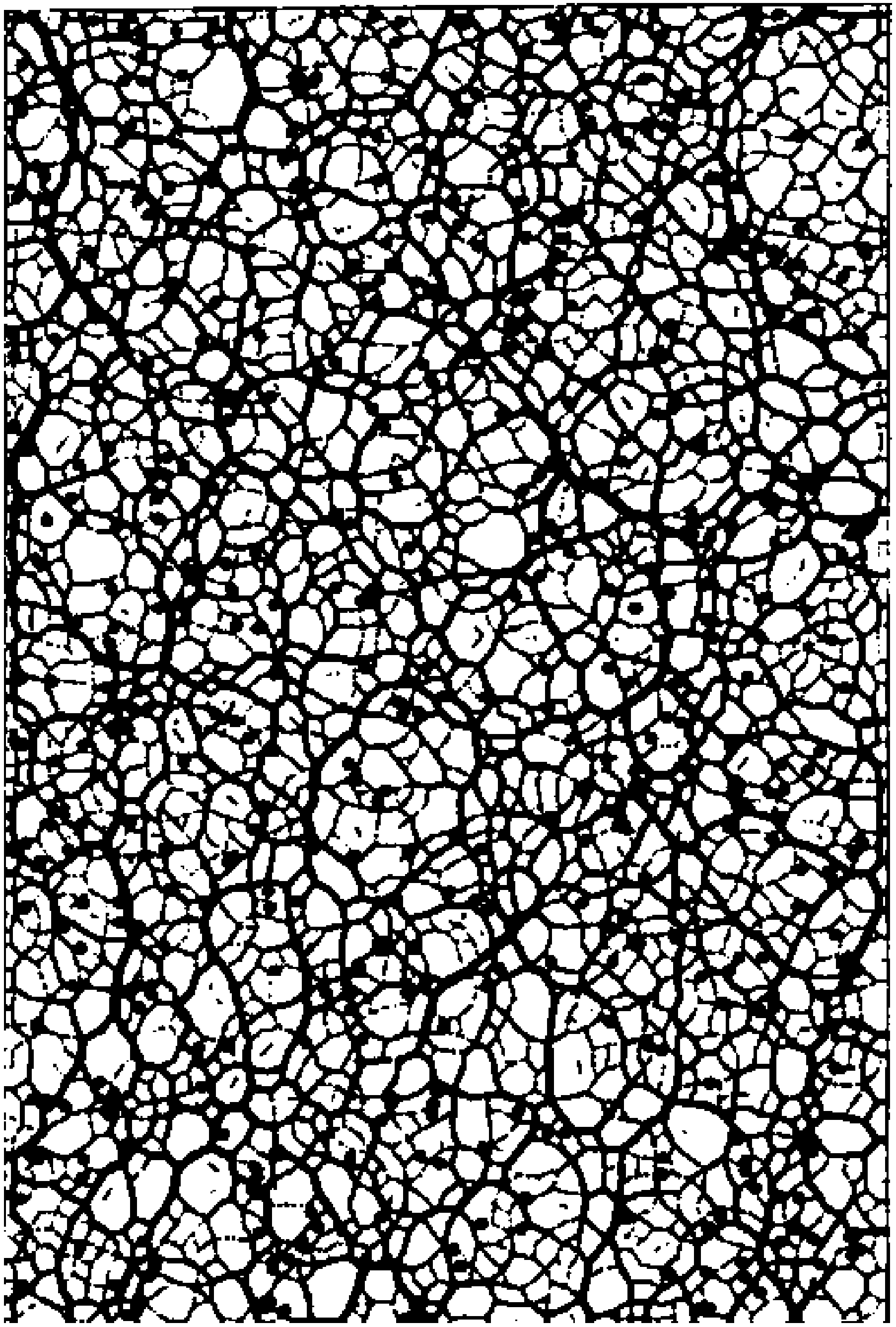
1. **Strong sub-network** (Force chains). Carries all the deviatoric stress, and a fraction of the isotropic stress.
2. **Weak sub-network**. Carries the remaining fraction of the isotropic stress. Behaves like a "frictional fluid". All shearing occurs in the weak network




Representative Volume Element - schematic.

FORCE CHAIN NETWORK SHOWING DISPLACEMENT POINTS UNDER SHEAR. NEARLY ALL ARE IN WEAK NETWORK.

RADJAI et al (1997)



# BI-MODAL MODEL-PREDICTIONS

- The tear drop parameter -alpha- is the fraction of the volume of the RVE in the strong sub-network.
- The spacing ratio parameter -gamma- is the determined by the ratio of the pressure in the strong sub-network to the mean pressure in the RVE.

# WHERE IS REYNOLD’S IDEA OF DILATANCY OF A GRANULAR MATERIAL INCLUDED?

NOWHERE!!!

