file: \barry\hyp.tex

Prague 2006
Hypoelasticity
by
Barry Bernstein

Hypoelasticity was introduced by Truesdell in 1955 as a (non-thermal) theory

involving rate type equations which show no time effects. I am told that it is used to describe behavior of asphalt. But Truesdell's interest was in generalizations of mathematical models: He was interested in theory for its own sake.

The idea of hypo-elasticity is basically simple. Let us look at a one dimensional situation, where ϵ is strain and σ is stress. Linear elasticity would then involve a relation of the form

$$\sigma = \mu\epsilon$$

where μ is a modulus — an elastic modulus. Since classical elasticity is to hold for small strains, we can think of this equation as holding when ϵ and σ are small perturbations about zero strain and zero stress respectively. Carrying this further, ϵ and σ could be replaced by their changes, $\Delta\epsilon$ and $\Delta\sigma$ respctively. Divide by the time interval Δt in which these changes occur, and with the usual limiting process, we arrive at

$$\dot{\sigma} = \mu\dot{\epsilon}$$

This form allows large changes, for which μ need not be a constant. Let μ depend on σ and we have the prototype for hypo-elasticity.

$$\dot{\sigma} = \mu(\sigma)\dot{\epsilon}$$

It turns out, then, once the proper three dimensional properly invariant equations are formulated, the resulting theory of hypoelasticity is one possible generalization of linear elasticity. Indeed, in the same year, 1955, Noll showed that every elastic material for which the stress-strain relation is invertible is also a hypo-elastic material. So, hypoelasticity appears to be more general than elasticity, a property which was very important to Truesdell and his school.

Indeed, Truesdell had a classification of the elasticity-hypoelasticity: The most general of these theories, according to Truesdell, was hypoelasticity. Less general was Cauchy elasticity, in which stress was a function of deformation gradient, but a strain energy function did not necessarily exist. At the lowest rank of the heierarchy stood Green elasticity, which was Cauchy elasticity with a strain energy. To summarize, here is the ranking and the rational of the terminology of Truesdell:

1. Hypoelasticity — less than elasticty –a rate theory.
2. Elasticity — Cauchy elasticity
3. Hyperelasticity — more than elasticity — Green elasticity

Although the embedding of a strain energy in one of the thermodynamic potentials (Hemholtz, Gibbs, etc.) is often not carried out, without a thermodynamic potential, it is unreasonable to expect to be able to construct a thermodynamic theory. Thus, according to this classification, it appears that only the hyper-elastic theory (of those under discussion) allows of a thermodynamic theory. Professor Rajagopal and I have just turned that last statement on its head by constructing thermodynamics for hypoelasticity.

Related to these matters is also plasticity, although the plasticity that I have in mind is one which deals with infinitesmal strain. Nevertheless, as I shall show you, plasticity can arise as a type of singular behavior in a rate theory with no time-dependence, as does hypoelasticity. In the process of this lecture, I shall also show you the construction of a thermodynamic theory of elasticity and plasticity which agrees with the first and second laws.

Let us now turn to the proper formulation of the hypoelastic equations. We shall use Cartesian tensor notation in which, although we write all indices as subscripts, we do sum out on repeated indices in a term.

First we must define some symbols: to this end:

σ_{ij}		stress tensor	
s_{ij}	$= \sigma_{ij} - \sigma_{kk}\delta ij/3$	stress deviator	
e_{ij}		infinitesmal strain	
ϵ_{ij}	$= e_{ij} - e_{kk}\delta_{ij}/3$	strain deviator	
v_i		velocity vector	
d_{ij}	$= (v_{i,j} + v_{j,i})/2$	rate of deformation	
ω_{ij}	$= (v_{i,j} - v_{j,i})/2$	rate of rotation	
p		pressure	
ρ		mass density$\upsilon = 1/\rho$	specific volume

