Jindřich Nečas Center for Mathematical Modeling
Lecture notes

Volume 5

Qualitative properties of solutions to partial differential equations

Volume edited by E. Feireisl, P. Kaplický and J. Málek
Dedicated to the memory of Professor Tetsuro Miyakawa

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VYDAVATELSTVÍ MATEMATICKO-FYZIKÁLNÍ FAKULTY
UNIVERZITY KARLOVY V PRAZE

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Qualitative properties of solutions to partial differential equations

Dedicated to the memory of Professor Tetsuro Miyakawa

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Abstract. The text provides a record of lectures given by the visitors of the Jindřich Nečas Center for Mathematical Modeling in academic years 2006-2009. The lecture notes are focused on qualitative properties of solutions to evolutionary equations.

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Preface

The aim of the present volume is to acquaint the interested reader with various qualitative properties of solutions to evolutionary equations. The topics written by leading experts in their respective fields are not necessarily related. A part of the volume consists of lecture notes of the international summer school EVEQ 2008, held in Prague, 16–20 June 2008. The contributions are presented in alphabetical order according to the name of the first author.

The article by *Dorin Bucur* documents a series of lectures delivered by the author at the Nečas Center for Mathematical Modeling in 2006 and 2007. Its aim is to study the behavior of solutions to certain partial differential equations posed on domains with the rough (rapidly oscillating) boundaries. *Grzegorz Karch* in his EVEQ lecture addresses a new topic, namely evolutionary equations with anomalous diffusion. The contribution of *Roger Lewandowski* is devoted to problems related to turbulence associated with fluid motions. The paper of *Andro Mikelić* is closely related to that of Dorin Bucur. It addresses the problem of effective boundary conditions on domains with rough boundaries. The final contribution to the volume is written by another EVEQ lecturer *Paolo Secchi* and his collaborators *Alessandro Morando* and *Paola Trebeschi*. Here, they present general results concerning regularity of solutions to hyperbolic systems with characteristic boundary.

We firmly believe that the fascinating variety of rather different topics covered in this volume will contribute to inspiring and motivating research studies in the future.

This volume is dedicated to the memory of *Tetsuro Miyakawa*. He visited the Nečas Center spending two months in Prague in fall 2008 as a senior lecturer. He gave a series of lectures on "On the existence and asymptotic behavior of dissipative 2D quasi-geostrophic flows" and we felt that the extended form of his lecture notes should be included in this volume. However, his sudden death makes this impossible.

We are very thankful to *Yoshiyuki Kagei* for a commemorative note with the complete list of research papers of Tetsuro Miyakawa.

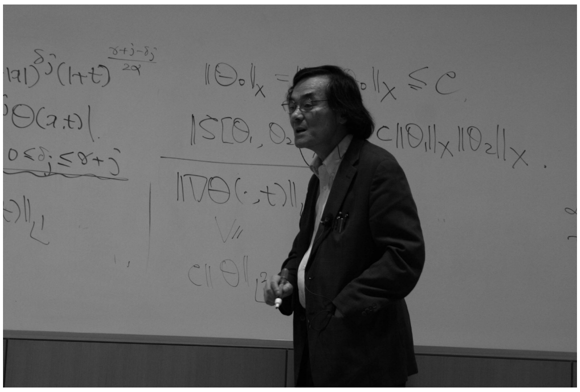
Prague, 30 June 2009
Eduard Feireisl
Petr Kaplický
Josef Málek

Tetsuro Miyakawa (1948–2009)

Tetsuro Miyakawa was born on March 10, 1948, in a small city in the middle-north part of Japan. He suddenly passed away on February 11, 2009. He made major contributions to the field of mathematical analysis of the incompressible Navier–Stokes equation. He analyzed this equation by his sophisticated technique with great insight and established significant results. He developed an L^p semigroup approach to the Navier-Stokes equation, which has become a fundamental framework in the analysis of this field. He introduced various function spaces suited to the analysis of the Navier–Stokes equation. One of his main contributions is found in the theory of weak solutions of the Navier–Stokes flows in exterior domains to which he devoted much of his energies in the prime of his life. His recent works concern with space-time asymptotic behavior of solutions in unbounded domains. It seems to me that his last interest was still in the decay properties of weak solutions in exterior domains.

He was very kind, especially to young people.

Fukuoka, 30 June 2009
Yoshiyuki Kagei



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Contents

Part 1

The rugosity effect

Dorin Bucur

2000 *Mathematics Subject Classification.* 35B40, 49Q10

Key words and phrases. geometric perturbation, partial differential equations, boundary behaviour, rough domains

Abstract. This paper surveys the series of lectures given by the author at the Nečas Center for Mathematical Modelling in 2006 and 2007. The main purpose is the study of the boundary behaviour of solutions of some partial differential equations in domains with rough boundaries. Several classical examples are recalled: the strange term "coming from somewhere else" of Cioranescu–Murat, Babuška's paradox, the Courant–Hilbert example and the rugosity effect in fluid dynamics. Some classical and recent results on the shape stability of partial differential equations with Dirichlet boundary conditions are presented. In particular we describe different ways to deal with the rugosity effect in fluid dynamics or contact mechanics.

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Contents

CHAPTER 1

Some classical examples

1. Introduction

The behaviour of the solutions of partial differential equations or the spectrum of some differential operators as a consequence of geometric domains perturbations is a classical question which has both theoretical and numerical issues. It is natural to expect that if Ω_ε is a "nice" perturbation of a smooth open set Ω, then the solution of some partial differential equation defined on Ω_ε converges to the solution of the same equation on Ω. While this is indeed a reasonable guess corresponding to the reality, there are many "simple" situations where dramatic changes can be produced by "small" geometric perturbations.

We recall some classical examples of such geometric perturbations and give the main tools for handling the particular case of Dirichlet boundary conditions and of the rugosity effect. We underline the fact that the Dirichlet boundary conditions are much easier to deal with than Neumann or Robin boundary conditions (see [7, 20, 25]). The rugosity effect can be seen as sort of effect of *partial* Dirichlet boundary conditions for vector valued solutions, which interact with the geometric perturbation.

In the sequel, we show how *small* geometric perturbations can produce huge effects on the solution of the partial differential equations, or on the spectrum of some differential operators. The word *small* is not clear and may have significantly different interpretations. Overall, the perturbations are certainly small in terms of Lebesgue measure but they have also some other features which at a first sight may lead to the false intuition that the perturbations would leave the behaviour of the partial differential equation unchanged.

2. The example of Cioranescu and Murat: a strange term coming from somewhere else

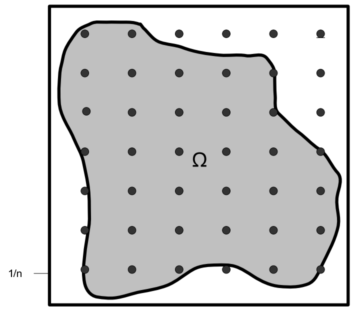
We consider an open set Ω contained in the unit square S in $\mathbb{R}^2$ and $f \in L^2(S)$. For every $n \in \mathbb{N}$ we introduce

$$C_n = \bigcup_{i,j=0}^{n} \overline{B}_{(i/n,j/n),r_n}, \qquad \Omega_n = \Omega \setminus C_n,$$

where $r_n = e^{-cn^2}$, $c > 0$ being a fixed positive constant.

If we denote by u_n the weak solution of

$$\begin{cases} -\Delta u_n = f \text{ in } \Omega_n \\ u_n \in H_0^1(\Omega_n). \end{cases} \tag{1.1}$$



$r_n = e^{-cn^2}$

one can prove that $u_n \rightharpoonup u$ weakly in $H_0^1(S)$, where u solves

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega \\ u \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

We refer the reader to [14] for a detailed proof of the passage to the limit as $n \to \infty$. The proof is elementary and comes from a direct computation as follows: one introduces the functions $z_n \in H^1(S)$:

$$z_n = \begin{cases} 0 & \text{on } C_n \\ \dfrac{\ln\sqrt{(x-i/n)^2 + (y-j/n)^2} + cn^2}{cn^2 - \ln(2n)} & \text{on } \overline{B}_{(i/n,j/n),1/2n} \setminus C_n \\ 1 & \text{on } S \setminus \bigcup_{i,j=0}^n \overline{B}_{(i/n,j/n),1/2n}. \end{cases}$$

Then, for every $\varphi \in C_0^\infty(\Omega)$, one can take $z_n\varphi$ as test function in equation (1.1). The passage to the limit for $n \to \infty$ can be performed completely to arrive to the weak form of (1.2).

The explanation of the fact that a union of small perforations of measure less than $\pi n^2 e^{-2cn^2}$ rapidly converging to zero can produce a huge effect on the equation can be completely understood in terms of Γ-convergence (see [19]). The effect is observed by the presence of the "strange term" cu in the limit equation. For a complete description of this phenomenon in relationship with optimal design problems we refer to the recent book [7].

3. Babuška's paradox

We consider the sequence $(P_n)_n$ of regular polygons with n edges, inscribed in the unit circle in $\mathbb{R}^2$. As $n \to \infty$, it is reasonable to expect that the solutions of (some) partial differential equations set on P_n would converge to the solution on the disc. This is indeed the case for some partial differential equations of second order, like the Laplace equation with homogeneous Dirichlet or Neumann boundary conditions (with a fixed admissible right hand side, see [7]).

Nevertheless, as Babuška noticed (see [2] and also [29]) this is not anymore the case for a fourth order equation of bi-laplacian type as equilibrium problems in the bending of simply supported Kirchhoff-Love plates (see for a detailed explanation [2] and also [29], [21]).

Precisely, we consider the constant force $f = 1$ and $0 \leq \sigma < \frac{1}{2}$. For every bounded Lipschitz open set $\Omega \subseteq \mathbb{R}^2$, the solution of the following minimization problem:

$$\min\{u \in H^2(\Omega) \cap H_0^1(\Omega) : \int_\Omega \frac{1}{2}|\Delta u|^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - u dx\}$$

is denoted u_Ω. Then u_Ω is a formal weak solution of the following partial differential equation

$$\begin{cases} \Delta^2 u = 1 \text{ in } \Omega \\ u = \Delta u - (1-\sigma)k\frac{u}{n} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

k being the curvature of the boundary.

It turns out that if Ω has a polygonal shape, as P_n does, then the term

$$\int_\Omega (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})dx$$

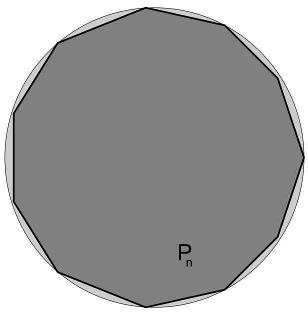
vanishes identically in the energy functional above (see [24, Lemma 2.2.2]). So that, the solution u_{P_n} is also solution of the minimization problem

$$\min\{u \in H^2(P_n) \cap H_0^1(P_n) : \int_{P_n} \frac{1}{2}|\Delta u|^2 - u dx\},$$

and formal weak solution of

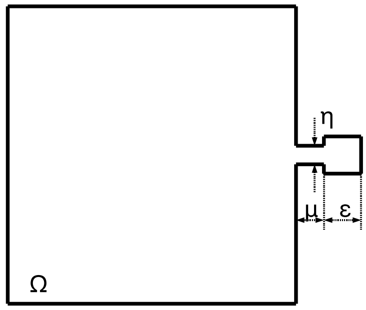
$$\begin{cases} \Delta^2 u = 1 \text{ in } P_n \\ u = \Delta u = 0 \text{ on } \partial P_n \end{cases} \tag{1.4}$$

When $n \to \infty$, one can notice that u_{P_n} converges in L^∞ to the solution of (1.4) on



The sequence $(P_n)_n$ of regular polygons "converges" to the disc

the disc, which is different from the solution of (1.3) on the disc. This means, that the approximation of the disc by the sequence of regular polygons $(P_n)_n$ for equation (1.3) does not hold! The implications of this non-stability result for equation (1.3) in numerical analysis are obvious.



The parameters ε, η, μ vanish with different speeds

4. The Courant–Hilbert example for the Neumann–Laplacian spectrum

One considers the Neumann–Laplacian eigenvalues associated to the following Lipschitz domain, which depends on the small parameters $\varepsilon, \eta, \mu > 0$. Precisely, the values of η, μ will be chosen dependently on ε. By abuse of notation, let us denote Ω_ε the perturbed domain and by Ω the limit square.

Since Ω_ε is Lipschitz, the spectrum of the Neumann Laplacian consists only of eigenvalues satisfying formally

$$\begin{cases} -\Delta u = \lambda_k(\Omega_\varepsilon) u \text{ in } \Omega_\varepsilon \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{1.5}$$

for some function $u \in H^1(\Omega)$, $u \not\equiv 0$. The eigenvalues can be ordered, counting their multiplicities

$$0 = \lambda_1(\Omega_\varepsilon) < \lambda_2(\Omega_\varepsilon) \leq ...$$

Using the continuous dependence of the eigenvalues for *smooth* domain perturbations (see [7, 15]) or, alternatively, the definition of the eigenvalues with the Rayleigh quotient, for every $c \in (0, \lambda_2(\Omega))$ and ε small enough, one can choose $\mu = \varepsilon$ and $\eta \in (0, \varepsilon)$ such that $\lambda_2(\Omega_\varepsilon) = c$.

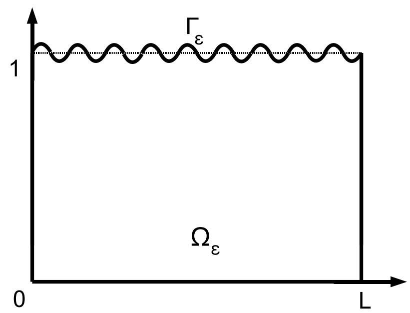
Consequently, when $\varepsilon \to 0$, the first nonzero eigenvalue of the Neumann-Laplacian on Ω_ε will converge to c, which is different from the first nonzero eigenvalue associated to Ω. The conclusion is that a "small" geometric perturbation of the square Ω leads to an uncontrollable behaviour of the Neumann-Laplacian spectrum (see [7] for details).

5. The rugosity effect

For simplicity, the Stokes equation with perfect slip boundary conditions (on a piece of the boundary) is considered in the 2D-rectangle $\Omega = (0, L) \times (0, 1)$. Roughly speaking, the rugosity effect is the following: a geometric perturbation of the boundary at a microscopic scale may transform perfect slip boundary conditions in total adherence. We refer the reader to [13] for a description of this phenomenon if the perturbation of the boundary has a periodic structure:

$$\Gamma_\varepsilon = \{(x, 1 + \varepsilon\varphi(\frac{x}{\varepsilon})) : x \in (0, L)\},$$

where $\varphi \in C^2[0, L]$, $\varphi(0) = \varphi(L)$, is extended by periodicity on $\mathbb{R}$.



Example of periodic rugosity. The amplitude ε of the perturbation vanishes.

This phenomenon occurs (in 2D) as soon as *some* rugosity is present (i.e. $\nabla\varphi \not\equiv 0$) in particular the boundary Γ_ε is not flat. This means for the periodic case above that $\varphi \not\equiv \varphi(0)$! It is a consequence of the oscillating normal in relationship with the non-penetration condition satisfied by the solutions $u_\varepsilon \cdot n_\varepsilon = 0$ on Γ_ε, where n_ε is the normal vector on the oscillating boundary.

Recent results in [9, 10, 11] give more hints on how arbitrary rugosity acts on the solution of a Stokes (or Navier-Stokes) equation, precisely by "driving" the flow on the boundary and by introducing some friction matrix.

In the next chapter we give some explanations of the rugosity effect, from the variational point of view. In particular one may use the results on the geometric perturbations for scalar elliptic equations with Dirichlet boundary conditions, since the perfect slip boundary conditions for vector valued PDEs can be seen as sort of partial Dirichlet boundary conditions for vector PDEs.

The influence of the rugosity in the presence of complete adherence is a different problem, and we refer the reader to [26]. In this case, the complete adherence is preserved in the limit, the challenge being to find better approximations of the solutions associated to the rough boundaries in a smooth domain where the complete adherence is replaced by a wall law (see also [4]).

CHAPTER 2

Variational analysis of the rugosity effect

1. Scalar elliptic equations with Dirichlet boundary conditions

Let $D \subseteq \mathbb{R}^N$ be a bounded open set, $f \in H^{-1}(D)$ (one can consider $f \in L^2(D)$ for simplicity) and Ω_ε be a *geometrical* perturbation of $\Omega \subseteq D$. We consider the Dirichlet problem for the Laplacian on the moving domain

$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } \Omega_\varepsilon \\ u_\varepsilon \in H^1_0(\Omega_\varepsilon). \end{cases} \tag{2.1}$$

The question we deal with is whether the convergence $u_\varepsilon \to u$ holds, and *in which norm*?

The following abstract result can be found in [7]. It gives a first elementary approach to study whether or not the solution of the Dirichlet problem (2.1) is stable for an arbitrary geometric perturbation. The main drawback of this (abstract) result is that for particular geometric perturbations of non-smooth sets it does not give a clear answer whether or not the solution is stable.

THEOREM 2.1. *Assertions (1) to (4) below are equivalent:*

(1) For every $f \in H^{-1}(D)$, $u_\varepsilon \to u$ in $H^1_0(D)$-strong;
(2) For $f = 1$, $u_\varepsilon \to u$ in $H^1_0(D)$-strong;
(3) $H^1_0(\Omega_\varepsilon)$ converges in the sense of Mosco to $H^1_0(\Omega)$, i.e.
 M1) *For all $\phi \in H^1_0(\Omega)$ there exists a sequence $\phi_\varepsilon \in H^1_0(\Omega_\varepsilon)$ such that ϕ_ε converges strongly in $H^1_0(D)$ to ϕ.*
 M2) *For every sequence $\phi_{\varepsilon_k} \in H^1_0(\Omega_{\varepsilon_k})$ weakly convergent in $H^1_0(D)$ to a function ϕ, $\phi \in H^1_0(\Omega)$.*
(4) If $F_\varepsilon : L^2(D) \to \mathbb{R} \cup \{+\infty\}$,

$$F_\varepsilon(u) = \begin{cases} \int_D |\nabla u|^2 dx & \text{if } u \in H^1_0(\Omega_\varepsilon) \\ +\infty & \text{otherwise} \end{cases}$$

then F_ε Γ-converges in $L^2(D)$ to F, i.e.

- *$\forall \phi_\varepsilon \to \phi$ in $L^2(D)$ then*

$$F(\phi) \le \liminf_{\varepsilon \to 0} F_\varepsilon(\phi_\varepsilon)$$

- *$\forall \phi \in L^2(D)$ there exists $\phi_\varepsilon \to \phi$ in $L^2(D)$ s.t.*

$$F(\phi) \ge \limsup_{\varepsilon \to 0} F_\varepsilon(\phi_\varepsilon)$$

REMARK 2.2. *From the previous theorem, it appears clearly that the solution of the equations with the right hand side equal to* 1 *plays a crucial role. For simplicity, let us denote w_ε the solutions for $f \equiv 1$. Assume now that $(\Omega_\varepsilon)_\varepsilon$ is a sequence of* *arbitrary open subsets of D and that for some $f \in H^{-1}(D)$ $u_\varepsilon \rightharpoonup u$ and $w_\varepsilon \rightharpoonup w$ weakly in $H_0^1(D)$. Here the limit set Ω is not given, so we wonder whether u and w are solutions on some set Ω? If such set exists, its identification would not be complicated since by the maximum principle one should have $\Omega = \{x : w(x) > 0\}$. This set may be quasi-open, in general.*

In practice, from the example of Cioranescu and Murat, one can notice that the set Ω may not exists because of the new term which appears: the strange term. In fact, one can formalise the emerging of this strange term (which in general will be a positive Borel measure, maybe infinite but absolute continuous with respect to capacity), and give a full interpretation through Γ-convergence arguments.

Let $\varphi \in C_0^\infty(D)$ and take $w_\varepsilon\varphi$ as test function in (2.1) on Ω_ε. Then (we integrate over D for simplicity)

$$\begin{aligned}\int_D f w_\varepsilon \varphi dx &= \int_D \nabla u_\varepsilon \nabla (w_\varepsilon \varphi) dx \\ &= \int_D \nabla u_\varepsilon \nabla \varphi w_\varepsilon dx + \int_D \nabla u_\varepsilon \nabla w_\varepsilon \varphi dx \\ &= \int_D \nabla u_\varepsilon \nabla \varphi w_\varepsilon dx - \int_D u_\varepsilon \nabla w_\varepsilon \nabla \varphi dx - \langle \Delta w_\varepsilon, \varphi u_\varepsilon \rangle_{H^{-1}\times H_0^1} \\ &= \int_D \nabla u_\varepsilon \nabla \varphi w_\varepsilon dx - \int_D u_\varepsilon \nabla w_\varepsilon \nabla \varphi dx + \int_D u_\varepsilon \varphi dx.\end{aligned}$$

Let $\varepsilon \to 0$ and use

$$-\int_D u \nabla w \nabla \varphi dx = \int_D \nabla u \nabla w \varphi dx + \langle \Delta w, u\varphi \rangle_{H^{-1}(D)\times H_0^1(D)}.$$

Consequently,

$$\int_D \nabla u \nabla (\varphi w) dx + \langle \Delta w + 1, u\varphi \rangle_{H^{-1}\times H_0^1} = \int_D f \varphi w dx. \tag{2.2}$$

But $\nu = \Delta w + 1 \geq 0$ in $\mathcal{D}'(D)$ is a non-negative Radon measure belonging to $H^{-1}(D)$. In fact, the positivity can be easily proven for smooth sets, and then use the weak convergence in $H^{-1}(D)$: $\Delta w_\varepsilon + 1 \rightharpoonup \Delta w + 1$.

We formally write

$$\int_D \nabla u \nabla (\varphi w) dx + \int_D u \varphi w d\mu = \int_D f \varphi w dx, \tag{2.3}$$

where μ is the Borel measure defined by

$$\mu(B) = \begin{cases} +\infty & \text{if } cap(B \cap \{w = 0\}) > 0 \\ \displaystyle\int_B \frac{1}{w} d\nu & \text{if } cap(B \cap \{w = 0\}) = 0. \end{cases} \tag{2.4}$$

Using the density of $\{w\varphi : \varphi \in C_0^\infty(D)\}$ in $H_0^1(D) \cap L^2(D,\mu)$, it turns out that u solves in a weak sense the following problem

$$\begin{cases} -\Delta u + u\mu = f \text{ in } D \\ u \in H_0^1(D) \cap L^2(D,\mu). \end{cases} \tag{2.5}$$

i.e.

$$\forall \varphi \in H_0^1(D) \cap L^2(D,\mu) \quad \int_D \nabla u \nabla \varphi dx + \int_D u \varphi d\mu = \int_D f \varphi dx.$$

In the case of the example of Cioranescu-Murat, the measure μ *equals* $c\mathcal{L}\lfloor_\Omega$ *and* $+\infty$ *on* $S \setminus \Omega$*, where* $\mathcal{L}$ *is the Lebesgue measure.*

This phenomenon, called relaxation, plays a crucial role in optimal design problems (see [7]). It can be formalised as follows, in terms of Γ*-convergence of the energy functionals (point (4) in Theorem 2.1).*

THEOREM 2.3. *Let* $(\Omega_\varepsilon)_\varepsilon$ *be an arbitrary sequence of open subsets of* D*. There exists a sub-sequence (still denoted using the same index) and a functional* $F : L^2(D) \to \mathbb{R} \cup \{+\infty\}$ *such that* F_ε Γ*-converges in* $L^2(D)$ *to* F*. Moreover,* F *can be represented as*

$$F(u) = \int_D |\nabla u|^2 dx + \int_D u^2 d\mu$$

where μ *is a positive Borel measure, absolutely continuous with respect to capacity.*

REMARK 2.4. *A way to prove this theorem (see Theorem 2.12 in the next paragraph for the vector case), is to prove in a first step the compactness result (which is of topological nature) and in a second step to use representation theorems in order to find the form of the* Γ*-limit functional.*

REMARK 2.5. *The measure* μ *above, is precisely the measure computed with the help of the solutions* w_ε *for the right hand side* $f \equiv 1$*.*

It is quite easy to notice that for every $f \in H^{-1}(D)$ *we have that*

$$F_\varepsilon(\cdot) - 2\langle f, \cdot\rangle_{H^{-1}(D)\times H_0^1(D)}$$

Γ*-converges to*

$$F(\cdot) - 2\langle f, \cdot\rangle_{H^{-1}(D)\times H_0^1(D)}.$$

As the Γ*-convergence implies the convergence of the minimizers of the functionals, one gets the strong convergence* $L^2(D)$ *(and weak* $H_0^1(D)$*) of* u_ε *to the solution of* (2.5) *for every admissible right hand side* f*. Notice the very important fact, that the measure* μ *is independent on* f*, being only an effect of the geometric perturbation.*

REMARK 2.6. *When the measure is known? The measure can be computed explicitly for very few geometric perturbations, often with periodic character. There are formulas giving in general the value of the measure in terms of the limits of local capacities of* $\Omega_\varepsilon^c \cap B$ *for a well chosen family of balls [16, 17]).*

REMARK 2.7. *Also notice, that for some particular geometric perturbations, e.g. when one of the assertions of Theorem 2.1 holds, the relaxation process does not occur, and so the measure* μ *coming from Theorem 2.3 corresponds to a (quasi)-open set* Ω*, i.e.* $\mu(A) = 0$ *if* $\operatorname{cap}(A \cap \Omega^c) = 0$ *and* $\mu(A) = +\infty$ *if* $\operatorname{cap}(A \cap \Omega^c) > 0$*.*

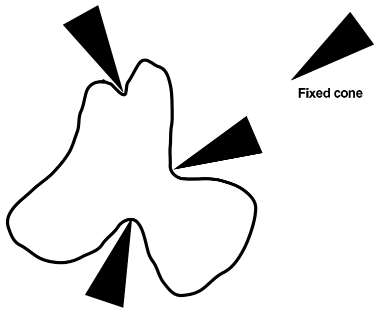
REMARK 2.8. *When classical stability holds? That means that in the limit no relaxation occurs and the (quasi)-open set* Ω *can be identified. Below are some situations when the geometric limit is identified.*

- *Increasing sequences of domains: this case is very easy, the geometric limit is the union of the open sets (direct use of Theorem 2.1).*
- *Decreasing sequences of domains: this problem is not so simple. Yet, what is the limit domain? The intersection of a decreasing sequence of open sets is not, in general, an open set. One may suspect that the interior of the intersection is the right limit, but the answer is not always affirmative.*

Keldysh gave the answer to this problem in 1962 [27], and introduced a new regularity concept, called stability (see [7] for an interpretation through Γ*-convergence).*

- *Perturbations satisfying some geometric constraints: if the domains satisfy a uniform geometric constraint forcing the boundary to avoid oscillations, or new holes to appear, than no relaxation occurs, and the limit set* Ω *can be identified by some geometric convergence, precisely in the Hausdorff complementary topology (see [7]).*

  Here is an example of a domain satisfying a pointwise cone condition: there exists a non trivial cone C *(of dimension* N *or* $N-1$ *) such that for every point* $x \in \partial\Omega_\varepsilon$ *there exists a cone congruent to* C *with vertex at* x *and lying in* Ω_ε^c*. If every* Ω_ε *satisfy this condition with the cone* C*, then no relaxation occurs, and the geometric limit can be identified. In* $\mathbb{R}^2$ *a* $2D$ *cone is a triangle and a* $1D$ *cone is a segment. This condition is related to a uniformity property of the Wiener criterion (see Theorem 2.9 below).*



Pointwise cone condition

- *Perturbations satisfying some topological constraints: in two dimensions of the space provided the number of the connected components of the complements* Ω_ε^c *is uniformly bounded (roughly speaking there is a uniformly bounded number of holes) the relaxation process does not hold and the limit can be identified in the Hausdorff complementary topology. This result is due to Šverák [28] and opened the way of intensive use of potential theory in understanding the behaviour of the solutions* u_ε *near the oscillating boundaries. In fact, in any other dimension of the space the topological constraint is not relevant. The "equivalent" constraint is a density property in terms of capacity (see [7]).*

The use of capacity estimates in terms of the Wiener criterion allows us to handle the local oscillations of the solutions (see [23], [7]). For the convenience of the reader we recall the definition of the capacity: let $E \subseteq D$ be two sets in R^N, such that D is open. The capacity of E in D is

$$\text{cap}(E, D) = \inf\{\int_D |\nabla u|^2 + |u|^2 dx, \quad u \in \mathcal{U}_{E,D}\}$$

where $\mathcal{U}_{E,D}$ stand for the class of all functions $u \in H^1_0(D)$ such that $u \geq 1$ a.e. in an open set containing E.

We recall the following result from [12] (see also [7]).

THEOREM 2.9. *Assume that Ω_ε converges in the Hausdorff complementary topology to some open set Ω and that there exists a function $g : (0,1] \times (0,1] \to (0,+\infty)$ such that*

$$\lim_{r\to 0} g(r,R) = +\infty$$

and for every $\varepsilon > 0, x \in \partial\Omega_\varepsilon, 0 < r < R < 1$ we have

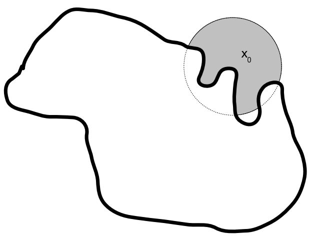
$$\int_r^R \frac{\mathrm{cap}(\Omega\varepsilon^c \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})} \frac{dt}{t} \geq g(r,R).$$

Then $u_\varepsilon \to u$ in $H^1_0(D)$.

REMARK 2.10. *Notice that this theorem involves a quantitative estimate of the complement of Ω^ε near the boundary and not its smoothness. A particular situation when this theorem can be applied, is the so called capacity density condition. For some positive constant c and for $t \in (0,r)$ independent on ε, the stronger estimate*

$$\frac{\mathrm{cap}(\Omega^c_\varepsilon \cap B_{x,t}, B_{x,2t})}{\mathrm{cap}(B_{x,t}, B_{x,2t})} \geq c$$

holds for every $x \in \partial\Omega_\varepsilon$.



The uniform minoration of the local capacity of the complement

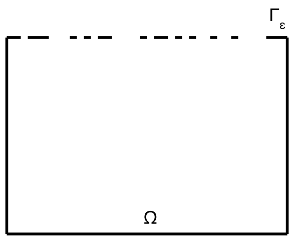
If ω is a smooth open subset of an $(N-1)$ dimensional manifold, such that $\{0\} \subseteq \omega \subseteq B(0,\frac{1}{2})$ and $F = \cup_{\alpha\in\mathbb{Z}^N} T_\alpha(\omega)$, then all the sets $(\varepsilon F)_\varepsilon$ satisfy uniformly a capacity density condition. Here $T_\alpha(\omega)$ is the translation of ω by the vector α.

REMARK 2.11. *Recent advances on the stability question involve convergence of solutions in L^∞. Indeed, for right hand sides $f \in L^{\frac{N}{2}+\varepsilon}(D)$, the solutions u_ε belong to $L^\infty(\Omega_\varepsilon)$ so that a natural question is to seek if u_ε converges to u in $L^\infty(D)$. This problem is not anymore of variational type and relies on the study of the oscillations near the boundaries related to some geometric information. A characterization of the stability is given in [6]. We refer the reader to [1, 3, 20] for more results concerning this question.*

2. The rugosity effect in fluid dynamics

2.1. The vector case: in a scalar setting... The rugosity effect can be seen as the influence of partial Dirichlet boundary conditions on the behaviour of the solutions of vector valued PDEs. In order to make the relationship with the scalar case, we give below an example of scalar equation with partial Dirichlet boundary conditions. Here the word "partial" is understood in a geometric sense: there are small regions with perfect support of a membrane (homogeneous Dirichlet boundary conditions) and small regions with free membrane boundary conditions.

We consider a rectangle $\Omega \subseteq \mathbb{R}^2$, $f \in L^2(\Omega)$ and a sequence of closed sets $\Gamma_\varepsilon \subseteq \partial\Omega$ (for example located on the upper edge Γ of Ω). We consider the Laplace equation with mixed Dirichlet and Neumann homogeneous boundary conditions.



$$\begin{cases} -\Delta u_\varepsilon = f \text{ in } \Omega \\ u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \\ \frac{\partial u_\varepsilon}{\partial n} = 0 \text{ on } \Gamma \setminus \Gamma_\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\Omega \setminus \Gamma \end{cases} \tag{2.6}$$

When $\varepsilon \to 0$, for a subsequence one has $u_\varepsilon \to u$ weakly in $H^1(D)$ and the limit u solves the same equation on Ω but with Robin boundary conditions on the upper edge! There exists a positive measure μ such that u solves in a weak sense

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \mu u = 0 \text{ on } \Gamma \\ u = 0 \text{ on } \partial\Omega \setminus \Gamma \end{cases} \tag{2.7}$$

This result fits precisely into the theory of the first section of this chapter. Indeed, one can formally reflect Ω and u_ε with respect to Γ, in Ω^r and u^r, respectively and obtain that u_ε together with its reflection, is solution of the Laplace equation with Dirichlet boundary conditions on $\Gamma_\varepsilon \cup \partial(\Omega \cup \Omega^r)$ in $\Omega \cup \Omega^r$. In this way, the Neumann b.c. can be ignored and all results of the previous section apply, thus the presence of the measure μ in the limit process.

2.2. The Stokes equation. For simplicity, we consider the following situation $\Omega = (0,1)^N \subseteq \mathbb{R}^N$, $N \geq 2$. Let us denote $T = (0,1)^{N-1}$ and a sequence of functions $\varphi_\varepsilon : T \to \mathbb{R}$ such that $\varphi_\varepsilon \in W^{1,\infty}(T)$, $\|\varphi_\varepsilon\|_\infty \leq \varepsilon$ and $\|\nabla\varphi_\varepsilon\|_\infty \leq M$, for some $M > 0$ independent on ε. If $x = (x_1, .., x_N) \in R^N$, by $\hat{x}$ we denote $\hat{x} = (x_1, .., x_{N-1})$. Then , we introduce the perturbed domains

$$\Omega_\varepsilon = \{x \in \mathbb{R}^N : \hat{x} \in T, 0 < x_N < 1 + \varphi_\varepsilon(\hat{x})\},$$

And denote $\Gamma_\varepsilon = \{x \in \mathbb{R}^N : \hat{x} \in T, x_N = 1 + \varphi_\varepsilon(\hat{x})\}$.