# MODELLING REYNOLDS DILATANCY

SPLIT VOLUME STRAIN RATE INTO TWO TERMS

$$\dot{e}^{P}_{V} = \dot{e}^{P}_{VC} + \dot{e}^{P}_{VI}$$

IRREVERSIBLE TERM PREDICTED BY FLOW RULE FOR ISOTROPIC MODEL

INDUCED TERM DUE TO REYNOLDS EFFECT

$$\dot{e}^{P}_{VI} = -\tan\theta \; \dot{e}^{P}_{\gamma}$$

$$\tan\psi = \tan\psi_{C} + \tan\theta$$

DILATANCY DUE TO RECOVERY OF “FROZEN ENERGY”

INDUCED DILATANCY ANGLE

# ENERGY BALANCE EQUATION

$$p\dot{e}_{V}^{P} + q\dot{e}_{\gamma}^{P} = p_{S}\dot{e}_{VC}^{P} + [p_{R}\dot{e}_{VI}^{P} + q_{R}\dot{e}_{\gamma}^{P}] + [p_{D}\dot{e}_{VC}^{P} + q_{D}\dot{e}_{\gamma}^{p}]$$

STORED WORK AND DISSIPATION
ONLY DEPENDS ON
IRREVERSIBLE
PLASTIC VOLUME STRAIN

WORK RATE ASSOCIATED
WITH REYNOLDS DILATANCY
IS ZERO

$$p_{R}\dot{e}_{VI}^{P} + q_{R}\dot{e}_{\gamma}^{P} = 0$$

KANATANI(1982), GODDARD(1990), HOULSBY(1993)
COLLINS & MUHUNTHAN (2003)