Of course δ_{ij} is the Kroneker delta, i.e. the unit matrix,

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

and the comma denotes partial differentiation. For example, if the spatial cartesian coordinates are x_i, $i = 1, 2, 3$,

$$p_{,i} \equiv \frac{\partial p}{\partial x_i}$$

$$v_{i,j} \equiv \frac{\partial v_i}{\partial x_j}$$

Let us define the hypoelastic equations. For these equations, there is a fourth order tensor valued function of σ_{ij},

$$A_{ijkl} = A_{jik\ell} = A_{ji\ell k}, \qquad A_{ijkk} = A_{kkij} = 0$$

such that the equations of hypoelasticity become

$$\dot{s}_{ij} = s_{ik}\omega_{jk} + s_{jk}\omega_{ik} + A_{ijk\ell}d_{k\ell}$$

In our treatment, we assume that there is also a relation of density or specific volume to pressure, p. Since this can be a standard type of relation, the discussion of it here will distract us from our treatment of the shear (stress deviator) and so we shall not deal with it here. where the dot stands for material derivative. The terms in ω_{ij} are needed for proper geometric invariance and if put on the right hand side, then together with the material derivative, they would make up what is called the corotational derivative. They are of second order and hence are dropped in the case of infinitesmal displacement, but they are necessary for finite strain.

In the general theory of hypoelasticity the assignment of the $A_{ijk\ell}$ as functions of stress is left open, except for requiring this assignment to be isotropic. Later we shall look at some restrictions on such assignment, these restrictions resulting from thermodynamic considerations.

Now let us change pace and talk about plasticity. In the theory of perfect plasticity, which is a theory of infinitesmal strain, the stress-strain relation is that of elasticity as long as the stress, or, actually some function of the stress remains below a critical value, perhaps known as the yield condition, and above which other conditions hold – e.g. the yield condition remaining constant. To be more specific, we deal with the von-Mises yield condition, which is as follows: There is a critical value $k > 0$ such that the yield condition is

$$s_{ij}s_{ij} = 2k^2$$

As long as $s_{ij}s_{ij} < 2k^2$, the stress-strain response is elastic. When equality is reached, the von-Mises condition continues to hold and a set of rate equations, known as the Prandtl-Reuss equations, takes over. This is a theory generally known as elasticity–perfect plasticity. But there is a concept which is not usually too well defined, namely loading and unloading. The Prandtl-Reuss equations hold at the yield condition during loading. During unloading, there is a return to elastic behavior, albeit with permanent set. But hold on – we will have a definition of loading and of unloading before our hour is up. And later, when we discuss thermodynamics, we shall have a definition which will also involve temperature.

We shall start with a simple one dimensional model: Let ϵ be shear strain and let s be shear stress. We consider the following constitutive equation:

$$\frac{ds}{d\epsilon} = \sqrt{1 - s^2}$$

This has solutions

$$s = \sin(\epsilon - c)$$

But it also has a two singular solutions:

$$s = 1 \qquad \text{and} \qquad s = -1$$

Now consider ϵ to be varied. As long as s remains between $s = -1$ and $s = 1$, This equation obeys a Lipshitz condition, and therefore follows a unique path in ϵ, s space. Such we may call an elastic regime. When s reaches one of the singular values, say $s = 1$, which it does exactly, not asymptotically at $\epsilon - c = \pi/2$, it stays there as long as ϵ is increasing. This is a plastic regime.

Now we are going to consider reversing the direction of $\dot{\epsilon}$, but first we need to say something more about the plastic regime. We need to introduce the concept of stability: When ϵ is increasing, and s is perturbed (only downwards is possible), s will come right back up to the yield value. Upon decreasing ϵ, however, s will continue to fall: the stable solution will be to return to an elastic regime, albeit in general not the one which in which it entered the yield condition in the first place.

Perhaps the sine function is not what most people have in mind for an elastic regime. Indeed, if we replace the equation by

$$\frac{ds}{dt} = \sqrt{1 - s^{2n}}$$

then the elastic regimes straighten out for large enough n. For example, $n = 64$ gives the following:

(show curve)

This is fine for the one-dimensional situation. For the three dimensional situation, we shall deal here only with the shear part and ask you to believe that the pressure-volume part can be added.

Consider now the strain deviator, ϵ_{ij}. We assume a strain potential function

$$\varphi(\alpha) \qquad \text{where} \qquad \alpha = \sqrt{\epsilon_{ij}\epsilon_{ij}}, \qquad s_{ij} = \frac{\partial \varphi}{\partial \epsilon_{ij}} = \varphi'(\alpha)\frac{\epsilon_{ij}}{\alpha}$$

and let

$$\beta = \sqrt{s_{ij}s_{ij}}$$

Then

$$\beta = \varphi'(\alpha) \qquad \text{which, when solved for alpha is expressed as } \alpha = \psi(\beta)$$

