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Hypoelasticity

by

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1-dimensional situation, where ϵ is strain and σ is stress. Linear

$$\sigma = \mu\epsilon$$

where μ is a modulus.

$$\dot{\sigma} = \mu\dot{\epsilon}$$

$$\dot{\sigma} = \mu(\sigma)\dot{\epsilon}$$

1. Hypoelasticity — less than elasticity –a rate theory.
2. Elasticity — Cauchy elasticity
3. Hyperelasticity — more than elasticity — Green elasticity

Let us now turn to the proper formulation of the hypoelastic equations. We shall use Cartesian tensor notation in which, although we write all indices as subscripts, we do sum out on repeated indices in a term.

First we must define some symbols: to this end:

σ_{ij}		stress tensor
s_{ij}	$= \sigma_{ij} - \sigma_{kk}\delta ij/3$	stress deviator
e_{ij}		infinitesmal strain
ϵ_{ij}	$= e_{ij} - e_{kk}\delta_{ij}/3$	strain deviator
v_i		velocity vector
d_{ij}	$= (v_{i,j} + v_{j,i})/2$	rate of deformation
ω_{ij}	$= (v_{i,j} - v_{j,i})/2$	rate of rotation
p		pressure
ρ		mass density
$\upsilon = 1/\rho$		specific volume

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

If the spatial cartesian coordinates are x_i, $i = 1, 2, 3$,

$$p_{,i} \equiv \frac{\partial p}{\partial x_i}$$

$$v_{i,j} \equiv \frac{\partial v_i}{\partial x_j}$$

$$A_{ijkl} = A_{jik\ell} = A_{ji\ell k}, \qquad A_{ijkk} = A_{kkij} = 0$$

$$\dot{s}_{ij} = s_{ik}\omega_{jk} + s_{jk}\omega_{ik} + A_{ijk\ell}d_{k\ell}$$

yield condition

$$s_{ij}s_{ij} = 2k^2$$

$\beta^2 = s_{ij}s_{ij} < 2k^2$, elastic.,

$$\frac{ds}{d\epsilon} = \sqrt{1 - s^2}$$

Solutions

$$s = \sin(\epsilon - c)$$

But it also has a two singular solutions:

$$s = 1 \qquad \text{and} \qquad s = -1$$

$-1 < s < 1$, Lipschitz – uniqueness

$s = \pm 1$, Lipschitz & uniqueness fails.

$$\frac{ds}{dt} = \sqrt{1 - s^{2n}}$$

$$\varphi(\alpha) \qquad \text{where} \qquad \alpha = \sqrt{\epsilon_{ij}\epsilon_{ij}},$$

$$s_{ij} = \frac{\partial \varphi}{\partial \epsilon_{ij}} = \varphi'(\alpha)\frac{\epsilon_{ij}}{\alpha}$$

and let

$$\beta = \sqrt{s_{ij}s_{ij}}$$

Then

$$\beta = \varphi'(\alpha)$$

which, when solved for alpha is expressed as

$$\alpha = \psi(\beta)$$

After a bit of manipulation, we get

$$\frac{\psi(\beta)}{\beta} s_{ij} = \epsilon_{ij}$$

$$\dot{s}_{ij} = \frac{\beta}{\psi(\beta)}\dot{\epsilon}_{ij} + \frac{1}{\beta^2}\left[\frac{1}{\psi'(\beta)} - \frac{\beta}{\psi(\beta)}\right] s_{ij}s_{k\ell}\dot{\epsilon}_{k\ell}$$

$$\frac{d}{dt}\beta^2 = \frac{2}{\psi'(\beta)} s_{ij}\dot{\epsilon}_{ij}$$

$$\psi(\beta) = \frac{k\sqrt{2}}{2\mu} \sin^{-1} \frac{\beta}{2k\sqrt{2}}$$

which results in

$$\frac{d\beta^2}{dt} = 4\mu\sqrt{1 - \beta^2/2k^2} s_{ij}\dot{\epsilon}_{ij}$$

Actually, we can play around with ψ: In fact, the way to do this is to pick $\psi'\beta$. Given a positive integer, n, we can pick ψ so that

$$\psi'(\beta) = \frac{1}{2\mu\sqrt{1 - (\beta^2/2k^2)^n}}$$

$$\frac{d\beta^2}{dt} = \sqrt{1 - (\beta^2/2k^2)^n} s_{ij}\dot{\epsilon}_{ij}$$

$$\dot{s}_{ij} = \frac{\beta}{\psi(\beta)}\dot{\epsilon}_{ij} + \frac{1}{\beta^2}\left[\frac{1}{\psi'(\beta)} - \frac{\beta}{\psi(\beta)}\right] s_{ij}s_{k\ell}\dot{\epsilon}_{k\ell}$$