# CYCLIC LOADING TESTS ON SAND

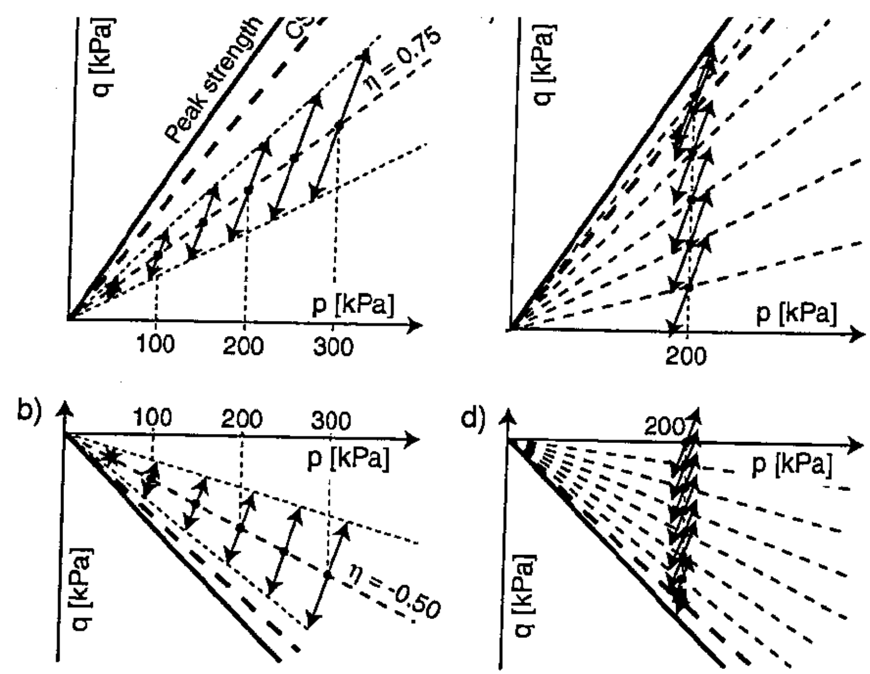
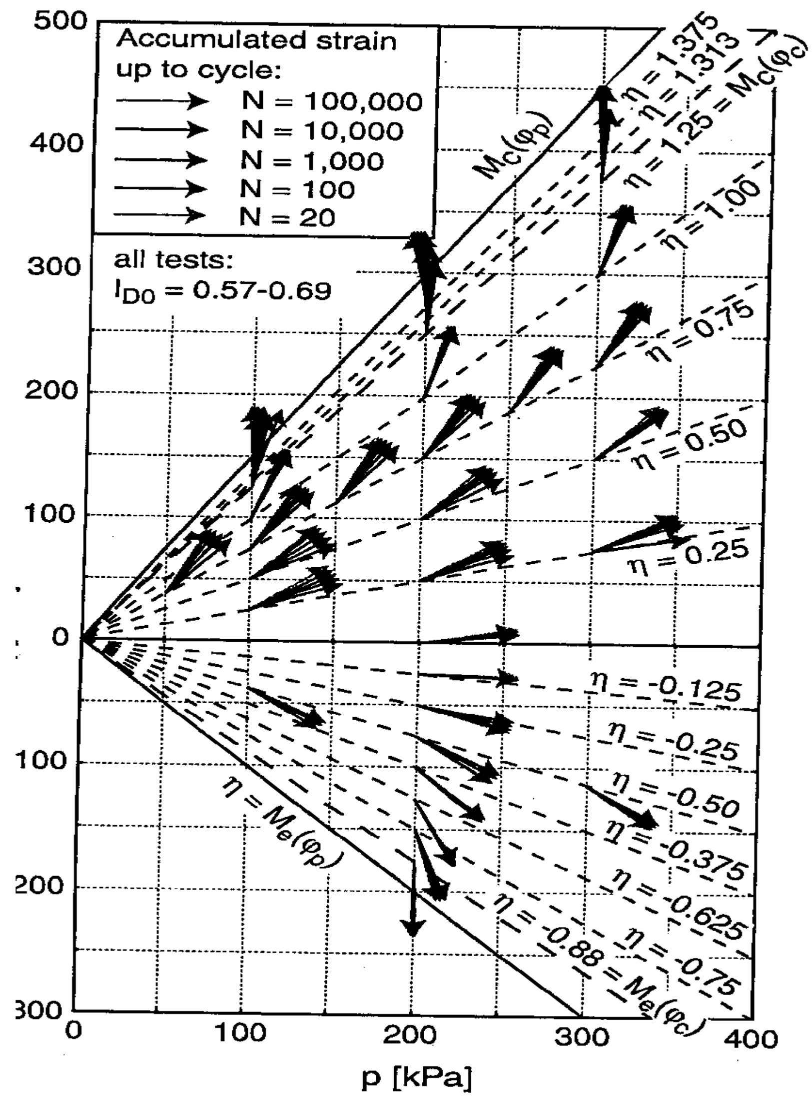
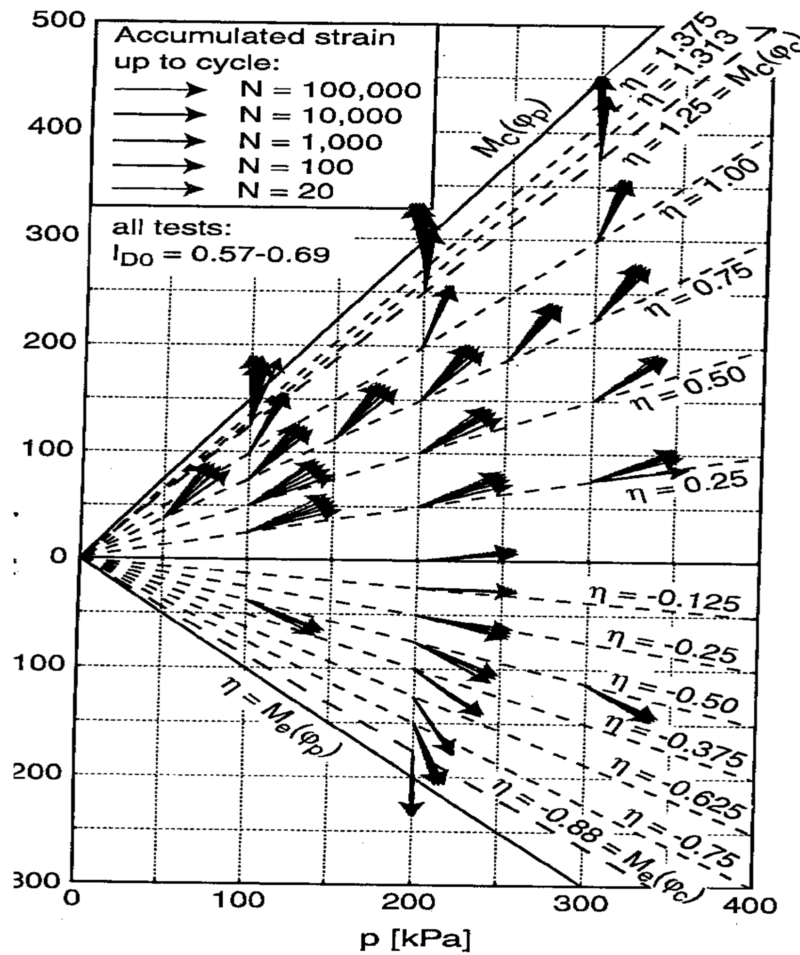



Figure 10. Cyclic stress paths in the tests with different average stresses $\sigma^{av}$