After a bit of manipulation, we get

$$\frac{\psi(\beta)}{\beta}s_{ij} = \epsilon_{ij}$$

But we want a rate equation, and so we take the time derivative and after some algebra, we arrive at the following rate equation:

$$\dot{s}_{ij} = \frac{\beta}{\psi(\beta)}\dot{\epsilon}_{ij} + \frac{1}{\beta^2}\left[\frac{1}{\psi'(\beta)} - \frac{\beta}{\psi(\beta)}\right] s_{ij}s_{k\ell}\dot{\epsilon}_{k\ell}$$

Multiplying both sides by s_{ij} and summing out on the indices yields

$$\frac{d}{dt}\beta^2 = \frac{2}{\psi'(\beta)} s_{ij}\dot{\epsilon}_{ij}$$

We can now choose the function $\psi(\beta)$ so that the equations are those of an elastic-plastic material. Indeed, first choose ψ to be

$$\psi(\beta) = \frac{k\sqrt{2}}{2\mu}\sin^{-1}\frac{\beta}{2k\sqrt{2}}$$

which results in

$$\frac{d\beta^2}{dt} = 4\mu\sqrt{1-\beta^2/2k^2}\, s_{ij}\dot{\epsilon}_{ij}$$

Thus, when β^2 reaches the value $2k^2$ which, as s did in the one-dimensional case, $d\beta^2/dt$ will be zero, and so the von-Mises yield condition will be reached. Loading now can be defined as $s_{ij}\dot{\epsilon}_{ij}$. Again, during loading, the yield condition will be stable. On the onset of unloading, however, the stable solution will be a return to an elastic regime.

Actually, we can play around with ψ: In fact, the way to do this is to pick $\psi'\beta$. Given a positive integer, n, we can pick ψ so that

$$\psi'(\beta) = \frac{1}{2\mu\sqrt{1-(\beta^2/2k^2)^n}}$$

which will straighten out the elastic regime curves. and give the yield behavior

$$\frac{d\beta^2}{dt} = \sqrt{1-(\beta^2/2k^2)^n}\, s_{ij}\dot{\epsilon}_{ij}$$

But there are still the equations for s_{ij}. Remember, they are

$$\dot{s}_{ij} = \frac{\beta}{\psi(\beta)}\dot{\epsilon}_{ij} + \frac{1}{\beta^2}\left[\frac{1}{\psi'(\beta)} - \frac{\beta}{\psi(\beta)}\right] s_{ij}s_{k\ell}\dot{\epsilon}_{k\ell}$$

During yield, we set $1/\psi'(\beta) = 0$, and $\beta^2 = 2k^2$ to get

$$\dot{s}_{ij} = 4\mu \frac{k\sqrt{2}}{\psi(k\sqrt{2})} \left[\dot{\epsilon}_{ij} - \frac{1}{2k^2} \right] s_{ij} s_{k\ell} \dot{\epsilon}_{k\ell}$$

which are the well known Prandtl-Reuss equations. And so we have recovered elasticity and plasticity of infinitesmal strain in a single set of equations.

Now let us see how all of this plays out thermodynamically. To construct the thermodynamics, we need to be sure what will be taken as state variables. Indeed, we shall take stress deviator s_{ij} and temperature T as state variables. I ask that you take my word that we can also include pressure or mass density as state variables in a fairly straightforward way, but doing so will just distract us now. So let us stick with the stress-deviator and temperature.

The thermodynamic potential which is most appropriate here is the Gibbs function, $G(s_{ij}, T)$, which we take to depend on s_{ij} through the invariant, β

$$G = G(\beta, T)$$

We write down some relationships, which we then replace by their time derivatives as rate equations. Indeed, we write

$$\epsilon_{ij} = -\frac{\partial G}{\partial s_{ij}}$$

$$\eta = \frac{-\partial G}{\partial T} \qquad (\eta \text{ is entropy density})$$

$$E = G + T\eta + s_{ij}\epsilon_{ij} \qquad \psi = -\frac{\partial G}{\partial \beta}$$

where η is entropy density, T is temperature, and E is internal energy. After some manipulation, we arrive at the following rate equation:

$$\dot{s}_{ij} = \frac{\beta}{\psi} + \frac{1}{\beta^2} \left[\frac{1}{\psi_\beta} - \frac{\beta}{\psi} \right] s_{ij} s_{k\ell} \dot{\epsilon}_{k\ell} - \frac{\psi_T}{\beta \psi_\beta} s_{ij} \dot{T}$$