Let $f \in L^2_{loc}(\mathbb{R}^N)$. We consider the Stokes equation on Ω_ε with perfect slip boundary conditions on Γ_ε and total adherence boundary conditions on $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$.

$$\begin{cases} -\text{div } \mathbf{D}[\mathbf{u}_\varepsilon] + \nabla p_\varepsilon &= \mathbf{f} \text{ in } \Omega_\varepsilon \\ \text{div } \mathbf{u}_\varepsilon &= 0 \text{ in } \Omega_\varepsilon \\ \mathbf{u}_\varepsilon \cdot \mathbf{n}_\varepsilon &= 0 \text{ on } \Gamma_\varepsilon \\ (\mathbf{D}[\mathbf{u}_\varepsilon] \cdot \mathbf{n}_\varepsilon)_{tan} &= 0 \text{ on } \Gamma_\varepsilon \\ \mathbf{u}_\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \end{cases} \tag{2.8}$$

It is easy to notice that the solutions $\mathbf{u}_\varepsilon \in H^1(\Omega_\varepsilon)$ are uniformly bounded, as a consequence of the uniform Korn inequality in the equi-Lipschitz domains Ω_ε. For a subsequence (still denoted using the same index) we have that

$$1_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \overset{L^2(\mathbb{R}^n)}{\rightarrow} 1_\Omega \mathbf{u}, \tag{2.9}$$

and

$$1_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \overset{L^2(\mathbb{R}^n)}{\rightharpoonup} 1_\Omega \nabla \mathbf{u}. \tag{2.10}$$

The question is: *what is the equation satisfied by* $\mathbf{u}$?

It is not complicated to observe that $\mathbf{u}$ satisfies in a weak sense the equation (by multiplication with test functions with free divergence in $H^1_0(\Omega)$)

$$-\text{div } \mathbf{D}[\mathbf{u}] + \nabla p = \mathbf{f} \text{ in } \Omega$$

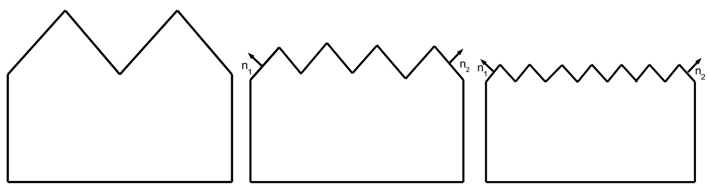
and

$$\text{div } \mathbf{u} = 0 \text{ in } \Omega,$$

in the sense of distributions. As well, on the part of $\partial\Omega$ which is not oscillating, namely $\partial\Omega \setminus \Gamma$, one gets immediately $\mathbf{u} = 0$.

Several approaches are available in the literature in order to understand the behaviour of the solution on the upper boundary.

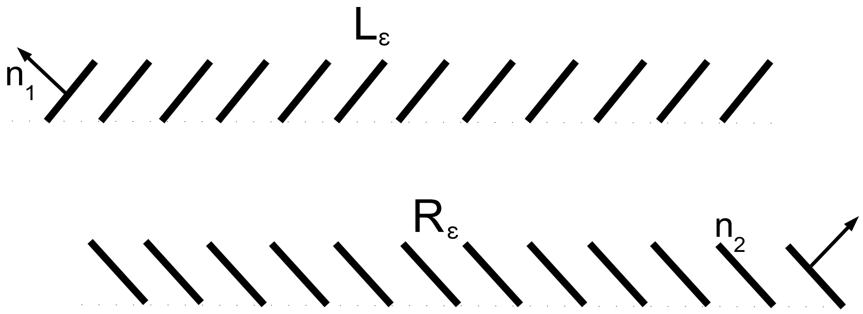
Below there is an intuitive justification of the rugosity phenomenon in $\mathbb{R}^2$. Let us consider the function $\varphi(x) = |x - \frac{1}{2}|$ defined on $[0, 1]$ and extended by periodicity on $\mathbb{R}$. Moreover, the upper boundaries Γ_ε of the two dimensional sets are given by the functions $\varphi_\varepsilon(x) = \varepsilon\varphi(\frac{x}{\varepsilon})$.



If we denote n_1 and n_2 the two normals at the boundaries, for every solution u_ε we have $u_\varepsilon \cdot n_1 = 0$ on L_ε and $u_\varepsilon \cdot n_2 = 0$ on R_ε (L_ε stands for the segments of Γ_ε which correspond to the locally increasing part of φ_ε and R_ε to the complement).

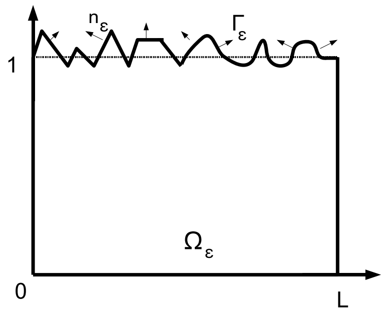
At this point, we use the vanishing information for the scalar H^1-functions $u_\varepsilon \cdot n_1$ and $u_\varepsilon \cdot n_2$. As pointed in the previous paragraph, both L_ε and R_ε satisfy a capacity density condition and converge in the Hausdorff metric to the segment $\Gamma = [0, 1] \times \{1\}$. Consequently

$$u \cdot n_1 = 0 \quad \text{and } u \cdot n_2 = 0 \quad \text{on } \Gamma.$$



As n_1 and n_2 are linearly independent, we conclude with $u = 0$ on Γ.

For general rugosity it is more difficult to follow the normals. Below we briefly describe four methods.



Example of "arbitrary" rugosity. The amplitude ε of the perturbation vanishes.

Method 1: use of Young measures. In order to handle the oscillations of the boundaries, a very efficient way to describe the limit(s) of $\nabla\phi_\varepsilon$ is the use of Young measures. We refer the reader to [22] for an introduction to Young measures. The passage to the limit of the impermeability condition $u_\varepsilon \cdot (\nabla\phi_\varepsilon, -1) = 0$ may give a substantial information provided that the support of the Young measures associated to the sequence $(\nabla\phi_\varepsilon)_\varepsilon$ is large enough. We refer the reader to [9] for a description of this method.

Here are some examples where the rugosity effect is produced under mild assumptions (see [9]).

- periodic boundaries of the form $\varphi_\varepsilon(x') = \varepsilon\varphi(\frac{x}{\varepsilon})$ for some Lipschitz function defined on $\mathcal{T}$;
- crystalline boundaries;
- riblets;
- etc.

Method 2: use of capacity estimates. This method relies on the previous paragraph on scalar functions. One may mimic the intuitive example above but, as normals vary, should work with cones of normals instead of discrete normals. For example, let us fix a vector n and denote by $C(n)$ a cone of axis n and opening ω.

Then, if for some point x we have $u_\varepsilon(x) \cdot n_\varepsilon(x) = 0$ and assume that $n_\varepsilon(x) \in C(n)$. We get

$$|u_\varepsilon(x) \cdot n| \leq |n - n_\varepsilon(x)||u_\varepsilon(x)|.$$

In order to make the idea clear, let us assume that u_ε are moreover uniformly bounded in L^∞, i.e. for some $M > 0$ and for every ε we have $|u_\varepsilon|_\infty \leq M$. Consequently, for the point x we have

$$|u_\varepsilon(x) \cdot n| \leq Mc(\omega),$$

where $c(\omega)$ depends only on the opening of the cone and vanishes for $\omega \to 0$. In particular, this means that $(|u_\varepsilon \cdot n| - Mc(\omega))^+$ vanishes at x, and in general on the region where the normals $n_\varepsilon(x)$ are defined and belong to the cone $C(n)$. Consequently, for the scalar sequence of functions $(|u_\varepsilon \cdot n| - Mc(\omega))^+$ we can fully use the scalar setting for Dirichlet Laplacian by estimating precisely in capacity the size of the region where the normals $n_\varepsilon(x)$ belong to $C(n)$. If this region satisfy a density capacity condition (which is likely to be the case for periodic boundaries and well chosen n) then in the limit we get $(|u \cdot n| - Mc(\omega))^+ = 0$ on Γ. Making $\omega \to 0$, we get $u \cdot n = 0$ on Γ.

In order to give a general frame, let us consider $V \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$. As in the scalar case, one can construct a measure supported on Γ which is associated to V and counts the energy effect of the asymptotical rugosity of $\partial\Omega_\varepsilon$ when $\varepsilon \to 0$, into the direction of the field V. The fact that the field V is fixed *a priori* allows, roughly speaking, to use the previous results for scalar functions by considering the family of scalar functions $(v_\varepsilon \cdot V)_\varepsilon$. Typically, the argument above for $V = n$ can be used. Nevertheless, in order to give a general framework and avoid unnecessary hypotheses as uniform boundedness in L^∞, one can formally consider energy functionals of the form $F_\varepsilon : L^2(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$,

$$F_\varepsilon(u) = \begin{cases} \int_{\mathbb{R}^N} |\nabla(u \cdot V)|^2 dx & \text{if } u \in H^1(\Omega_\varepsilon), u \cdot n_\varepsilon = 0 \text{ on } \Gamma_\varepsilon, u = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon \\ +\infty & \text{otherwise} \end{cases}$$

and to investigate their inferior Γ-limit.

We consider the family $\mathcal{M}_V$ of positive Borel measures, absolutely continuous with respect to the capacity, such that for every sequence $v_{\varepsilon_k} \in H^1(\Omega_{\varepsilon_k}, \mathbb{R}^N), v_{\varepsilon_k} \cdot n_{\varepsilon_k} = 0$ *on* Γ_{ε_k} $v_{\varepsilon_k} = 0$ on $\partial\Omega_{\varepsilon_k} \setminus \Gamma_{\varepsilon_k}$ and such that $v_{\varepsilon_k} \to v$ in the sense of relations (2.9)-(2.10), then

$$\int_D |\nabla(v \cdot V)|^2 dx + \int_D (v \cdot V)^2 d\mu \leq \liminf_{k \to \infty} \int_D |\nabla(v_{\varepsilon_k} \cdot V)|^2 dx.$$

The equality $v_{\varepsilon_k} \cdot n_{\varepsilon_k} = 0$ is understood pointwise where the normal exists and for a quasi continuous representative of v.

Since at least the zero measure can be considered above, $\mathcal{M}_V \neq \emptyset$ so that

$$\mu_V = \sup\{\mu : \mu \in \mathcal{M}_V\}$$

is well defined.

The measure μ_V is supported on Γ and takes into account precisely the rugosity effect on $\partial\Omega$ in the direction of the field V from an energetic point of view. If, as in the scalar case, one can prove that $\mu = \infty_\Gamma$, then we get $u \cdot V = 0$ on Γ, so that the flow is orthogonal to V on Γ. This argument works properly in several cases when computations can be carried out, e.g. the periodical case.

Method 3: uniform estimates. Let us denote $U_\varepsilon = (0,1)^{N-1} \times \{1-2\varepsilon\}$. Provided some uniformity on the rugosities φ_ε, one can prove the existence of a constant $C > 0$, independent on ε such that for every solution of the Stokes equation (2.8), we have

$$\int_{U_\varepsilon} |u_\varepsilon|^2 d\sigma \leq C\varepsilon \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.$$

Of course, if such an estimate holds and since the solutions $(u_\varepsilon)_\varepsilon$ have uniformly bounded energy, then as $\varepsilon \to 0$ one gets $u = 0$ on Γ.

We refer to [8, 13] for estimates of this kind in the periodic case, and to [5] for improvements of the periodic case, if the Lipschitz hypothesis is removed.

Method 4: representation by Γ-convergence. In order to find the general form of the limit problem, in [11] it is used an approach based on Γ-convergence.

THEOREM 2.12. *Let $\varepsilon \to 0$ and let $\mathbf{f} \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ be given. Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be the family of (weak) solutions to the Stokes equation* (2.8) *in Ω_ε.*

Then, at least for a suitable subsequence we have

$$1_{\Omega_\varepsilon}\mathbf{u}_\varepsilon \to 1_\Omega \mathbf{u} \ \textit{(strongly) in } L^2(\mathbb{R}^N, \mathbb{R}^N),$$
$$1_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon \to 1_\Omega \nabla\mathbf{u} \ \textit{weakly in } L^2(\mathbb{R}^N, \mathbb{R}^{N\times N}),$$

and there exists a suitable trio $\{\mu, A, \mathcal{V}\}$ independent of the driving force $\mathbf{f}$ *such that*

- *μ is a capacitary measure concentrated on Γ*
- *$\{\mathcal{V}\}_{x\in\Gamma}$ is a family of vector subspaces in $\mathbb{R}^{N-1}$*
- *A is a positive symmetric matrix function A defined on Γ*

and $\mathbf{u}$ *is a solution in Ω of the Stokes equation with friction-driven b.c.*

$$\left\{\begin{array}{rcl} -div\,\mathbf{D}[\mathbf{u}] + \nabla p &=& \mathbf{f} \ in\, \Omega \\ div\,\mathbf{u} &=& 0 \ in\, \Omega \\ \mathbf{u}(x) &\in& V(x) \ for\ q.e.\ x \in \Gamma \\ \Big[\mathbf{D}[\mathbf{u}]\cdot\mathbf{n} + \mu A\mathbf{u}\Big]\cdot\mathbf{v} &=& 0 \ for\ any\ \mathbf{v} \in V(x),\ x\in\Gamma \\ \mathbf{u}(x) &=& 0 \ for\ q.e.\ x \in \partial\Omega\setminus\Gamma. \end{array}\right. \tag{2.11}$$

The sense in which u solves the equation (2.11) is the following: u is solution of the minimization of

$$\mathcal{J}(\mathbf{v}) := \frac{1}{2}\int_\Omega \Big(|\mathbf{D}[\mathbf{v}]|^2 + |\mathbf{v}|^2\Big)dx + \frac{1}{2}\int_{\partial\Omega} \mathbf{v}^T A\mathbf{v}\,d\mu - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx, \tag{2.12}$$

on

$$\Big\{\mathbf{v} \in H^1(\Omega, R^N) \ \Big|\ \text{div}\,\mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}(x) \in V(x) \text{ for q. e. } x \in \Gamma, \mathbf{v} = 0 \text{ on } \partial\Omega\setminus\Gamma\Big\}.$$

PROOF. The main steps of the proof are the following:

Step 1. introduce energy functionals involving the boundary constraint: $\mathbf{u}_\varepsilon \cdot n_\varepsilon = 0$ and remove incompressibility condition;

Step 2. use representation results of the Γ-limit for vector valued functionals (see [18] and also [16, 17] for scalar or vector equations for Dirichlet boundary conditions);

Step 3. prove that the measure is concentrated on the surface;
Step 4. use a diagonal argument in order to handle the incompressibility condition.

□

This theorem gives the general form of the limit problem, but in any particular situation, specific computations should be carried out in order to identify the trio $\{\mu, A, \mathcal{V}\}$.

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Part 2

Nonlinear evolution equations with anomalous diffusion

Grzegorz Karch

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ABSTRACT. This is the review article on nonlinear pseudodifferential equations involving Lévy semigroup generators–used in physical models where the diffusive behavior is affected by hopping and trapping phenomena. In first chapter, properties of Lévy generators are discussed. Results on the large time asymptotics of solutions to the fractal Burgers equation are presented in Chapter 2. A generalization of the Kardar–Parisi–Zhang equation modeling the ballistic rain of particles onto the surface is discussed in Chapter 3. In the last chapter, some other classes of nonlinear evolution equations with Lévy operators are briefly described. These are the lectures notes presented by the author at EVEQ 2008—International Summer School on Evolution Equations Prague, Czech Republic, June 16-20, 2008.

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Contents

CHAPTER 1

Lévy operator

1. Probabilistic motivations – Wiener and Lévy processes

In 1827 the Scottish botanist Robert Brown observed that pollen grains suspended in liquid performed an irregular motion, caused by the random collisions with the molecules of the liquid, see Figure 1. The hits occur a large number of times in any small interval of time, independently of each other and the effect of a particular hit is small compared to the total effect. The physical theory of this motion (and the probabilistic derivation of the heat equation, see (1.2)) was set up by Einstein in 1905. All those facts suggest that this motion is random, and has the following properties:

(i) it has independent increments;
(ii) increments are Gaussian random variables;
(iii) the motion is continuous.

Property (i) means that the displacements of a pollen particle over disjoint time intervals are independent random variables. Property (ii) is not surprising in view of the central-limit theorem.

To describe this motion mathematically, we recall first that a random variable $X : \Omega \to \mathbb{R}$ is called Gaussian with mean m and variance σ^2 (and one uses the

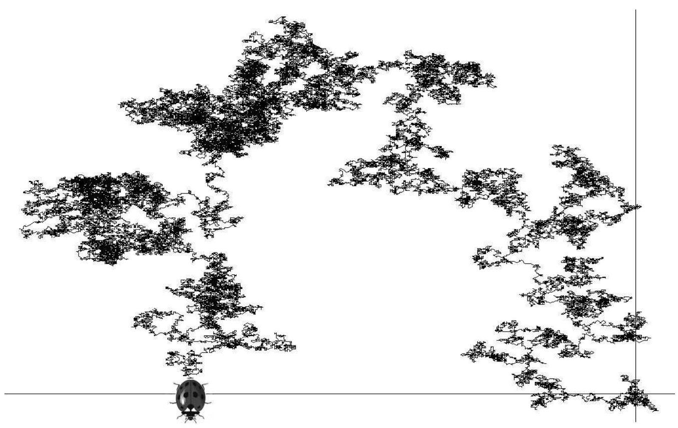

FIGURE 1. Starting at the origin trajectory of a Brownian motion.

notation $X \sim \mathcal{N}(m, \sigma^2)$) if, for every Borel set $A \subset \mathbb{R}$

$$P(\{\omega \in \Omega \,:\, X(\omega) \in A\}) = \int_A \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big)\, dx.$$

A random variable $X = (X_1, ..., X_n) : \Omega \to \mathbb{R}^n$ is called Gaussian if all linear combinations of the random variables X_k, $k = 1, ..., n$, are Gaussian.

Norbert Wiener proposed to model the Brownian motion by a continuous time *stochastic process* $\{W(t)\}_{t\geq 0}$ (see Definition 1.1, below). Here, $W(t, \omega)$ is a random variables for each $t \geq 0$ which is interpreted as the position at time t of the pollen grain ω.

DEFINITION 1.1. The stochastic process $\{W(t)\}_{t\geq 0}$ is called the Wiener process, if it fulfils the following conditions

- $W(0) = 0$ with probability equal to one: $P(\{\omega \in \Omega : W(0, \omega) = 0\}) = 1$,
- $W(t)$ has independent increments: for every sequence $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $W(t_0), W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$ are independent,
- trajectories of W are continuous with probability equal to one
- for all $0 \leq s \leq t$, we have $W_t - W_s \sim \mathcal{N}(0, t-s)$.

It is possible to prove that such processes exist and probabilists have studied systematically their properties, see the book by Revuz and Yor [55].

Now, for every $x \in \mathbb{R}^n$ and every function $u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ we define the average

$$u(x,t) = \mathbb{E}(u_0(x + W(t))) = \int_{\mathbb{R}^n} u_0(x+y)\, \mathcal{N}(0,t)(dy), \tag{1.1}$$

where "$\mathbb{E}$" denotes the mathematical expectation and

$$\mathcal{N}(0,t)(dy) = (2\pi t)^{-n/2} e^{-|x|^2/(2t)}\, dy$$

is the probability measure on $\mathbb{R}^n$ called the *centered Gaussian measure*. Here, the process $x + W(t)$ denotes the Wiener process (or Brownian motion) started at x. By a direct calculation, it is possible to check that the function $u = u(x,t)$ from (1.1) is the solution of the initial value problem for the heat equation

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u \quad \text{for} \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{1.2}$$

In other words, in (1.1), we obtained a solution of the heat equation starting a Wiener process at each point $x \in \mathbb{R}^n$ and computing the average (the mathematical expectation) of all trajectories started at x.

However, there are several examples from fluid mechanics, solid state physics, polymer chemistry, and mathematical finance leading to non-Gaussian processes where the trajectories are no longer continuous (they have jumps as shown on Figure 1). Such phenomena appear to be well modeled by Lévy processes (named after the French mathematician Paul Lévy), where the assumption on the continuity of trajectories from the definition of a Wiener process is replaced by the more general notion of continuity in probability.

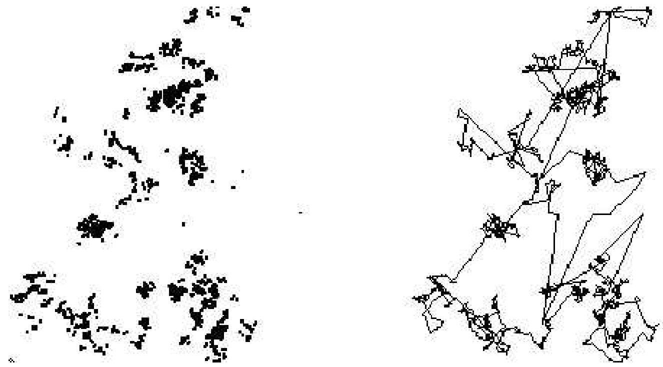

FIGURE 2. Two pictures of the same trajectory of a pure jump Lévy process. On the right hand side, points of jumps of this trajectory were connected by straight lines in order to make the motion more visible.

DEFINITION 1.2. The stochastic process $\{X(t)\}_{t\geq 0}$ on the probability space (Ω, F, P) is called the *Lévy process* if it fulfils the following conditions:

- $X(0) = 0$ with probability equal to one,
- $X(t)$ has independent increments,
- the probability distribution of $X(s+t) - X(s)$ is independent of s,
- the process $X(t)$ is continuous in probability, namely, $\lim_{s\to t} P(|X_s - X_t| > \varepsilon) = 0$.

Note that the mathematical assumption on the continuity in probability admits Lévy processes having trajectories with jumps (see Remark 1.13). We refer the reader to the review articles by Applebaum [6] and Woyczyński [59] for several applications of Lévy processes as well as to the book by Bertoin [12] for mathematical results.

Now, with a given Lévy process $X(t)$, we associate the family of probability measures μ^t on $\mathbb{R}^n$ defined by the formula

$$\int_A \mu^t(dy) \equiv P(\{\omega \in \Omega : X(t,\omega) \in A\})$$

for every Borel set $A \subset \mathbb{R}^n$. Next, similarly as in the case of a Wiener process, for every $u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we define the function

$$u(x,t) = \mathbb{E}(u_0(x + X(t))) = \int_{\mathbb{R}^n} u_0(x+y)\, \mu^t(dy) \tag{1.3}$$

where $x + X(t)$ is a Lévy process started at x.

In the next section, using purely non-probabilistic language, we shall identify the initial value problem satisfied by the function $u = u(x,t)$.

2. Convolution semigroup of measures and Lévy operator

As it was explained in the previous section, the chaotic motion described by the Wiener process or, more generally, by the Lévy process can be described (in a purely analytic way) by the family of probability measures $\{\mu^t\}_{t\geq 0}$ on $\mathbb{R}^n$ with the properties stated in the following definition.

DEFINITION 1.3. The family of nonnegative Borel measures $\{\mu^t\}_{t\geq 0}$ on $\mathbb{R}^n$ is called *the convolution semigroup* if

(1) $\mu^t(\mathbb{R}^n) = 1$ for all $t \geq 0$;
(2) $\mu^s * \mu^t = \mu^{t+s}$ for $s, t \geq 0$ and $\mu^0 = \delta_0$ (the Dirac delta)
(3) $\mu^t \to \delta_0$ vaguely as $t \to 0$, namely,

$$\int_{\mathbb{R}^n} \varphi(y)\ \mu^t(dy) \to \varphi(0) \quad \text{as} \quad t \to 0$$

for every test function $\varphi \in C_c(\mathbb{R}^n)$ (smooth and compactly supported).

Obviously, we deal with probability measures by condition (1). Item (2) is the analytic way to say that the increments of the corresponding stochastic process are independent. The continuity in probability of the process is encoded in (3).

The following theorem results directly from Definition 1.3.

THEOREM 1.4. *Let $\{\mu^t\}_{t\geq 0}$ be a convolution semigroup of measure on $\mathbb{R}^n$. There exists a function $a : \mathbb{R}^n \to \mathcal{C}$ such that the equality $\widehat{\mu}^t(\xi) = (2\pi)^{-n/2}e^{-ta(\xi)}$ holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$.*

PROOF. Recall that the Fourier transform of a measures is defined as

$$\widehat{\mu}^t(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\mu^t(dx).$$

Now, for fixed $\xi \in \mathbb{R}^n$ we consider the mapping $\phi_\xi : [0, \infty) \mapsto \mathcal{C}$ defined by

$$\phi_\xi(t) = (2\pi)^{n/2}\widehat{\mu}^t(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi}\ \mu^t(dx).$$

By condition (ii) from Definition 1.3, we obtain

$$\phi_\xi(s + t) = \phi_\xi(t)\phi_\xi(s), \tag{1.4}$$

because the Fourier transform changes a convolution into a product. Moreover, the convergence stated in (3) of Definition 1.3 implies $\lim_{t\to 0} \phi_\xi(t) = 1$.

The functional equation (1.4) has the well-known unique (continuous at zero) solution. Hence, for every $\xi \in \mathbb{R}^n$ there is a unique complex number $a(\xi)$ such that $\phi_\xi(t) = e^{-ta(\xi)}$ for all $t \geq 0$. □

DEFINITION 1.5. The function $a = a(\xi)$ obtained in Theorem 1.4 is called *the symbol* of the convolution semigroup of measures $\{\mu^t\}_{t\geq 0}$.

Now, we are in a position to define the pseudodifferential operator, which plays the main role in these lecture notes.

DEFINITION 1.6. *Lévy operator* $\mathcal{L}$ is the pseudodifferential operator with the symbol $a = a(\xi)$ corresponding to a certain convolution semigroup of measures. In other words, $\widehat{\mathcal{L}v}(\xi) = a(\xi)\widehat{v}(\xi)$.

Let us now explain the connection between the convolution semigroup and the corresponding initial value problem with Lévy operator. This is the first step toward studying evolution equations with Lévy operators.

THEOREM 1.7. *Denote by* $a = a(\xi)$ *the symbol of the convolution semigroup* $\{\mu^t\}_{t\geq 0}$ *in* $\mathbb{R}^n$. *For every sufficiently regular (bounded) function* $u_0 = u_0(x)$ *the convolution*

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)\,\mu^t(dy). \tag{1.5}$$

is the solution of the initial value problem

$$u_t = -\mathcal{L}u, \quad x \in \mathbb{R}^n, \ \ t \geq 0 \tag{1.6}$$

$$u(x,0) = u_0(x). \tag{1.7}$$

PROOF. If we compute the Fourier transform of the function in (1.5) and of the equation (1.6), we see that, for every $\xi \in \mathbb{R}^n$, the function

$$\widehat{u}(\xi,t) = (2\pi)^{-n/2} e^{-a(\xi)} \widehat{u}_0(\xi)$$

(*cf.* Theorem 1.4) is the solution of the ordinary differential equation

$$\widehat{u}_t(\xi,t) = -a(\xi)\widehat{u}(\xi,t)$$

supplemented with the initial datum $\widehat{u}_0(\xi)$. □

The initial value problem (1.6)-(1.7) describes so-called *anomalous diffusion*.

REMARK 1.8. *Notice that the convolution stated in* (1.5) *differs from the convolutions in* (1.1) *and in* (1.3). *Obviously, both expressions are equivalent because it suffices to replace any probability measure* $\mu^t(dy)$ *by* $\mu^t(-dy)$. *In this work, we prefer to use the standard notation from* (1.5).

REMARK 1.9. *Using a more sophisticated language, one can say that the operator* $-\mathcal{L}$ *generates a strongly continuous semigroup* $e^{-t\mathcal{L}}$ *of linear operators on* $L^2(\mathbb{R})$ *given by (1.5). This is the sub-Markovian semigroup, namely,*

$$0 \leq v \leq 1 \quad \textit{implies} \quad 0 \leq e^{-t\mathcal{L}} v \leq 1$$

almost everywhere (see e.g. [37, Chapter 4] for more details).

EXAMPLE 1.10. *There is the well-known connection between the Cauchy problem for the heat equation*

$$\begin{aligned} u_t &= \Delta u, \quad x \in \mathbb{R}^n, \ \ t \geq 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.8}$$

and the following convolution semigroup ("dy" means the Lebesgue measure)

$$\mu^t(dy) = (4\pi t)^{-n/2} e^{-|y|^2/(4t)}\, dy \quad \textit{for all} \quad t > 0.$$

Indeed, the solution of the initial value problem (1.8) (for not too bad initial conditions) has the form

$$u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)(4\pi t)^{-n/2} e^{-|y|^2/(4t)}\, dy.$$

In this case, we have the equality $\widehat{\mu^t}(\xi) = (2\pi)^{-n/2} e^{-t|\xi|^2}$ *from which we immediately obtain the symbol* $a(\xi) = |\xi|^2$ *of this convolution semigroup and the corresponding Lévy operator* $\mathcal{L} = -\Delta$.

EXAMPLE 1.11. *Now, let us show that, for every fixed $b \in \mathbb{R}^n$, the first order differential operator $\mathcal{L} = b \cdot \nabla$ is the Lévy operator with the symbol $a(\xi) = ib \cdot \xi$. Indeed, in this case, we should consider the initial value problem for the transport equation*

$$u_t + b \cdot \nabla u = 0, \quad u(x,0) = u_0(x) \tag{1.9}$$

with the well-known solution $u(x,t) = u_0(x - bt)$. Note that this solution takes the form from (1.5) for the convolution semigroup of measures

$$\mu^t(dx) = \delta_{tb} \quad \textit{(the Dirac delta at tb).}$$

It is possible to characterize all Lévy operators.

THEOREM 1.12 (Lévy–Khinchin formula). *Assume that $a : \mathbb{R}^n \to \mathcal{C}$ is the symbol of a certain convolution semigroup of measures on $\mathbb{R}^n$. Then there exist*

- *a constant $c \geq 0$,*
- *a vector $b \in \mathbb{R}^n$,*
- *a symmetric positive semidefinite quadratic form q on $\mathbb{R}^n$*

$$q(\xi) = \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k,$$

- *a nonnegative Borel measure Π on $\mathbb{R}^n$ satisfying $\Pi(\{0\}) = 0$ and*

$$\int_{\mathbb{R}^n} \min(1, |\eta|^2)\, \Pi(d\eta) < \infty \tag{1.10}$$

such that the following representation is valid

$$a(\xi) = ib \cdot \xi + q(\xi) + \int_{\mathbb{R}^n} \Big(1 - e^{-i\eta\xi} - i\eta\xi 1\!\!1_{\{|\eta|<1\}}(\eta)\Big)\, \Pi(d\eta). \tag{1.11}$$

Moreover, this representation is unique.

In other words, taking into account Theorem 1.4, we may reformulate the Lévy–Khinchin Theorem 1.12 as follows: the Fourier transform of any convolution semigroup $\{\mu^t\}_{t\geq 0}$ of measures on $\mathbb{R}^n$ is of the form $\widehat{\mu}^t(\xi) = (2\pi)^{-n/2}e^{-ta(\xi)}$ where the symbol $a = a(\xi)$ is given by (1.11). One should also remember the reverse implication: for every c, b, q, Π as in Theorem 1.12, the function $a = a(\xi)$ in (1.11) is the symbol of certain convolution semigroup of measures (see [37, Thm. 3.7.8]) hence the corresponding pseudodifferential operator is a Lévy operator.

Here, we skip the long proof of Theorem (1.12) and we refer the reader to [37, Ch. 3.7] for an analytic reasoning (based on properties of the Fourier transform of a measure) which leads to representation (1.11). However, to understand deeper this representation, one should look at probabilistic arguments which lead to Theorem 1.12. We sketch and discuss them in Remark 1.13, below.

Now, let us emphasize that, since every Lévy operator is defined by the Fourier transform as $\widehat{\mathcal{L}u}(\xi) = a(\xi)\widehat{u}(\xi)$, using the explicit form of the symbol $a = a(\xi)$ given in (1.11) and inverting the Fourier transform we obtain the most general form of the Lévy operator:

$$\begin{aligned}\mathcal{L}u(x) =& b\cdot\nabla u(x) - \sum_{j,k=1}^{n} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k}\\ &- \int_{\mathbb{R}^n} \big(u(x-\eta) - u(x) - \eta\cdot\nabla u(x) 1\!\!\mathrm{I}_{\{|\eta|<1\}}(\eta)\big)\, \Pi(d\eta).\end{aligned} \tag{1.12}$$

The first term on the right-hand side of (1.12) corresponds to the transport operator recalled in Example 1.11. Note that the matrix $(a_{jk})_{j,k=1}^n$ is assumed to be nonnegative-definite; if it is not degenerate, a linear change of the variables transforms the second term in (1.12) into the usual Laplacian $-\Delta$ on $\mathbb{R}^n$ which corresponds to the Brownian part of the diffusion modeled by $\mathcal{L}$. The integral on the right-hand side of (1.12) is called *the pure jump part* of the Lévy operator and *the Lévy measure* Π describes the statistical properties of jumps of the corresponding Lévy process.

Remark 1.13. *In the study of evolution equations with Lévy operator, it is useful to keep in mind probabilistic arguments which lead to the Lévy–Khinchin formula (1.12). The probabilistic proof of Theorem 1.12 consists in showing that any Lévy process* $\{X(t)\}_{t\ge 0}$ *(cf. Definition 1.2) can be expressed as the sum of three independent Lévy processes*

$$X(t) = X^{(1)}(t) + X^{(2)}(t) + X^{(3)}(t),$$

where

- $X^{(1)}$ *is a linear transform of a Brownian motion with drift*
- $X^{(2)}$ *is a compound Poisson process having only jumps of size at least* 1*,*
- $X^{(3)}$ *is a pure-jump martingale only with jumps of size less than* 1*.*

Moreover, this decomposition is unique.

Note that the process $X^{(1)}$ *has continuous trajectories almost surely, and is expressed by the first and the second term on the right-hand side of (1.12). Now, we should decompose the integral term in (1.12) into two parts: the integral describing large jumps* $|\eta|\ge 1$ *modeled by Poisson process* $X^{(2)}$ *and to the integral corresponding to the pure-jump martingale* $X^{(3)}$ *for small jumps* $|\eta|<1$*.*

Details of this proof, which can be understood by non-probabilists, can be found in the first chapter of the excellent book by J. Bertoin [12].

3. Fractional Laplacian

Let us now present the most important example of the Lévy operator which will often appear in these lectures. Choosing, in formula (1.12), $b=0$, $a_{jk}=0$ for all $j,k\in\{1,...,n\}$, and the following Lévy measure

$$\Pi(d\eta) = \frac{C(\alpha)}{|\eta|^{n+\alpha}} \quad \text{with} \quad \alpha\in(0,2) \tag{1.13}$$

and with a certain explicit constant $C=C(\alpha)>0$ we obtain the so-called α-stable anomalous diffusion operator

$$\mathcal{L} = (-\Delta)^{\alpha/2} \quad \text{with the symbol} \quad a(\xi)=|\xi|^\alpha \quad \text{for} \quad 0<\alpha\le 2. \tag{1.14}$$

Using the symmetry of the Lévy measure, we can rewrite (1.12) in this particular case as

$$(-\Delta)^{\alpha/2}u(x) = -C(\alpha)\lim_{\varepsilon\to 0}\int_{|\eta|\geq\varepsilon}\frac{u(x-\eta)-u(x)}{|\eta|^{n+\alpha}}\,d\eta. \tag{1.15}$$

Calculations based only on the properties of the Fourier transform which shows the equivalence of definitions (1.14) and (1.15) can be also found *e.g.* in [27, Thm. 1].