During yield, $1/\psi'(\beta) = 0$, and $\beta^2 = 2k^2$ to get

$$\dot{s}_{ij} = 4\mu\frac{k\sqrt{2}}{\psi(k\sqrt{2})}\left[\dot{\epsilon}_{ij} - \frac{1}{2k^2}\right] s_{ij}s_{k\ell}\dot{\epsilon}_{k\ell}$$

s_{ij} and T state variables.

Gibbs function, $G(s_{ij}, T)$, which we take to depend on s_{ij} through the invariant, β

$$G = G(\beta, T)$$

$$\begin{aligned}
\epsilon_{ij} &= -\frac{\partial G}{\partial s_{ij}} \\
\eta &= \frac{-\partial G}{\partial T} \qquad\qquad (\eta \text{ is entropy density}) \\
E &= G + T\eta + s_{ij}\epsilon_{ij} \\
\psi &= -\frac{\partial G}{\partial \beta}
\end{aligned}$$

η entropy density, T temperature, E is internal energy.

$$\dot{s}_{ij} = \frac{\beta}{\psi} + \frac{1}{\beta^2}\left[\frac{1}{\psi_\beta} - \frac{\beta}{\psi}\right] s_{ij}s_{k\ell}\dot{\epsilon}_{k\ell} - \frac{\psi_T}{\beta\psi_\beta}s_{ij}\dot{T}$$

Energy balance

$$\dot{E} = s_{ij}\dot{\epsilon}_{ij} - q_{i,i}$$

q_i heat flux vector

$$S_{ij} = \frac{s_{ij}}{k\sqrt{2}}$$
$$H(B,T) = G(\beta,T)$$
$$\Theta(B,T) = \psi(\beta,T)$$

$$\dot{S}_{ij} = \frac{B}{\Theta}\dot{\epsilon}_{ij} + \frac{1}{B^2}\left[\frac{1}{\Theta_B} - \frac{B}{\Theta}\right] S_{ij}S_{k\ell}\dot{\epsilon}_{k\ell} - \frac{\Theta_T}{B\Theta_B} S_{ij}\dot{T}$$

and the eqution for B becomes

$$\frac{dB^2}{dt} = \frac{2}{\Theta_B}\left[S_{ij}\dot{\epsilon}_{ij} - B\Theta_B\dot{T}\right]$$

yield condition is $B = 1$. last equation

$$\frac{dB^2}{dt} = \frac{4\mu}{k\sqrt{2}}\sqrt{1 - B^{2n}}\left[S_{k\ell}\dot{\epsilon}_{k\ell} - B\Theta_T\dot{T}\right]$$

$$\left[S_{k\ell}\dot{\epsilon}_{k\ell} - B\Theta_T\dot{T}\right] > 0 \quad \text{loading}$$
$$\left[S_{k\ell}\dot{\epsilon}_{k\ell} - B\Theta_T\dot{T}\right] < 0 \quad \text{unloading}$$

$B^2 < 1$, chain rule applies

$$\dot{E} = s_{k\ell}\dot{\epsilon}_{k\ell} + T\dot{\eta}$$

which, with placed energy balance,

$$\dot{E} = s_{ij}\dot{\epsilon}_{ij} - q_{i,i}$$

results in

$$\rho T\dot{\eta} + q_{i,i} = 0$$

no entropy production except due to heat flow. During yield, $B = 1$, First we set B to the constant value, $B = 1$ and then differentiate $G = G(1, T)$

$$\dot{E} = k\sqrt{2}\Theta_T\dot{T} + T\dot{\eta}$$

Result

$$\boxed{T\dot{\eta} + q_{i,i} = k\sqrt{2}\left[S_{ij}\dot{\epsilon}_{ij} - B\Theta_T\dot{T}\right]}$$

$q_iT_{,i} \leq 0$ so that we get always

$$\dot{\eta} + \frac{q_{i,i}}{T} = \dot{\eta} + \left(\frac{q_i}{T}\right)_{,i} + \frac{q_iT_{,i}}{T^2}$$

So positive entropy production and positive thermal conductivity ($q_iT_{,i} \leq 0$) gives agreement with the second law. The theory agrees with the laws of thermodynamics.