We now use the energy balance equation, which is the following:

$$\dot{E} = s_{ij}\dot{\epsilon}_{ij} - q_{i,i}$$

where q_i is the heat flux vector. And we need to non-dimensionalize our equations. To this end let $k(T)$ be the temperature dependent yield stress and let

$$\begin{aligned}
&S_{ij} = \frac{s_{ij}}{k\sqrt{2}}\\
&H(B,T) = G(\beta,T)\\
&\Theta(B,T) = \psi(\beta,T)
\end{aligned}$$

in which case the rate equation becomes

$$\dot{S}_{ij} = \frac{B}{\Theta}\dot{\epsilon}_{ij} + \frac{1}{B^2}\left[\frac{1}{\Theta_B} - \frac{B}{\Theta}\right] S_{ij}S_{k\ell}\dot{\epsilon}_{k\ell} - \frac{\Theta_T}{B\Theta_B}S_{ij}\dot{T}$$

and the eqution for B becomes

$$\frac{dB^2}{dt} = \frac{2}{\Theta_B}\left[S_{ij}\dot{\epsilon}_{ij} - B\Theta_B\dot{T}\right]$$

So the yield condition is $B = 1$. If we take ψ as before, we get for this last equation

$$\frac{dB^2}{dt} = \frac{4\mu}{k\sqrt{2}}\sqrt{1-B^{2n}}\left[S_{k\ell}\dot{\epsilon}_{k\ell} - B\Theta_T\dot{T}\right]$$

This tells us that yield is stable when the epression in the brackets is positive and unstable when the expression in the brackets is negative. So we have a definition of loading and unloading

$$\begin{aligned}
&\left[S_{k\ell}\dot{\epsilon}_{k\ell} - B\Theta_T\dot{T}\right] > 0 \quad \text{loading}\\
&\left[S_{k\ell}\dot{\epsilon}_{k\ell} - B\Theta_T\dot{T}\right] < 0 \quad \text{unloading}
\end{aligned}$$

But this is not the end: If If we go back to the balance of energy equation, we may consider two conditions; 1) $B^2 < 1$, in which case the chain rule of calculus applies. One obtains

$$\dot{E} = s_{k\ell}\dot{\epsilon}_{k\ell} + T\dot{\eta}$$

which, when placed into the energy balance,

$$\dot{E} = s_{ij}\dot{\epsilon}_{ij} - q_{i,i}$$

results in

$$\rho T\dot{\eta} + q_{i,i} = 0$$

So there is no entropy production except that due to heat flow.

On the other hand, during yield, $B = 1$, we have to differentiate differently: First we set B to the constant value, $B = 2$ and then differentiate. So $G = G(1, T)$ depends only on temperature for this analysis. When we do so, we get

$$\dot{E} = k\sqrt{2}\Theta_T \dot{T} + T\dot{\eta}$$

which then results in the following

$$\boxed{T\dot{\eta} + q_{i,i} = k\sqrt{2}\left[S_{ij}\dot{\epsilon}_{ij} - B\Theta_T \dot{T}\right]}$$

Isn't this interesting? The condition for stability of yield now comes into the entropy production. If during yield if the expression in brackets is positive, (laoding), yield is stable and produces entropy: The second law is upheld by positive entropy production. On the other hand, should this expression become negative, (unloading), the stable solution dictates a return to an elastic regime, in which case there is zero entropy production. We need add only positive thermal conductivity, $q_i T_{,i} \le 0$ so that we get always

$$\dot{\eta} + \frac{q_{i,i}}{T} = \dot{\eta} + \left(\frac{q_i}{T}\right)_{,i} + \frac{q_i T_{,i}}{T^2}$$

So positive entropy production and positive thermal conductivity ($q_i T_{,i} \le 0$) gives agreement with the second law. The theory agrees with the laws of thermodynamics.

Now to hypoelasticity. What we have done in the case of small strain can be carried over to large strain, with appropriate modifications. Indeed, let us consider what we need to do for a thermodynamic approach: We need to

- Choose our state variables
- Choose the appropriate thermodynamic potential in terms of which we will state conservation of energy
- Choose the rate equation for stress (deviator)
- Establish that the second law holds.

In 1984, Olsen and Bernstein did just that for a thermodynamic set of equations which follows pretty much what was just done for the case of infinitesmal strain, but which treats large deformation hypoelasticity. We now present the essence of this contribution.