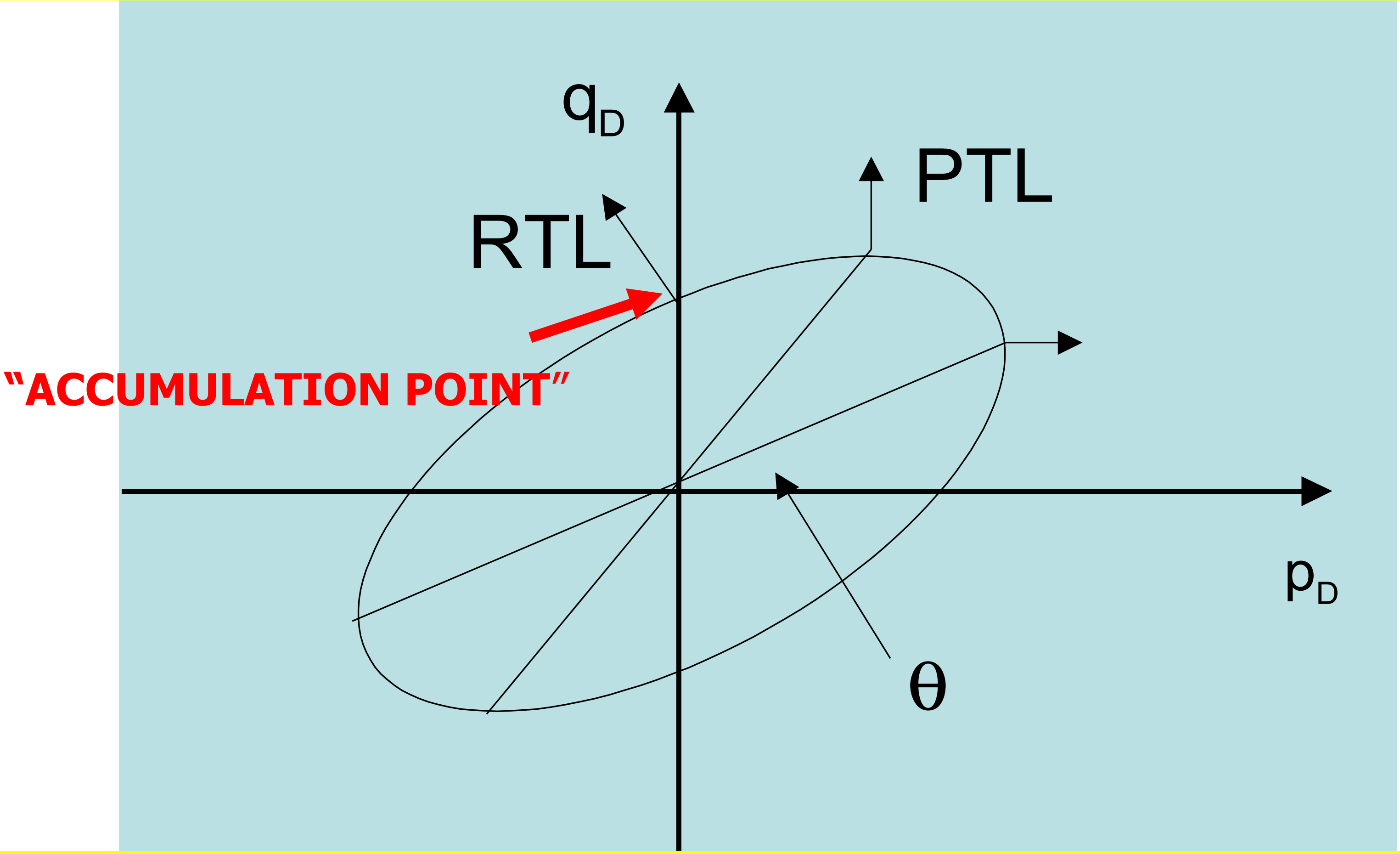


# FOR SIMPLITY AND ILLUSTRATIVE PURPOSES WE TAKE MCC AS "BASE" ISOTROPIC MODEL

CYCLIC LOADING EXPERIMENTS ON SANDS, SHOW THAT THE ACCUMULATE PLASTIC STRAIN INCREMENTS ARE, APPROXIMATELY, NORMAL TO A CAM-CLAY TYPE YIELD LOCUS.