The corresponding convolution semigroup of measures has a density $\mu^t(dx) = p_\alpha(x,t)\,dx$ for all $t>0$, where the function $p_\alpha(x,t)$ can be computed via the Fourier transform $\widehat{p}_\alpha(\xi,t) = e^{-t|\xi|^\alpha}$ (*c.f.* Theorem 1.4). In particular,

$$p_\alpha(x,t) = t^{-n/\alpha}P_\alpha(xt^{-1/\alpha}), \tag{1.16}$$

where P_α is the inverse Fourier transform of $e^{-|\xi|^\alpha}$ (see [37, Ch. 3] for more details). It is well known that for every $\alpha\in(0,2)$ the function P_α is smooth, nonnegative, and satisfies the estimates

$$0 < P_\alpha(x) \leq C(1+|x|)^{-(\alpha+n)} \qquad \text{and} \qquad |\nabla P_\alpha(x)| \leq C(1+|x|)^{-(\alpha+n+1)} \tag{1.17}$$

for a constant C and all $x\in\mathbb{R}^n$. Moreover,

$$P_\alpha(x) = c_0|x|^{-(\alpha+n)} + O\left(|x|^{-(2\alpha+n)}\right), \qquad \text{as } |x|\to\infty, \tag{1.18}$$

and

$$\nabla P_\alpha(x) = -c_1\,x|x|^{-(\alpha+n+2)} + O\left(|x|^{-(2\alpha+n+1)}\right), \qquad \text{as } |x|\to\infty, \tag{1.19}$$

where

$$c_0 = \alpha 2^{\alpha-1}\pi^{-(n+2)/2}\sin(\alpha\pi/2)\Gamma\Big(\frac{\alpha+n}{2}\Big)\Gamma\Big(\frac{\alpha}{2}\Big),$$

and

$$c_1 = 2\pi\alpha 2^{\alpha-1}\pi^{-(n+4)/2}\sin(\alpha\pi/2)\Gamma\Big(\frac{\alpha+n+2}{2}\Big)\Gamma\Big(\frac{\alpha}{2}\Big).$$

We refer to [21] for a proof of the formula (1.18) with the explicit constant c_0. The optimality of the estimate of the lower order term in (1.18) is due Kolokoltsov [46, Eq. (2.13)], where higher order expansions of P_α are also computed. The proof of the asymptotic expression (1.19) and the value of c_1 can be deduced from (1.18) using an identity by Bogdan and Jakubowski [22, Eq. (11)].

4. Maximum principle

In this section and in the next one, we recall properties of Lévy operators which are useful in the study of nonlinear equations. We begin with the maximum principle which is well known in the case of elliptic and parabolic problems. Here, we present results for Lévy operators, but they can be formulated in a much more general case of generators of Feller semigroups, see [37, Sec. 4.5].

DEFINITION 1.14. We say that the operator $(A, D(A))$ satisfies the *positive maximum principle* if for any $\varphi\in D(A)$ the fact that $0\leq\varphi(x_0) = \sup_{x\in\mathbb{R}^n}\varphi(x)$ for some $x_0\in\mathbb{R}^n$ implies $A\varphi(x_0)\leq 0$.

REMARK 1.15. *Obviously, the operators $A\varphi = \varphi''$ and , more generally, $A\varphi = \Delta\varphi$ satisfy the positive maximum principle.*

THEOREM 1.16. *Denote by $\mathcal{L}$ the Lévy diffusion operator. Then $A = -\mathcal{L}$ satisfies the positive maximum principle.*

PROOF. We present two different arguments which are based on different properties of the Lévy operator $\mathcal{L}$. Let $\varphi \in D(\mathcal{L})$ and assume that

$$0 \leq \varphi(x_0) = \sup_{x \in \mathbb{R}^n} \varphi(x) \quad \text{for some} \quad x_0 \in \mathbb{R}^n.$$

First argument. Using the Lévy–Khinchin representation (1.12) we obtain that the following quantity

$$\begin{aligned} & -\mathcal{L}\varphi(x_0) \\ = \quad & -b \cdot \nabla\varphi(x_0) + \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 \varphi(x_0)}{\partial x_j \partial x_k} \\ & + \int_{\mathbb{R}^n} \left(\varphi(x_0 - \eta) - \varphi(x_0) - \sum_{j=1}^{n} \eta_j \frac{\partial \varphi(x_0)}{\partial x_j} 1\!\!1_{\{|\eta|<1\}}(\eta) \right) \Pi(d\eta) \end{aligned}$$

is nonpositive because the first term on the right-hand side is equal to zero since x_0 is the point of the maximum of φ, the second term is nonpositive due to the property of the matrix $\{a_{jk}\}_{j,k=1}^{n}$ (see Theorem 1.12), and the integrand of the third term is nonpositive because $\varphi(x_0 - \eta) \leq \varphi(x_0)$ for all $\eta \in \mathbb{R}^n$.

Second argument. Recall that, by Theorem 1.7, the solution of the problem

$$\begin{aligned} & u_t = -\mathcal{L}u, \quad x \in \mathbb{R}^n, \ \ t \geq 0, \\ & u(x, 0) = \varphi(x) \end{aligned}$$

is given by

$$u(x, t) = \int_{\mathbb{R}^n} \varphi(x - y) \mu^t(dy).$$

Hence, by the definition of the derivative ∂_t, we have

$$-\mathcal{L}\varphi(x_0) = \lim_{t \to 0^+} \frac{u(x_0, t) - \varphi(x_0)}{t}.$$

Now, the right-hand side is nonpositive for any $t > 0$ because $\int_{\mathbb{R}^n} \mu^t(dy) = 1$ and because

$$u(x_0, t) - \varphi(x_0) = \int_{\mathbb{R}^n} \Big(\varphi(x_0 - y) - \varphi(x_0) \Big) \mu^t(dy) \leq 0$$

by the definition of x_0 and since the measures μ^t are nonnegative. □

Next, we prove an analogous result for bounded functions which not necessarily attain their points of global maximum. Here, we follow an argument from [27, Thm. 2].

LEMMA 1.17. *Let $\varphi \in C_b^2(\mathbb{R}^n)$. Assume that the sequence $\{x_n\}_{n \geq 1} \subset \mathbb{R}^n$ satisfies $\varphi(x_n) \to \sup_{x \in \mathbb{R}^n} \varphi(x)$. Then*

$$\lim_{n \to \infty} \nabla\varphi(x_n) = 0 \qquad \textit{and} \qquad \limsup_{n \to \infty} -\mathcal{L}\varphi(x_n) \leq 0.$$

PROOF. By the assumption, the matrix $D^2\varphi$ has bounded coefficients, hence there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \varphi(x) \geq \varphi(x_n + z) \geq \varphi(x_n) + \nabla\varphi(x_n)\cdot z - C|z|^2. \tag{1.20}$$

Since the sequence $\nabla\varphi(x_n)$ is bounded, passing to the subsequence, we can assume that $\nabla\varphi(x_n) \to p$. Consequently, passing to the limit in (1.20) we obtain the inequality

$$0 \geq p\cdot z - C|z|^2 \quad \text{for every} \quad z\in\mathbb{R}^n.$$

Choosing $z = tp$ and letting $t \to 0^+$, we get $p = 0$.

Now, we prove that $\limsup_{n\to\infty} -\mathcal{L}\varphi(x_n) \leq 0$. Note first that, by the definition of the sequence $\{x_n\}_n$, we have

$$\varphi(x_n + z) - \varphi(x_n) \leq \sup_{x\in\mathbb{R}^n} \varphi - \varphi(x_n) \to 0 \quad \text{as} \quad n\to\infty.$$

Hence $\limsup_{n\to\infty}\Big(\varphi(x_n+z) - \varphi(x_n)\Big) \leq 0$ and equivalently for $\nabla\varphi(x_n)\to 0$,

$$\limsup_{n\to\infty}\Big(\varphi(x_n+z) - \varphi(x_n) - \nabla\varphi(x_n)\cdot z\Big) \leq 0.$$

Finally, it suffices to use the Fatou lemma in the expression

$$\mathcal{L}\varphi(x_n) = \int_{\mathbb{R}^n} \big(\varphi(x_n - z) - u(x_n) - z\cdot\nabla\varphi(x_n)1\!\!1_{\{|z|<1\}}(z)\big)\,\Pi(dz),$$

because the Lévy measure Π is nonnegative. □

We are in a position to prove the main comparison principle for equations with Lévy operators.

THEOREM 1.18. *Assume that* $u\in C_b(\mathbb{R}^n\times[0,T])\cap C_b^2(\mathbb{R}^n\times[\varepsilon,T])$ *is the solution of the equation*

$$u_t = -\mathcal{L}u + v(x,t)\cdot\nabla u, \tag{1.21}$$

where $\mathcal{L}$ *is the Lévy operator represented by (1.12) and* $v = v(x,t)$ *is a given and sufficiently regular function with values in* $\mathbb{R}^n$. *Then*

$$u(x,0)\leq 0 \quad \textit{implies} \quad u(x,t)\leq 0 \quad \textit{for all}, x\in\mathbb{R}^n, t\in[0,T].$$

PROOF. We extract this proof from [27, Proof of Prop. 2].

It is an easy exercise using assumptions imposed on u to show that the function

$$\Phi(t) = \sup_{x\in\mathbb{R}^n} u(x,t)$$

is well-defined and continuous. Our goal is to show that Φ is locally Lipschitz and $\Phi'(t)\leq 0$ almost everywhere.

To show the Lipschitz continuity of Φ, for every $\varepsilon > 0$ we chose x_ε such that

$$\sup_{x\in\mathbb{R}^n} u(x,t) = u(x_\varepsilon,t) + \varepsilon.$$

Now, we fix $t, s\in I$, where $I\subset(0,T)$ is a bounded and closed interval and we suppose (without loss of generality) that $\Phi(t)\geq\Phi(s)$. Using the definition of Φ

and regularity of u we obtain

$$\begin{aligned}0 \le \Phi(t) - \Phi(s) &= \sup_{x\in\mathbb{R}^n} u(x,t) - \sup_{x\in\mathbb{R}^n} u(x,s)\\ &\le \varepsilon + u(x_\varepsilon, t) - u(x_\varepsilon, s)\\ &\le \varepsilon + \sup_{x\in\mathbb{R}^n} |u(x,t) - u(x,s)|\\ &\le \varepsilon + |t-s| \sup_{x\in\mathbb{R}^n, t\in I} |\nabla_t u(x,t)|.\end{aligned}$$

Since $\varepsilon > 0$ and $t, s \in I$ are arbitrary, we immediately obtain that the function Φ is locally Lipschitz hence, by the Rademacher theorem, differentiable almost everywhere, as well.

Let us now differentiate $\Phi(t) = \sup_{x\in\mathbb{R}^n} u(x,t)$ with respect to $t > 0$. By the Taylor expansion, for $0 < s < t$, we have

$$u(x,t) \le u(x, t-s) + s u_t(x,t) + Cs^2.$$

Hence, using equation (1.21), we obtain

$$u(x,t) \le \sup_x u(x,t-s) + s\Big(- \mathcal{L}u(x,t) + v(x,t)\nabla u(x,t)\Big) + Cs^2. \tag{1.22}$$

Substituting in (1.22) $x = x_n$, where $u(x_n, t) \to \sup_x u(x,t)$ as $n \to \infty$, passing to the limit using Lemma 1.17, we obtain the inequality

$$\sup_x u(x,t) \le \sup_x u(x,t-s) + Cs^2$$

which can be transformed into

$$\frac{\Phi(t) - \Phi(s)}{s} \le Cs.$$

For $s \searrow 0$, we obtain $\Phi'(t) \le 0$ in those t, where Φ is differentiable. □

5. Integration by parts and the Lévy operator

We have seen in the previous section that any pseudodifferential operator given by the Lévy-Khinchin formula (1.12) satisfies the maximum principle typical for the Laplace operator. Now, we present other properties of Lévy operators which will allow us to show energy-type estimates for solutions of some evolution equations. It is worth to emphasize that equalities and inequalities, proved in the case of Laplacian integrating by parts, can be generalized for any Lévy operator by using suitable convex inequalities.

Let us illustrate this phenomenon by proving the Kato inequality.

THEOREM 1.19 (Kato inequality for Laplacian). *For every $\varphi \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (-\Delta\varphi)\,\mathrm{sgn}\,\varphi\, dx \ge 0.$$

PROOF. Let us begin with the following smooth approximation of the sign function

$$g_\varepsilon(s) = \frac{d}{ds}\left(\sqrt{\varepsilon + s^2}\right) = \frac{s}{\sqrt{\varepsilon + s^2}}.$$

Note that $g'_\varepsilon(s) \geq 0$ and $g_\varepsilon(s) \to \operatorname{sgn} s$ as $\varepsilon \to 0$. Now, we integrate by parts to obtain

$$\int_{\mathbb{R}^n} (-\Delta\varphi)\, g_\varepsilon(\varphi)\, dx = \int_{\mathbb{R}^n} |\nabla\varphi|^2\, g'_\varepsilon(\varphi)\, dx \geq 0.$$

To complete the proof, it suffices to pass to the limit $\varepsilon \to 0$ on the left-hand side using the Lebesgue dominated convergence theorem. □

THEOREM 1.20 (Kato inequality for Lévy operator). *For every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and for every Lévy operator represented by (1.12), we have*

$$\int_{\mathbb{R}^n} (\mathcal{L}\varphi) \operatorname{sgn}\varphi\, dx \geq 0.$$

PROOF. According to Definitions 1.6 and 1.3 we denote by $\{\mu^t\}_{t\geq 0}$ the convolution semigroup of measures corresponding to the Lévy operator $\mathcal{L}$. Recall that

$$e^{-t\mathcal{L}}u_0(x) \equiv u(x,t) = \int_{\mathbb{R}^n} u_0(x-y)\, \mu^t(dx) \tag{1.23}$$

is the solution of the initial value problem (1.6)-(1.7). In particular, we have

$$\mathcal{L}\varphi = \lim_{t\to 0^+} \frac{\varphi - e^{-t\mathcal{L}}\varphi}{t}.$$

Consequently, it suffices to show that, for every $t > 0$, we have the inequality

$$\int_{\mathbb{R}^n} (\varphi - e^{-t\mathcal{L}}\varphi) \operatorname{sgn}\varphi\, dx \geq 0$$

which is equivalent to

$$\int_{\mathbb{R}^n} |\varphi|\, dx \geq \int_{\mathbb{R}^n} (e^{-t\mathcal{L}}\varphi) \operatorname{sgn}\varphi\, dx. \tag{1.24}$$

We complete the proof of inequality (1.24) by using the formula (1.23), the Fubini theorem, and the fact that μ^t is the probability measure for every $t \geq 0$ as follows

$$\left| \int_{\mathbb{R}^n} (e^{-t\mathcal{L}}\varphi) \operatorname{sgn}\varphi\, dx \right| \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x-y)|\, \mu^t(dy)\, dx = \int_{\mathbb{R}^n} |\varphi|\, dx.$$

□

Let us present the next result which looks like an integration by parts for any Lévy operator.

THEOREM 1.21 (Strook–Varopoulos inequality). *Assume that $\mathcal{L}$ is a Lévy operator. For every $p \in (1,\infty)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$ we have*

$$4\frac{p-1}{p^2} \int_{\mathbb{R}^n} (\mathcal{L}\varphi^{p/2})\, \varphi^{p/2}\, dx \leq \int_{\mathbb{R}^n} (\mathcal{L}\varphi)\, \varphi^{p-1}\, dx. \tag{1.25}$$

REMARK 1.22. *Note that, for $\mathcal{L} = b\cdot\nabla$ with any fixed $b \in \mathbb{R}^n$, both sides of the Strook–Varopoulos inequality (1.25) are equal to 0. On the other hand, if $\mathcal{L} = -\Delta$,* *we integrate by parts to obtain the equality*

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta\varphi)\,\varphi^{p-1}\,dx &= (p-1)\int_{\mathbb{R}^n} |\nabla\varphi|^2\,\varphi^{p-2}\,dx\\ &= (p-1)\int_{\mathbb{R}^n} |\nabla\varphi\,\varphi^{p/2-1}|^2\,dx\\ &= 4\frac{p-1}{p^2}\int_{\mathbb{R}^n} |\nabla\varphi^{p/2}|^2\,dx.\end{aligned}$$

Sketch of proof of Theorem 1.21. Inequality (1.25) was proved by Strook [56] and Varopoulos [57]. We also refer the reader to the review article by Liskevich and A. Semenov [48] (the preprint is available on the V.A. Liskevisch webpage) for the proof of this inequality in the case of much more general Markov semigroups. Here, we emphasize the main steps of the proof of (1.25), only.

Step 1. Let $\alpha > 0$ and $\beta > 0$ be such that $\alpha + \beta = 2$. Then the following inequality

$$(x^\alpha - y^\alpha)(x^\beta - y^\beta) \geq \alpha\beta(x-y)^2$$

holds true for all $x \geq 0$ and $y \geq 0$.

Step 2. As before, we use the relation

$$\int_{\mathbb{R}^n} (\mathcal{L}f)\,g\,dx = \lim_{t\to 0^+}\frac{1}{t}\int_{\mathbb{R}^n} (f - e^{-t\mathcal{L}}f)\,g\,dx,$$

valid for all $f, g \in D(\mathcal{L})$.

Step 3. We use inequality from Step 1 and formula (1.5) (remember that μ^t is a probability measure) to show

$$\int_{\mathbb{R}^n} (f^\alpha - e^{-t\mathcal{L}}f^\alpha)\,f^\beta\,dx \geq \alpha\beta\int_{\mathbb{R}^n} (f - e^{-t\mathcal{L}}f)\,f\,dx$$

for every $f \in D(\mathcal{L})$ such that $f \geq 0$ and for $\alpha + \beta = 2$.

Step 4. Finally, we substitute in the inequality form Step 3

$$f = \varphi^{p/2},\quad \alpha = \frac{2}{p},\quad \beta = 2 - \frac{2}{p},\quad \alpha\beta = 4\frac{p-1}{p^2},$$

and, after dividing by t, we pass to the limit $t \to 0^+$ to conclude the proof. □

Remark 1.23 (General Strook–Varopoulos inequality). *The Kato inequality combined with the Strook–Varopoulos inequality give the following estimate*

$$\frac{4(p-1)}{p^2}\langle \mathcal{L}|\varphi|^{p/2}, |\varphi|^{p/2}\rangle \leq \langle \mathcal{L}\varphi, |\varphi|^{p-1}\operatorname{sgn}\varphi\rangle \tag{1.26}$$

for every $\varphi \in D(\mathcal{L})$. This inequality is more suitable for studying sign changing solutions.

Theorem 1.24 (Convexity inequality, see *e.g.* [25, 27, 40]). *Let $u \in C_b^2(\mathbb{R}^n)$ and $g \in C^2(\mathbb{R})$ be a convex function. Then*

$$\mathcal{L}g(u) \leq g'(u)\mathcal{L}u. \tag{1.27}$$

Remark 1.25. *Note that, in the one dimensional case, for $\mathcal{L} = -\partial_x^2$ we have*

$$-(g(u))_{xx} = -g''(u)u_x^2 - g'(u)u_{xx} \leq -g'(u)u_{xx} \quad \text{since } g'' \geq 0.$$

PROOF OF THEOREM 1.24. The convexity of the function g leads to the inequality

$$g(u(x-\eta)) - g(u(x)) \geq g'(u(x))[u(x-\eta) - u(x)],$$

which can by immediately reformulate as follows

$$g(u(x-\eta)) - g(u(x)) - \eta \cdot \nabla g(u(x)) \geq g'(u(x))[u(x-\eta) - u(x) - \eta \cdot \nabla u(x)]$$

for any $\eta \in \mathbb{R}^n$. To complete the proof, it suffices to apply the Lévy-Khinchin form of any Lévy operator given in (1.12). □

Now, we state an important application of the convexity inequality (1.27).

COROLLARY 1.26. *Let $g \in C^2(\mathbb{R})$ be a convex function. Assume $g(u) \in D(\mathcal{L})$ and $\mathcal{L}g(u) \in L^1(\mathbb{R}^n)$. Then*

$$0 \left(= \int_{\mathbb{R}^n} \mathcal{L}g(u(x))\, dx \right) \leq \int_{\mathbb{R}^n} g'(u(x)) \mathcal{L}u(x)\, dx.$$

PROOF. Denoting $v(x) = g(u(x))$ and using properties of the (inverse) Fourier transform we obtain

$$\int_{\mathbb{R}^n} \mathcal{L}v(x)\, dx = \int_{\mathbb{R}^n} (a\, \widehat{v})\check{}(x)\, dx = (2\pi)^{n/2} a(0)\widehat{v}(0) = 0,$$

because $a(0) = 0$ (*cf.* (1.11)). Now, it suffices to apply inequality (1.27). □

COROLLARY 1.27. *Any Lévy diffusion operator $\mathcal{L}$ satisfies*

$$\int_{\mathbb{R}^n} (\mathcal{L}u) \Big((u-k)_+ \Big)^p\, dx \geq 0$$

for each $1 < p < \infty$ and all constants $k \geq 0$.

REMARK 1.28. *Note that the general Strook–Varopoulos inequality*

$$C(p) \langle \mathcal{L}|\varphi|^{p/2}, |\varphi|^{p/2} \rangle \leq \langle \mathcal{L}\varphi, |\varphi|^{p-1} \operatorname{sgn} \varphi \rangle$$

can be obtained immediately from the convexity inequality (1.27), applied with the convex function $g(\varphi) = |\varphi|^{p/2}$ for $p > 2$. Here, however, we have got the non-optimal constant

$$C(p) = \frac{2}{p} \quad \left(\leq \frac{4(p-1)}{p^2} \right) \quad \text{for every} \quad p > 2.$$

We conclude this section by the proof of a particular case of the Gagliardo–Nirenberg inequality. The proof of the following theorem uses an argument from the celebrated work by Nash [53] where, on page 935, the author emphasized that this argument was shown to him by E.M. Stein.

THEOREM 1.29 (Nash inequality). *Let $0 < \alpha$. There exists a constant $C_N > 0$ such that*

$$\|w\|_2^{2(1+\alpha)} \leq C_N \|\Lambda^{\alpha/2} w\|_2^2 \|w\|_1^{2\alpha} \tag{1.28}$$

for all functions $w = w(x)$ satisfying $w \in L^1(\mathbb{R})$ and $\Lambda^{\alpha/2} w \in L^2(\mathbb{R})$.

PROOF. For every $R > 0$, we decompose the L^2-norm of the Fourier transform of w as follows

$$\begin{aligned}\|w\|_2^2 &= C \int_{\mathbb{R}} |\widehat{w}(\xi)|^2 \, d\xi \\ &\le C\|\widehat{w}\|_\infty^2 \int_{|\xi|\le R} d\xi + CR^{-\alpha} \int_{|\xi|>R} |\xi|^\alpha |\widehat{w}(\xi)|^2 \, d\xi \\ &\le CR\|w\|_1^2 + CR^{-\alpha}\|\Lambda^{\alpha/2} w\|_2^2.\end{aligned}$$

Choosing $R = \left(\|\Lambda^{\alpha/2}w\|_2^2/\|w\|_1^2\right)^{1/(1+\alpha)}$ we obtain (1.28). □

CHAPTER 2

Fractal Burgers equation

1. Statement of the problem

To see properties of a Lévy operator "in action", we present recent results on the asymptotic behavior of solutions of the Cauchy problem for the nonlocal conservation law

$$u_t + \Lambda^\alpha u + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

$$u(0, x) = u_0(x) \tag{2.2}$$

where $\Lambda^\alpha = (-\partial^2/\partial x^2)^{\alpha/2}$ is the Lévy operator defined via the Fourier transform $\widehat{(\Lambda^\alpha v)}(\xi) = |\xi|^\alpha \widehat{v}(\xi)$, see Section 3.

Remark 2.1. *Following [13], we will call equation (2.1) the fractal Burgers equation. There are two reasons for using here the word "fractal". We want to emphasize the fractal nature of the symmetric α-stable stochastic process which corresponds to the operator Λ^α. In this sense, the usual viscous Burgers equation (i.e. (2.1) with $\alpha = 2$) should be also called the fractal Burgers equation. Moreover, we would like to distinguish our equation (2.1) from the fractional Burgers equation with the fractional derivative with respect to time t.*

Equations of this type appear in the study of growing interfaces in the presence of self-similar hopping surface diffusion [49]. Moreover, in their recent papers, Jourdain, Méléard, and Woyczynski [38, 39] gave probabilistic motivations to study equations with the anomalous diffusion, when Laplacian (corresponding to the Wiener process) is replaced by a more general pseudodifferential operator generating the Lévy process. In particular, the authors of [38] studied problem (2.1)-(2.2), where the initial condition u_0 is assumed to be a nonconstant function with bounded variation on $\mathbb{R}$. In other words, a.e. on $\mathbb{R}$,

$$u_0(x) = c + \int_{-\infty}^{x} m(dy) = c + H * m(x) \tag{2.3}$$

with $c \in \mathbb{R}$, m being a finite signed measure on $\mathbb{R}$, and $H(y)$ denoting the unit step function $\mathbb{1}_{\{y \geq 0\}}$. Observe that the gradient $v(x, t) = u_x(x, t)$ satisfies

$$v_t + \Lambda^\alpha v + (vH * v)_x = 0, \quad v(\cdot, 0) = m. \tag{2.4}$$

If m is a probability measure on $\mathbb{R}$, the equation (2.4) is a nonlinear Fokker-Planck equation. In the case of an arbitrary finite signed measure, the authors of [38] associate (2.4) with a suitable nonlinear martingale problem. Next, they study the convergence of systems of particles with jumps as the number of particles tends to $+\infty$. As a consequence, the weighted empirical cumulative distribution functions of the particles converge to the solution of the martingale problem connected to (2.4). This phenomena is called *the propagation of chaos* for problem (2.1)–(2.2) and we refer the reader to [38] for more details and additional references.

Motivated by the results from [38], we study problem (2.1)–(2.2) under the crucial assumption $\alpha \in (1,2)$ and with the initial condition of the form (2.3). In our main result, we assume that u_0 is a function satisfying

$$u_0 - u_- \in L^1((-\infty,0)) \quad \text{and} \quad u_0 - u_+ \in L^1((0,+\infty)) \quad \text{with} \quad u_- < u_+, \tag{2.5}$$

where $u_- = c$ and $u_+ - u_- = \int_{\mathbb{R}} m(dx)$.

REMARK 2.2. *For $c = 0$ and a probability measure m, the function u_0 is called the probability cumulative distribution function.*

2. Viscous conservation laws and rarefaction waves

It is well known [31, 35, 50] that the asymptotic profile as $t \to \infty$ of solutions of the viscous Burgers equation

$$u_t - u_{xx} + uu_x = 0 \tag{2.6}$$

(*i.e.* equation (2.1) with $\alpha = 2$) supplemented with an initial datum satisfying (2.5) is given by the so-called rarefaction wave. This is the continuous function

$$w^R(x,t) = W^R(x/t) = \begin{cases} u_-\,, & x/t \le u_-, \\ x/t\,, & u_- \le x/t \le u_+, \\ u_+\,, & x/t \ge u_+, \end{cases} \tag{2.7}$$

which is the unique entropy solution of following Riemann problem

$$w^R_t + w^R w^R_x = 0, \tag{2.8}$$

$$w^R(x,0) = w^R_0(x) = \begin{cases} u_-\,, & x < 0, \\ u_+\,, & x > 0. \end{cases} \tag{2.9}$$

Below, we use the *smooth approximations of rarefaction waves*, namely, the solutions of the following Cauchy problem

$$w_t - w_{xx} + ww_x = 0, \tag{2.10}$$

$$w(x,0) = w_0(x) = \begin{cases} u_-\,, & x < 0, \\ u_+\,, & x > 0. \end{cases}. \tag{2.11}$$

LEMMA 2.3. *Let $u_- < u_+$. Problem (2.10)–(2.11) has the unique, smooth, global-in-time solution $w(x,t)$ satisfying*

i) $u_- < w(t,x) < u_+$ and $w_x(t,x) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$;

ii) for every $p \in [1,\ \infty]$, there exists a constant $C = C(p,u_-,u_+) > 0$ such that

$$\|w_x(t)\|_p \le Ct^{-1+1/p}, \quad \|w_{xx}(t)\|_p \le Ct^{-3/2+1/(2p)}$$

and

$$\|w(t) - w^R(t)\|_p \le Ct^{-(1-1/p)/2},$$

for all $t > 0$, where $w^R(x,t)$ is the rarefaction wave (2.7).

All results stated in Lemma 2.3 are deduced from the explicit formula for solutions of (2.10)–(2.11) and detailed calculations can be found in [31] with some additional improvements contained in [44, Section 3].

Finally, let us necessarily recall the fundamental paper of Il'in and Oleinik [35] who showed the convergence toward rarefaction waves of solutions of the nonlinear equation $u_t - u_{xx} + f(u)_x = 0$ under strict convexity assumption imposed on f. That idea was next extended in several different directions and we refer the reader, *e.g.*, to [31, 50, 51, 54] for an overview of know results and additional references.

3. Existence o solutions

The basic questions on the existence and the uniqueness of solutions of problem (2.1)–(2.2) with $\alpha \in (1, 2)$ have been answered in the papers [26, 27].

THEOREM 2.4. ([26, Thm. 1.1], [27, Thm. 7]) *Let $\alpha \in (1, 2)$ and $u_0 \in L^\infty(\mathbb{R})$. There exists the unique solution $u = u(x, t)$ to problem (2.1)–(2.2) in the following sense: for all $T > 0$,*

$$u \in C_b((0, T) \times \mathbb{R}) \; and, \; for \; all \; a \in (0, T), \; u \in C_b^\infty((a, T) \times \mathbb{R}),$$
$$u \; satisfies \; (2.1) \; on \; (0, T) \times \mathbb{R},$$
$$u(t, \cdot) \to u_0 \; in \; L^\infty(\mathbb{R}) \; weak - * \; as \; t \to 0.$$

Moreover, the following inequality holds true

$$\|u(t)\|_\infty \le \|u_0\|_\infty \quad for \; all \quad t > 0. \tag{2.12}$$

REMARK 2.5. *Notice that $L^\infty(\mathbb{R})$ is not a separable Banach space. Hence, the statement $u(t, \cdot) \to u_0$ in $L^\infty(\mathbb{R})$ weak–$*$ means that, for every $\varphi \in L^1(\mathbb{R})$, we have $\int_{\mathbb{R}} \big(u(x, t) - u_0(x)\big)\varphi(x)\, dx \to 0$ as $t \to 0$.*

The proof of Theorem 2.4 is based on the Banach fixed point argument applied to the integral formulation of the Cauchy problem (2.1)–(2.2)

$$u(t) = S_\alpha(t)u_0 - \int_0^t S_\alpha(t - \tau)u(\tau)u_x(\tau)\, d\tau, \tag{2.13}$$

where

$$S_\alpha(t)u_0 = p_\alpha(t) * u_0(x). \tag{2.14}$$

Here, the fundamental solution $p_\alpha(x, t)$ of the linear equation $\partial_t v + \Lambda^\alpha v = 0$ can be computed via the Fourier transform $\widehat{p}_\alpha(\xi, t) = e^{-t|\xi|^\alpha}$ and its properties are discussed in Section 3. Hence, by the Young inequality for the convolution, we obtain the estimates

$$\|S_\alpha(t)v\|_p \le Ct^{-(1-1/p)/\alpha}\|v\|_1, \tag{2.15}$$

$$\|(S_\alpha(t)v)_x\|_p \le Ct^{-(1-1/p)/\alpha-1/\alpha}\|v\|_1 \tag{2.16}$$

for every $p \in [1, \infty]$ and all $t > 0$. Moreover, we can replace v in (2.15) and in (2.16) by any signed measure m. In that case, $\|v\|_1$ should be replaced by $\|m\|$.

Note also that inequality (2.12) is the immediate consequence of Theorem 1.18.

Now, let us deal with $\alpha \in (0, 1]$. It was shown in [2] (see also [45]) that solutions of the initial value problem (2.1)–(2.2) can become discontinuous in finite time if $0 < \alpha < 1$. Hence, in order to deal with discontinuous solutions, the notion of entropy solutions in the sense of Kruzhkov was extended by Alibaud [1] to nonlocal problem (2.1)–(2.2).

THEOREM 2.6 ([1]). *Assume that $0 < \alpha \leq 1$ and $u_0 \in L^\infty(\mathbb{R})$. There exists the unique entropy solution $u = u(x,t)$ to the Cauchy problem (2.1)–(2.2). This solution u belongs to $C([0,\infty); L^1_{loc}(\mathbb{R}))$ and satisfies $u(0) = u_0$. Moreover, we have the following maximum principle:* $\operatorname{essinf} u_0 \leq u \leq \operatorname{esssup} u_0$.

The occurrence of discontinuities in finite time of entropy solutions of (2.1)–(2.2) with $\alpha = 1$ seems to be not clear. Regularity results have recently been obtained [24, 45, 52] for a large class of initial conditions, that does unfortunately not include general L^∞ initial data. Nevertheless, Theorem 2.6 provides the existence and the uniqueness of global-in-time the entropy solution even for the critical case $\alpha = 1$.

4. Decay estimates

Due to possible singularities of solutions of (2.1)–(2.2) with $\alpha \in (0,1)$, from now on, we study solutions of the Cauchy problem for the regularized fractal Burgers equation with $\varepsilon > 0$ if $\alpha \in (0,1]$ and $\varepsilon = 0$ for $\alpha \in (1,2)$

$$u^\varepsilon_t + \Lambda^\alpha u^\varepsilon - \varepsilon u^\varepsilon_{xx} + u^\varepsilon u^\varepsilon_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.17}$$

$$u^\varepsilon(x,0) = u_0(x). \tag{2.18}$$

The procedure now is, roughly speaking, to make the asymptotic study of u^ε with stability estimates uniform in ε. Next, we pass to the limit $\varepsilon \to 0$ using the theory developed in [3] in order to obtain for solutions of (2.1)–(2.2). Most of the results of this section are inspired from [41] and, when it is the case, the reader is referred to precise proofs in that paper.

One can show (as in Theorem 2.4) that problem (2.17)–(2.18) admits the unique global-in-time smooth solution that satisfies the maximum principle. If, moreover, the initial datum u_0 can be written in the form (2.3) for a constant $c \in \mathbb{R}$ and a signed finite measure m on $\mathbb{R}$, the solution $u^\varepsilon = u^\varepsilon(x,t)$ of problem (2.17)–(2.18) satisfies $u^\varepsilon_x \in C((0,T]; L^p(\mathbb{R}))$ for each $1 \leq p \leq \infty$ and every $T > 0$. Here, for the proofs of those properites, one should follow [41, Thm. 2.2].

Main properties of $u^\varepsilon_x(x,t)$ are contained in the following theorem.