- Choose our state variables
- Choose the appropriate thermodynamic potential in terms of which we will state conservation of energy
- Choose the rate equation for stress (deviator)
- Establish that the second law holds.

1984, Olsen and Bernstein

$$E = G - s_{ij}\frac{\partial G}{\partial s_{ij}} + T\eta$$

Energy balance

$$-\rho s_{ij}\left(\frac{\partial G}{\partial s_{ij}}\right)^{\cdot} = s_{ij}d_{ij}$$

becomes

$$\rho T\dot{\eta} + q_{i,i} = 0$$

or

$$\rho\dot{\eta} + \left(\frac{q_i}{T}\right)_{,i} = -\frac{q_i T_{,i}}{T^2}$$

$q_i T_{,i} \leq 0$ is assumed, gives agreement with second law. Let $\psi = -\partial G/\partial\beta$ One arrives at constitutive eq.

$$\dot{s}_{ij} - s_{ik}\omega_{jk} - s_{jk}\omega_{ik}$$
$$= \frac{1}{\rho}\left[\frac{1}{\beta^2}\left(\frac{1}{\psi_\beta} - \frac{\beta}{\psi}\right)s_{ij}s_{k\ell} + \frac{\beta}{\psi}\varphi_{ijk\ell}\right]d_{k\ell} - \frac{1}{\beta}\frac{\psi_T}{\psi_\beta}\dot{T}s_{ij}$$

where

$$\varphi_{ijk\ell} = \frac{1}{2}\delta_{ij}\delta_{k\ell} + \frac{1}{2}\delta_{jk}\delta_{i\ell} - \frac{1}{3}\delta_{ij}\delta_{k\ell}$$

Bernstein and Olsen - model not elastic.

ZAMP Thermodynamics of Hypoelasticity, by

B. Bernstein and K. Rajagopal.

The stress deviator is traceless, $s_{ii} = 0$. This relation uses up one of its three invariants. Of the other two, we have dealt with β. We have one more, which we call γ.

$$\beta = \sqrt{s_{ij}s_{ij}}$$
$$\gamma = \sqrt[3]{s_{ik}s_{k\ell}s_{\ell j}}$$

So that $G = G(\beta, \gamma, T)$

- The constituive relation

  $$\dot{s}_{ij} = s_{ik}\omega_{jk} + s_{jk}\omega_{ik} + \upsilon A_{ijk\ell}d_{k\ell} + a_{ij}\dot{T} + \dot{\upsilon}b_{ij}$$

- The Gibbs function depending on s_{ij} and T.
- conservation of energy
- If we put all this into our equations and define

$$\varphi_{ij} = -\frac{\partial G}{\partial s_{ij}}$$

$$\rho T\dot{\eta} + q_{i,i} - \rho r = s_{ij}d_{ij} - \rho s_{ij}\dot{\varphi}_{ij}$$

For second law to hold

$$s_{ij}d_{ij} - \rho s_{ij}\dot{\varphi}_{ij} \geq 0$$

$s_{ij}d_{ij} - \rho s_{ij}\dot{\varphi}_{ij} = 0$
For $A_{ijk\ell}$ we have the following