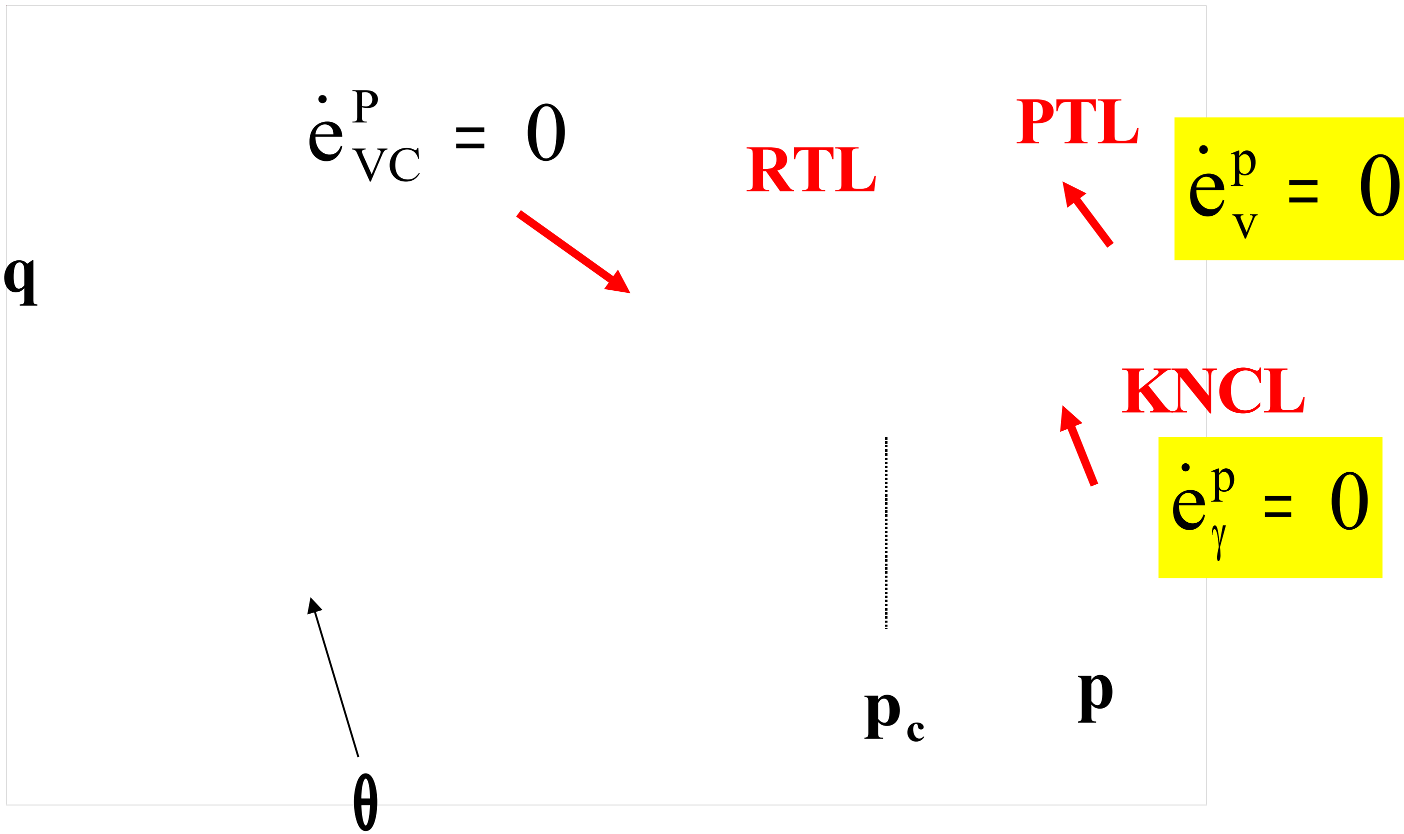


# ANISOTROPIC DISSIPATIVE YIELD LOCUS

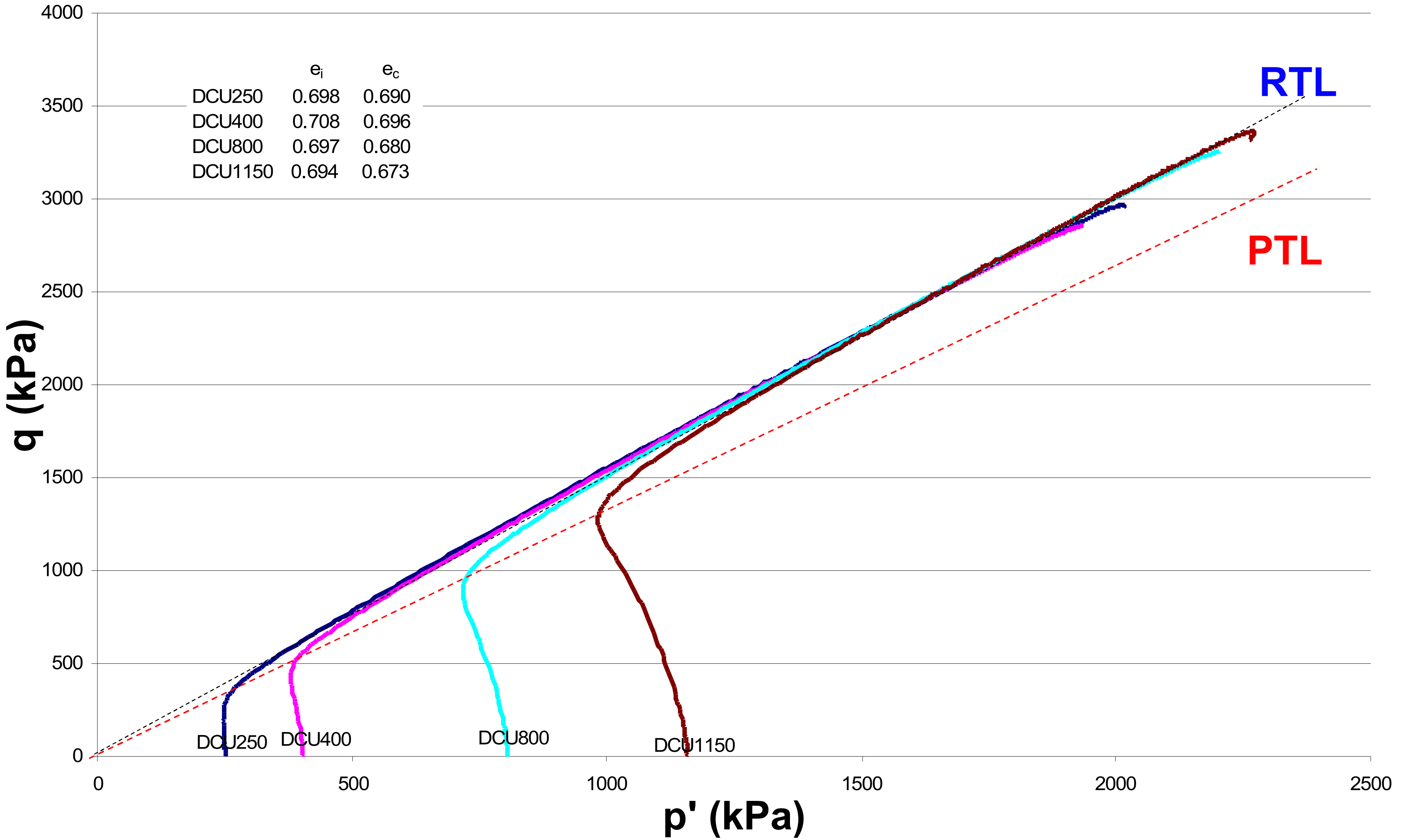



THE CRITICAL STATE LINE (CSL) OF ISOTROPIC MODEL SPLITS INTO TWO:THE REYNOLDS-TAYLOR (RTL) and PHASE TRANSITION LINES (PTL)