THEOREM 2.7. *Assume that $0 < \alpha \leq 2$, $\varepsilon > 0$, and u_0 is of the form (2.3) with $c \in \mathbb{R}$ and a finite <u>nonnegative</u> measure $m(dx)$ on $\mathbb{R}$. Denote by $u^\varepsilon = u^\varepsilon(x,t)$ the unique solution of problem (2.17)–(2.18). Then*

(i) $u^\varepsilon_x(x,t) \geq 0$ for all $x \in \mathbb{R}$ and $t > 0$,
(i) for every $p \in [1,\infty]$ there exists $C = C(p) > 0$ <u>independent</u> of ε such that

$$\|u^\varepsilon_x(t)\|_p \leq C(p) \min\left\{ t^{-(1/\alpha)(1-1/p)}\|m\|, t^{-(1-1/p)}\|m\|^{1/p} \right\} \tag{2.19}$$

for all $t > 0$

SKETCH OF PROOF. To prove this result, it suffices to modify slightly the argument from [41, Thm. 2.3] as follows. We write the equation for $v = u^\varepsilon_x$

$$v_t + \Lambda^\alpha v - \varepsilon v_{xx} + (u^\varepsilon u^\varepsilon_x)_x = 0 \tag{2.20}$$

and we note that, due to the Kato inequality (*c.f.* Theorems 1.19 and 1.20), we have the "good" sign of the following quantities

$$-\varepsilon \int_{\mathbb{R}} v_{xx}(x,t)\varphi(v(x))\,dx \geq 0 \quad \text{and} \quad \int_{\mathbb{R}} \Lambda^\alpha v(x,t)\varphi(v(x))\,dx \geq 0$$

for any nondecreasing function φ. Hence, to prove Theorem 2.7 it suffices to rewrite all inequalities from [41, Proof of Thm. 2.3] skipping each term containing ε. Here, we recall that argument proving inequality (2.19) for $p = 2$, only.

For $v = u_x^\varepsilon \geq 0$, we multiply equation (2.20) by v and integrate over $\mathbb{R}$:

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \varepsilon \int_{\mathbb{R}} (v_x)^2\,dx + \int_{\mathbb{R}} v\Lambda^\alpha v\,dx + \frac{1}{2}\int_{\mathbb{R}} v^3\,dx = 0. \tag{2.21}$$

Note that second, third, and forth term of identity (2.21) are nonnegative. Let us use the third term and skip the other two. Applying Nash inequality (1.28) to estimate the third term of (2.21) we obtain

$$\frac{d}{dt}\|v(t)\|_2^2 + 2C_N^{-1}\|m\|^{-2\alpha}\|v(t)\|_2^{2(1+\alpha)} \leq 0,$$

which, after integration, leads to

$$\|v(t)\|_2 \leq C_1\|m\|t^{-1/(2\alpha)} \quad \text{with} \quad C_1 = (C_N/2\alpha)^{1/(2\alpha)}.$$

This is the first decay estimate on the right-hand side of (2.19) with $p = 2$. To show the second inequality in (2.19), one should proceed analogously using the term $\int_{\mathbb{R}} v^3\,dx$.

The idea of the proof of (2.19) for $p \neq 2$ is similar and uses Strook-Varopoulos inequality (1.25) combined with Nash inequality (1.28), see [41] for more details. To show Theorem 2.7.i, one should apply either the comparison principle from Theorem 1.18 (see [27]) or an energy argument based on Corollary 1.27 (see [41, Thm. 2.3]) . □

In the study of the large time asymptotics to (2.1)–(2.2), we also need the following asymptotic stability result.

THEOREM 2.8. *Let $\alpha \in (0,2)$. Assume that u^ε and $\widetilde{u^\varepsilon}$ are two solutions of the regularized problem (2.17)–(2.18) with initial conditions u_0 and $\widetilde{u}_0$ of the form (2.3), the both of with finite signed measures m and $\widetilde{m}$, respectively. Suppose, moreover, that the measure $\widetilde{m}$ is nonnegative and $u_0 - \widetilde{u}_0 \in L^1(\mathbb{R})$. Then, for every $p \in [1,\infty]$ there exists a constant $C = C(p) > 0$ independent of ε such that for all $t > 0$*

$$\|u^\varepsilon(t) - \widetilde{u^\varepsilon}(t)\|_p \leq Ct^{-(1-1/p)/\alpha}\|u_0 - \widetilde{u}_0\|_1\,.$$

PROOF. Here, it suffices to copy calculations from [41, Proofs of Thm. 2.2 and Lemma 3.1] skipping each term containing ε as it was explained in the proof of Theorem 2.7. □

5. Convergence toward rarefaction waves for $\alpha \in (1,2)$

Now, we are in a position to state the result for the large time asymptotics of solutions of (2.1)–(2.2) with $\alpha \in (1,2)$. Here, we use estimates from the previous section assuming that $\varepsilon = 0$.

THEOREM 2.9 ([41]). *Let $\alpha \in (1, 2)$ and the initial datum u_0 be of the form (2.3) with $c \in \mathbb{R}$ and m being a finite measure on $\mathbb{R}$ (not necessarily nonnegative). We assume, moreover, the (2.5) holds true. For every $p \in \left(\frac{3-\alpha}{\alpha-1}, \infty\right]$ there exists $C > 0$ independent of t such that*

$$\|u(t) - w^R(t)\|_p \leq Ct^{-[\alpha-1-(3-\alpha)/p]/2} \log(2 + t)$$

for all $t > 0$.

PROOF. In view of Lemma 2.3, we my replace the rarefaction wave $w^R(x,t)$ by its smooth approximation $w = w(x,t)$. Next, using the Gagliardo-Nirenberg inequality we have

$$\|u(t) - w(t)\|_p \quad \leq C\Big(\|u_x(t)\|_\infty + \|w_x(t)\|_\infty\Big)^a \|u(t) - w(t)\|_{p_0}^{1-a}$$

for every $1 < p_0 < p < \infty$. Since both quantities $\|u_x(t)\|_\infty$ and $\|w_x(t)\|_\infty$ decay in time by (2.19), the proof is completed by applying Lemma 2.10, stated below. □

LEMMA 2.10. *For $p_0 = (3 - \alpha)/(\alpha - 1)$, the following estimate is valid*

$$\|u(t) - w(t)\|_{p_0} \leq C \log(2 + t).$$

PROOF. The function $v = u - w$ satisfies

$$v_t + \Lambda^\alpha v + \frac{1}{2}[v^2 + 2vw]_x = -\Lambda^\alpha w + w_{xx}.$$

We multiply this equation by $|v|^{p-2}v$ and we integrate over $\mathbb{R}$ to obtain

$$\frac{1}{p}\frac{d}{dt}\int |v|^p \, dx + \int (\Lambda^\alpha v)(|v|^{p-2}v) \, dx + \frac{1}{2}\int [v^2 + 2vw]_x |v|^{p-2}v \, dx$$
$$= \int (-\Lambda^\alpha w + w_{xx})(|v|^{p-2}v) \, dx.$$

The second and the third term on the left hand side are nonnegative, hence we skip them. Using the Hölder inequality on the right-hand side we obtain the following differential inequality

$$\frac{d}{dt}\|v(t)\|_p^p \leq p\left(\|\Lambda^\alpha w(t)\|_p + \|w_{xx}(t)\|_p\right)\|v(t)\|_p^{p-1},$$

which, after integration, leads to

$$\|v(t)\|_p \leq \|v(t_0)\|_p + \int_{t_0}^t \|\Lambda^\alpha w(\tau)\|_p + \|w_{xx}(\tau)\|_p \, d\tau.$$

Now, we use the decay properties of the smooth approximation of rarefaction waves from Lemma 2.3 to complete the proof (see [41, Lemma 3.3] for more details). □

6. Self-similar solution for $\alpha = 1$

Using the uniqueness result from [1] (see Theorem 2.6 above) combined with a standard scaling technique, one can show that equation (2.1) with $\alpha = 1$ has self-similar solutions.

THEOREM 2.11. *Assume $\alpha = 1$. The unique entropy solution $U = U(x,t)$ of the initial value problem (2.1)–(2.2) with the initial condition*

$$U_0(x) = \begin{cases} u_- , & x < 0, \\ u_+ , & x > 0, \end{cases} \tag{2.22}$$

is self-similar, i.e. it has the form $U(x,t) = U(x/t, 1)$ for all $x \in \mathbb{R}$ and all $t > 0$.

Note that the function U_0 from (2.22) is of the form (2.3) for the measure $m := (u_+ - u_-)\delta_0$ (here, δ_0 denotes the Dirac delta at 0).

In [3], we show that the self-similar profile $U(x,1)$ from Theorem 2.11 enjoys the following properties:

- Regularity: $U(\cdot, 1)$ is Lipschitz-continuous.
- Monotonicity: $U(\cdot, 1)$ is non-decreasing with $\lim_{x\to\pm\infty} U(x,1) = u_\pm$.
- Symmetry: For all $y \in \mathbb{R}$, we have the equality
$$U(\overline{c} + y, 1) = \overline{c} - U(\overline{c} - y, 1), \quad \text{where} \quad \overline{c} := \frac{u_- + u_+}{2}.$$
- Convexity: $U(\cdot, 1)$ is convex (resp. concave) on $(-\infty, \overline{c}]$ (resp. on $[\overline{c}, +\infty)$).
- Decay at infinity: We have $U_x(x,1) \sim \frac{u_+ - u_-}{2\pi^2} |x|^{-2}$, as $|x| \to \infty$.

This self-similar solution $U = U(x,t)$ describes the large time asymptotics of other solutions of (2.1)–(2.2) with $\alpha = 1$.

THEOREM 2.12. *Let $\alpha = 1$. Let $u = u(x,t)$ be the entropy solution of problem (2.1)–(2.2) corresponding to the initial condition of the form (2.3) satisfying (2.5). Denote by $U = U(x,t)$ the self-similar solution from Theorem 2.11. For every $p \in [1, \infty]$ there exists a constant $C = C(p) > 0$ such that*

$$\|u(t) - U(t)\|_p \le C t^{-(1-1/p)} \|u_0 - U_0\|_1$$

for all $t > 0$.

PROOF. This result is the immediate consequence of Theorem 2.8 by passing to the limit $\varepsilon \to 0$. □

We refer the reader to [3] for more details concerning self-similar solutions of equation (2.1).

7. Linear asymptotics for $0 < \alpha < 1$

In the case where $\alpha < 1$, the Duhamel principle (2.13) combined with the decay estimates (2.19) allow us to show that the nonlinearity in (2.1) is negligible in the asymptotic expansion of solutions.

THEOREM 2.13 ([3]). *Let $0 < \alpha < 1$ and $u = u(x,t)$ be the entropy solution of (2.1)–(2.2) corresponding to the initial condition of the form (2.3) satisfying (2.5). Denote by $S_\alpha(t)u_0$ the solution of the linear initial value problem $u_t + \Lambda^\alpha u = 0$, $u(x,0) = u_0(x)$. For every $p \in \left(\frac{1}{1-\alpha}, \infty\right]$ there exists $C = C(p) > 0$ such that*

$$\|u(t) - S_\alpha(t)u_0\|_p \le C \|u_0\|_\infty \|m\| t^{1-(1/\alpha)(1-1/p)} \tag{2.23}$$

for all $t > 0$.

It follows from the proof of Theorem 2.13 that inequality (2.23) is valid for every $p \in [1, \infty]$. Its right-hand-side, however, decays for $p \in \left(\frac{1}{1-\alpha}, \infty\right]$, only.

Actually, the asymptotic term $S_\alpha(t)u_0$ in (2.23) can be written in a self-similar way.

Corollary 2.14. *Under the assumptions of Theorem 2.13, we have*

$$\left\| c + H_\alpha(t) \int_{\mathbb{R}} m(dx) - u(t) \right\|_p \to 0 \qquad \textit{as} \quad t \to \infty,$$

where $H_\alpha(x,t) := \int_{-\infty}^{x} p_\alpha(y,t)\, dy$ *and* $p_\alpha(x,t)$ *is the fundamental solution of the linear equation* $u_t + \Lambda^\alpha u = 0$. *Moreover, if we assume in addition that* $\int_{\mathbb{R}} |x||m|(dx) < \infty$, *then we have the following rate of convergence*

$$\left\| c + H_\alpha(t) \int_{\mathbb{R}} m(dx) - u(t) \right\|_p \leq C \, \|u_0\|_\infty \left(\|m\| + \int |x||m|(dx) \right) t^{1-(1/\alpha)(1-1/p)}, \quad (2.24)$$

for some constant $C = C(p)$.

Notice that $c + H_\alpha(x,t) \int m(dx)$ is nothing else than the solution of problem $u_t + \Lambda^\alpha u = 0$, $u(x,0) = U_0(x)$ (U_0 being defined in (2.22)). It is well-known that this solution is effectively self-similar with the scaling $H_\alpha(x,t) = H_\alpha\left(xt^{-1/\alpha}, 1\right)$, see also the homogeneity property (1.16).

8. Probabilistic summary

Let us summarize our results on large time behavior of solutions of the initial value problem (2.1)–(2.2). In the case $\alpha > 1$, the diffusive term in (2.1) is negligible in the asymptotic expansion of solutions (see Theorem 2.9), whereas in the case $\alpha < 1$, the nonlinear convection term does not appear in the asymptotics of solutions (*cf.* Theorem 2.13). In the case $\alpha = 1$, both terms have to be taken into account (*cf.* Theorem 2.12).

To conclude, let us emphasize the probabilistic meaning of these results. We have already mentioned that the solution u of (2.1)–(2.2) supplemented with the initial datum of the form (2.3) with $c = 0$ and with a probability measure m on $\mathbb{R}$ is the cumulative distribution function for every $t \geq 0$. This family of probabilities defined by problem (2.1)–(2.2) converges, as $t \to \infty$, toward

- the uniform distribution on the interval $[0, t]$ if $1 < \alpha \leq 2$ (see Theorem 2.9);
- the one parameter family of new laws constructed in Theorem 2.11 if $\alpha = 1$ (see Theorem 2.12);
- the symmetric α-stable laws $p_\alpha(t)$ if $0 < \alpha < 1$ (*cf.* Theorem 2.13 and Corollary 2.14).

CHAPTER 3

Fractal Hamilton–Jacobi–KPZ equations

1. Kardar, Parisi & Zhang and Lévy operators

The well-known Kardar–Parisi–Zhang (KPZ) equation

$$h_t = \nu\Delta h + \frac{\lambda}{2}|\nabla h|^2$$

was derived in [43] as a model for growing random interfaces. Recall that the interface is parameterized here by the transformation $\Sigma(t) = (x, y, z = h(x, y, t))$, so that $h = h(x, y, t)$ is the surface elevation function, $\nu > 0$ is identified in [43] as a "surface tension" or "high diffusion coefficient", Δ and ∇ stand, respectively, for the usual Laplacian and gradient differential operators in spatial variables, and $\lambda \in \mathbb{R}$ scales the intensity of the ballistic rain of particles onto the surface.

An alternative, first-principles derivation of the KPZ equation (*cf.* [49], for more detailed information and additional references) makes three points:

- The Laplacian term can be interpreted as a result of the surface transport of adsorbed particles caused by the standard Brownian diffusion;
- In several experimental situations a hopping mechanism of surface transport is present which necessitates augmentation of the Laplacian by a nonlocal term modeled by a Lévy stochastic process;
- The quadratic nonlinearity is a result of truncation of a series expansion of a more general, physically justified, nonlinear even function.

These observations lead us to consider in this paper a nonlinear nonlocal equation of the form

$$u_t = -\mathcal{L}u + \lambda|\nabla u|^q, \tag{3.1}$$

where the Lévy diffusion operator defined in (1.12). In this chapter, we assume (for the sake of the simplicity of the exposition) that there is no transport term in the Lévy operator (1.12), namely $b = 0$. Recall also that if the matrix $(a_{jk})_{j,k=1}^n$ in (1.12) is not degenerate, a linear change of the variables transforms the corresponding term in (1.12) into the usual Laplacian $-\Delta$ on $\mathbb{R}^n$.

Relaxing the assumptions that led to the quadratic expression in the classical KPZ equation, the nonlinear term in (3.1) has the form

$$\lambda|\nabla u|^q = \lambda\left(|\partial_{x_1}u|^2 + ... + |\partial_{x_n}u|^2\right)^{q/2},$$

where q is a constant parameter. To study the interaction of the "strength" of the nonlocal Lévy diffusion parameterized by the Lévy measure Π, with the "strength" of the nonlinear term, parameterized by λ and q, we consider in (3.1) the whole range, $1 < q < \infty$, of the nonlinearity exponent.

Finally, as far as the intensity parameter $\lambda \in \mathbb{R}$ is concerned, we distinguish two cases:

- *The deposition case:* Here, $\lambda > 0$ characterizes the intensity of the ballistic deposition of particles on the evolving interface,
- *The evaporation case:* Here, $\lambda < 0$, and the model displays a time-decay of the total "mass" $M(t) = \int_{\mathbb{R}^n} u(x,t)\ dx$ of the solution (*cf.* equation (3.12), below).

Equation (3.1) will be supplemented with the nonnegative initial datum,

$$u(x,0) = u_0(x), \tag{3.2}$$

and our standing assumptions are that $u_0 \in W^{1,\infty}(\mathbb{R}^n)$, and $u_0 - K \in L^1(\mathbb{R}^n)$, for some constant $K \in \mathbb{R}$; as usual, W, with some superscripts, stands for various Sobolev spaces.

REMARK 3.1. *The long-time behavior of solutions of the viscous Hamilton-Jacobi equation $u_t = \Delta u + \lambda|\nabla u|^q$, with $\lambda \in \mathbb{R}$, and $q > 0$, has been studied by many authors, see e.g. [5, 9, 10, 11, 30, 47], and the references therein. The dynamics of solutions of this equation is governed by two competing effects, one resulting from the diffusive term Δu, and the other corresponding to the "hyperbolic" nonlinearity $|\nabla u|^q$. The above-cited papers aimed at explaining how the interplay of these two effects influences the large-time behavior of solutions depending on the values of q and the initial data.*

Below, we are going to present results from [42] where we follow strategy from Remark 3.1, as well. Hence, in [42], we want to understand the interaction of the diffusive nonlocal Lévy operator (1.12) with the power-type nonlinearity. Our results can be viewed as extensions of some of the above-quoted work. However, their physical context is quite different and, to prove them, new mathematical tools have to be developed.

2. Assumptions and preliminary results

The basic assumption throughout the paper is that the Lévy operator $\mathcal{L}$ is a "perturbation" of the fractional Laplacian $(-\Delta)^{\alpha/2}$ (see Section 3) or, more precisely, that it satisfies the following condition:

- The symbol a of the operator $\mathcal{L}$ can be written in the form

$$a(\xi) = \ell|\xi|^\alpha + k(\xi), \tag{3.3}$$

  where $\ell > 0$, $\alpha \in (0,2]$. and the pseudodifferential operator $\mathcal{K}$, corresponding to the symbol k, generates a strongly continuous semigroup of operators on $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, with norms uniformly bounded in t.

Observe that, without loss of generality (rescaling the spatial variable x), we can assume that the scaling constant ℓ in (3.3) is equal to 1. Also, note that the above assumptions on the operator $\mathcal{K}$ are satisfied if the Fourier transform of the function $e^{-tk(\xi)}$ is in $L^1(\mathbb{R}^n)$, for every $t > 0$, and its L^1-norm is uniformly bounded in t.

The study of the large time behavior of solutions of the nonlinear problem (3.1)-(3.2) will necessitate the following supplementary asymptotic condition on the Lévy operator $\mathcal{L}$:

- The symbol $k = k(\xi)$ appearing in (3.3) satisfies the condition

$$\lim_{\xi \to 0} \frac{k(\xi)}{|\xi|^\alpha} = 0. \tag{3.4}$$

The assumptions (3.3) and (3.4) are fulfilled, *e.g.*, by *multifractal diffusion operators*

$$\mathcal{L} = -a_0 \Delta + \sum_{j=1}^{k} a_j (-\Delta)^{\alpha_j/2},$$

with $a_0 \geq 0$, $a_j > 0$, $1 < \alpha_j < 2$, and $\alpha = \min_{1 \leq j \leq k} \alpha_j$, but, more generally, one can consider here

$$\mathcal{L} = (-\Delta)^{\alpha/2} + \mathcal{K},$$

where $\mathcal{K}$ is a generator of another Lévy semigroup. Nonlinear conservation laws with such nonlocal operators were studied in [16, 17, 18].

In view of the assumption (3.3) imposed on its symbol $a(\xi)$, the semigroup $e^{-t\mathcal{L}}$ satisfies the following decay estimates analogous to those in (2.15)–(2.16) (*cf.* [18, Sec. 2], for details):

$$\|e^{-t\mathcal{L}} v\|_p \leq Ct^{-n(1-1/p)/\alpha} \|v\|_1, \tag{3.5}$$

$$\|\nabla e^{-t\mathcal{L}} v\|_p \leq Ct^{-n(1-1/p)/\alpha - 1/\alpha} \|v\|_1, \tag{3.6}$$

for each $p \in [1, \infty]$, all $t > 0$, and a constant C depending only on p and n. The sub-Markovian property of $e^{-t\mathcal{L}}$ implies that, for every $p \in [1, \infty]$,

$$\|e^{-t\mathcal{L}} v\|_p \leq \|v\|_p. \tag{3.7}$$

Moreover, for each $p \in [1, \infty]$, we have

$$\|\nabla e^{-t\mathcal{L}} v\|_p \leq Ct^{-1/\alpha} \|v\|_p. \tag{3.8}$$

Let us also note that under the assumption (3.4), the large time behavior of $e^{-t\mathcal{L}}$ is described by the fundamental solution $p_\alpha(x, t)$, defined in (1.16), of the linear equation $u_t + (-\Delta)^{\alpha/2} u = 0$, see [42, Lemma 6.1] for more details.

We are now in a position to present our results concerning the nonlinear problem (3.1)-(3.2) starting with the fundamental problems of the existence, the uniqueness, and the regularity of solutions. Note that at this stage no restrictions are imposed on the sign of the parameter λ and the initial datum u_0. Consequently, all results of Theorem 3.2 are valid for both the deposition and the evaporation cases.

THEOREM 3.2. *Assume that the symbol $a = a(\xi)$ of the Lévy operator $\mathcal{L}$ satisfies condition (3.3) with an $\alpha \in (1, 2]$. Then, for every $u_0 \in W^{1,\infty}(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, and $T > 0$, problem (3.1)-(3.2) has the unique solution u in the space $\mathcal{X} = C([0, T), W^{1,\infty}(\mathbb{R}^n))$.*

If, additionally, there exists a constant $K \in \mathbb{R}$ such that $u_0 - K \in L^1(\mathbb{R}^n)$, then

$$u - K \in C([0, T], L^1(\mathbb{R}^n)), \quad \text{and} \quad \sup_{0 < t \leq T} t^{1/\alpha} \|\nabla u(t)\|_1 < \infty. \tag{3.9}$$

Moreover, for all $t \geq 0$,

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \text{and} \quad \|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty, \tag{3.10}$$

and the following comparison principle is valid: for any two initial data satisfying condition, for all $x \in \mathbb{R}^n$, $u_0(x) \le \tilde{u}_0(x)$, *the corresponding solutions satisfy the inequality* $u(x,t) \le \tilde{u}(x,t)$, *for all* $x \in \mathbb{R}^n$, *and* $t \ge 0$.

REMARK 3.3. *Note that if* u *is a solution of (3.1) then so is* $u - K$, *for any constant* $K \in \mathbb{R}$. *Hence, without loss of generality, in what follows we will assume that* $K = 0$.

In a recent publication, Droniou and Imbert [27] study a nonlinear-nonlocal viscous Hamilton-Jacobi equation of the form

$$u_t + (-\Delta)^{\alpha/2}u + F(t, x, u, \nabla u) = 0.$$

For $\alpha \in (0, 2)$, and under very general assumptions on the nonlinearity, they construct a unique, global-in-time viscosity solution for initial data from $W^{1,\infty}(\mathbb{R}^n)$, and emphasize (*cf.* [27, Remark 5]) that an analogous result can be obtained in the case of more general nonlocal operators, including those given by formula (1.12). That unique solution also satisfies the maximum principle (*cf.* Theorem 1.18) which implies inequalities (3.10), and the comparison principle contained in Theorem 3.2. Finally, the L^1-property of solutions stated in (3.9) (under the additional assumption $u_0 - K \in L^1(\mathbb{R}^n)$) is proved in [42, Section 3]. Here we only mention that the reasoning used in the construction of solutions of (3.1)-(3.2) involves the integral (mild) equation

$$u(t) = e^{-t\mathcal{L}}u_0 + \lambda \int_0^t e^{-(t-\tau)\mathcal{L}} |\nabla u(\tau)|^q \, d\tau, \tag{3.11}$$

estimates (3.10), and the Banach fixed point argument.

3. Large time asymptotics – the deposition case

Once the global-in-time solution u is constructed, it is natural to ask questions about its behavior as $t \to \infty$. From now onwards, equation (3.1) will be supplemented with the nonnegative integrable initial datum (3.2). In view of Theorem 3.2, the standing assumption $u_0 \in W^{1,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ allows as to define the "mass" of the solution of (3.1)-(3.2) by the formula

$$\begin{aligned} M(t) = \|u(t)\|_1 &= \int_{\mathbb{R}^n} u(x,t) \, dx \\ &= \int_{\mathbb{R}^n} u_0(x) \, dx + \lambda \int_0^t \int_{\mathbb{R}^n} |\nabla u(x,s)|^q \, dxds \end{aligned} \tag{3.12}$$

To show last equality, note that since, for every $t \ge 0$, μ^t in the representation (1.5) is a probability measure it follows from the Fubini theorem, and from the representation (1.5), that

$$\int_{\mathbb{R}^n} e^{-t\mathcal{L}}u_0(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u_0(x-y) \, \mu^t(dy) dx = \int_{\mathbb{R}^n} u_0(y) \, dy,$$

and, similarly,

$$\int_{\mathbb{R}^n} \int_0^t e^{-(t-\tau)\mathcal{L}} |\nabla u(x,\tau)|^q \, d\tau dx = \int_0^t \int_{\mathbb{R}^n} |\nabla u(x,\tau)|^q \, dxd\tau.$$

Hence, identity (3.12) is immediately obtained from equation (3.11) by integrating it with respect to x.

The large-time behavior of $M(t)$ is one of the principal objects of presented in this chapter. It turns out that in the deposition case, *i.e.*, for $\lambda > 0$, the function $M(t)$ is nondecreasing in t (*cf.* equation (3.12)) and, for sufficiently small q, escapes to $+\infty$, as $t \to \infty$.

THEOREM 3.4. *Let* $\lambda > 0$, $1 < q \leq \frac{n+\alpha}{n+1}$, *and suppose that the symbol* a *of the Lévy operator* $\mathcal{L}$ *satisfies conditions (3.3) and (3.4) with* $\alpha \in (1,2]$. *If* $u = u(x,t)$ *is a solution of (3.1) with an initial datum satisfying conditions* $0 \leq u_0 \in L^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$, *and* $u_0 \not\equiv 0$, *then* $\lim_{t\to\infty} M(t) = +\infty$.

When q is greater that the critical exponent $(n+\alpha)/(n+1)$, we are able to show that, for sufficiently small initial data, the mass $M(t)$ is uniformly bounded in time.

THEOREM 3.5. *Let* $\lambda > 0$, $q > \frac{n+\alpha}{n+1}$, *and suppose that the symbol* a *of the Lévy operator* $\mathcal{L}$ *satisfies conditions (3.3) and (3.4) with* $\alpha \in (1,2]$. *If, either* $\|u_0\|_1$ *or* $\|\nabla u_0\|_\infty$ *is sufficiently small, then* $\lim_{t\to\infty} M(t) = M_\infty < \infty$.

The smallness assumption from Theorem 3.5 can be easily formulated if we limit ourselves to $\mathcal{L} = (-\Delta)^{\alpha/2}$. In this case, it suffices to assume that the quantity $\|u_0\|_1 \|\nabla u_0\|_\infty^{(q(n+1)-\alpha-n)/(\alpha-1)}$ is smaller than a given (and small) number independent of u_0. This fact, for $\alpha = 2$, is in perfect agreement with the assumption imposed in [47]. To see this result, note that, for every $R > 0$, the equation $u_t = -(-\Delta)^{\alpha/2} u + \lambda |\nabla u|^q$ is invariant under rescaling $u_R(x,t) = R^b u(Rx, R^\alpha t)$, with $b = (\alpha - q)/(q-1)$. Choosing $R = \|\nabla u_0\|_\infty^{-1/(1+b)}$ we immediately obtain $\|\nabla u_{0,R}\|_\infty = R^{1+b} \|\nabla u_0\|_\infty = 1$. Hence, the conclusion follows from the smallness assumption imposed on $\|u_{0,R}\|_1$ in Theorem 3.5 and from the identity $\|u_{0,R}\|_1 = \|u_0\|_1 R^{b-n} = \|u_0\|_1 \|\nabla u_0\|_\infty^{(q(n+1)-\alpha-n)/(\alpha-1)}$.

If the Lévy operator $\mathcal{L}$ has a non-degenerate Brownian part, and if $q \geq 2$, we can strengthen the assertion of Theorem 3.5 and show that the mass of every solution (not necessary small) is bounded as $t \to \infty$.

THEOREM 3.6. *Let* $\lambda > 0$, $q \geq 2$, *and suppose that the Lévy diffusion operator* $\mathcal{L}$ *has a non-degenerate Brownian part. Then, each nonnegative solution of (3.1)-(3.2) with an initial datum* $u_0 \in W^{1,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ *has the mass* $M(t) = \int_{\mathbb{R}^n} u(x,t)\,dx$ *increasing to a finite limit* M_∞, *as* $t \to \infty$.

The smallness assumption imposed in Theorem 3.5 seems to be necessary. Indeed, for $\mathcal{L} = -\Delta$, it is known that if $\lambda > 0$, and $(n+2)/(n+1) < q < 2$, then there exists a solution of (3.1)-(3.2) such that $\lim_{t\to\infty} M(t) = +\infty$ (cf. [11] and [9, Thm. 2.4]). Moreover, if $\|u_0\|_1$ and $\|\nabla u_0\|_\infty$ are “large”, then the large-time behavior of the corresponding solution is dominated by the nonlinear term ([9]), and one can expect that $M_\infty = \infty$. We conjecture that analogous results hold true at least for the α-stable operator (fractional Laplacian) $\mathcal{L} = (-\Delta)^{\alpha/2}$, and for q satisfying the inequality $(n+\alpha)/(n+1) < q < \alpha$. We also conjecture that the critical exponent $q = 2$, for $\mathcal{L} = -\Delta$, should be replaced by $q = \alpha$ if $\mathcal{L}$ has a nontrivial α-stable part. In this case, for $q \geq \alpha$, we expect that, as $t \to \infty$, the mass of any nonnegative solution converges to a finite limit, just like in Theorem 3.6. Our expectation is that the proof of this conjecture can be based on a reasoning similar to that contained in the proof of Theorem 3.6. However, at this time, we were unable to obtain those estimates in a more general case.

4. Large time asymptotics – the evaporation case

In the evaporation case, $\lambda < 0$, the mass $M(t)$ is a nonincreasing function of t (see equation (3.12)), and the question, answered in the next two theorems, is when it decays to 0 and when it decays to a positive constant.

THEOREM 3.7. *Let $\lambda < 0$, $1 \le q \le \frac{n+\alpha}{n+1}$, and suppose that the symbol a of the Lévy operator $\mathcal{L}$ satisfies conditions (3.3) and (3.4). If u is a nonnegative solution of (3.1)-(3.2) with an initial datum satisfying $0 \le u_0 \in W^{1,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then $\lim_{t\to\infty} M(t) = 0$.*

Again, when q is greater than the critical exponent, the diffusion effects prevails for large times and, as $t \to \infty$, the mass $M(t)$ converges to a positive limit.

THEOREM 3.8. *Let $\lambda < 0$, $q > \frac{n+\alpha}{n+1}$, and suppose that the symbol a of the Lévy operator $\mathcal{L}$ satisfies condition (3.3). If u is a nonnegative solution of (3.1)-(3.2) with an initial datum satisfying $0 \le u_0 \in W^{1,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then $\lim_{t\to\infty} M(t) = M_\infty > 0$.*

The proof of Theorem 3.8 is based on the decay estimates of $\|\nabla u(t)\|_p$ proven in [42, Thm. 3.8]. However, as was the case for $\lambda > 0$, we can significantly simplify that reasoning for Lévy operators $\mathcal{L}$ with nondegenerate Brownian part, and $q \ge 2$; see [42, Remark 5.3].

Our final result shows that when the mass $M(t)$ tends to a finite limit M_∞, as $t \to \infty$, the solutions of problem (3.1)-(3.2) display a self-similar asymptotics dictated by the fundamental solution of the linear equation $u_t + (-\Delta)^{\alpha/2} u = 0$ which is given by the formula

$$p_\alpha(x,t) = t^{-n/\alpha} p_\alpha(xt^{-1/\alpha}, 1) = \frac{1}{(2\pi)^{/2}} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t|\xi|^\alpha}\, d\xi, \tag{3.13}$$

see Section 3. More precisely, we have

THEOREM 3.9. *Let $u = u(x,t)$ be a solution of problem (3.1)-(3.2) with $u_0 \in L^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$, and suppose that the symbol a of the Lévy operator $\mathcal{L}$ satisfies conditions (3.3) and (3.4). If $\lim_{t\to\infty} M(t) = M_\infty$ exists and is finite, then*

$$\lim_{t\to\infty} \|u(t) - M_\infty p_\alpha(t)\|_1 = 0. \tag{3.14}$$

If, additionally,

$$\|u(t)\|_p \le Ct^{-n(1-1/p)/\alpha}, \tag{3.15}$$

for some $p \in (1,\infty]$, all $t > 0$, and a constant C independent of t, then, for every $r \in [1,p)$,

$$\lim_{t\to\infty} t^{n(1-1/r)/\alpha} \|u(t) - M_\infty p_\alpha(t)\|_r = 0. \tag{3.16}$$

REMARK 3.10. *Note that, in the case $M_\infty = 0$, the results of Theorem 3.9 only give that, as $t \to \infty$, $\|u(t)\|_r$ decays to 0 faster than $t^{-n(1-1/r)/\alpha}$.*

CHAPTER 4

Other equations with Lévy operator

1. Lévy conservation laws

In this section, we present asymptotic results for the Cauchy problem for nonlinear pseudodifferential equations of the form

$$u_t + \mathcal{L}u + \nabla \mathcal{N} u = 0, \quad u(x,0) = u_0(x), \tag{4.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \geq 0$, $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, $-\mathcal{L}$ is a (linear) generator of a symmetric positive semigroup $e^{-t\mathcal{L}}$ on $L^1(\mathbb{R}^n)$, with the symbol defined by the Lévy–Khintchine formula (1.12).