Choose as state variables the stress deviator and temperature. Then assume given a Gibbs free energy function $G(s_{ij}, T)$ which depends only of the invariant $\beta = \sqrt{s_{ij}s_{ij}}$. Again, let $\eta = -\partial G/\partial T$ be the entropy density. The internal energy will then be given by

$$E = G - s_{ij}\frac{\partial G}{\partial s_{ij}} + T\eta$$

Note that this time, we do not assume that the derivative of the Gibbs function with respect to stress is strain. Instead, let these derivatives remain as functions of stress and temperature, but we make the terms involving s_{ij} balance balance the work terms in the energy balance, i.e.

$$-\rho s_{ij}\left(\frac{\partial G}{\partial s_{ij}}\right)^{\cdot} = s_{ij}d_{ij}$$

Then the balance of energy becomes

$$\rho T\dot{\eta} + q_{i,i} = 0$$

or

$$\rho\dot{\eta} + \left(\frac{q_i}{T}\right)_{,i} = -\frac{q_iT_{,i}}{T^2}$$

which, when postitive thermal conductivity is $q_iT_{,i}$ is assumed, gives agreement with the second law. It remains only to complete the algebra and to replace the time derivative of stress by the corotational derivative and to define $\psi = -\partial G/\partial\beta$ to arrive at the constitutive equation

$$\begin{aligned}\dot{s}_{ij} - s_{ik}\omega_{jk} - s_{jk}\omega_{ik}&\\ &= \frac{1}{\rho}\left[\frac{1}{\beta^2}\left(\frac{1}{\psi_\beta} - \frac{\beta}{\psi}\right)s_{ij}s_{k\ell} + \frac{\beta}{\psi}\varphi_{ijk\ell}\right]d_{k\ell} - \frac{1}{\beta}\frac{\psi_T}{\psi_\beta}\dot{T}s_{ij}\end{aligned}$$

where

$$\varphi_{ijk\ell} = \frac{1}{2}\delta_{ij}\delta_{k\ell} + \frac{1}{2}\delta_{jk}\delta_{i\ell} - \frac{1}{3}\delta_{ij}\delta_{k\ell}$$

Bernstein and Olsen were also able to show that this model was definitely not elastic by applying conditions derived by in 1960.

The latest addition to these studies is to be published in ZAMP under the title

of Thermodynamics of Hypoelasticity, by B. Bernstein and K. Rajagopal.

The stress deviator is traceless, $s_{ii} = 0$. This relation uses up one of its three invariants. Of the other two, we have dealt with β. We have one more, which we call γ.

$$\beta = \sqrt{s_{ij}s_{ij}}$$
$$\gamma = \sqrt[3]{s_{ik}s_{k\ell}s_{\ell j}}$$

So that G depends on s_{ij} through its two invariants, β and γ. We proceed pretty much as before, except that we make two assumptions and ask when they tie together.

- The constituive relation

$$\dot{s}_{ij} = s_{ik}\omega_{jk} + s_{jk}\omega_{ik} + \upsilon A_{ijk\ell}d_{k\ell} + a_{ij}\dot{T} + \dot{\upsilon}b_{ij}$$

- The Gibbs function depending on s_{ij} and T.
- conservation of energy
- If we put all this into our equations and define

$$\varphi_{ij} = -\frac{\partial G}{\partial s_{ij}}$$

we get after some manipulation

$$\rho T\dot{\eta} + q_{i,i} - \rho r = s_{ij}d_{ij} - \rho s_{ij}\dot{\varphi}_{ij}$$

So, if we want the second law to hold, we need to have

$$s_{ij}d_{ij} - \rho s_{ij}\dot{\varphi}_{ij} \geq 0$$

But if the right hand side of this is > 0, we could reverse the path and thus make it < 0. So the only way we are going to satisfy the second law is to make it zero.

After using the constitutive equation in this last relationship and doing some tensor algegra, we come up with restrictions on the $A_{ijk\ell}$ which make this last condition hold. For $A_{ijk\ell}$ we have the following