# TRUE STRESS SPACE




**eg ROTATED MODIFIED CAM CLAY MODEL. WHEN ON RTL MATERIAL IS BEHAVING IN MANNER ENVISGED BY REYNOLDS**




UNDRAINED TESTS ON DENSE QUARTZ SAND

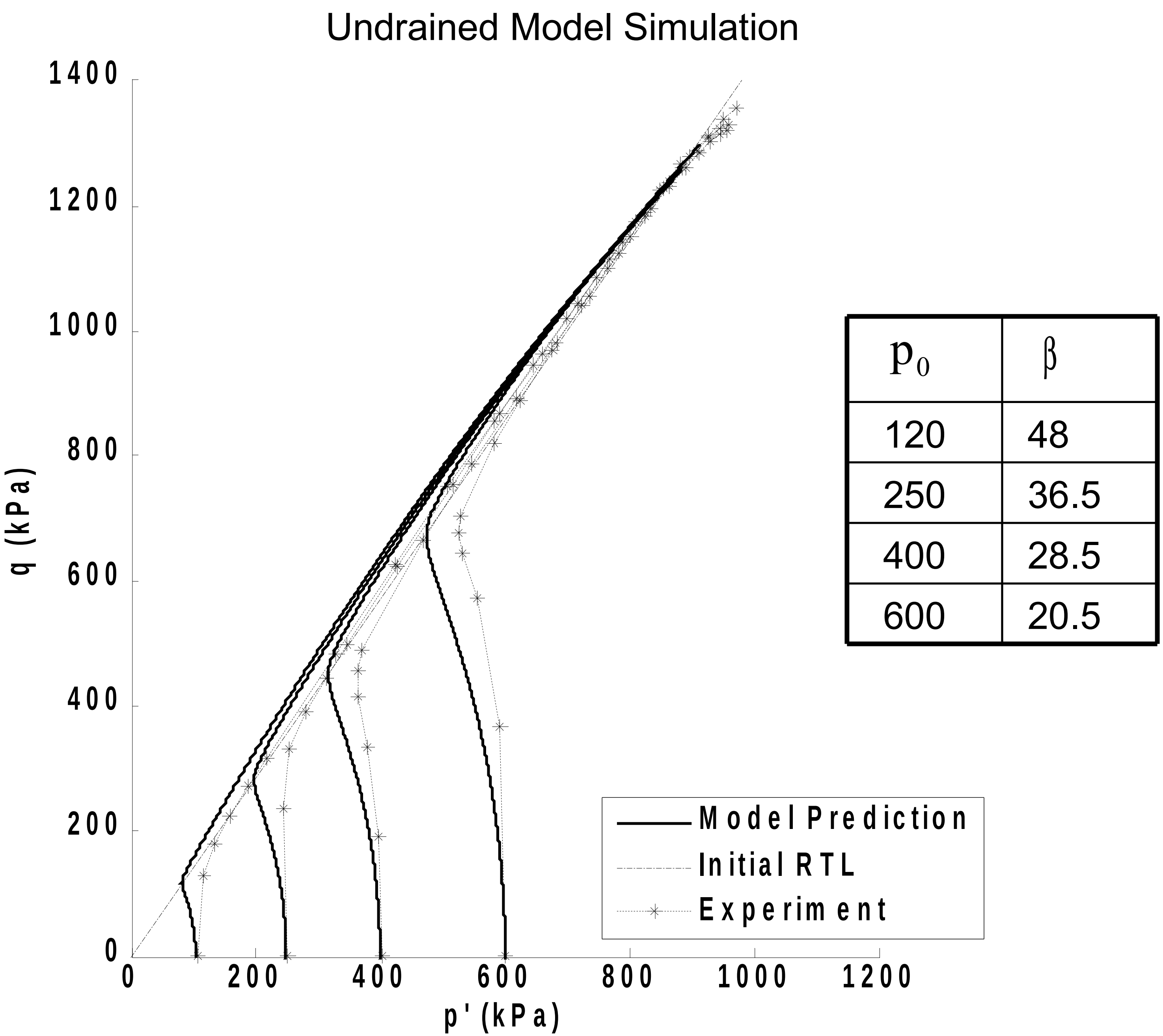

Undrained Model Simulation
q (kPa)
p' (kPa)
0
200
400
600
800
1000
1200
1400
0
200
400
600
800
1000
1200
p0