The solutions of the Cauchy problem (4.4) have to be understood in some weak sense and several options are here available and have been studied in the papers quoted below. For the sake of this presentation let us just say that as the *mild* solution of (4.4) we mean a solution of the integral equation

$$u(t) = e^{-t\mathcal{L}} u_0 - \int_0^t \nabla \cdot e^{-(t-\tau)\mathcal{L}} (\mathcal{N} u)(\tau)\, d\tau, \tag{4.2}$$

motivated by the classical Duhamel formula.

Such equations are used in physical models where the diffusive behavior is affected by hopping, trapping and other nonlocal, but possibly self-similar, phenomena (see, *e.g.*, [7, 8, 23, 29, 58, 59]).

Recently, the questions of existence, uniqueness, regularity, and temporal asymptotics have been studied for certain special cases of equation (4.4), in particular, the *fractal Burgers equation* (see, [14]),

$$u_t + (-\Delta)^{\alpha/2} u + c \cdot \nabla(u|u|^{r-1}) = 0, \quad c \in \mathbb{R}^n, \tag{4.3}$$

and the one-dimensional *multifractal conservation laws* (see [16]),

$$u_t + \mathcal{L}u + f(u)_x = 0, \tag{4.4}$$

with the *multifractal operator*

$$\mathcal{L} = -a_0 \Delta + \sum_{j=1}^{k} a_j (-\Delta)^{\alpha_j/2}, \tag{4.5}$$

$0 < \alpha_j < 2$, $\;a_j > 0, j = 0, 1, \ldots, k$, where $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, is the fractional Laplacian defined is defined in Section 3. All these equations are generalizations of the classical Burgers equation

$$u_t - u_{xx} + (u^2)_x = 0. \tag{4.6}$$

Let us briefly sketch our results from [16, 17, 18] in the particular case of the Cauchy problem

$$u_t + (-\Delta)^{\alpha/2} u + b \cdot \nabla \left(u|u|^q\right) = 0, \quad u(x,0) = u_0(x). \tag{4.7}$$

Intuitively speaking, our results from [16, 17] have shown that, for q sufficiently large, the first order asymptotics (as $t \to \infty$) for solutions of (4.7) is essentially linear.

THEOREM 4.1 (Linear asymptotics). *Assume that $\alpha \in (1,2)$ and $q > 1$. Let u be the solution of the Cauchy problem (4.7). Suppose that the initial datum satisfies*

$$u_0 \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} u_0(x)\, dx = M$$

for some fixed $M \in \mathbb{R}$. If $q > (\alpha - 1)/n$, then then

$$t^{n(1-1/p)/\alpha} \|u(t) - M p_\alpha(t)\|_p \to 0 \quad \text{as} \quad t \to \infty,$$

for each $p \in [1, \infty]$, where $p_\alpha(x,t)$ is defined in (1.16).

By contrast with the results in Theorem 4.1, let us note, that the first order asymptotics of solutions of the Cauchy problem for the Burgers equation (4.6) is described by the relation

$$t^{(1-1/p)/2} \|u(t) - U_M(t)\|_p \to 0, \qquad \text{as} \qquad t \to \infty,$$

where

$$U_M(x,t) = t^{-1/2} \exp(-x^2/4t) \left(K(M) + \frac{1}{2} \int_0^{x/\sqrt{t}} \exp(-\xi^2/4)\, d\xi \right)^{-1}$$

is the, so-called, source solution such that $u(x,0) = M\delta_0$. It is easy to verify that this solution is self-similar, *i.e.*, $U_M(x,t) = t^{-1/2} U(xt^{-1/2}, 1)$. Thus, the long time behavior of solutions of the classical Burgers equation is genuinely nonlinear, i.e., it is not determined by the asymptotics of the linear heat equation.

As it turns out that genuinely nonlinear behavior of the Burgers equation is due to the precisely matched balancing influence of the regularizing Laplacian diffusion operator and the gradient-steepening quadratic inertial nonlinearity.

The next result finds such a matching critical nonlinearity exponent for the nonlocal multifractal Burgers equation.

THEOREM 4.2 (Nonlinear asymptotics). *Assume that $\alpha \in (1,2)$ and $q > 0$. Let u be the solution of the Cauchy problem (4.7). Suppose that*

$$u_0 \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} u_0(x)\, dx = M.$$

If $q = (\alpha - 1)/n$, then

$$t^{n(1-1/p)/\alpha} \|u(t) - U_M(t)\|_p \to 0 \quad \text{as} \quad t \to \infty,$$

for each $p \in [1, \infty]$, where $U_M(x,t) = t^{-n/\alpha} U_M(xt^{-1/\alpha}, 1)$ is the unique self-similar solution of the equation

$$u_t + (-\Delta)^{\alpha/2} u + b \cdot \nabla (u|u|^{(\alpha-1)/n}) = 0$$

with the initial datum $M\delta_0$.

2. Nonlocal equation in dislocation dynamics

Dislocations are line defects in crystals whose typical length is $\sim 10^{-6}$ m and their thickness is $\sim 10^{-9}$ m. When the material is submitted to shear stresses, these lines can move in the crystallographic planes and this dynamics can be observed using electron microscopy. The elementary mechanisms at the origin of the deformation of monocrystals are rather well understood, however, many questions concerning the plastic behavior of materials containing a high density of defects are still open. Hence, in recent years, new physical theories describing the collective behavior of dislocations have been developed and numerical simulations of dislocations have been performed. We refer the reader to the recent publications [4, 36] for the comprehensive references about modeling of dislocation dynamics.

One possible (simplified) model of the dislocation dynamics is given by the system of ODEs

$$\dot{y}_i = F - V_0'(y_i) - \sum_{j\in\{1,\dots,N\}\setminus\{i\}} V'(y_i - y_j) \quad \text{for} \quad i = 1, \dots, N, \tag{4.8}$$

where F is a given constant force, V_0 is a given potential and V is a potential of two-body interactions. One can think of y_i as the position of dislocation straight lines. In this model, dislocations can be of two types, $+$ or $-$, depending on the sign of their Burgers vector (see the book by Hirth and Lothe [34] for a physical definition of the Burgers vector).

Self-similar solutions (*i.e.* solutions of the form $y_i(t) = g(t)Y_i$ with constant Y_i) of system (4.8) with the particular potential $V'(z) = \frac{1}{z}$ as well as their role in the asymptotic behavior of other solutions of (4.8) were studied by Head in [32]. More recently, Forcadel *et al.* showed in [28, Th. 8.1] that, under suitable assumptions on V_0 and V in (4.8), the rescaled "cumulative distribution function"

$$\rho^\varepsilon(x,t) = \varepsilon\left(-\frac{1}{2} + \sum_{i=1}^{N} H\left(x - \varepsilon y_i\left(\frac{t}{\varepsilon}\right)\right)\right) \tag{4.9}$$

(where H is the Heaviside function) satisfies (as a discontinuous viscosity solution) the following nonlocal eikonal equation

$$\rho_t^\varepsilon(x,t) = \left(c\Big(\frac{x}{\varepsilon}\Big) + M^\varepsilon\left(\frac{\rho^\varepsilon(\cdot,t)}{\varepsilon}\right)(x)\right)|\rho_x^\varepsilon(x,t)| \tag{4.10}$$

for $(x,t) \in \mathbb{R} \times (0,+\infty)$, with $c(y) = V_0'(y) - F$. Here, M^ε is the nonlocal operator defined by

$$M^\varepsilon(U)(x) = \int_{\mathbb{R}} J(z)\, E\big(U(x+\varepsilon z) - U(x)\big)\, dz, \tag{4.11}$$

where $J(z) = V''(z)$ on $\mathbb{R}\setminus\{0\}$ and E is the modification of the integer part: $E(r) = k + 1/2$ if $k \le r < k+1$. Note that the nonlocal operator M^ε describes the interactions between dislocation lines, hence, interactions are completely characterized by the kernel J.

Next, under the assumption that the kernel J is a sufficiently smooth, even, nonnegative function with the following behavior at infinity

$$J(z) = \frac{1}{|z|^2} \quad \text{for all} \quad |z| \ge R_0 \tag{4.12}$$

and for some $R_0 > 0$, the rescaled cumulative distribution function ρ^ε, defined in (4.9), is proved to converge (*cf.* [28, Th. 2.5]) towards the unique solution of the corresponding initial value problem for nonlinear diffusion equation

$$u_t = \widetilde{H}(-\Lambda u, u_x), \tag{4.13}$$

where the Hamiltonian $\widetilde{H}$ is a continuous function and Λ is a Lévy operator of order 1. It is defined for any function $U \in C_b^2(\mathbb{R})$ and for $r > 0$ by the formula

$$-\Lambda U(x) = C(1) \int_{\mathbb{R}} \Big(U(x+z) - U(x) - zU'(x)\mathbf{1}_{\{|z|\le r\}} \Big) \frac{1}{|z|^2}\, dz \tag{4.14}$$

with a constant $C(1) > 0$. Finally, in the particular case of $c \equiv 0$ in (2.6), we have $\widetilde{H}(L,p) = L|p|$ (*cf.* [28, Th. 2.6]) which allows us to rewrite equation (4.13) in the form

$$u_t + |u_x|\Lambda u = 0. \tag{4.15}$$

One can show that the definition of Λ is independent of $r > 0$, hence, we fix $r = 1$. In fact, for suitably chosen $C(1)$, $\Lambda = \Lambda^1 = \big(-\partial^2/\partial x^2\big)^{1/2}$ is the pseudodifferential operator defined in the Fourier variables by $\widehat{(\Lambda^1 w)}(\xi) = |\xi|\widehat{w}(\xi)$ (*cf.* formula (4.21) below). In this particular case, equation (4.15) is an integrated form of a model studied by Head [33] for the self-dynamics of a dislocation density represented by u_x. Indeed, denoting $v = u_x$ we may rewrite equation (4.15) as

$$v_t + (|v|\mathcal{H}v)_x = 0, \tag{4.16}$$

where $\mathcal{H}$ is the Hilbert transform defined by

$$\widehat{(\mathcal{H}v)}(\xi) = -i\ \mathrm{sgn}(\xi)\ \widehat{v}(\xi). \tag{4.17}$$

Let us recall two well known properties of this transform

$$\mathcal{H}v(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{v(y)}{x-y}\, dy \quad \text{and} \quad \Lambda^1 v = \mathcal{H}v_x. \tag{4.18}$$

Motivated by physics described above, in [15], we study the following initial value problem for the nonlinear and nonlocal equation involving $u = u(x,t)$

$$u_t = -|u_x|\ \Lambda^\alpha u \quad \text{on} \quad \mathbb{R}\times(0,+\infty), \tag{4.19}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{4.20}$$

where the assumptions on the initial datum u_0 will be precised later. Here, for $\alpha \in (0,2)$, $\Lambda^\alpha = \big(-\partial^2/\partial x^2\big)^{\alpha/2}$ is the pseudodifferential operator discussed in Section 3. Recall that the operator Λ^α has the Lévy–Khintchine integral representation for every $\alpha \in (0,2)$

$$-\Lambda^\alpha w(x) = C(\alpha) \int_{\mathbb{R}} \big(w(x+z) - w(x) - zw'(x)\mathbf{1}_{\{|z|\le 1\}} \big) \frac{dz}{|z|^{1+\alpha}}, \tag{4.21}$$

where $C(\alpha) > 0$ is a constant. This formula (discussed in Chapter 1 for functions w in the Schwartz space) allows us to extend the definition of Λ^α to functions which are bounded and sufficiently smooth, however, not necessarily decaying at infinity.

As we have already explained (*cf.* equation (4.15)), in the particular case $\alpha = 1$, equation (4.19) is a mean field model that has been derived rigorously in [28] as the limit of a system of particles in interactions (*cf.* (4.8)) with forces $V'(z) = \frac{1}{z}$. Here, the density u_x means the positive density $|u_x|$ of dislocations of type of the sign of u_x. Moreover, the occurrence of the absolute value $|u_x|$ in the equation allows the vanishing of dislocation particles of the opposite sign. In the work [15], we study the general case $\alpha \in (0, 2)$ that could be seen as a mean field model of particles modeled by system (4.8) with repulsive interactions $V'(z) = \frac{1}{z^\alpha}$.

Here, we would like also to keep in mind that (4.19) is the simplest nonlinear anomalous diffusion model (described by the Lévy operator Λ^α) which degenerates for $u_x = 0$.

In work [15], we construct explicitly the self-similar solution of (4.19)-(4.20) and we prove its asymptotic stability. Moreover, we show the existence and the uniqueness of viscosity solutions of (4.19)-(4.20) as well as decay estimates using properties of the Lévy operator Λ^α presented in Chapter 1.

FIGURE 1. Wrocław. The view of the Grunwaldzki Bridge

From: http://wikitravel.org/en/Wroclaw

Wrocław in Polish, formally known as **Breslau** in German, is a large undiscovered gem of a city in southwestern Poland in the historic region of Silesia. It boasts fascinating architecture, many rivers and bridges, and a lively and metropolitan cultural scene. It is a city with a troubled past, having seen much violence and devastation, and was almost completely destroyed during the end of the Second World War. However, it has been brilliantly restored and can now be counted amongst the highlights of Poland, and all of Central Europe. As Poland rushes headlong into further integration with the rest of Europe, now is the time to visit before the tourist hordes (and high prices) arrive. Read Norman Davies' and Roger Moorhouse's Microcosm: Portrait of a Central European City to understand the complicated history of the town.

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Part 3

On a continuous deconvolution equation

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Abstract. We introduce in this paper the notion of "Continuous Deconvolution Equation" in a 3D periodic case. We first show how to derive this new equation from the Van Cittert algorithm. Next we show many mathematical properties of the solution to this equation. Finally, we show how to use it to introduce a new turbulence model for high Reynolds number flows.

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The deconvolution equation (1.10) has been discovered following a talk between Roger Lewandowski and Edriss Titi during the stay of R. Lewandowski in Weizmann Institute (Israel) in September 2007. It has been fixed by both during the stay of E. Titi in IRMAR, Rennes (France) in June 2008. R. Lewandowski also warmly thanks Weizmann Institute for the hospitality as well as E. Titi for many stimulating discussions.

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Contents

CHAPTER 1

Introduction and main facts

1. General orientation

It is well known since Kolmogorov's work [25], that to simulate an incompressible 3D turbulent flow using the Navier–Stokes equations,

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0, \end{cases} \tag{1.1}$$

requires about $Nb = Re^{9/4}$ points in a numerical grid (details are available in [19] or [35]). Here, $Re = UL/\nu$ denotes the Reynolds number (see a rigorous definition in 4.1.1, below in the text).

For realistic flows, such as those involved in mechanical engineering or in geophysics, Re is of order 10^8-10^{10}, sometimes much more. Therefore, the number of points Nb necessary for the simulation is huge and the amount of memory computational algorithms need distinctively exceeds memory size of most powerful modern computers. This is why one needs "turbulent models" in order to reduce the appropriate number of grid points, and to simulate at least averages of turbulent flows.

There are two main families of turbulent models: statistical models, such as the well known k-ε model (see in [31] and [35]), and Large Eddy Simulations models (see in [9] and [36]), known as "LES models". This paper deals with LES models family. The idea behind LES is to simulate the "large scale" of the flow, trying to keep energy information on the "small" scales. Eddy viscosities are mostly involved in those models.

Many models also emerged without eddy viscosity, such as Bardina's models [3] or related (see in [32], [28], [27] [26]), as well as the family of α-models and related (see for instance in [17], [21], [24], [13], [12]). All of them are still considered as LES models. They mainly aim to regularize the nonlinear term $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ in the Navier–Stokes equations.

This idea takes inspiration in the work of Jean Leray in 1934 [30]. At this time computers did not exist and people were not thinking about numerical simulations of flows past aircraft wings or about numerical simulations in weather forecast. They were mostly trying to find analytical solutions to the 3D Navier–Stokes equations in cases of laminar flows or where geometrical symmetries occur as well as where special 2D approximations were legitimate, the general case remaining out of reach. Such calculus is well explained in the famous book by G. Batchelor [4]. Therefore the question is whether the Navier–Stokes equations in the general case have a solution or do not have a solution even if it is not possible to give analytical formula for these solutions.

Jean Leray showed the existence of what we call today "a dissipative weak solution" to the Navier–Stokes equation in the whole space $\mathbb{R}^3$ (see definition 4.2 below in the text). To do this, he first constructed approximated smooth solutions to the Navier–Stokes equations. Secondly, using some compactness arguments, he considered the limit of a subsequence, showing that this limit is a dissipative weak solution, called formerly "turbulent solution". By "dissipative" solution we mean a distributional solution satisfying the energy inequality (see (4.6) below in the text).

We still do not know if there is a unique dissipative solution in the general case, and also if it does or does not develop singularities in finite time. The question of singularities for particular dissipative solutions called "suitable weak solutions", is studied in the very famous paper by Caffarelli–Kohn–Nirenberg [11].

2. Towards the models

To build approximated smooth solutions, J. Leray got the idea to replace the nonlinear term $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ by $((\mathbf{u}\star\rho_\varepsilon)\cdot\nabla)\,\mathbf{u}$, where $(\rho_\varepsilon)_{\varepsilon>0}$ is a sequence of mollifiers: doing like this, he introduced the first LES models without knowing it, a long time before Smagorinsky published his first paper in 1953 [37], Smagorinsky being often considered as a main pioneer of LES. This idea of smoothing the nonlinear term can be generalized in many other cases, such as in the periodic case we consider in this paper. In this case, one can regularize the Navier–Stokes equations by using the so called "Helmholtz equation".

Let $\mathbf{u}$ be an incompressible periodic field $\mathbf{u}$ ($\nabla\cdot\mathbf{u} = 0$), the mean value of which, $m(\mathbf{u})$ (see (2.2) below in the text), being equal to zero. Notice that in the rest of the paper, all fields we consider will have a zero mean value for compatibility reasons. We do not mention it every time so far no risk of confusion occurs. Such a field $\mathbf{u}$ being given, let us consider the Stokes problem

$$\begin{cases} A\overline{\mathbf{u}} = -\alpha^2\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}} + \nabla\pi = \mathbf{u}, \\ \nabla\cdot\overline{\mathbf{u}} = 0. \end{cases} \tag{1.2}$$

The parameter α is the "small parameter". It is generally agreed that α must be taken about the numerical grid size in numerical simulations, even if this claim is sometimes subject to caution.

The Leray-α model is the one where the nonlinear term in the Navier–Stokes equations is regularized by taking $(\overline{\mathbf{u}}\cdot\nabla)\,\mathbf{u}$ in place of $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$. The Bardina's model of order zero is the one where one replaces the nonlinear term by $\overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}$. The solutions of these approximated Navier–Stokes equations are supposed to give approximations of mean values of pressure and velocity fields. To see this, let us take average of (1.1). We get the following "right" equation for $\overline{\mathbf{u}}$,

$$\begin{cases} \partial_t\overline{\mathbf{u}} + \overline{(\mathbf{u}\cdot\nabla)\,\mathbf{u}} - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}}, \\ \nabla\cdot\overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}}(0,\mathbf{x}) = \overline{\mathbf{u}}_0, \end{cases} \tag{1.3}$$

that we can rewrite as

$$\begin{cases} \partial_t \overline{\mathbf{u}} + B_\alpha(\overline{\mathbf{u}}, \overline{\mathbf{u}}) - \nu \Delta \overline{\mathbf{u}} + \nabla \overline{p} = \overline{\mathbf{f}} + B_\alpha(\overline{\mathbf{u}}, \overline{\mathbf{u}}) - \overline{(\mathbf{u} \cdot \nabla)\, \mathbf{u}}, \\ \nabla \cdot \overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}}(0, \mathbf{x}) = \overline{\mathbf{u}}_0, \end{cases} \tag{1.4}$$

where $B_\alpha(\overline{\mathbf{u}}, \overline{\mathbf{u}})$ is a nonlinear term depending on α and that is “regular enough”. In the model, $B_\alpha(\overline{\mathbf{u}}, \overline{\mathbf{u}})$ must replace $\overline{(\mathbf{u} \cdot \nabla)\, \mathbf{u}}$, and $R_\alpha = B_\alpha(\overline{\mathbf{u}}, \overline{\mathbf{u}}) - \overline{(\mathbf{u} \cdot \nabla)\, \mathbf{u}}$ is a residual stress that we neglect for more or less legitimate physical or numerical reasons. Then the principle of the model consists in simulating flows by computing an approximation of $\overline{\mathbf{u}}$ and $\overline{p}$, denoted by $\mathbf{u}_\alpha$ and p_α, being a solution to

$$\begin{cases} \partial_t \mathbf{u}_\alpha + B_\alpha(\mathbf{u}_\alpha, \mathbf{u}_\alpha) - \nu \Delta \mathbf{u}_\alpha + \nabla p_\alpha = \overline{\mathbf{f}}, \\ \nabla \cdot \mathbf{u}_\alpha = 0, \\ \mathbf{u}_\alpha(0, \mathbf{x}) = \overline{\mathbf{u}}_0. \end{cases} \tag{1.5}$$

Such a model is relevant if:

- B_α correctly filters high frequencies and describes with accuracy low frequencies.
- System (1.5) has a unique “smooth enough” solution when $\mathbf{u}_0 \in L^2_{loc}$ (therefore $\overline{\mathbf{u}}_0 \in H^2_{loc}$). By “smooth enough” we mean $\mathbf{u} \in L^\infty([0,T], (H^1_{loc})^3) \cap L^2([0,T], (H^2_{loc})^3)$, $p \in L^2([0,T], H^1_{loc})$, in any time interval $[0,T]$.
- The unique solution $(\mathbf{u}_\alpha, p_\alpha)$ to (1.5) satisfies an energy balance like (4.15) (and not only an energy inequality like (4.6), see below in the text), for α fixed.
- There is a subsequence of the sequence $(\mathbf{u}_\alpha, p_\alpha)_{\alpha>0}$ which converges (in a certain sense) to a dissipative weak solution to (1.1) when α goes to zero.

We must say that there are many B_α such that the last three points on the list above are satisfied. But in order to use these equations to simulate realistic flows we must check the first point. Unfortunately there is no rigorous definition that can make this point precise, see also linked notion of “cut frequency”.

3. Approximate deconvolution models

In 1999 and later, Adams and Stolz ([1], [39], [38], [2]) were considering “the Bardina’s model of order zero” where $B_\alpha(\mathbf{u}, \mathbf{u}) = \overline{\nabla \cdot (\mathbf{u} \otimes \mathbf{u})} = \overline{(\mathbf{u} \cdot \nabla)\mathbf{u}}$. In order to improve reconstruction of the right field in numerical simulations, they got the idea to apply a “deconvolution operator D_N”. To do it, they introduced a parameter N of deconvolution, using the discrete “van Cittert algorithm” (see in [10]),

$$\begin{cases} \mathbf{w}_0 = \overline{\mathbf{u}}, \\ \mathbf{w}_{N+1} = \mathbf{w}_N + (\overline{\mathbf{u}} - A^{-1}\mathbf{w}_N), \end{cases} \tag{1.6}$$

where the operator A is defined in (1.2). The deconvolution operator is defined by $H_N(\mathbf{u}) = \mathbf{w}_N = D_N(\overline{\mathbf{u}})$. It is fixed such that for given incompressible field $\mathbf{u}$, $H_N(\mathbf{u}) = D_N(\overline{\mathbf{u}})$ goes to $\mathbf{u}$ in a certain sense (see Section 1 in Chapter 3 below). Therefore, the model consists in replacing the nonlinear term by

$$B_{\alpha,N}(\mathbf{u}, \mathbf{u}) = \overline{\nabla \cdot (D_N(\mathbf{u}) \otimes D_N(\mathbf{u}))},$$

that gives model

$$\begin{cases} \partial_t \mathbf{u}_{\alpha,N} + \overline{\nabla \cdot (D_N(\mathbf{u}_{\alpha,N}) \otimes D_N(\mathbf{u}_{\alpha,N}))} - \nu\Delta\mathbf{u}_{\alpha,N} + \nabla p_{\alpha,N} = D_N(\overline{\mathbf{f}}), \\ \nabla \cdot \mathbf{u}_{\alpha,N} = 0, \\ \mathbf{u}_{\alpha,N}(0,\mathbf{x}) = D_N(\overline{\mathbf{u}}_0) = H_N(\mathbf{u}_0). \end{cases} \tag{1.7}$$

This model is called an "Approximate deconvolution model". Existence, regularity and uniqueness of a solution to this model for general deconvolution order N, were proved by Dunca–Epshteyn in 2006 [16]. The case $N = 0$ was already studied in detail before in [26], [27], [32]. Questions of accuracy and error estimates were also studied in [28] for general order of deconvolution N.

The exciting point in model (1.7) is that it formally "converges" to the right averaged Navier–Stokes equations (1.3) when N goes to infinity and α is fixed. A suitable choice of the deconvolution order N combined with a suitable choice of α, gives hope that we can approach with a good accuracy the right average of the real field, defined by the Navier–Stokes equations (expecting uniqueness of the dissipative solution).

Therefore, we had to investigate the problem of the convergence of $(\mathbf{u}_{\alpha,N}, p_{\alpha,N})_{N\in\mathbb{N}}$ to a solution of the mean Navier–Stokes equations (1.3) when N goes to infinity. This problem is very tough, and we got very recently ideas how to solve it [8]. Earlier, in [29], we got an idea to introduce a simplified deconvolution model, where the nonlinear term is $(H_N(\mathbf{u}) \cdot \nabla)\,\mathbf{u}$

$$\begin{cases} \partial_t \mathbf{u}_{\alpha,N} + (H_N(\mathbf{u}_{\alpha,N}) \cdot \nabla)\,\mathbf{u}_{\alpha,N} - \nu\Delta\mathbf{u}_{\alpha,N} + \nabla p_{\alpha,N} = H_N(\mathbf{f}), \\ \nabla \cdot \mathbf{u}_{\alpha,N} = 0, \\ \mathbf{u}_{\alpha,N}(0,\mathbf{x}) = H_N(\mathbf{u}_0). \end{cases} \tag{1.8}$$

In [29] we proved existence, uniqueness and regularity of a solution $(\mathbf{u}_{\alpha,N}, p_{\alpha,N})$ to (1.8), and also that a subsequence of the sequence $(\mathbf{u}_{\alpha,N}, p_{\alpha,N})_{N\in\mathbb{N}}$ converges, in a certain sense, to a dissipative weak solution of the Navier–Stokes equations for a fixed α, when N goes to infinity.

4. The deconvolution equation and outline of the remainder

All the models we have shown above have been well studied in the periodic case. This calls for the question of adapting them in cases of realistic boundary conditions.

We have considered an ocean forced by the atmosphere, under the rigid lid hypothesis with a mean flux condition on the surface (see in [31]). As we started working on this question, it appeared soon that we were not able to do the job for the Adams–Stolz deconvolution model (1.7), often known as ADM model. Indeed, if we keep the natural boundary condition on the surface, we cannot write an identity like

$$\int_\Omega \overline{\nabla \cdot (D_N(\mathbf{u}) \otimes D_N(\mathbf{u}))} \cdot \mathbf{u} = 0,$$

though it is the key to get the $L^2([0,T],(H^2)^3) \cap L^\infty([0,T],(H^1)^3)$ estimate in the periodic case. Therefore even modeling of the boundary condition remains an open problem in task to derive an ADM model which fits with the physics and has good mathematical properties.

Facing the difficulty in the question of boundary conditions in model (1.7), we turned to another deconvolution model we have in hand, the model (1.8), although we take ADM model (1.7) for the best one in this class of models. Indeed, (1.7) really approaches the averaged Navier–Stokes equations for high deconvolution's order making it a right LES model, at least formally, when model (1.8) approaches the right Navier–Stokes equations, fading the role of α, a fact we cannot physically interpret, although it shows a good numerical behavior (see in [5]).

We next thought that fixing the van Cittert algorithm with realistic boundary conditions would be easy. Unfortunately, we had troubles when rewriting it in the form (1.6), precisely because of the boundary conditions. This is why we decided to replace the van Cittert algorithm by a continuous variational problem. Our key observation is that this algorithm can be written in the form

$$-\alpha^2 \left(\frac{\Delta \mathbf{w}_{N+1} - \Delta \mathbf{w}_N}{\delta\tau} \right) + \mathbf{w}_{N+1} + \nabla \pi_{N+1} = \mathbf{u}, \tag{1.9}$$

with $\delta\tau = 1$. This is precisely the finite difference equation corresponding to the continuous equation

$$\begin{cases} -\alpha^2 \Delta \left(\dfrac{\partial \mathbf{w}}{\partial \tau} \right) + \mathbf{w} + \nabla \pi = \mathbf{u}, \\ \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}(0, \mathbf{x}) = \overline{\mathbf{u}}. \end{cases} \tag{1.10}$$

We set

$$H_\tau(\mathbf{u}) = \mathbf{w}(\tau, \mathbf{x}).$$

The parameter τ is a non dimensional parameter. We call it "deconvolution parameter". Equation (1.10) is called the "deconvolution equation". The corresponding LES model becomes

$$\begin{cases} \partial_t \mathbf{u}_{\alpha,\tau} + (H_\tau(\mathbf{u}_{\alpha,\tau}) \cdot \nabla)\, \mathbf{u}_{\alpha,\tau} - \nu \Delta \mathbf{u}_{\alpha,\tau} + \nabla p_{\alpha,\tau} = H_\tau(\mathbf{f}), \\ \nabla \cdot \mathbf{u}_{\alpha,\tau} = 0, \\ \mathbf{u}_{\alpha,\tau}(0, \mathbf{x}) = H_\tau(\mathbf{u}_0). \end{cases} \tag{1.11}$$

This model appears first in [7] and [6], in the case of the ocean. It also constitutes a part of the PhD thesis of A.-C. Bennis [5], who made very good numerical tests in 2D cases with the software FreeFem++ [23], showing that this model deserves further numerical investigations in realistic 3D situations, compared with *in situ data*, a work which remains to be done.

The goal of the rest of this paper is to study in detail the deconvolution equation and the related model (1.11) in the 3D periodic case. For pedagogical reasons and for the simplicity, we study the deconvolution equation in the scalar case. By virtue of periodicity, we can express the solution of this equation in terms of Fourier's series. The same analysis holds for incompressible 3D fields.

We next show existence and uniqueness of a solution $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$ to problem (1.11) for α and τ fixed, the solution being "regular enough". We finish the paper by showing that there exists a sequence τ_n which goes to infinity when n goes to infinity, and such that the sequence $(\mathbf{u}_{\alpha,\tau_n}, p_{\alpha,\tau_n})_{n\in\mathbb{N}}$ converges to a dissipative weak solution to the Navier–Stokes equations when n goes to infinity, always when α is fixed.

The rest of the paper is organized as follows. We start by giving some mathematical tools such as function spaces we are working with and the Helmholtz equation. We next turn to the study of the continuous deconvolution equation. As we have already said, for the sake of simplicity and clarity we will show results in the scalar case, so far the generalization to incompressible fields is straightforward. In a last section, we will study the model (1.11) and prove the announced results.

CHAPTER 2

Mathematical tools

1. General background

Let $L \in \mathbb{R}_+^\star$, $\Omega = [0, L]^3 \subset \mathbb{R}^3$. By $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ we denote the orthonormal basis in $\mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ denotes a point in $\mathbb{R}^3$. Let us first start with some basic definitions.

Definition 2.1.

(1) A function $u : \mathbb{R}^3 \to \mathbb{C}$ is said to be Ω-periodic if and only if for all $\mathbf{x} \in \mathbb{R}^3$, for all $(p, q, r) \in \mathbb{Z}^3$ one has $u(\mathbf{x} + L(p\,\mathbf{e}_1 + q\,\mathbf{e}_2 + r\,\mathbf{e}_3)) = u(\mathbf{x})$.

(2) $\mathcal{D}_{per}$ denotes all functions Ω-periodic of class C^∞.

(3) We put $\mathcal{T}_3 = 2\pi\mathbb{Z}^3/L$. Let $\mathbb{T}_3$ be the torus defined by $\mathbb{T}_3 = \left(\mathbb{R}^3/\mathcal{T}_3\right)$.

(4) When $p \in [1, \infty[$, by $L^p(\mathbb{T}_3)$ we denote the function space defined by $L^p(\mathbb{T}_3) = \{u : \mathbb{R}^3 \to \mathbb{C},\, u \in L^p_{loc}(\mathbb{R}^3),\, u \text{ is } \Omega\text{−periodic}\}$, endowed with the norm $||u||_{0,p} = \left(\frac{1}{L^3}\int_{\mathbb{T}_3} |u(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}$. When $p = 2$, $L^2(\mathbb{T}_3)$ is a Hermitian space with the Hermitian product

$$(u, v) = \frac{1}{L^3}\int_{\mathbb{T}_3} u(x)\overline{v}(x)dx. \tag{2.1}$$

(5) Let $u \in L^1(\mathbb{T}_3)$. We put $m(u) = \int_\Omega u(\mathbf{x})d\mathbf{x}$.

(6) Let $s \in \mathbb{R}^+$. By $H^s_{per,0}(\mathbb{R}^3)$ we denote the space

$$H^s_{per,0}(\mathbb{R}^3) = \{u : \mathbb{R}^3 \to \mathbb{C},\ u \in H^s_{loc}(\mathbb{R}^3), u \text{ is } \Omega - \text{periodic},\ m(u) = 0\}. \tag{2.2}$$

The space $H^s_{per,0}(\mathbb{R}^3)$ is endowed by the induced topology of the classical space $H^s(\mathbb{T}_3)$.

(7) For $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{T}_3$, we put $|\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2$, $|k|_\infty = \sup_i |k_i|$, $I_n = \{\mathbf{k} \in \mathcal{T}_3;\ |\mathbf{k}|_\infty \le n\}$.

(8) We say that a Ω-periodic function P is a trigonometric polynomial if there exists $n \in \mathbb{N}$ and coefficients $a_\mathbf{k}$, $\mathbf{k} \in I_n$, and such that $P = \sum_{\mathbf{k}\in I_n} a_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$. The degree of P is the greatest q such that there is a $\mathbf{k}$ with $|\mathbf{k}|_\infty = q$ and $a_\mathbf{k} \ne 0$.

(9) By V_n we denote the finite dimensional space of all trigonometric polynomials of degree less than n with mean value equal to zero,

$$V_n = \{u = \sum_{\mathbf{k}\in I_n} u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},\ u_\mathbf{0} = 0\},$$

and $\mathbb{P}_n$ the orthogonal projection from $L^2(\mathbb{T}_3)$ onto its closed subspace V_n.

(10) Let us put $\mathcal{I}_3 = \mathcal{T}_3^\star = (2\pi\mathbb{Z}^3/L) \setminus \{0\}$.