$$\begin{aligned}\tilde{A}_{ijk\ell} =& A_1\delta_{ij}\delta_{k\ell} + \frac{A_2}{2}\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right) + A_3\delta_{ij}s_{k\ell} + A_4 s_{ij}\delta_{k\ell} \\ &+ \frac{A_5}{4}\left(\delta_{ik}s_{j\ell} + \delta_{jk}s_{i\ell} + \delta_{i\ell}s_{jk} + \delta_{j\ell}s_{ik}\right) + A_6\delta_{ij}s^2_{k\ell} + A_7\delta_{k\ell}s^2_{ij} \\ &+ \frac{A_8}{4}\left(\delta_{ik}s^2_{j\ell} + \delta_{jk}s^2_{i\ell} + \delta_{i\ell}s^2_{jk} + \delta_{j\ell}s^2_{ik}\right) \\ &+ A_9 s_{ij}s_{k\ell} + \frac{A_{10}}{2}\left(s_{ik}s_{j\ell} + s_{jk}s_{i\ell}\right) \\ &+ A_{11}s_{ij}s^2_{k\ell} + A_{12}s^2_{ij}s_{k\ell} + \frac{A_{13}}{4}\left(s_{ik}s^2_{j\ell} + s_{jk}s^2_{i\ell} + s_{i\ell}s^2_{jk} + s_{j\ell}s^2_{ik}\right) \\ &+ A_{14}s^2_{ij}s^2_{k\ell} + \frac{A_{15}}{2}\left(s^2_{ik}s^2_{j\ell} + s^2_{jk}s^2_{i\ell}\right)\end{aligned}$$

After making it traceless by subtracting its traces, and using that last relation, we come up with the following restrictions on the A_k: Let P and Q be defined respectively by

$$\begin{aligned}P &\overset{\text{def}}{=} -G_{\beta\beta} - \frac{\gamma}{\beta}G_{\beta\gamma} \\ Q &\overset{\text{def}}{=} -\frac{\beta}{\gamma^2}G_{\gamma\beta} - \frac{1}{\gamma}G_{\gamma\gamma}\end{aligned}$$

Then these are conditions that must be satisfied:

$$\begin{aligned}&PA_2 + Q\frac{\beta^2}{6}A_5 + \left(P\frac{\beta^2}{2} + Q\frac{\gamma^3}{3}\right)A_8 + \left(P\beta^2 + Q\gamma^3\right)A_9 \\ &+ \left(P\frac{\beta^2}{2} + Q\frac{\gamma^3}{3}\right)A_{10} + \left(P\gamma^3 + Q\frac{\beta^4}{6}\right)A_{12} + \left(P\frac{\gamma^3}{3} + Q\frac{\beta^4}{12}\right)A_{13} \\ &+ \left(P\frac{\beta^4}{4} + 2Q\frac{\beta^2\gamma^3}{9}\right)A_{15} = 1\end{aligned}$$

and

$$\begin{aligned}&QA_2 + PA_5 + Q\frac{\beta^2}{6}A_8 + Q\frac{\beta^2}{6}A_{10} + \left(P\beta^2 + Q\gamma^3\right)A_{11} \\ &+ \left(P\frac{\beta^2}{2} + Q\frac{\gamma^3}{3}\right)A_{13} + \left(P\gamma^3 + Q\frac{\beta^4}{6}\right)A_{14} \\ &+ \left(P\frac{\gamma^3}{3} + Q\frac{\beta^4}{12}\right)A_{15}xxx = 0\end{aligned}$$

These are two conditions on the ten coefficients, a grossly undetermined system. Actually, it is even more so. Because of the volume of material presented here in a relatively short time, I have not discussed the restrictions on the temperature and volume terms. In fact, these add two more restrictions, but also add four more coefficients. And all of these coefficients are functions of the invariants, the temperature and, actually, also the volume.

The Bernstein-Olsen results fit in very nicely by choosing

$$A_2 = \frac{\beta}{\psi} \qquad \text{and} \qquad A_9 = \frac{1}{\beta^2}\left[\frac{1}{\psi_\beta} - \frac{\beta}{\psi}\right]$$

Concluding Remarks

- Hypoelasticity is a rate theory of stress versus strain.
- Hypoelasticity is considered more general than Cauchy elasticity, in which there stress is a function of strain.
- Nevertheless under appropriate restrictions, one may construct a thermodynamic theory of hypoelasticity.
- The theory presented here involves a severely underdetermined set of restrictions which arise from the requirement that the theory fit the laws of thermodynamics.
- Because hypoelasticity is path reversible, obeying the second law, as formulated here, gives no entropy production except that due to heat flow, as in an elastic theory.
- The theory of elasticity and perfect plasticity can be formulated in a single set of equations. These can be made into a thermodynamic theory in which loading and unloading is well defined. The definition even includes dependence on temperature changes.
- In the elasticity-plasticity equations given here, agreement with the second law follows from the assumption that the stable path will be taken both in loading and unloading.