β
120
48
250
36.5
400
28.5
600
20.5
Model Prediction
Initial RTL
Experiment

FIG 4a

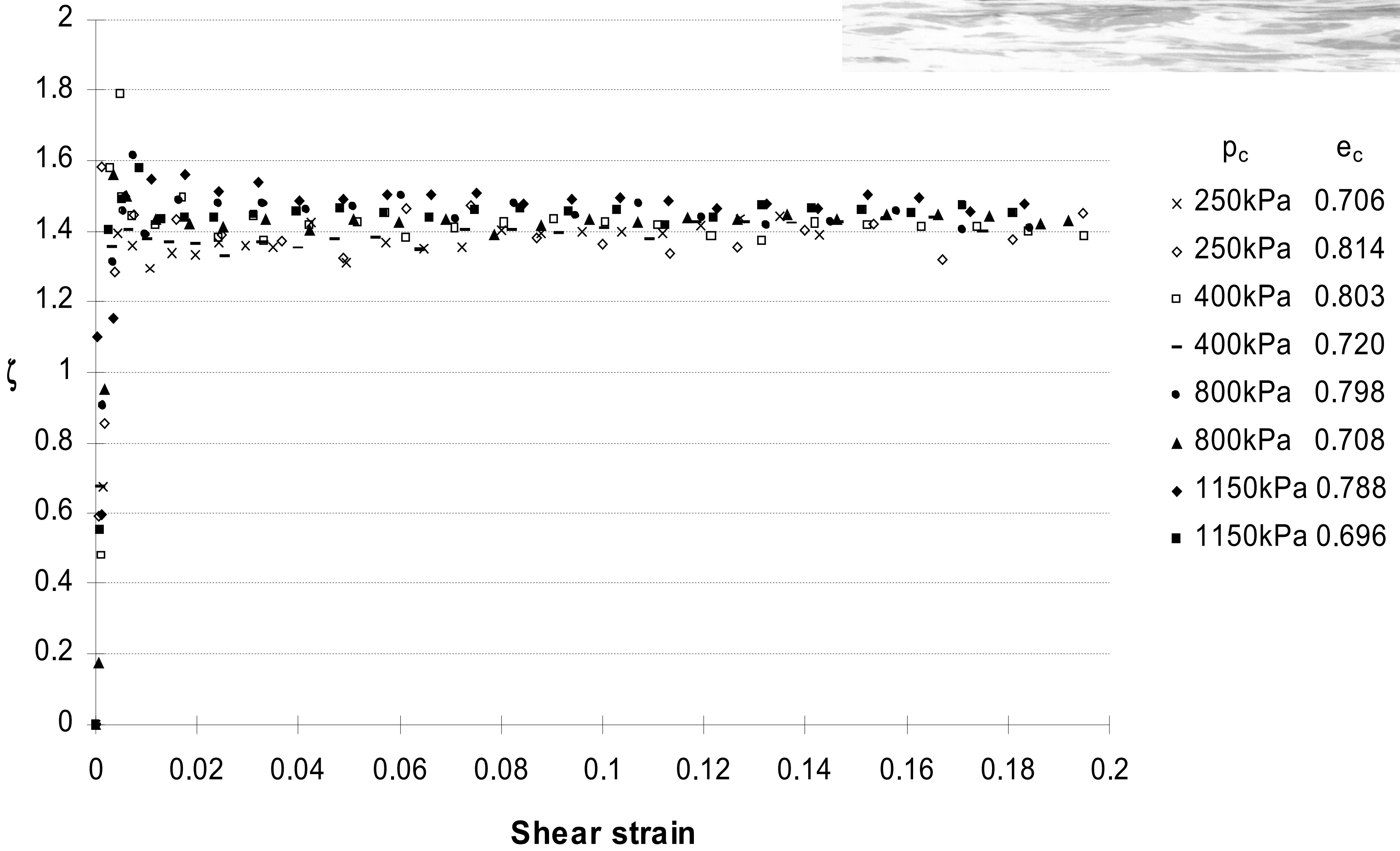
# REYNOLDS TAYLOR LINE

An important prediction from the theory is that before reaching the critical state, it first another state-the “Reynolds- Taylor State” .When the specimen is on the RTL:

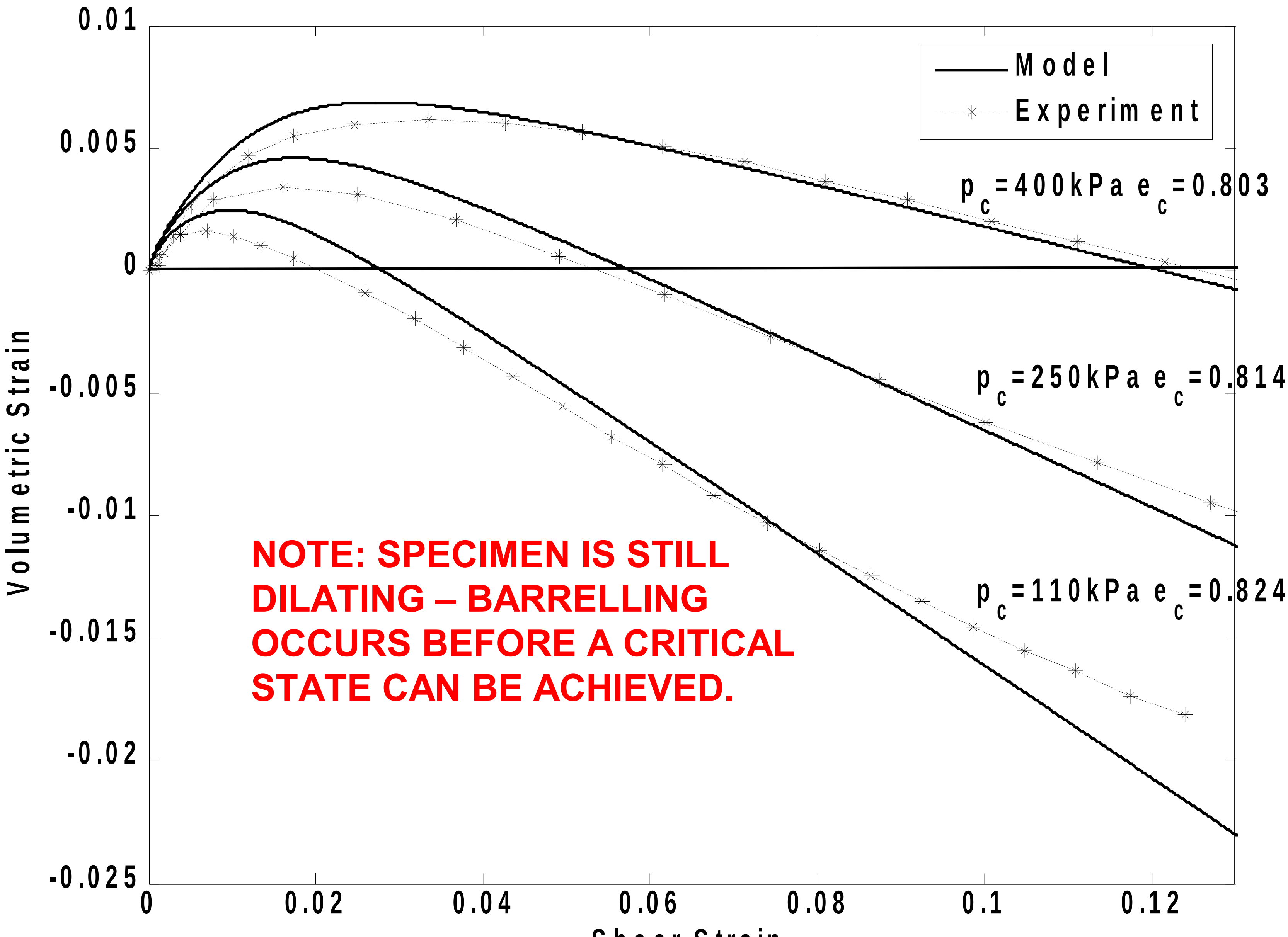
$$\zeta \equiv \frac{q}{p} - \tan\psi = M$$

The difference between the stress ratio and the tangent of the dilation angle is constant.

# DRAINED RESULTS FOR PAKIRI BEACH SAND

# DRAINED TESTS AND SIMULATIONS OF PAKIRI SAND



# ONGOING & FUTURE RESEARCH

- THREE DIMENSIONAL MODELS
- INTRODUCE THE BEEN & JEFFERIES STATE PARAMETER.
- MICRO-POLAR MODELS
- COSSERAT MODELS
- USE OF PROBABILITY DISTIBUTIONS
- FINITE STRAIN MODELS (simple shear)

# DAWN AT PAKIRI BEACH

THANK YOU! ANY QUESTIONS?