A real number s being given, we consider the function space $\mathbb{H}_s$ defined by

$$\mathbb{H}_s = \left\{ u : \mathbb{R}^3 \to \mathbb{C},\ u = \sum_{\mathbf{k} \in \mathcal{T}_3} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},\ u_{\mathbf{0}} = 0,\ \sum_{\mathbf{k} \in \mathcal{T}_3} |\mathbf{k}|^{2s} |u_{\mathbf{k}}|^2 < \infty \right\}. \tag{2.3}$$

We put

$$||u||_{s,2} = \left(\sum_{\mathbf{k} \in \mathcal{I}_3} |\mathbf{k}|^{2s} |u_{\mathbf{k}}|^2 \right)^{\frac{1}{2}}, \quad (u,v)_s = \sum_{\mathbf{k} \in \mathcal{I}_3} |\mathbf{k}|^{2s} u_{\mathbf{k}} \overline{v}_{\mathbf{k}}. \tag{2.4}$$

In the formula above, $\overline{v}_{\mathbf{k}}$ stands for the complex conjugate of $v_{\mathbf{k}}$. The following can be proved (see in [33])

- For all $s \geq 0$, the space $\mathbb{H}_s$ is a Hermitian space, isomorphic to space $H^s_{per,0}(\mathbb{R}^3)$.
- One always has $(\mathbb{H}_s)' = \mathbb{H}_{-s}$,

DEFINITION 2.2. Let $s \geq 0$ and $\mathbb{H}_s^{\mathbb{R}}$ be a closed subset of $\mathbb{H}_s$ made of all real valued functions $u \in \mathbb{H}_s$,

$$\mathbb{H}_s^{\mathbb{R}} = \{ u \in \mathbb{H}_s,\ \forall \mathbf{x} \in \mathbb{T}_3,\ u(\mathbf{x}) = \overline{u(\mathbf{x})} \}. \tag{2.5}$$

2. Basic Helmholtz filtration

Let $\alpha > 0$, $s \geq 0$, $u \in \mathbb{H}_s$ and let $\overline{u} \in \mathbb{H}_{s+2}$ be the unique solution to the equation

$$-\alpha^2 \Delta \overline{u} + \overline{u} = u. \tag{2.6}$$

We are aware that $\overline{u}$ could be confused with the complex conjugate of u instead of the solution of the Helmholtz equation (2.6). Unfortunately, this is also the usual notation used by many authors working on the topic. This is why we decided to keep the notations like that, expecting that no confusion will occur. We also shall denote by A the operator

$$A : \left\{ \begin{array}{l} \mathbb{H}_{s+2} \longrightarrow \mathbb{H}_s, \\ w \longrightarrow -\alpha^2 \Delta w + w. \end{array} \right. \tag{2.7}$$

Therefore, one has

$$\overline{u} = A^{-1} u. \tag{2.8}$$

It is easily checked that if $u = \sum_{\mathbf{k} \in \mathcal{T}_3} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$, then

$$\overline{u} = \sum_{\mathbf{k} \in \mathcal{T}_3} \frac{u_{\mathbf{k}}}{1 + \alpha^2 |\mathbf{k}|^2} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{2.9}$$

Formula (2.9) yields easily estimates

$$||\overline{u}||_{s+2,2} \leq \frac{1}{\alpha^2} ||u||_{s,2}, \qquad ||\overline{u} - u||_{s,2} \leq \alpha ||u||_{s+1,2}. \tag{2.10}$$

We will sometimes use notation $\overline{u}_\alpha$ instead of $\overline{u}$, if we need to recall the dependence on parameter α.

THEOREM 2.3. *Assume $u \in \mathbb{H}_s$. Then the sequence $(\overline{u}_\alpha)_{\alpha > 0}$ converges strongly to u in the space $\mathbb{H}_s$.*

PROOF. By definition, one has

$$||\overline{u} - u||_{s,2}^2 = \sum_{\mathbf{k}\in\mathcal{I}_3} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^2 |\mathbf{k}|^{2s}|u_{\mathbf{k}}|^2.$$

Let $\varepsilon > 0$. As $u \in \mathbb{H}_s$, there exists N be such that

$$\sum_{\mathbf{k}\in\mathcal{I}_3\setminus I_N} |\mathbf{k}|^{2s}|u_{\mathbf{k}}|^2 \leq \frac{\varepsilon}{2},$$

and since $\alpha^2|\mathbf{k}|^2/(1+\alpha^2|\mathbf{k}|^2) \leq 1$,

$$\mathbf{I}_N = \sum_{\mathbf{k}\in\mathcal{I}_3\setminus I_N} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^2 |\mathbf{k}|^{2s}|u_{\mathbf{k}}|^2 \leq \frac{\varepsilon}{2}.$$

On the other hand, because the set I_N is finite,

$$\lim_{\alpha\to 0} \sum_{\mathbf{k}\in I_N^\star} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^2 |\mathbf{k}|^{2s}|u_{\mathbf{k}}|^2 = 0.$$

Therefore, there exists $\alpha_0 > 0$ be such that for each $\alpha \in]0, \alpha_0[$ one has

$$\mathbf{J}_N = \sum_{\mathbf{k}\in I_N^\star} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^2 |\mathbf{k}|^{2s}|u_{\mathbf{k}}|^2 \leq \frac{\varepsilon}{2}.$$

As $||\overline{u} - u||_{s,2}^2 = \mathbf{I}_N + \mathbf{J}_N$, then for all $\alpha \in]0, \alpha_0[$, one has $||\overline{u} - u||_{s,2}^2 \leq \varepsilon$ ending the proof like that. □

CHAPTER 3

From discrete to continuous deconvolution operator

1. The van Cittert algorithm

Let us consider the operator

$$D_N = \sum_{n=0}^{N} (I - A^{-1})^n.$$

We introduce the operator

$$H_N(u) = D_N(\overline{u}). \tag{3.1}$$

A straightforward calculation gives

$$H_N\left(\sum_{\mathbf{k}\in\mathcal{I}_3} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\right) = \sum_{\mathbf{k}\in\mathcal{I}_3}\left(1 - \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^{N+1}\right) u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{3.2}$$

One can prove the following (see in [29]):

- Let $s \in \mathbb{R}$, $u \in \mathbb{H}_s$. Then $H_N(u) \in \mathbb{H}_{s+2}$ and $||H_N(u)||_{s+2,2} \leq C(N,\alpha)||u||_{s,2}$, where $C(N,\alpha)$ blows up when α goes to zero and/or N goes to infinity. This is due to the fact
$$\left(1 - \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^{N+1}\right) \approx \frac{N+1}{\alpha^2|\mathbf{k}|^2} \quad \text{as} \quad |\mathbf{k}|_\infty \to \infty.$$
- The operator H_N maps continuously $\mathbb{H}_s$ into $\mathbb{H}_s$ and $||H_N||_{\mathcal{L}(\mathbb{H}_s)} = 1$.
- Let $u \in \mathbb{H}_s$. Then the sequence $(H_N(u))_{N\in\mathbb{N}}$ converges strongly to u in $\mathbb{H}_s$ when N goes to infinity.

Let us put $w_0 = \overline{u}$, $\overline{u}_N = H_N(u)$. We now show how one can compute each w_N thanks to the van Cittert algorithm (see also in [10]), starting from the definition

$$w_N = \sum_{n=0}^{N} (I - A^{-1})^n \overline{u}. \tag{3.3}$$

When A^{-1} acts on both sides in (3.3), one gets

$$\begin{aligned} A^{-1} w_N = \sum_{n=0}^{N} A^{-1}(I - A^{-1})^n \overline{u} &= -\sum_{n=0}^{N} (I - A^{-1})^{n+1}\overline{u} + \sum_{n=0}^{N} (I - A^{-1})^n \overline{u} \\ &= -w_{N+1} + \overline{u} + w_N. \end{aligned} \tag{3.4}$$

In summary, the van Cittert algorithm is the following:

$$\begin{cases} w_0 = \overline{u}, \\ w_{N+1} = w_N + (\overline{u} - A^{-1}w_N). \end{cases} \tag{3.5}$$

2. The continuous deconvolution equation

Applying A on both sides of (3.5) yields

$$Aw_{N+1} - Aw_N + w_N = A\overline{u} = u.$$

Using the definition of A, $Aw = -\alpha^2\Delta w + w$, one deduces the equality

$$-\alpha^2(\Delta w_{N+1} - \Delta w_N) + w_{N+1} = u. \tag{3.6}$$

Here is the analogy. Let $\delta\tau > 0$ be a real number and consider the equation

$$-\alpha^2\left(\frac{\Delta w_{N+1} - \Delta w_N}{\delta\tau}\right) + w_{N+1} = u. \tag{3.7}$$

We notice the following facts:

- equation (3.6) is a special case of equation (3.7) when $\delta\tau = 1$,
- equation (3.7) is a finite difference scheme that corresponds to the equation satisfied by the variable $w = w(\tau, \mathbf{x})$, $\tau > 0$,

$$\begin{cases} -\alpha^2\Delta\left(\dfrac{\partial w}{\partial \tau}\right) + w = u, \\ w(0, \mathbf{x}) = \overline{u}(\mathbf{x}), \end{cases} \tag{3.8}$$

with the zero mean condition $m(w) = 0$ so far u also satisfies $m(u) = 0$ as well as $m(\overline{u}) = 0$. We call equation (3.8) *the continuous deconvolution equation*. The parameter τ is dimensionless. We call it *the deconvolution parameter*.

Before doing anything, we first make change of variable $v(\tau, \mathbf{x}) = w(\tau, \mathbf{x}) - u(\mathbf{x})$. The variable v is solution to the equation

$$\begin{cases} -\alpha^2\Delta\left(\dfrac{\partial v}{\partial \tau}\right) + v = 0, \\ v(0, \mathbf{x}) = \overline{u}(\mathbf{x}) - u(\mathbf{x}), \end{cases} \tag{3.9}$$

with periodic boundary conditions. We also keep in mind that we impose all variables to have a zero mean value over a cell, a fact we shall not recall every time.

In the rest of this section, we will study in detail the solution of problem (3.9) and thus problem (3.8) that we will solve completely. To do this, we will express the solution in terms of Fourier series.

We search for a solution $v(\tau, \mathbf{x})$ as

$$v(\tau, \mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{I}_3} v_{\mathbf{k}}(\tau)e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, \tag{3.10}$$

with initial condition, with obvious notation,

$$v_{\mathbf{k}}(0) = -\frac{\alpha^2|\mathbf{k}|^2}{1 + \alpha^2|\mathbf{k}|^2}u_{\mathbf{k}} = (\overline{u} - u)_{\mathbf{k}}. \tag{3.11}$$

We deduce that each mode at frequency $\mathbf{k}$ satisfies the differential equation

$$\begin{cases} \alpha^2|\mathbf{k}|^2 \dfrac{dv_{\mathbf{k}}}{d\tau} + v_{\mathbf{k}} = 0, \\ v_{\mathbf{k}}(0) = (\overline{u} - u)_{\mathbf{k}}. \end{cases} \tag{3.12}$$

We deduce that

$$v_{\mathbf{k}}(\tau) = (\overline{u} - u)_{\mathbf{k}} e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}}. \tag{3.13}$$

Therefore, the general solution to problem (3.8) is

$$w(\tau, \mathbf{x}) = u(\mathbf{x}) - \sum_{\mathbf{k}\in\mathcal{I}_3} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2} \right) u_{\mathbf{k}} e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, \tag{3.14}$$

where

$$u = \sum_{\mathbf{k}\in\mathcal{I}_3} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}.$$

3. Various properties of the deconvolution equation

We now prove general properties satisfied by the solution of the deconvolution equation, using either equation (3.8) itself, or formula (3.14).

In the following, we put

$$H_\tau(u) = H_\tau(u)(\tau, \mathbf{x}) = w(\tau, \mathbf{x}), \tag{3.15}$$

where $v(\tau, \mathbf{x})$ is the solution to equation (3.8).

LEMMA 3.1. *Let $s \in \mathbb{R}$, $u \in \mathbb{H}_s$. Then for all $\tau \geq 0$, $H_\tau(u) \in \mathbb{H}_s$ and*

$$||H_\tau(u)||_{s,2} \leq 2||u||_{s,2}. \tag{3.16}$$

PROOF. Since one has for every $\tau \geq 0$ and every $\mathbf{k} \in \mathcal{I}_3$

$$0 \leq \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2} \right) e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}} \leq 1,$$

the result is a direct consequence of (3.14). □

LEMMA 3.2. *Let $\alpha > 0$ be fixed, $s \in \mathbb{R}$ and $u \in \mathbb{H}_s$. Then $(H_\tau(u))_{\tau>0}$ converges strongly to u in $\mathbb{H}_s$, when $\tau \to \infty$.*

PROOF. One has

$$u - H_\tau(u) = \sum_{\mathbf{k}\in\mathcal{I}_3} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2} \right) u_{\mathbf{k}} e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},$$

which yields

$$||u - H_\tau(u)||_{s,2}^2 = \sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} \left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2} \right)^2 |u_{\mathbf{k}}|^2 e^{-\frac{2\tau}{\alpha^2|\mathbf{k}|^2}} \leq e^{-\frac{2\tau}{\alpha^2}} ||u||_{s,2}.$$

Therefore, $\lim_{\tau\to 0} ||u - H_\tau(u)||_{s,2} = 0$, and the proof is finished. □

LEMMA 3.3. *Let $\alpha > 0$ and $\tau \geq 0$ be fixed, $s \in \mathbb{R}$ and $u \in \mathbb{H}_s$. Then $H_\tau(u) \in \mathbb{H}_{s+2}$ and one has*

$$||H_\tau(u)||_{s+2,2} \leq \frac{C(L)(1+\tau)}{\alpha^2} ||u||_{s,2}, \tag{3.17}$$

where $C(L)$ is a constant which only depends on the box size L.

PROOF. Let us write equation (3.8) in the form

$$-\alpha^2 \Delta \frac{\partial H_\tau(u)}{\partial \tau} = u - H_\tau(u).$$

Since we already know that $u - H_\tau(u) \in \mathbb{H}_s$, we deduce from the standard elliptic theory that

$$\frac{\partial H_\tau(u)}{\partial \tau} \in \mathbb{H}_{s+2}, \quad \left|\left|\frac{\partial H_\tau(u)}{\partial \tau}\right|\right|_{s+2,2} \leq \frac{C(L)}{\alpha^2}||u - H_\tau(u)||_{s,2} \leq \frac{3C(L)}{\alpha^2}||u||_{s,2} \tag{3.18}$$

We now write

$$H_\tau(u) = \overline{u} + \int_0^\tau \frac{\partial H_{\tau'}(u)}{\partial \tau'} d\tau',$$

The result is a consequence of (3.18) combined with (2.10). □

4. An additional convergence result

We finish this section devoted to the continuous deconvolution equation by a convergence result. Indeed, when one studies existence result for some variational problem such as the Navier–Stokes equations and related, one usually must prove some compactness or continuity result. In all cases, there is one moment when one faces the question of studying a sequence $(u_n)_{n\in\mathbb{N}}$ of approximated solutions which converges to some u in a certain sense, and one must identify the equation satisfied by u.

The problem we are working with uses the operator $u \to H_\tau(u)$. Among many compactness results that we potentially can prove, we will restrict ourself to one we will use in the next section.

We are studying evolution problems. Therefore the functions (and later the fields) we consider are time dependent, that means $u = u(t, \mathbf{x})$ for $\mathbf{x} \in \mathbb{T}_3$ and t belongs to a time interval $[0, T]$. Let $s \geq 0$; the space $L^2([0, T], \mathbb{H}_s)$ can easily be described to be a set of all functions $u : \mathbb{T}_3 \to \mathbb{C}$ that can be decomposed as Fourier series (see in [33])

$$u = \sum_{\mathbf{k}\in\mathcal{I}_3} u_{\mathbf{k}}(t) e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} \text{ be such that } ||u||^2_{L^2([0,T],\mathbb{H}_s)} = \sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} \int_0^T |u_{\mathbf{k}}(t)|^2 dt < \infty.$$

LEMMA 3.4. *Let $\alpha > 0$ and $\tau > 0$ be fixed. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2([0, T], \mathbb{H}_s)$ which converges strongly to u in the space $L^2([0, T], \mathbb{H}_s)$. Therefore, $(H_\tau(u_n))_{n\in\mathbb{N}}$ converges to $H_\tau(u)$ strongly in $L^2([0, T], \mathbb{H}_s)$ when $n \to \infty$.*

PROOF. We use formula (3.14) to estimate $H_\tau(u_n) - H_\tau(u)$. Therefore one has, with obvious notations

$$H_\tau(u_n) - H_\tau(u) = u_n - u + \sum_{\mathbf{k}\in\mathcal{I}_3} \left(\frac{\alpha^2|\mathbf{k}|^2}{1 + \alpha^2|\mathbf{k}|^2}\right)(u_{\mathbf{k},n} - u_{\mathbf{k}}) e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2} + \mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{3.19}$$

This yields the estimate

$$||H_\tau(u_n) - H_\tau(u)||_{L^2([0,T],\mathbb{H}_s)} \leq 2||u_n - u||_{L^2([0,T],\mathbb{H}_s)}, \tag{3.20}$$

because

$$\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right) e^{-\frac{\tau}{\alpha^2|\mathbf{k}|^2}} \leq 1.$$

The result is then a direct consequence of (3.20). □

CHAPTER 4

Application to the Navier–Stokes equations

1. Dissipative solutions to the Navier–Stokes equations

Let us start by writing again the Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \\ m(\mathbf{u}) = \mathbf{0}, \quad m(p) = 0, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0 \end{cases} \tag{4.1}$$

Here, $\mathbf{u}$ stands for the velocity and p for the pressure, and they are both the unknowns. Since the fields are real valued and periodic, one can consider them as fields from $\mathbb{T}_3$ to $\mathbb{R}^3$ for the velocity, from $\mathbb{T}_3$ to $\mathbb{R}$ for the pressure. The right hand side $\mathbf{f}$ is a datum of the problem as well as the kinematic viscosity $\nu > 0$. Recall that

$$m(\mathbf{u}) = \int_\Omega \mathbf{u}(t,\mathbf{x})d\mathbf{x}, \quad m(p) = \int_\Omega p(t,\mathbf{x})d\mathbf{x}.$$

Recall that for fields satisfying $\nabla\cdot\mathbf{u} = 0$, one always has $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot(\mathbf{u}\otimes\mathbf{u})$. We shall use sometimes this identity when we need it, without special warnings. Let us recall some facts and notation.

Definition 4.1.

(1) The Reynolds number Re is defined as $Re = UL/\nu$, where L is the box size, U is a typical velocity scale, for instance

$$U = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{L^3}\int_\Omega |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}\right)^{1/2} dt,$$

where lim stands for the generalized Banach limit (see in [14], [15] and [18]).

(2) Let $s \geq 0$. We set

$$\mathbf{IH}_s = \{\mathbf{u} \in (\mathbb{H}_s^{\mathbb{R}})^3, \quad \nabla\cdot\mathbf{u} = 0\}.$$

The space $\mathbf{IH}_s$ is a closed subset of $(\mathbb{H}_s)^3$ and contains real valued vector fields, see (2.2), endowed with the Hermitian product, for $\mathbf{u} = (u_1,u_2,u_3)$, $\mathbf{v} = (v_1,v_2,v_3)$, $(\mathbf{u},\mathbf{v})_s = (u_1,v_1)_s + (u_1,v_1)_s + (u_1,v_1)_s$ (see (2.4)). We still denote $||\mathbf{u}||_{s,2} = (||u_1||_{s,2}^2 + ||u_2||_{s,2}^2 + ||u_3||_{s,2}^2)^{1/2}$.

(3) We put $W^{-1,p'}(\mathbb{T}_3) = (W^{1,p}(\mathbb{T}_3))'$ for $1/p + 1/p' = 1$, $p \geq 1$. We also put $\mathbf{IH}_{-s} = (\mathbf{IH}_s)'$ for $s \geq 0$.

(4) The usual case we keep in mind for the data in the Navier–Stokes equations, is the case $\mathbf{u}_0 \in \mathbf{IH}_0$ and $\mathbf{f} \in L^2([0,T],(H^1(\mathbb{T}_3)^3)')$, noting that $(H^1(\mathbb{T}_3)^3)' \subset \mathbf{IH}_{-1}$.

Definition 4.2. We say that $(\mathbf{u}, p)$ is a dissipative solution to the Navier–Stokes equations (4.1) in time interval $[0,T]$ if:

(1) The following holds:

$$\mathbf{u} \in L^2([0,T], \mathbb{H}_1) \cap L^\infty([0,T], \mathbb{H}_0), \tag{4.2}$$

$$p \in L^{5/3}([0,T] \times \mathbb{T}_3), \tag{4.3}$$

$$\partial_t \mathbf{u} \in L^{5/3}([0,T], (W^{-1,5/3}(\mathbb{T}_3))^3). \tag{4.4}$$

(2) $\lim_{t\to 0} ||\mathbf{u}(t,\cdot) - \mathbf{u}_0(\cdot)||_{0,2} = 0$

(3) $\forall\, \mathbf{v} \in L^{5/2}([0,T], W^{1,5/2}(\mathbb{T}_3)^3)$ one has for all $t \in [0,T]$,

$$(\partial_t \mathbf{u}, \mathbf{v}) - \int_0^t \int_{\mathbb{T}_3} \mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} + \nu \int_0^t \int_{\mathbb{T}_3} \nabla \mathbf{u} : \nabla \mathbf{v}\, d\mathbf{x} dt' - \int_0^t \int_{\mathbb{T}_3} p\,(\nabla \cdot \mathbf{v}) = \int_0^t (\mathbf{f}, \mathbf{v}), \tag{4.5}$$

where $(\cdot,\cdot)$ stands here for the duality product between $W^{1,5/2}(\mathbb{T}_3)^3$ and $W^{-1,5/3}(\mathbb{T}_3)^3$, noting that $(H^1(\mathbb{T}_3)^3)' \subset W^{-1,5/3}(\mathbb{T}_3)^3$.

(4) The energy inequality holds, for all $t \in [0,T]$,

$$\frac{1}{2}\int_{\mathbb{T}_3} |\mathbf{u}(t,\mathbf{x})|^2 + \nu \int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}(t',\mathbf{x})|^2 d\mathbf{x} dt \le \frac{1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \int_0^t (\mathbf{f}, \mathbf{u}) dt', \tag{4.6}$$

where $(\cdot,\cdot)$ stands here for the duality product between $\mathbb{H}_1$ and $\mathbb{H}_{-1}$, noting that $(H^1(\mathbb{T}_3)^3)' \subset \mathbb{H}_{-1}$.

Remark 4.3. *This definition makes sense once $\mathbf{u}_0 \in \mathbb{H}_0$ and $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$, giving a sense to the integrals on the right hand side of* (4.5) *and* (4.6). *Moreover, by interpolation we see that the regularity conditions in point (1) in the definition above, make sure that $\mathbf{u} \in L^{10/3}([0,T] \times \mathbb{T}_3)^3$. When combining this fact with the regularity for $\partial_t \mathbf{u}$, we see that all integrals on the right hand side of* (4.5) *are well defined.*

Remark 4.4. *The condition imposed on the pressure, $p \in L^{5/3}([0,T] \times \mathbb{T}_3)$, is directly satisfied when we already have the estimate $\mathbf{u} \in L^\infty([0,T], \mathbb{H}_0) \cap L^2([0,T], \mathbb{H}_1)$. Indeed, when one takes the divergence of the momentum equation formally using $\nabla \cdot \mathbf{u} = 0$ (included in the definition of the function space $\mathbb{H}_s$ in 4.1.2), we get the following equation for the pressure*

$$-\Delta p = \nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u})). \tag{4.7}$$

Now, by using Hölder's inequality, it is easy to check that $\mathbf{u} \in L^\infty([0,T], \mathbb{H}_0) \cap L^2([0,T], \mathbb{H}_1)$ implies $\mathbf{u} \in L^{10/3}([0,T] \times \mathbb{T}_3)^3$. Therefore,

$$\nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u})) \in L^{5/3}([0,T], W^{-2,5/3})$$

and by the standard elliptic theory it follows $p \in L^{5/3}([0,T] \times \mathbb{T}_3)$.

Let us recall a result due to J. Leray [30].

THEOREM 4.5. *Assume that* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T],(H^1(\mathbb{T}_3)^3)')$. *Then the Navier–Stokes equations* (4.1) *have a dissipative solution.*

We still do not know whether

- this solution is unique,
- if it develops singularities in finite time, even if $\mathbf{u}_0$ and $\mathbf{f}$ are smooth.

2. The deconvolution model

The deconvolution equation for incompressible fields takes the form

$$\begin{cases} -\alpha^2 \Delta \left(\dfrac{\partial \mathbf{w}}{\partial \tau} \right) + \mathbf{w} + \nabla \pi = \mathbf{u}, \\ \nabla \cdot \mathbf{w} = 0, \\ m(\mathbf{w}) = 0, \quad m(\pi) = 0, \\ \mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}}, \end{cases} \tag{4.8}$$

where $\mathbf{u}$ is such that $m(\mathbf{u}) = \nabla \cdot \mathbf{u} = 0$, and $\overline{\mathbf{u}}$ is the solution of the Stokes problem

$$\begin{cases} A\overline{\mathbf{u}} = -\alpha^2 \Delta \overline{\mathbf{u}} + \overline{\mathbf{u}} + \nabla \xi = \mathbf{u}, \\ \nabla \cdot \overline{\mathbf{u}} = 0, \\ m(\overline{\mathbf{u}}) = 0, \quad m(\xi) = 0. \end{cases} \tag{4.9}$$

In the equations above, π and ξ are necessary Lagrange multipliers, involved because of the zero divergence constraint. In the following we set

$$H_\tau(\mathbf{u})(t,\mathbf{x}) = w(\tau, t, \mathbf{x}),$$

where $w(\tau, t, \mathbf{x})$ is the solution for the deconvolution parameter τ at a fixed time t. Of course $H_0(\mathbf{u}) = \overline{\mathbf{u}}$. A straightforward adaptation of the results of Section 3 in Chapter 3 combined with classical results related to the Stokes problem (see [22]) yield that Lagrange multipliers π and ξ are both equal to zero, and that the following facts are satisfied:

(1) Let $\mathbf{u} \in L^\infty([0,T], \mathbb{H}_0)$. Then for all $\tau \geq 0$, $H_\tau(\mathbf{u}) \in L^\infty([0,T], \mathbb{H}_2)$ and one has

$$\sup_{t\geq 0} ||H_\tau(\mathbf{u})||_{2,2} \leq C \sup_{t \geq 0} ||\mathbf{u}||_{0,2}, \tag{4.10}$$

where the constant C depends on τ and blows up when τ goes to infinity. Thanks to Sobolev injection theorem, we deduce from (4.10) that in addition

$$\begin{aligned} H_\tau(\mathbf{u}) \in L^\infty([0,T] \times \mathbb{T}_3)^3, \\ ||H_\tau(\mathbf{u})||_{L^\infty([0,T]\times\mathbb{T}_3)^3} \leq C(\tau, \alpha, \sup_{t\geq 0} ||\mathbf{u}||_{0,2}). \end{aligned} \tag{4.11}$$

(2) Let $\mathbf{u} \in L^2([0,T], \mathbb{H}_1)$. Then the following estimate holds:

$$\int_0^T ||\mathbf{u}(t,\cdot) - H_\tau(\mathbf{u})(t,\cdot)||_{1,2}^2 dt \leq e^{-\frac{2\tau}{\alpha^2}} \int_0^T ||\mathbf{u}(t,\cdot)||_{1,2}^2 dt. \tag{4.12}$$

In particular, the sequence $(H_\tau(\mathbf{u}))_{\tau>0}$ goes strongly to $\mathbf{u}$ in the space $L^2([0,T], \mathbb{H}_1)$ when τ goes to infinity and $\alpha > 0$ is fixed.

Let us consider the problem

$$\begin{cases} \partial_t \mathbf{u}_{\alpha,\tau} + (H_\tau(\mathbf{u}_{\alpha,\tau}) \cdot \nabla)\, \mathbf{u}_{\alpha,\tau} - \nu\Delta \mathbf{u}_{\alpha,\tau} + \nabla p_{\alpha,\tau} = H_\tau(\mathbf{f}), \\ \nabla \cdot \mathbf{u}_{\alpha,\tau} = 0, \\ m(\mathbf{u}_{\alpha,\tau}) = 0, \quad m(p_{\alpha,\tau}) = 0, \\ \mathbf{u}_{\alpha,\tau}(0, \mathbf{x}) = H_\tau(\mathbf{u}_0), \end{cases} \tag{4.13}$$

with periodic boundary conditions.

Definition 4.6. We say that $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$ is a weak solution to Problem (4.13) if the following properties are satisfied:

(1) $\mathbf{u}_{\alpha,\tau} \in L^\infty([0,T], \mathbb{H}_1) \cap L^2([0,T], \mathbb{H}_1)$, $\partial_t \mathbf{u}_{\alpha,\tau} \in (L^2([0,T] \times \mathbb{T}_3))^3$, $p \in L^2([0,T], \mathbb{H}_1)$,
(2) $\lim_{t\to 0} ||\mathbf{u}_{\alpha,\tau}(t,\cdot) - H_\tau(\mathbf{u}_0)||_{0,2} = 0$,
(3) $\forall\, \mathbf{v} \in L^2([0,T], (H^1(\mathbb{T}_3)^3))$,

$$\int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{u}_{\alpha,\tau} \cdot \mathbf{v} + \int_0^T \int_{\mathbb{T}_3} (H_\tau(\mathbf{u}_{\alpha,\tau}) \cdot \nabla) \mathbf{u}_{\alpha,\tau} \cdot \mathbf{v} \\ + \nu \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{u}_{\alpha,\tau} : \nabla \mathbf{v} + \int_0^T \int_{\mathbb{T}_3} \nabla p \cdot \mathbf{v} = \int_0^T \int_{\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{v}, \tag{4.14}$$

(4) the following energy balance holds for all $t \in [0,T]$,

$$\frac{1}{2} \int_{\mathbb{T}_3} |\mathbf{u}_{\alpha,\tau}(t,\mathbf{x})|^2 d\mathbf{x} + \nu \int_0^t \int_{\mathbb{T}_3} |\nabla \mathbf{u}_{\alpha,\tau}(t',\mathbf{x})|^2 d\mathbf{x} dt' \\ = \frac{1}{2} \int_{\mathbb{T}_3} |H_\tau(\mathbf{u}_0)(\mathbf{x})|^2 d\mathbf{x} + \int_0^t \int_{\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{u}_{\alpha,\tau} d\mathbf{x} dt'. \tag{4.15}$$

We now prove the following two results.

Theorem 4.7. *Assume that* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$. *Then Problem* (4.13) *admits a unique weak solution* $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$.

Theorem 4.8. *Assume that* $\mathbf{u}_0 \in \mathbb{H}_0$ *and* $\mathbf{f} \in L^2([0,T], (H^1(\mathbb{T}_3)^3)')$. *Then there exists a sequence* $(\tau_n)_{n\in\mathbb{N}}$ *which goes to infinity when* n *goes to infinity and such that the sequence* $(\mathbf{u}_{\alpha,\tau_n}, p_{\alpha,\tau_n})_{n\in\mathbb{N}}$ *goes to a dissipative weak solution of the Navier–Stokes equations.*

Proof of Theorem 4.7. A complete proof of Theorem 4.7 would use the Galerkin method. We construct approximations as solutions of variational problems in the finite dimensional spaces V_n, thanks to the Cauchy–Lipchitz theorem. Afterwards we derive estimates in order to finally pass to the limit. To make the paper easy and not too difficult, we bypass the construction of approximations in finite dimensional spaces, a procedure we have already completed for similar models (see for instance in [29]). The general Galerkin method is well explained in the famous book by J. L. Lions published in 1969 [34]. Therefore, we concentrate our effort on two main points that make the result true:

- a priori estimates,
- the compactness property and how to pass to the limit in the equations.

2.1. A priori estimates. For the simplicity, we write $(\mathbf{u}, p)$ instead of $(\mathbf{u}_{\alpha,\tau}, p_{\alpha,\tau})$. We perform computations assuming that $(\mathbf{u}, p)$ are enough regular to validate the integrations by parts we do. We also keep in mind that the boundary terms compensate each other in the integrations by parts, thanks to the periodicity. Therefore no boundary terms occur in these computations.

As usual, we take $\mathbf{u}$ as a test function in (4.13), and we integrate by parts on $\mathbb{T}_3$ and on the time interval $[0, t]$ for some $t \in [0, T]$, using $\nabla \cdot \mathbf{u} = 0$ as well as $\nabla \cdot (H_\tau(\mathbf{u})) = 0$. We get in particular

$$\int_{\mathbb{T}_3} (H_\tau(\mathbf{u}) \cdot \nabla)\, \mathbf{u} \cdot \mathbf{u} = 0,$$

and therefore

$$\frac{1}{2}\int_{\{t\}\times\mathbb{T}_3} |\mathbf{u}|^2 + \nu \int_{[0,t]\times\mathbb{T}_3} |\nabla \mathbf{u}|^2 = \frac{1}{2}\int_{\mathbb{T}_3} |H_\tau(\mathbf{u}_0)|^2 + \int_{[0,t]\times\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{u}. \quad (4.16)$$

As $\mathbf{u}_0 \in \mathbb{H}_0$, $H_\tau(\mathbf{u}_0) \in \mathbb{H}_2$, and recall that $||H_\tau(\mathbf{u}_0)||_{0,2} \le 2||\mathbf{u}_0||_{0,2}$. Similarly,

$$\left| \int_{[0,t]\times\mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{u} \right| \le C||\mathbf{f}||_{-1,2}||\mathbf{u}||_{1,2},$$

where again C do not depend on τ and α. Here and in the rest, we still denote the norm on $(H^1(\mathbb{T}_3)^3)'$ by $||\cdot||_{-1,2}$. Therefore, (4.16) yields

$$\sup_{t\in[0,T]} \int_{\{t\}\times\mathbb{T}_3} |\mathbf{u}|^2 \le C(||\mathbf{u}_0||_{0,2}, ||\mathbf{f}||_{-1,2}), \quad (4.17)$$

$$\int_{[0,t]\times\mathbb{T}_3} |\nabla \mathbf{u}|^2 \le C(||\mathbf{u}_0||_{0,2}, ||\mathbf{f}||_{-1,2}, \nu). \quad (4.18)$$

Next, we use fact (1) ($H_\tau(\mathbf{u}) \in L^\infty([0,T]\times\mathbb{T}_3)^3$) and estimate (4.11) together with (4.17). This yields in particular

$$\mathbb{A} = (H_\tau(\mathbf{u})\cdot\nabla)\mathbf{u} \in L^2([0,T]\times\mathbb{T}_3)^3, \quad ||\mathbb{A}||_{L^2([0,T]\times\mathbb{T}_3)^3} \le C(\tau, \alpha, ||\mathbf{u}_0||_{0,2}, ||\mathbf{f}||_{-1,2}). \quad (4.19)$$

Let us now take $\partial_t \mathbf{u}$ as a test function in equation (4.13), and we integrate on $[0,t]\times\mathbb{T}_3$, using $\nabla\cdot(\partial_t \mathbf{u}) = 0$. Therefore we get

$$\int_{[0,t]\times\mathbb{T}_3} |\partial_t \mathbf{u}|^2 + \frac{1}{2}\int_{\{t\}\times\mathbb{T}_3} |\nabla \mathbf{u}|^2 = \frac{1}{2}\int_{\mathbb{T}_3} |\nabla H_\tau(\mathbf{u}_0)|^2 + \int_{[0,t]\times\mathbb{T}_3} \mathbb{A}\cdot\partial_t \mathbf{u} + \int_{[0,t]\times\mathbb{T}_3} H_\tau(\mathbf{f})\cdot\partial_t \mathbf{u} \quad (4.20)$$

Since $H_\tau(\mathbf{u}_0) \in \mathbb{H}_2$ and $H_\tau(\mathbf{f}) \in L^2([0,T], H^1(\mathbb{T}_3)^3)$, using (4.19) combined with Cauchy–Schwarz and Young inequalities, we deduce from (4.20)

$$\int_{[0,t]\times\mathbb{T}_3} |\partial_t \mathbf{u}|^2 \le C(\tau, \alpha, ||\mathbf{u}_0||_{0,2}, ||\mathbf{f}||_{-1,2}), \quad (4.21)$$

$$\sup_{t\in[0,T]} \int_{\{t\}\times\mathbb{T}_3} |\nabla \mathbf{u}|^2 \le C(\tau, \alpha, ||\mathbf{u}_0||_{0,2}, ||\mathbf{f}||_{-1,2}). \quad (4.22)$$

In other words $\partial_t \mathbf{u} \in L^2([0,T]\times\mathbb{T}_3)^3$ and $\mathbf{u} \in L^\infty([0,T], \mathbb{H}_1)$. In fact, one easily verifies that $\partial_t \mathbf{u} \in L^2([0,T], \mathbb{H}_0)$.