$$\begin{aligned}
\tilde{A}_{ijk\ell} =& A_1\delta_{ij}\delta_{k\ell} + \frac{A_2}{2}\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right) \\
&+ A_3\delta_{ij}s_{k\ell} + A_4 s_{ij}\delta_{k\ell} \\
&+ \frac{A_5}{4}\left(\delta_{ik}s_{j\ell} + \delta_{jk}s_{i\ell} + \delta_{i\ell}s_{jk} + \delta_{j\ell}s_{ik}\right) \\
&+ A_6\delta_{ij}s^2_{k\ell} + A_7\delta_{k\ell}s^2_{ij} \\
&+ \frac{A_8}{4}\left(\delta_{ik}s^2_{j\ell} + \delta_{jk}s^2_{i\ell} + \delta_{i\ell}s^2_{jk} + \delta_{j\ell}s^2_{ik}\right) \\
&+ A_9 s_{ij}s_{k\ell} + \frac{A_{10}}{2}\left(s_{ik}s_{j\ell} + s_{jk}s_{i\ell}\right) \\
&+ A_{11}s_{ij}s^2_{k\ell} + A_{12}s^2_{ij}s_{k\ell} \\
&+ \frac{A_{13}}{4}\left(s_{ik}s^2_{j\ell} + s_{jk}s^2_{i\ell} + s_{i\ell}s^2_{jk} + s_{j\ell}s^2_{ik}\right) \\
&+ A_{14}s^2_{ij}s^2_{k\ell} + \frac{A_{15}}{2}\left(s^2_{ik}s^2_{j\ell} + s^2_{jk}s^2_{i\ell}\right)
\end{aligned}$$

$$P \stackrel{\text{def}}{=} -G_{\beta\beta} - \frac{\gamma}{\beta}G_{\beta\gamma}$$

$$Q \stackrel{\text{def}}{=} -\frac{\beta}{\gamma^2}G_{\gamma\beta} - \frac{1}{\gamma}G_{\gamma\gamma}$$

Then these are conditions that must be satisfied:

$$
\begin{aligned}
&PA_2 + Q\frac{\beta^2}{6}A_5 + \left(P\frac{\beta^2}{2} + Q\frac{\gamma^3}{3}\right)A_8 \\
&+ \left(P\beta^2 + Q\gamma^3\right)A_9 \\
&+ \left(P\frac{\beta^2}{2} + Q\frac{\gamma^3}{3}\right)A_{10} + \left(P\gamma^3 + Q\frac{\beta^4}{6}\right)A_{12} \\
&+ \left(P\frac{\gamma^3}{3} + Q\frac{\beta^4}{12}\right)A_{13} \\
&+ \left(P\frac{\beta^4}{4} + 2Q\frac{\beta^2\gamma^3}{9}\right)A_{15} = 1
\end{aligned}
$$

and

$$
\begin{aligned}
&QA_2 + PA_5 + Q\frac{\beta^2}{6}A_8 + Q\frac{\beta^2}{6}A_{10} + \left(P\beta^2 + Q\gamma^3\right)A_{11} \\
&+ \left(P\frac{\beta^2}{2} + Q\frac{\gamma^3}{3}\right)A_{13} + \left(P\gamma^3 + Q\frac{\beta^4}{6}\right)A_{14} \\
&+ \left(P\frac{\gamma^3}{3} + Q\frac{\beta^4}{12}\right)A_{15} = 0
\end{aligned}
$$

2 conds on 10 coefficients — Underdetermined.

The Bernstein-Olsen results; choose

$$A_2 = \frac{\beta}{\psi} \qquad \text{and} \qquad A_9 = \frac{1}{\beta^2}\left[\frac{1}{\psi_\beta} - \frac{\beta}{\psi}\right]$$

Concluding Remarks

- Hypoelasticity is a rate theory of stress versus strain.
- Hypoelasticity is considered more general than Cauchy elasticity, in which there stress is a function of strain.
- Nevertheless under appropriate restrictions, one may construct a thermodynamic theory of hypoelasticity.
- The theory presented here involves a severely underdetermined set of restrictions which arise from the requirement that the theory fit the laws of thermodynamics.
- Because hypoelasticity is path reversible, obeying the second law, as formulated here, gives no entropy production except that due to heat flow, as in an elastic theory.

- The theory of elasticity and perfect plasticity can be formulated in a single set of equations. These can be made into a thermodynamic theory in which loading and unloading is well defined. The definition even includes dependence on temperature changes.
- In the elasticity-plasticity equations given here, agreement with the second law follows from the assumption that the stable path will be taken both in loading and unloading.