We now get a bound for $\mathbf{u}$ in the space $L^2([0,T],\mathbb{H}_2)$. For it, let us consider a fixed $t \in [0,T]$ and let us write the Navier–Stokes equations (4.13) in the form of Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = H_\tau(\mathbf{f}) - \mathbb{A} - \partial_t \mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \\ m(\mathbf{u}) = 0, \quad m(p) = 0. \end{cases} \tag{4.23}$$

Classical results on the Stokes problem yield the estimate

$$\begin{aligned} ||\mathbf{u}||^2_{\mathbb{H}_2} + ||p||^2_{H^1(\mathbb{T}_3)} &\le C_1(\nu)||H_\tau(\mathbf{f}) - \mathbb{A} - \partial_t\mathbf{u}||^2_{L^2(\mathbb{T}_3)} \\ &\le C_2(\nu)(||H_\tau(\mathbf{f})||^2_{L^2(\mathbb{T}_3)} + ||\mathbb{A}||^2_{L^2(\mathbb{T}_3)} + ||\partial_t\mathbf{u}||^2_{L^2(\mathbb{T}_3)}) \end{aligned} \tag{4.24}$$

We now integrate (4.24) with respect to time. We get

$$||\mathbf{u}||_{L^2([0,T],\mathbb{H}_2)} + ||p||_{L^2([0,T],H^1(\mathbb{T}_3))} \le C(\nu,\tau,\alpha,||\mathbf{u}_0||_{0,2},||\mathbf{f}||_{-1,2}), \tag{4.25}$$

where we have used the regularizing effect of H_τ and estimates (4.19) and (4.21).

In summary, we get:

(1) $\mathbf{u}$ is in $L^2([0,T],\mathbb{H}_1)\cap L^\infty([0,T],\mathbb{H}_0)$ and therefore $p \in L^{5/3}([0,T],\mathbb{L}_{5/3})$ and $\partial_t\mathbf{u} \in L^{5/3}([0,T],W^{-1,5/3}(\mathbb{T}_3)^3)$. *The bounds only depend on the data* ν, $||\mathbf{u}_0||_{0,2}$ *and* $||\mathbf{f}||_{-1,2}$.

(2) $\mathbf{u} \in L^2([0,T],\mathbb{H}_2) \cap L^\infty([0,T],\mathbb{H}_1)$ and $p \in L^2([0,T],H^1(\mathbb{T}_3))$. The bounds depend on the data ν, $||\mathbf{u}_0||_{0,2}$ and $||\mathbf{f}||_{-1,2}$ as well as on the deconvolution parameter τ and the filtration parameter α. In particular these bounds blow up when τ goes to infinity and/or α goes to zero.

(3) $\partial_t\mathbf{u} \in L^2([0,T],\mathbb{H}_0)$. The bounds depend on the data ν, $||\mathbf{u}_0||_{0,2}$ and $||\mathbf{f}||_{-1,2}$ as well as on the deconvolution parameter τ and the filtration parameter α.

2.2. Compactness property. Let us now consider a sequence $(u_n,p_n)_{n\in\mathbb{N}}$ of “smooth” solutions to problem (4.13). We aim to show that we can extract from this sequence a subsequence which converges in a certain sense to a solution of problem (4.13), when n goes to infinity.

Fact (2) makes sure that we can extract a subsequence, still denoted $(u_n,p_n)_{n\in\mathbb{N}}$, such that

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{weakly in } L^2([0,T],\mathbb{H}_2), \tag{4.26}$$

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{weakly-star in } L^\infty([0,T],\mathbb{H}_1), \tag{4.27}$$

$$p_n \longrightarrow p \quad \text{weakly in } L^2([0,T],H^1(\mathbb{T}_3)). \tag{4.28}$$

Let us now find a strong compactness property. We have the following

$$\mathbb{H}_2 \subset \mathbb{H}_1 \subset \mathbb{H}_0,$$

the injections being continuous, compact and dense. We know that $(\partial_t\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],\mathbb{H}_0)$ while $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],\mathbb{H}_2)$ We deduce from Aubin–Lions lemma (see in [34]) that

$$\mathbf{u}_n \longrightarrow \mathbf{u} \quad \text{strongly in } L^2([0,T],\mathbb{H}_1). \tag{4.29}$$

Finally, it is easily checked that we can extract an other subsequence such that

$$\partial_t\mathbf{u}_n \longrightarrow \partial_t\mathbf{u} \quad \text{weakly in } L^2([0,T]\times\mathbb{T}_3)^3. \tag{4.30}$$

Notice that the limit (u, p) satisfies points (1), (2) and (3) on the list above.

It remains to show that (u, p) is a solution to problem (4.13). Let us start with the initial data, writing

$$\mathbf{u}_n(t) = H_\tau(\mathbf{u}_0) + \int_0^t \partial_t \mathbf{u}_n dt'.$$

It is easy to pass to limit here in $L^2([0,T] \times \mathbb{T}_3)^3$, to get for free relation

$$\mathbf{u}(t) = H_\tau(\mathbf{u}_0) + \int_0^t \partial_t \mathbf{u} \, dt',$$

that tells us that $\mathbf{u} \in C^0([0,T], \mathbb{H}_0)$ and that $\mathbf{u}(0, \mathbf{x}) = H_\tau(\mathbf{u}_0(\mathbf{x}))$. In fact we have a much better result since $\mathbf{u} \in C^0([0,T], \mathbb{H}_1)$. The proof is left to the reader.

Let us now pass to the limit in the momentum equation. Let $\mathbf{v} \in L^2([0,T], \mathbb{H}_1)$ be a test vector field. One obviously has—when n goes to infinity—

$$\begin{aligned} \int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{u}_n \cdot \mathbf{v} &\longrightarrow \int_0^T \int_{\mathbb{T}_3} \partial_t \mathbf{u} \cdot \mathbf{v}, \\ \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{u}_n : \nabla \mathbf{v} &\longrightarrow \int_0^T \int_{\mathbb{T}_3} \nabla \mathbf{u} : \nabla \mathbf{v}, \\ \int_0^T \int_{\mathbb{T}_3} p_n (\nabla \cdot \mathbf{u}) &\longrightarrow \int_0^T \int_{\mathbb{T}_3} p \, (\nabla \cdot \mathbf{u}), \end{aligned} \tag{4.31}$$

where we have used the identity

$$\int_{\mathbb{T}_3} p \, (\nabla \cdot \mathbf{u}) = -\int_{\mathbb{T}_3} \nabla p \cdot \mathbf{v}.$$

It remains to treat the term $(H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n$ which constitutes the novelty. This is why we focus our attention on it. Let us remark that $(\nabla \mathbf{u}_n)_{n \in \mathbb{N}}$ goes strongly to $\nabla \mathbf{u}$ in the space $L^2([0,T] \times \mathbb{T}_3)^9$. On the other hand, applying Lemma 3.4, we get that $(H_\tau(\mathbf{u}_n))_{n \in \mathbb{N}}$ converges to $H_\tau(\mathbf{u})$ in $L^2([0,T] \times \mathbb{T}_3)^9$ when n goes to infinity. Therefore the sequence $((H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n)_{n \in \mathbb{N}}$ goes strongly to $(H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u}$ in $L^1([0,T] \times \mathbb{T}_3)^3$. Finally, since the sequence $((H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n)_{n \in \mathbb{N}}$ is bounded in $L^2([0,T] \times \mathbb{T}_3)^3$, it converges weakly, up to a subsequence, to some $\mathbf{g}$ in $L^2([0,T] \times \mathbb{T}_3)^3$. The result above and uniqueness of the limit allows us to claim that $\mathbf{g} = (H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u}$. Consequently

$$\int_0^T \int_{\mathbb{T}_3} (H_\tau(\mathbf{u}_n) \cdot \nabla)\mathbf{u}_n \cdot \mathbf{v} \longrightarrow \int_0^T \int_{\mathbb{T}_3} (H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u} \cdot \mathbf{v}.$$

In summary, (u, p) satisfies:

(1) $u \in L^2([0,T], \mathbb{H}_2) \cap L^\infty([0,T], \mathbb{H}_1)$, $p \in L^2([0,T], H^1(\mathbb{T}_3))$,
(2) $\lim_{t \to 0} ||u(t, \cdot) - H_\tau(\mathbf{u}_0)||_{0,2} = 0$,
(3) $\forall \mathbf{v} \in L^2([0,T], \mathbb{H}_1)$:

$$\int_{[0,T] \times \mathbb{T}_3} [\partial_t \mathbf{u} \cdot \mathbf{v} + (H_\tau(\mathbf{u}) \cdot \nabla)\mathbf{u} \cdot \mathbf{v} + \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v}] + \int_{[0,T] \times \mathbb{T}_3} \nabla p \cdot \mathbf{v} = \int_{[0,T] \times \mathbb{T}_3} H_\tau(\mathbf{f}) \cdot \mathbf{v}. \tag{4.32}$$

Uniqueness is proven exactly like in [29], and we skip the details. Moreover, taking $\mathbf{u}$ as a test vector field, which is a legitimate operation, and integrating in space and time using $\nabla \cdot \mathbf{u} = 0$ yields the energy equality

$$\frac{1}{2}\int_{\{t\}\times\mathbb{T}_3} |\mathbf{u}|^2 + \nu \int_{[0,T]\times\mathbb{T}_3} |\nabla \mathbf{u}|^2 = \frac{1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0|^2 + \int_{[0,T]\times\mathbb{T}_3} \mathbf{f}\cdot\mathbf{u}.$$

Therefore, (u, p) is a smooth weak solution to problem (4.13), which concludes the proof of Theorem 4.7. □

Proof of Theorem 4.8. We finish the paper by proving the convergence result when τ goes to infinity. We note that for solution (u_τ, p_τ) the grid parameter α is fixed. In this case, we only can use estimates (4.17) and (4.18). We also use estimate (4.12). Let us first write the equation for the pressure:

$$-\Delta p_\tau = \nabla \cdot (\nabla \cdot (H_\tau(\mathbf{u}_\tau) \otimes \mathbf{u}_\tau)). \tag{4.33}$$

This yields, by interpolation combining (4.17), (4.18) and (4.12), existence of a constant $C = C(\nu, ||\mathbf{u}_0||_{0,2}, ||f||_{-1,2})$ such that

$$||p_\tau||_{L^{5/3}([0,T]\times\mathbb{T}_3)} \leq C. \tag{4.34}$$

When writing

$$\partial_t \mathbf{u}_\tau = -\nabla \cdot (H_\tau(\mathbf{u}_\tau) \otimes \mathbf{u}_\tau) + \nu\Delta\mathbf{u}_\tau - \nabla p_\tau + H_\tau(\mathbf{f}), \tag{4.35}$$

we obtain the existence of a constant $C = C(\nu, ||\mathbf{u}_0||_{0,2}, ||f||_{-1,2})$ such that

$$||\partial_t \mathbf{u}_\tau||_{L^{5/3}([0,T],W^{-1,5/3}(\mathbb{T}_3)^3)} \leq C. \tag{4.36}$$

We are now well prepared to pass to the limit. Thanks to all these bounds, there exists $(\tau_n)_{n\in\mathbb{N}}$ which goes to infinity when n goes to infinity and such that there exists $\mathbf{u} \in L^2([0,T], \mathbb{H}_1) \cap L^\infty([0,T], \mathbb{H}_0)$ and $p \in L^{5/3}([0,T] \times \mathbb{T}_3)$ such that

$$\mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \quad \text{weakly in } L^2([0,T], \mathbb{H}_1), \tag{4.37}$$

$$\mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \quad \text{weakly star in } L^\infty([0,T], \mathbb{H}_0), \tag{4.38}$$

$$p_{\tau_n} \longrightarrow p \quad \text{weakly in } L^{5/3}([0,T] \times \mathbb{T}_3), \tag{4.39}$$

when n goes to infinity. We must prove that (u, p) is a dissipative weak solution to the Navier–Stokes equations.

Let us start with the compactness result derived from Aubin–Lions lemma. We have

$$H^1(\mathbb{T}_3) \subset L^{10/3}(\mathbb{T}_3) \subset W^{-1,5/3}(\mathbb{T}_3),$$

the injections being dense and continuous, the first one being compact (since $10/3 < 6$, 6 being the critical exponent in the 3D case). Therefore, applying again Aubin–Lions lemma using the bound on $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ in $L^2([0,T], \mathbb{H}_1) \subset L^2([0,T], H^1(\mathbb{T}_3)^3)$ and the bound on $(\partial_t \mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ in $L^{5/3}([0,T], W^{-1,5/3}(\mathbb{T}_3)^3)$, we see that $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ is compact in $L^{5/3}([0,T], L^{10/3}(\mathbb{T}_3)^3)$. Then we have in particular

$$\mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \text{ strongly in } L^{5/3}([0,T] \times \mathbb{T}_3)^3. \tag{4.40}$$

Using Egorov's theorem combined with Lebesgue inverse theorem, we deduce from (4.40) combined with the bound in $L^{10/3}$ that

$$\forall q < 10/3 : \mathbf{u}_{\tau_n} \longrightarrow \mathbf{u} \text{ strongly in } L^q([0,T] \times \mathbb{T}_3)^3. \tag{4.41}$$

Let us again consider $(\partial_t \mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$. The bound (4.36) authorizes us to extract a subsequence (still using the same notation) and such that

$$\partial_t \mathbf{u}_{\tau_n} \longrightarrow \mathbf{g} \text{ weakly in } L^{5/3}([0,T], W^{-1,5/3})^3. \tag{4.42}$$

We must prove that $\mathbf{g} = \partial_t \mathbf{u}$. Let φ be a C^∞ field defined on $[0,T]\times\mathbb{T}_3$ and such that $\varphi(0,\mathbf{x}) = \varphi(T,\mathbf{x}) = 0$. Then one has

$$\int_{[0,T]\times\mathbb{T}_3} \partial_t \mathbf{u}_{\tau_n} \cdot \varphi = -\int_{[0,T]\times\mathbb{T}_3} \mathbf{u}_{\tau_n} \cdot \partial_t \varphi.$$

Passing to the limit in this equality using (4.42) yields

$$\int_{[0,T]\times\mathbb{T}_3} \mathbf{g} \cdot \varphi = -\int_{[0,T]\times\mathbb{T}_3} \mathbf{u} \cdot \partial_t \varphi,$$

which tells us that $\mathbf{g} = \mathbf{u}$ in the distributional sense, and also in L^p sense by uniqueness of the limit.

From now, $\mathbf{v} \in L^{5/2}([0,T], W^{1,5/2}(\mathbb{T}_3)^3)$ is a fixed test vector field. We have the obvious following convergences when n goes to infinity,

$$\begin{aligned}
\int_Q \partial_t \mathbf{u}_{\tau_n} \cdot \mathbf{v} &\longrightarrow (\partial_t \mathbf{u} \cdot \mathbf{v}),\\
\int_Q \nabla \mathbf{u}_{\tau_n} : \nabla \mathbf{v} &\longrightarrow \int_Q \nabla \mathbf{u} : \nabla \mathbf{v},\\
\int_Q p_{\tau_n} (\nabla\cdot\mathbf{v}) &\longrightarrow \int_Q p\,(\nabla\cdot\mathbf{v}),\\
\int_Q H_{\tau_n}(\mathbf{f})\cdot\mathbf{v} &\longrightarrow \int_Q \mathbf{f}\cdot\mathbf{v},
\end{aligned} \tag{4.43}$$

where $Q = [0,T]\times\mathbb{T}_3$ for the simplicity, $(\cdot,\cdot)$ stands for the duality product between $L^{5/2}([0,T], W^{1,5/2}(\mathbb{T}_3)^3)$ and $L^{5/3}([0,T], W^{-1,5/3}(\mathbb{T}_3)^3)$, and where we also have used Lemma 3.4.

We now have to deal with the nonlinear term. We first notice that $(H_{\tau_n}(\mathbf{u}_n)\otimes \mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ is bounded in $L^{5/3}(Q)^9$. Thus—up to a subsequence—it converges weakly in $L^{5/3}(Q)^9$ to a guy named $\mathbf{h}$ for the time being. That means

$$\int_Q H_{\tau_n}(\mathbf{u}_n)\otimes\mathbf{u}_{\tau_n} : \nabla\mathbf{v} \longrightarrow \int_Q \mathbf{h} : \nabla \mathbf{v}. \tag{4.44}$$

The challenge is to prove that $\mathbf{h} = \mathbf{u}\otimes\mathbf{u}$. We already know that $\mathbf{u}_{\tau_n}$ converges to $\mathbf{u}$ strongly in $L^{10/3-\varepsilon}(Q)$ ($\varepsilon > 0$ and as usual "small"). Let us study the sequence $H_{\tau_n}(\mathbf{u}_n)$. It obviously converges to $\mathbf{u}$ but we must specify in which space and in which topology. We shall work in a L^2 space type ($2 < 10/3$). We can write

$$H_{\tau_n}(\mathbf{u}_n) - \mathbf{u} = H_{\tau_n}(\mathbf{u}_n - \mathbf{u}) + H_{\tau_n}(\mathbf{u}) - \mathbf{u}.$$

Thanks to (3.16), we have for any fixed time t,

$$||H_{\tau_n}(\mathbf{u}_n - \mathbf{u})(t,\cdot)||_{0,2}^2 \le 2||(\mathbf{u}_n - \mathbf{u})(t,\cdot)||_{0,2}^2,$$

an inequality that we integrate on the time interval $[0,T]$. This ensures that the sequence $(H_{\tau_n}(\mathbf{u}_n - \mathbf{u}))_{n\in\mathbb{N}}$ converges to zero in $L^2(Q)^3$ when n goes to infinity.

Applying Lemma 3.4, we deduce that the sequence $(H_{\tau_n}(\mathbf{u}) - \mathbf{u})_{n\in\mathbb{N}}$ converges to zero in $L^2(Q)^3$ when n goes to infinity.

In summary, we obtain the convergence of $(H_{\tau_n}(\mathbf{u}_n) \otimes \mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ to $\mathbf{u}\otimes\mathbf{u}$ in $L^1(Q)^3$, making sure that $\mathbf{h} = \mathbf{u}\otimes\mathbf{u}$ and also thanks to (4.44),

$$\int_Q H_{\tau_n}(\mathbf{u}_n) \otimes \mathbf{u}_{\tau_n} : \nabla\mathbf{v} \longrightarrow \int_Q \mathbf{u}\otimes\mathbf{u} : \nabla\mathbf{v}. \tag{4.45}$$

In conclusion, $(\mathbf{u}, p)$ satisfies (4.5). Point 1 in definition (4.2) is already checked. To conclude our proof, it remains to prove points 2 (initial data) and 4 (energy inequality). We start with the energy inequality.

We already know that $(\mathbf{u}_{\tau_n}, p_{\tau_n})$ satisfies the energy equality (4.15). Let $0 \le t_1 < t_2 \le T$, and integrate (4.15) on the time interval $[t_1, t_2]$. We get

$$\begin{aligned} \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}_{\tau_n}(t,\mathbf{x})|^2 d\mathbf{x}dt + \nu \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}_3} |\nabla\mathbf{u}_{\tau_n}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt \\ = \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |H_{\tau_n}(\mathbf{u}_0)(\mathbf{x})|^2 d\mathbf{x} + \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}_3} H_{\tau_n}(\mathbf{f})\cdot\mathbf{u}_{\tau_n} d\mathbf{x}dt'dt. \end{aligned} \tag{4.46}$$

Because $(H_{\tau_n}(\mathbf{f}))_{n\in\mathbb{N}}$ converges strongly to $\mathbf{f}$ in $L^2([0,T], (H^1(\mathbb{T}_3)^3)')$ while $(\mathbf{u}_{\tau_n})_{n\in N}$ converges weakly to $\mathbf{u}$ in $L^2([0,T], \mathbb{H}_1)$, the standard arguments yield

$$\int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}_3} H_{\tau_n}(\mathbf{f})\cdot\mathbf{u}_{\tau_n} d\mathbf{x}dt'dt \longrightarrow \int_{t_1}^{t_2} (\mathbf{f},\mathbf{u})dt. \tag{4.47}$$

Analogous arguments also tell

$$\frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |H_{\tau_n}(\mathbf{u}_0)(\mathbf{x})|^2 d\mathbf{x} \longrightarrow \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x}. \tag{4.48}$$

As we know that $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ goes to $\mathbf{u}$ strongly in $L^2(Q)^3$, we have

$$\frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}_{\tau_n}(t,\mathbf{x})|^2 d\mathbf{x}dt \longrightarrow \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}dt. \tag{4.49}$$

Finally, by the standard arguments in analysis (see for instance in [31]), the weak convergence of $(\mathbf{u}_{\tau_n})_{n\in\mathbb{N}}$ to $\mathbf{u}$ in $L^2([0,T], \mathbb{H}_1)$ yields

$$\int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}_3} |\nabla\mathbf{u}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt \le \liminf_{n\in\mathbb{N}} \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}_3} |\nabla\mathbf{u}_{\tau_n}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt. \tag{4.50}$$

When one combines (4.46) together with (4.47),(4.48), (4.49) and (4.49), we obtain

$$\begin{aligned} \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}_3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}dt + \nu \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}_3} |\nabla\mathbf{u}(t',\mathbf{x})|^2 d\mathbf{x}dt'dt \\ \le \frac{t_2-t_1}{2}\int_{\mathbb{T}_3} |\mathbf{u}_0(\mathbf{x})|^2 d\mathbf{x} + \int_{t_1}^{t_2} (\mathbf{f},\mathbf{u})dt, \end{aligned} \tag{4.51}$$

an inequality which holds for every t_1, t_2 such that $0 \le t_1 < t_2 \le T$. We deduce that $\mathbf{u}$ satisfies the energy inequality (4.15).

To finish the proof, we have to study the initial data. Let us first notice that $\mathbf{u}(t,\cdot)_{t>0}$ is bounded in $L^2(\mathbb{T}_3)^3$. Therefore, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ which converges to 0 and a field $\mathbf{k} \in L^2(\mathbb{T}_3)^3$ such that $\mathbf{u}(t_n, \cdot)_{n\in N}$ converges weakly in $L^2(\mathbb{T}_3)^3$ to $\mathbf{k}$. The first task is to prove that $\mathbf{k} = \mathbf{u}_0$. We start from the equality

$$\mathbf{u}_{\tau_n}(t, \cdot) = H_{\tau_n}(\mathbf{u}_0) + \int_0^t \partial_t \mathbf{u}_{\tau_n} \, dt', \tag{4.52}$$

an equality that we consider in the space $W^{-1,5/3}(\mathbb{T}_3)^3$. Using a straightforward variant of Lemma 3.4 and the convergence results proved above, we can pass to the limit in (4.52), to get in $W^{-1,5/3}(\mathbb{T}_3)^3$,

$$\mathbf{u}(t, \cdot) = \mathbf{u}_0 + \int_0^t \partial_t \mathbf{u} \, dt'. \tag{4.53}$$

Because $\partial_t \mathbf{u} \in L^{5/3}([0, T], W^{-1,5/3}(\mathbb{T}_3)^3) \subset L^1([0, T], W^{-1,5/3}(\mathbb{T}_3)^3)$, this last equality says that $\mathbf{u}(0, \cdot) = \mathbf{u}_0$ at least in $W^{-1,5/3}(\mathbb{T}_3)^3$, and consequently in $L^2(\mathbb{T}_3)^3$. Therefore we have $\mathbf{k} = \mathbf{u}_0$. Since the limit is unique, we deduce that the whole sequence $\mathbf{u}(t, \cdot)_{t>0}$ converges weakly in $\mathbb{H}_0$ to $\mathbf{u}_0$ when t goes to zero. Moreover, one has

$$||\mathbf{u}||_{0,2} \leq \liminf_{t\to 0} ||\mathbf{u}(t, \cdot)||_{0,2}. \tag{4.54}$$

On the other hand, when one lets t go to zero in the energy inequality, we get

$$\limsup_{t\to 0} ||\mathbf{u}(t, \cdot)||_{0,2} \leq ||\mathbf{u}_0||_{0,2}. \tag{4.55}$$

We deduce that

$$\lim_{t\to 0} ||\mathbf{u}(t, \cdot)||_{0,2} = ||\mathbf{u}_0||_{0,2},$$

which combined with the weak convergence yields

$$\lim_{t\to 0} ||\mathbf{u}_0 - \mathbf{u}(t, \cdot)||_{0,2} = 0. \tag{4.56}$$

This concludes the question concerning initial data and also the proof of Theorem 4.8. □

REMARK 4.9. *Without too much effort, one can prove that the approximated velocity in model (4.13) lies in the space $C([0, T], \mathbb{H}_1)$. Concerning the Navier–Stokes equation, it is well known that the trajectories are continuous in $L^2(\mathbb{T}_3)^3$ with respect to its weak topology. Nevertheless, one may wonder about the strong continuity of the trajectory at $t = 0$ that we have proved here. This approach indeed seems not to be usual in the folklore of the Navier–Stokes equations. However, it fits with the famous local regularity result due to Fujita–Kato [20].*

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Part 4

Rough boundaries and wall laws

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Abstract. We consider Laplace and Stokes operators in domains with rough boundaries and search for an effective boundary condition. The method of homogenization, coupled with the boundary layers, is used to obtain it. In the case of the homogeneous Dirichlet condition at the rough boundary, the effective law is Navier's slip condition, used in the computations of viscous flows in complex geometries. The corresponding effective coefficient is determined by upscaling. It is given by solving an appropriate boundary layer problem. Finally we address application to the drag reduction. In this review article we will explain how those results are obtained, give precise references for technical details and present open problems.

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Contents

CHAPTER 1

Rough boundaries and wall laws

1. Introduction

Boundary value problems involving rough boundaries arise in many applications, like flows on surfaces with fine longitudinal ribs, rough periodic surface diffraction, cracks for elastic bodies in such situations etc.

An important class of problems is modeling reinforcement by thin layers with oscillating thickness (see e.g. Buttazzo and Kohn [25] and references therein). Reinforcement is described by an important contrast in the coefficients and in the Γ-limit a Robin type boundary condition, with coefficients of order 1, is obtained. Its value is calculated using finite cell auxiliary problems.

Next we can mention homogenization of elliptic problems with the Neumann boundary conditions in domain with rapidly oscillating locally periodic boundaries, depending on small parameter. For more details we refer to [27].

The main goal of this review is to discuss the effective boundary conditions for the Laplace equation and the Stokes system with homogeneous Dirichlet condition at the rough boundary.

In fluid mechanics the widely accepted boundary condition for viscous flows is the no-slip condition, expressing that fluid velocity is zero at an immobile solid boundary. It is only justified where the molecular viscosity is concerned. Since the fluid cannot penetrate the solid, its normal velocity is equal to zero. This is the condition of non-penetration. To the contrary, the absence of slip is not very intuitive. For the Newtonian fluids, it was established experimentally and contested even by Navier himself (see [44]). He claimed that the slip velocity should be proportional to the shear stress. The kinetic-theory calculations have confirmed Navier’s boundary condition, but they give the slip length proportional to the mean free path divided by the continuum length (see [47]). For practical purposes it means a zero slip length, justifying the use of the no-slip condition.

In many cases of practical significance the boundary is rough. An example is complex boundaries in the geophysical fluid dynamics. Compared with the characteristic size of a computational domain, such boundaries could be considered as rough. Other examples involve sea bottoms of random roughness and artificial bodies with periodic distribution of small bumps. A numerical simulation of the flow problems in the presence of a rough boundary is very difficult since it requires many mesh nodes and handling of many data. For computational purposes, an artificial smooth boundary, close to the original one, is taken and the equations are solved in the new domain. This way the rough boundary is avoided, but the boundary conditions at the artificial boundary are not given by the physical principles. It is clear that the non-penetration condition $v \cdot n = 0$ should be kept, but there are no reasons to keep the full no-slip condition. Usually it is supposed that the shear stress is a non-linear function F of the tangential velocity. F is determined empirically and its form varies for different problems. Such relations are called the *wall laws* and classical Navier's condition is one example. Another well-known example is modeling of the turbulent boundary layer close to the rough surface by a *logarithmic velocity profile*

$$v_\tau = \sqrt{\frac{\tau_w}{\rho}} \left(\frac{1}{\kappa} \ln \left(\frac{y}{\mu} \sqrt{\frac{\tau_w}{\rho}} \right) + C^+(k_s^+) \right) \tag{1.1}$$

where v_τ is the tangential velocity, y is the vertical coordinate and τ_w the shear stress. ρ denotes the density and μ the viscosity. $\kappa \approx 0.41$ is the von Kármán's constant and C^+ is a function of the ratio k_s^+ of the roughness height k_s and the thin wall sublayer thickness $\delta_v = \frac{\mu}{v_\tau}$. For more details we refer to the book of Schlichting [49].

Justifying the logarithmic velocity profile in the overlap layer is mathematically out of reach for the moment. Nevertheless, after recent results [35] and [37] we are able to justify the Navier's condition for the laminar incompressible viscous flows over periodic rough boundaries. In [37] the Navier law was obtained for the Couette turbulent boundary layer. We note generalization to random rough boundaries in [15].

In the text which follows, we are going to give a review of rigorous results on Navier's condition.

Somewhat related problem is the homogenization of the Poisson equation in a domain with a periodic oscillating boundary and we start by discussing that situation.

2. Wall law for Poisson's equation with the homogeneous Dirichlet condition at the rough boundary

In our knowledge, mathematically rigorous investigations of the effective wall laws started with the paper by Achdou and Pironneau [1]. They considered Poisson's equation in a ring with many small holes close to the exterior boundary. They create an oscillating perforated annular layer close to the outer boundary. The amplitude and the period of the oscillations are of order ε and the homogeneous Dirichlet condition is imposed on the solution. In the paper by Achdou and Pironneau [1] the homogenized problem was derived. The rough boundary was replaced by a smooth artificial one and the corresponding wall law was the Robin boundary condition, saying that the effective solution u was proportional to the characteristic roughness ε times its normal derivative. The proportionality constant was calculated using an auxiliary problem for Laplace's operator in a finite cell. Nevertheless, in [1] the conductivity of the thin layer close to the boundary is not small and, contrary to [25], the homogenized boundary condition contains an ε. Consequently, it is not clear that using the finite cell for the auxiliary problem gives the the H^1-error estimate from [1]. Despite this slight criticism, the reference [1] is a pioneering work since it was first to point out that a) keeping homogeneous Dirichlet boundary condition gives an approximation; b) the wall law is a correction of the previous approximation and c) the wall laws are valid for curved rough boundaries.

The readable error estimate for the wall laws, in the case of Poisson's equation and the flat rough boundary is in the paper by Allaire and Amar [4]. They considered a rectangular domain having one face which was a periodic repetition of $\varepsilon\Gamma_g$ and the same boundary value problem as in [1] except periodic lateral conditions. Then they introduced the following auxiliary boundary layer problem in the infinite strip $\Gamma_g \times]0, +\infty[$:

Find a harmonic function ψ, $\nabla\psi \in L^2$, periodic in $y' = (y_1, \dots, y_{n-1})$ and having a value on Γ_g equal to its parametric form. The classical theory (see e.g. [46] or [39]) gives existence of a unique solution which decays exponentially to a constant d. The conclusion of [4] was that the homogenized solution $\bar{u}^\varepsilon$ obeyed the wall law $\bar{u}^\varepsilon = \varepsilon d \frac{\partial \bar{u}^\varepsilon}{\partial x_n}$ on the artificial boundary and gave an interior H^1-approximation of order $\varepsilon^{3/2}$. We note the difference in determination of the proportionality constant in the wall law between papers [1] and [4].

It should be pointed out that there is a similarity between the homogenization of Poisson's equation in partially perforated domain and obtaining wall laws for the same equations in presence of rough boundaries. In [30] an effective Robin condition, analogous to one from [1] and [4] was obtained for the artificial boundary in the case of partially perforated domain.

Other important work on Laplace's operator came from the team around Y. Amirat and J. Simon. They were interested in the question if presence of the roughness diminishes the hydrodynamical drag. We will be back to this question in Section 4. In [7] and [8] they undertook study on the Couette flow over a rough plate. For the special case of longitudinal grooves, the problem is reduced to the Laplace operator. This research for the case of Laplace operator and for complicated roughness was continued in the doctoral thesis [28] and articles [11], [12] and [21].

Even if the homogeneous Dirichlet condition at the rough boundary is meaningful mostly for flow problems, it makes sense to study the case of Poisson equation. Following Bechert and Bartenwerfer [17] we can interpret it as simplified Stokes system for longitudinal ribs at the outer boundary. Mathematically, it is much easier to treat Laplace's operator than technically complicated Stokes system. We start with a simple problem, which would serve us to present the main ideas.

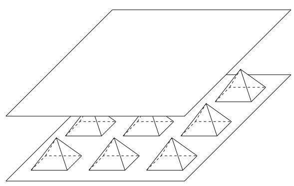

FIGURE 1. Domain Ω^ε with the rough boundary $\mathcal{B}^\varepsilon$.

2.1. The geometry and statement of the model problem. We consider the Poisson equation in a domain $\Omega^\varepsilon = P \cup \Sigma \cup R^\varepsilon$ consisting of the parallelepiped $P = (0, L_1) \times (0, L_2) \times (0, L_3)$, the interface $\Sigma = (0, L_1) \times (0, L_2) \times \{0\}$ and the layer of roughness $R^\varepsilon = \big(\cup_{\{k \in \mathbb{Z}^2\}} \varepsilon \left(Y + (k_1 b_1, k_2 b_2, -b_3)\right)\big) \cap ((0, L_1) \times (0, L_2) \times (-\varepsilon b_3, 0))$. The canonical cell of roughness $Y \subset (0, b_1) \times (0, b_2) \times (0, b_3)$ is defined in Subsection 2.2. Let $\Upsilon = \partial Y \setminus \Sigma$. For simplicity we suppose that $L_1/(\varepsilon b_1)$ and $L_2/(\varepsilon b_2)$

are integers. Let $\mathcal{I} = \{k \in \mathbb{Z}^2 : 0 \le k_1 \le L_1/b_1; 0 \le k_2 \le L_2/b_2\}$. Then, the rough boundary is $\mathcal{B}^\varepsilon = \cup_{\{k\in\mathcal{I}\}}\varepsilon\big(\Upsilon + (k_1b_1, k_2b_2, -b_3)\big)$. It consists of a large number of periodically distributed humps of characteristic length and amplitude ε, small compared with a characteristic length of the macroscopic domain. Finally, let $\Sigma_2 = (0, L_1) \times (0, L_2) \times \{L_3\}$.

We suppose that $f \in C^\infty(\overline{\Omega^\varepsilon})$, periodic in (x_1, x_2) with period (L_1, L_2), and consider the following problem:

$$
\begin{aligned}
-\Delta v^\varepsilon &= f && \text{in } \Omega^\varepsilon, && (1.2)\\
v^\varepsilon &= 0 && \text{on } \mathcal{B}^\varepsilon \cup \Sigma_2, && (1.3)\\
v^\varepsilon & && \text{is periodic in } (x_1, x_2) \text{ with period } (L_1, L_2). && (1.4)
\end{aligned}
$$

Obviously problem (1.2)–(1.4) admits a unique solution in $H(\Omega^\varepsilon)$, where

$$
\begin{aligned}
H(\Omega^\varepsilon) = \{\varphi \in H^1(\Omega^\varepsilon) :\ & \varphi = 0 \text{ on } \mathcal{B}^\varepsilon \cup \Sigma_2,\\
& \varphi \text{ is periodic in } x' = (x_1, x_2) \text{ with period } (L_1, L_2)\}. \quad (1.5)
\end{aligned}
$$

By elliptic regularity, $v^\varepsilon \in C^\infty(\Omega^\varepsilon)$. Every element of $H(\Omega^\varepsilon)$ is extended by zero to $(0, L_1) \times (0, L_2) \times (-b_3, 0) \setminus R^\varepsilon$.

STEP 1: Zero order approximation

We consider the problem

$$
\begin{aligned}
-\Delta u_0 &= f && \text{in } P, && (1.6)\\
u_0 &= 0 && \text{on } \Sigma \cup \Sigma_2, && (1.7)\\
u_0 & && \text{is periodic in } (x_1, x_2) \text{ with period } (L_1, L_2). && (1.8)
\end{aligned}
$$

Obviously problem (1.6)–(1.8) admits a unique solution in $H(P)$ and, after extension by zero to $(0, L_1)\times(0, L_2)\times(-b_3, 0)$, it is also an element of $H(\Omega^\varepsilon)$. Obviously

$$v^\varepsilon \rightharpoonup u_0, \text{ weakly in } H(P).$$

We wish to have an error estimate.

First we need estimates of the L^2-norms of the function in a domain and at a boundary using the L^2-norm of the gradient. Here the geometrical structure is used in essential way. We have:

Proposition 1.1. *Let $\varphi \in H(\Omega^\varepsilon)$. Then we have*

$$\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2}\|\nabla_x\varphi\|_{L^2(\Omega^\varepsilon\setminus P)^3}, \quad (1.9)$$

$$\|\varphi\|_{L^2(\Omega^\varepsilon\setminus P)} \le C\varepsilon\|\nabla_x\varphi\|_{L^2(\Omega^\varepsilon\setminus P)^3}. \quad (1.10)$$

This result is well-known and we give its proof only for the comfort of the reader.

Proof. Let $\tilde\varphi(y) = \varphi(\varepsilon y)$, $y \in Y+(k_1, k_2, -b_3)$. Then $\tilde\varphi \in H^1(Y+(k_1, k_2, -b_3))$, $\forall k$, and $\varphi = 0$ on $\Upsilon + (k_1, k_2, -b_3)$. Therefore by the trace theorem and the Poincaré's inequality

$$\int_{\{y_3=0\}\cap\bar{Y}+(k_1,k_2)} |\ \tilde\varphi(\tilde y, 0)\ |^2\ d\tilde y \le C\int_{Y+(k_1,k_2,-b_3)} |\ \nabla_y\tilde\varphi\ |^2\ dy.$$

Change of variables and summation over k gives

$$\left(\int_{\Sigma} |\ \varphi(\tilde{x}, 0)\ |^2 \ \mathrm{d}\tilde{x} \right)^{1/2} \leq C\varepsilon^{1/2} \left(\int_{R^\varepsilon} |\ \nabla_x \varphi(x)\ |^2 \ \mathrm{d}x \right)^{1/2}$$

and (1.9) is proved.

Inequality (1.10) is well-known (see e.g. Sanchez-Palencia [48]). □

Next we introduce $w = v^\varepsilon - u_0$. Then we have

$$-\Delta w = \begin{cases} 0, & \text{in } P, \\ f, & \text{in } R^\varepsilon, \end{cases} \tag{1.11}$$

and $w \in H(\Omega^\varepsilon)$ satisfies the variational equation

$$-\int_{\Sigma} \frac{\partial u_0}{\partial x_3} \varphi \ \mathrm{d}S + \int_{\Omega^\varepsilon} \nabla w \nabla \varphi \ \mathrm{d}x = \int_{R^\varepsilon} f \varphi \ \mathrm{d}x, \quad \forall \varphi \in H(\Omega^\varepsilon). \tag{1.12}$$

After testing (1.12) by $\varphi = w$, and using Proposition 1.1 we get

$$\int_{\Omega^\varepsilon} |\nabla w|^2 \ \mathrm{d}x \leq \left| \int_{R^\varepsilon} f w \ \mathrm{d}x \right| + \left| \int_{\Sigma} \frac{\partial u_0}{\partial x_3} w \ \mathrm{d}S \right| \leq C\sqrt{\varepsilon} ||w||_{L^2(R^\varepsilon)}. \tag{1.13}$$

We conclude that

$$||\nabla(v^\varepsilon - u_0)||_{L^2(\Omega^\varepsilon)} \leq C\sqrt{\varepsilon}. \tag{1.14}$$

Could we get some more precise error estimates? Answer is positive. First, after recalling that the total variation of ∇w is given by

$$\int_{\Omega^\varepsilon} |\nabla w| \ \mathrm{d}x = \sup \left\{ \int_{\Omega^\varepsilon} w \operatorname{div} \mathbf{s} \ \mathrm{d}x : \mathbf{s} \in C_0^1(\Omega^\varepsilon; \mathbb{R}^3), \ |\mathbf{s}(x)| \leq 1, \ \forall x \in \Omega^\varepsilon \right\},$$

we conclude that

$$||v^\varepsilon - u_0||_{BV(\Omega^\varepsilon)} \leq C\varepsilon. \tag{1.15}$$

Next, we need the notion of the *very weak solution* of the Poisson equation:

DEFINITION 1.2. Function $B \in L^2(P)$ is called a very weak solution of the problem

$$\begin{aligned} -\Delta B &= G \in H^{-1}(P) && \text{in } P \\ B &= \xi \in L^2(\Sigma \cup \Sigma_2) && \text{on } \Sigma \cup \Sigma_2 \\ B & && \text{is periodic in } (x_1, x_2) \text{ with period } (L_1, L_2). \end{aligned} \tag{1.16}$$

if

$$-\int_P B \Delta \varphi \ \mathrm{d}x - \int_{\Sigma_2} \frac{\partial \varphi}{\partial x_3} \xi \ \mathrm{d}S + \int_{\Sigma} \frac{\partial \varphi}{\partial x_3} \xi \ \mathrm{d}S = \int_P G \varphi \ \mathrm{d}x, \quad \forall \varphi \in H(P) \cap C^2(\bar{P}).$$

We recall the following result on very weak solutions to Poisson equation, which is easily proved using transposition:

LEMMA 1.3. *The problem (1.16) has a unique very weak solution such that*

$$\begin{aligned} ||B||_{L^2(\Sigma)} &\leq C \left(||\xi||_{L^2(\Sigma \cup \Sigma_2)} + ||G||_{H^{-1}(P)} \right), \\ ||B||_{L^2(P)} &\leq C \left(||\xi||_{L^2(\Sigma \cup \Sigma_2)} + ||G||_{H^{-1}(P)} \right). \end{aligned} \tag{1.17}$$

Direct consequence of Lemma 1.3 is the estimate

$$\begin{aligned} ||v^\varepsilon - u_0||_{L^2(\Sigma)} &\le C\varepsilon, \\ ||v^\varepsilon - u_0||_{L^2(P)} &\le C\varepsilon. \end{aligned} \tag{1.18}$$

Now we see that if we want to have a better estimate, an additional correction is needed.

2.2. Laplace's boundary layer. The effects of roughness occur in a thin layer surrounding the rough boundary. In this subsection we construct the 3D boundary layer, which will be used in taking into account the effects of roughness.

We start by prescribing the geometry of the layer. Let b_j, $j = 1, 2, 3$ be three positive constants. Let $Z = (0, b_1) \times (0, b_2) \times (0, b_3)$ and let Υ be a Lipschitz surface $y_3 = \Upsilon(y_1, y_2)$, taking values between 0 and b_3. We suppose that the rough surface $\cup_{k\in\mathbb{Z}^2} \big(\Upsilon + (k_1 b_1, k_2 b_2, 0)\big)$ is also a Lipschitz surface. We introduce the canonical cell of roughness (the canonical hump) by $Y = \{y \in Z : b_3 > y_3 > \max\{0, \Upsilon(y_1, y_2)\}\}$.

The crucial role is played by an auxiliary problem. It reads as follows:

Find β that solves

$$-\Delta_y \beta = 0 \qquad \text{in } Z^+ \cup (Y - b_3\vec{e}_3) \tag{1.19}$$

$$\begin{aligned} [\beta]_S (\cdot, 0) &= 0 \\ \left[\frac{\partial \beta}{\partial y_3}\right]_S (\cdot, 0) &= 1 \end{aligned} \tag{1.20}$$

$$\beta = 0 \qquad \text{on } (\Upsilon - b_3\vec{e}_3), \tag{1.21}$$

$$\beta \qquad \text{is } y' = (y_1, y_2)\text{-periodic}, \tag{1.22}$$

where $S = (0, b_1) \times (0, b_2) \times \{0\}$, $Z^+ = (0, b_1) \times (0, b_2) \times (0, +\infty)$, and $Z_{\text{bl}} = Z^+ \cup S \cup (Y - b_3\vec{e}_3)$.

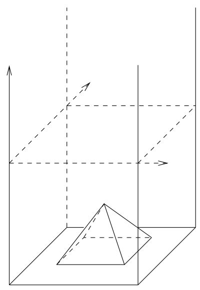

FIGURE 2. Boundary layer containing the canonical roughness.

Let $V = \big\{z \in L^2_{loc}(Z_{\text{bl}}) : \nabla_y z \in L^2(Z_{\text{bl}})^3; z = 0 \text{ on } (\Upsilon - b_3\vec{e}_3);\ \text{and } z \text{ is } y' = (y_1, y_2)\text{-periodic}\big\}$. Then, by Lax–Milgram lemma, there is a unique $\beta \in V$ satisfying

$$\int_{Z_{\text{bl}}} \nabla\beta\nabla\varphi \,\mathrm{d}y = -\int_S \varphi \,\mathrm{d}y_1\mathrm{d}y_2, \qquad \forall \varphi \in V. \tag{1.23}$$

By the elliptic theory, any variational solution β to (1.19)–(1.22) satisfies $\beta \in V \cap C^\infty(Z^+ \cup (Y - b_3\vec{e}_3))$.

LEMMA 1.4. *For every $y_3 > 0$ we have*

$$\int_0^{b_1}\int_0^{b_2} \beta(y_1,y_2,y_3)\,\mathrm{d}y_1\mathrm{d}y_2 = C^{\mathrm{bl}} = \int_S \beta\,\mathrm{d}y_1\mathrm{d}y_2 = -\int_{Z_{\mathrm{bl}}} |\nabla\beta(y)|^2\,\mathrm{d}y < 0. \tag{1.24}$$

Next, let $a > 0$ and let β^a be the solution for (1.19)–(1.22) with S replaced by $S_a = (0,b_1)\times(0,b_2)\times\{a\}$ and Z^+ by $Z^+_a = (0,b_1)\times(0,b_2)\times(a,+\infty)$. Then we have

$$C^{a,\mathrm{bl}} = \int_0^{b_1}\int_0^{b_2} \beta^a(y_1,y_2,a)\,\mathrm{d}y_1 = C^{\mathrm{bl}} - ab_1b_2. \tag{1.25}$$

PROOF. Integration of the equation (1.19) over the section, gives for any $y_3 > a$

$$\frac{\mathrm{d}^2}{\mathrm{d}y_3^2}\int_0^{b_1}\int_0^{b_2} \beta^a(y_1,y_2,y_3)\,\mathrm{d}y_1\mathrm{d}y_2 = 0 \quad \text{on } (a,+\infty). \tag{1.26}$$

Since $\beta^a \in V$, we conclude that $\int_0^{b_1}\int_0^{b_2}\beta^a(y_1,y_2,y_3)\,\mathrm{d}y_1\mathrm{d}y_2$ is constant on $(a,+\infty)$. Then the variational equation (1.23) yields (1.24).

Next we have

$$C^{a,\mathrm{bl}} = \int_0^{b_1}\int_0^{b_2} \beta^a(y_1,y_2,c)\,\mathrm{d}y_1\mathrm{d}y_2, \quad \forall c \geq a.$$

Let $0 \leq c_1 < a < c_2$. Integration of the equation (1.19) over (c_1,c_2) gives

$$\int_0^{b_1}\int_0^{b_2} \left\{ \frac{\partial\beta^a}{\partial y_3}(y_1,y_2,c_2) - \frac{\partial\beta^a}{\partial y_3}(y_1,y_2,a+0) \right. \\ \left. + \frac{\partial\beta^a}{\partial y_3}(y_1,y_2,a-0) - \frac{\partial\beta^a}{\partial y_3}(y_1,y_2,c_1) \right\} \mathrm{d}y_1\mathrm{d}y_2 = 0.$$

Hence from (1.20) and (1.26) we get

$$\frac{\mathrm{d}}{\mathrm{d}y_3}\int_0^{b_1}\int_0^{b_2} \beta^a(y_1,y_2,y_3)\,\mathrm{d}y_1\mathrm{d}y_2 = -b_1b_2, \quad \text{for } c_1 < y_3 < a$$

and

$$\int_0^{b_1}\int_0^{b_2} \beta^a(y_1,y_2,y_3)\,\mathrm{d}y_1\mathrm{d}y_2 = (a-y_3)b_1b_2 + C^{a,\mathrm{bl}}, \quad \text{for } 0 \leq y_3 \leq a. \tag{1.27}$$

The variational equation for $\beta^a - \beta$ reads

$$\int_{Z_{\mathrm{bl}}} \nabla(\beta^a-\beta)\nabla\varphi\,\mathrm{d}y = -\int_0^{b_1}\int_0^{b_2} (\varphi(y_1,y_2,a)-\varphi(y_1,y_2,0))\,\mathrm{d}y_1\mathrm{d}y_2, \quad \forall\varphi\in V.$$

Testing with $\varphi = \beta^a - \beta$ and using (1.27) yields

$$\int_{Z_{\mathrm{bl}}} |\nabla(\beta^a-\beta)|^2\,\mathrm{d}y = -\int_0^{b_1}\int_0^{b_2} (\beta^a(y_1,y_2,a)-\beta^a(y_1,y_2,0))\,\mathrm{d}y_1\mathrm{d}y_2 = ab_1b_2.$$

From the other hand

$$\int_{Z_{\mathrm{bl}}} |\nabla(\beta^a-\beta)|^2\,\mathrm{d}y = \int_{Z_{\mathrm{bl}}} |\nabla\beta^a|^2\,\mathrm{d}y + \int_{Z_{\mathrm{bl}}} |\nabla\beta|^2\,\mathrm{d}y - 2\int_{Z_{\mathrm{bl}}} \nabla\beta^a\nabla\beta\,\mathrm{d}y \\ = C^{\mathrm{bl}} - C^{a,\mathrm{bl}}$$

and formula (1.25) is proved. □

Next we search to establish the exponential decay. For the Laplace operator the result is known for long time. General reference for the decay of solutions to boundary layer problems corresponding to the operator $-\operatorname{div}(A\nabla u)$, with bounded and positively definite matrix A is [46], where a Saint Venant type estimate was proved. A very readable direct proof for similar setting and covering our situation, is in [4] and in [6]. Nevertheless one of the first known proofs for the case of second order elliptic operators in divergence form is in [39]. Here we will present the main steps of that approach from late seventies.

This early result is based on the following Tartar's lemma:

LEMMA 1.5. *(Tartar's lemma) Let V and V_0 be two real Hilbert spaces such that $V_0 \subset V$ with continuous injection. Let a be a continuous bilinear form on $V \times V_0$ and M a surjective continuous linear map between V and V_0. We assume that*

$$a(u, Mu) \geq \alpha ||u||_V^2, \quad \alpha > 0, \quad \forall u \in V \tag{1.28}$$

and $f \in V_0'$. Then there exists a unique $u \in V$ such that

$$a(u, v) = < f, v >_{V_0', V_0}, \quad \forall v \in V_0. \tag{1.29}$$

PROOF. For the proof see [39]. We note that this is a variant of Lax–Milgram lemma. □

Now we suppose that

$$\begin{aligned} A = A(y) \text{ is a matrix such that } A(y)\xi \cdot \xi \geq C_A |\xi|^2, \text{ a.e. and}& \\ ||A_{ij}||_\infty \leq \bar{C}_A \ ; \ g \in H^1_{per}(S);\ e^{\delta_0 y_3} f \in L^2(Z^+) \text{ for some } \delta_0 > 0,& \end{aligned} \tag{1.30}$$

and consider the problem

$$-\operatorname{div}_y(A(y)\nabla_y \beta) = f \qquad \text{in } Z^+, \tag{1.31}$$

$$\beta = g \qquad \text{on } S, \tag{1.32}$$

$$\beta \qquad \text{is } y' = (y_1, y_2)\text{-periodic}. \tag{1.33}$$

We have the following result

PROPOSITION 1.6. *Under conditions (1.30) the problem (1.31)–(1.33) admits a unique solution such that for some $\delta \in (0, \delta_0)$ we have*

$$\begin{aligned} \int_0^\infty \int_0^{b_1} \int_0^{b_2} e^{2\delta y_3} |\nabla_y \beta|^2 \, \mathrm{d}y < +\infty,& \\ \int_0^\infty \int_0^{b_1} \int_0^{b_2} e^{2\delta y_3} \left| \beta - \frac{1}{b_1 b_2} \int_0^{b_1} \int_0^{b_2} \beta(t, y_3) \, \mathrm{d}t \right|^2 \mathrm{d}y < +\infty.& \end{aligned} \tag{1.34}$$

PROOF. We just repeat the main steps from the proof from [39]. It relies on Tartar's lemma.

We introduce the spaces V and V_0 by

$$V = \{z \in L^2_{loc}((0, +\infty); H^1_{per}((0, b_1) \times (0, b_2))) : e^{\delta y_3} \nabla z \in L^2(Z^+) \text{ and } z|_S = 0\},$$

$$V_0 = \{z \in V : e^{\delta y_3} z \in L^2(Z^+)\}.$$

the associated bilinear form is

$$a(u,v)=\int_{Z^+} A\nabla u\nabla(e^{2\delta y_3}v)\,\mathrm{d}y,\quad u\in V,\ v\in V_0, \tag{1.35}$$

and the linear form is

$$< f,v>_{V_0',V_0}=\int_{Z^+} e^{2\delta y_3} f v\,\mathrm{d}y,\qquad v\in V_0. \tag{1.36}$$

Obviously, the linear form is continuous for $\delta\le\delta_0$. Same property holds for the bilinear form a.

In the next step we introduce the operator M by setting

$$Mu(y)=u(y)-\frac{2\delta}{b_1b_2}\int_0^{y_3}\int_0^{b_1}\int_0^{b_2} e^{-2\delta(y_3-t)}u(y_1,y_2,t)\,\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}t. \tag{1.37}$$

Using Poincaré's inequality in $H^1_{per}((0,b_1)\times(0,b_2))$ we get

$$e^{\delta y_3}Mu\in L^2(Z^+)\quad\text{and}\quad Mu\in V_0\quad\text{for}\quad \delta<\delta_0. \tag{1.38}$$

We note that M is surjective since the equation $Mu=v$, $v\in V_0$, admits a solution $u=v+2\delta\int_0^{y_3}< v>_{(0,b_1)\times(0,b_2)}(t)\;dt\in V$.

Concerning ellipticity, a direct calculation yields

$$a(u,Mu)\ge(\alpha-2\delta C_P||A||_\infty)||e^{\delta y_3}\nabla u||_{L^2(Z^+)}, \tag{1.39}$$

where C_p is the constant in Poincaré's inequality in $H^1_{per}((0,b_1)\times(0,b_2))$. Therefore, for $\delta<\min\left\{\delta_0,\frac{\alpha}{2C_p}\frac{1}{||A||_\infty)}\right\}$ we have the ellipticity and the Proposition is proved. □

Next, by refining the result of Proposition 1.6 we get the pointwise exponential decay, as in [46].

2.3. Rigorous derivation of the wall law. After constructing the boundary layer, we are ready for passing to the next order

STEP 2: Next order correction

From the proof of (1.13) we see that the main contribution comes from the term corresponding to the artificial interface Σ. Therefore one should eliminate the term $\int_\Sigma\frac{\partial u_0}{\partial x_3}\varphi\;dS$. The correction is given through a new unknown $u^{\mathrm{bl},\varepsilon}$ and we search for $u^{\mathrm{bl},\varepsilon}\in H(\Omega^\varepsilon)$ such that

$$\int_\Sigma\frac{\partial u_0}{\partial x_3}\varphi\;dS+\int_{\Omega^\varepsilon}\nabla u^{\mathrm{bl},\varepsilon}\nabla\varphi\,\mathrm{d}x=0,\qquad\forall\varphi\in H(\Omega^\varepsilon). \tag{1.40}$$

Since the geometry is periodic this problem can be written as

$$\sum_{\{k\in\mathbb{Z}^2\;:\;(\varepsilon k_1,\varepsilon k_2)\in(0,L_1)\times(0,L_2)\}}\left\{\int_{\Upsilon+(\varepsilon k_1b_1,\varepsilon k_2b_2)}\frac{\partial u_0}{\partial x_3}\bigg|_{x_3=0}\varphi|_{x_3=0}\;\mathrm{d}S\right.$$
$$\left.+\int_{\varepsilon Z_{\mathrm{bl}}+(\varepsilon k_1b_1,\varepsilon k_2b_2,0)}\nabla u^{\mathrm{bl},\varepsilon}\nabla\varphi\,\mathrm{d}x\right\}=0. \tag{1.41}$$

For $\frac{\partial u_0}{\partial x_3}\Big|_\Sigma$ constant, by uniqueness, the solution to (1.41) would read $u^{\mathrm{bl},\varepsilon} = \varepsilon\beta\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_3}\Big|_\Sigma$, where β is the solution for (1.23). In general this is not the case, but this is the candidate for a good approximation. Also, the boundary layer function β does not satisfy the homogeneous Dirichlet boundary condition at Σ_2. In order to have correct boundary condition we introduce an auxiliary function v by

$$\begin{aligned} -\Delta v &= 0 && \text{in P},\\ v &= \frac{\partial u_0}{\partial x_3}\Big|_\Sigma && \text{on } \Sigma,\\ v &= 0 && \text{on } \Sigma_2,\\ v & && \text{is } (y_1, y_2)\text{-periodic.} \end{aligned} \tag{1.42}$$

Therefore we search for $u^{\mathrm{bl},\varepsilon}$ in the form

$$u^{\mathrm{bl},\varepsilon} = \varepsilon\left(\left(\beta\left(\frac{x}{\varepsilon}\right) - \frac{C^{\mathrm{bl}}}{b_1 b_2}H(x_3)\right)\frac{\partial u_0}{\partial x_3}\Big|_\Sigma + \frac{C^{\mathrm{bl}}}{b_1 b_2}v(x)H(x_3)\right) - w_\varepsilon, \tag{1.43}$$

where $C^{\mathrm{bl}} < 0$ is a uniquely determined constant such that $e^{\delta y_3}\left(\beta(y) - \frac{C^{\mathrm{bl}}}{b_1 b_2}\right) \in L^2(Z^+)$ (the boundary layer tail). By Proposition 1.6 we know that such constant exists and is uniquely determined.

Next by direct calculation, as in [35], we get

- $\operatorname{div}\left(\nabla\left(\beta\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_3}\Big|_\Sigma\right)\right)$ is bounded by $C\varepsilon^{3/2}$ in H^{-1}.
- Jump of the normal derivative of εv at Σ leads also to a term which is bounded by $C\varepsilon^{3/2}$ in H^{-1}.
- Corresponding terms in R^ε are even smaller.

Then after testing by $w_\varepsilon = v^\varepsilon - u_0 + \varepsilon\left(\left(\beta\left(\frac{x}{\varepsilon}\right) - \frac{C^{\mathrm{bl}}}{b_1 b_2}H(x_3)\right)\frac{\partial u_0}{\partial x_3}\Big|_\Sigma + \frac{C^{\mathrm{bl}}}{b_1 b_2}v(x)H(x_3)\right)$, we get that

$$\begin{aligned} ||\nabla w_\varepsilon||_{L^2(\Omega^\varepsilon)} &\le C\varepsilon^{3/2},\\ ||w_\varepsilon||_{L^2(\Sigma)} + ||w_\varepsilon||_{L^2(\Omega)} &\le C\varepsilon^2. \end{aligned} \tag{1.44}$$

STEP 3: Derivation of the wall law

Having obtained a good approximation for the solution of the original problem, we get the wall law. We start by a formal derivation:

At the interface Σ we have

$$\frac{\partial v^\varepsilon}{\partial x_3} = \frac{\partial u_0}{\partial x_3} - \frac{\partial\beta\left(\frac{x}{\varepsilon}\right)}{\partial x_3}\frac{\partial u_0}{\partial x_3} + O(\varepsilon)$$

and

$$\frac{v^\varepsilon}{\varepsilon} = \frac{u_0}{\varepsilon} - \frac{\partial u_0}{\partial x_3}\beta\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

After averaging, and using that $C^{\mathrm{bl}} = \int_0^{b_1}\int_0^{b_2}\beta(y_1, y_2, 0)\,\mathrm{d}y_1\mathrm{d}y_2$ and that the mean of the normal derivative is zero, we obtain the familiar form of *the wall law*

$$u^{\mathrm{eff}} = -\varepsilon\frac{C^{\mathrm{bl}}}{b_1 b_2}\frac{\partial u^{\mathrm{eff}}}{\partial x_3} \quad \text{on } \Sigma, \tag{1.45}$$

where u^{eff} is the average over the impurities and $C^{\text{bl}} < 0$ is defined by (1.24). The higher order terms are neglected.

Let us now give a rigorous justification of *the wall law* (1.45). First we introduce the effective problem:

$$\begin{aligned} -\Delta u^{\text{eff}} &= f && \text{in } P \\ u^{\text{eff}} &= -\varepsilon \frac{C^{\text{bl}}}{b_1 b_2} \frac{\partial u^{\text{eff}}}{\partial x_3} = \varepsilon \frac{C^{\text{bl}}}{b_1 b_2} \frac{\partial u^{\text{eff}}}{\partial \mathbf{n}} && \text{on } \Sigma, \\ u^{\text{eff}} &= 0 && \text{on } \Sigma_2, \\ u^{\text{eff}} & && \text{is } (y_1, y_2)\text{-periodic.} \end{aligned} \tag{1.46}$$

How close is u^{eff} to v^ε? In the difference

$$v^\varepsilon - u^{\text{eff}} = w_\varepsilon + u_0 - \varepsilon \left(\left(\beta \left(\frac{x}{\varepsilon} \right) - \frac{C^{\text{bl}}}{b_1 b_2} H(x_3) \right) \frac{\partial u_0}{\partial x_3} \Big|_\Sigma + \frac{C^{\text{bl}}}{b_1 b_2} v(x) H(x_3) \right) - u^{\text{eff}},$$

the error estimate (1.44) implies that w_ε is negligible. Next $\varepsilon \left(\beta \left(\frac{x}{\varepsilon} \right) - \frac{C^{\text{bl}}}{b_1 b_2} \right) \frac{\partial u_0}{\partial x_3} \Big|_\Sigma$ is $O(\varepsilon^{3/2})$ in $L^2(P)$ and $O(\varepsilon^2)$ in $L^1(P)$. Therefore it is enough to consider the function $z_\varepsilon = u_0 - \varepsilon \frac{C^{\text{bl}}}{b_1 b_2} v(x) - u^{\text{eff}}$. What do we know about this function?

First, we have $\Delta \left(u_0 - \varepsilon \frac{C^{\text{bl}}}{b_1 b_2} v(x) - u^{\text{eff}} \right) = 0$ in P. Then on the lateral boundaries and on Σ_2 it satisfies homogeneous boundary conditions. Finally on Σ we have

$$z_\varepsilon = -\varepsilon \frac{C^{\text{bl}}}{b_1 b_2} \frac{\partial z_\varepsilon}{\partial x_3} + \varepsilon^2 \left(\frac{C^{\text{bl}}}{b_1 b_2} \right)^2 \frac{\partial v}{\partial x_3}.$$

Hence z_ε solves the variational equation

$$\int_P \nabla z_\varepsilon \nabla \varphi \, \mathrm{d}x - \frac{b_1 b_2}{\varepsilon C^{\text{bl}}} \int_\Sigma z_\varepsilon \varphi \, \mathrm{d}S = -\frac{\varepsilon C^{\text{bl}}}{b_1 b_2} \int_\Sigma \frac{\partial v}{\partial x_3} \varphi \, \mathrm{d}S, \quad \forall \varphi \in H(P). \tag{1.47}$$

Testing (1.47) by $\varphi = z_\varepsilon$ yields

$$||\nabla z_\varepsilon||_{L^2(P)} \le C\varepsilon^{3/2}, \quad ||z_\varepsilon||_{L^2(\Sigma)} \le C\varepsilon^2 \quad \text{and} \quad ||z_\varepsilon||_{L^2(P)} \le C\varepsilon^2. \tag{1.48}$$

Using (1.44), (1.48) and estimates for the boundary layer β we conclude that

$$\begin{aligned} ||v^\varepsilon - u^{\text{eff}}||_{L^2(P)} &\le C\varepsilon^{3/2}, \\ ||v^\varepsilon - u^{\text{eff}}||_{H^1_{loc}(P)} &\le C\varepsilon^{3/2}, \\ ||v^\varepsilon - u^{\text{eff}}||_{L^1(P)} &\le C\varepsilon^2. \end{aligned} \tag{1.49}$$

Note that the approximation on Σ is not good. In fact the boundary layer is concentrated around Σ and there is a price to pay for neglecting it.

STEP 4: Invariance of the wall law

It remains to prove that translation of the artificial boundary of order $O(\varepsilon)$ does not change our effective solution. We have established in Lemma 1.4 the formula (1.25), showing how the boundary tail changes with translation of the artificial interface for a. Next using the smoothness of u^{eff} we find out that $u^{\text{eff}}(\cdot, x_3 - a\varepsilon)$ satisfies the wall law at $x_3 = a$ with error $O(\varepsilon^2)$. Now if f does not depend on x_3, we see that *the translation of the artificial boundary* at $O(\varepsilon)$ changes the result at order $O(\varepsilon^2)$. Things are more complicated if f depends on x_3.

2.4. Some further questions: almost periodic rough boundaries and curved rough boundaries. In the above sections the roughness was *periodic*. This corresponds to uniformly distributed rough elements. This is acceptable for industrially produced surfaces. Natural rough surfaces contain random irregularly distributed roughness elements.

In applications it is important to derive wall laws for random surfaces. The natural question to be raised is if our construction still works in that case. In estimates we were using Poincaré's inequality and clearly one should impose that our roughness layer does not become of large size with positive probability. But the real difficulty is linked to construction of boundary layers without periodicity assumption.

In this direction there is a recent progress for flow problems (see e.g. [15]), but still there are open questions.

Let us discuss the question of decay at infinity of boundary layers which is crucial for our estimates. We will follow the results by Amar et al from [5].

For sake of simplicity, we shall work in $\mathbb{R}^2$. Our equation will be posed in the half space $\Pi = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, whose boundary $\partial\Pi$ is the real axis $\{(x,y) \in \mathbb{R}^2 : y = 0\}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function, which is almost-periodic in the sense of Bohr (simply, almost-periodic), which means that for every $\delta > 0$, there exists a strictly positive number $\ell_\delta > 0$ such that for every real interval of length ℓ_δ there exists a number τ_δ satisfying $\sup_{x\in\mathbb{R}} |h(x+\tau_\delta) - h(x)| \leq \delta$. A well known reference on almost-periodic functions is the book [19].

For any almost-periodic function h, the asymptotic average

$$M[h] = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} h(x)\,\mathrm{d}x$$

is well defined. Furthermore we can associate with h its generalized Fourier series, given by

$$h(x) \sim \sum_{\lambda\in\mathbb{R}} \tilde{h}(\lambda) e^{i\lambda x}, \qquad \tilde{h}(\lambda) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} h(x) e^{-i\lambda x}\,\mathrm{d}x.$$

The number $\tilde{h}(\lambda)$ is the Fourier coefficient of h associated to the frequency λ. It is well known that there exists *at most a countable set* of frequencies for which the Fourier coefficients are different from zero. Also the Parseval identity holds.

Now, in analogy with the periodic case and with almost-periodic data on $\partial\Pi$, we expect to find solutions to Laplace equation that are almost-periodic in the x variable and decay to a certain constant, say d, as y tends to infinity. In the periodic case d was equal to the average of h. In the almost-periodic case, d is given by the asymptotic average $M[h]$, that we may fix to be zero without loss of generality. In analogy with the periodic case, we introduce the following space of weakly decaying functions

$$L^2_{ap}(\Pi) = \Big\{\psi : \ x \to \psi(x,y) \text{ is almost-periodic } \forall y \geq 0,$$
$$||\psi||^2 = \int_0^{+\infty} \left[\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \psi^2(x,y)\,\mathrm{d}x \right] \mathrm{d}y = \int_0^{+\infty} M\left[\psi^2\right](y)\,\mathrm{d}y < +\infty \Big\} \tag{1.50}$$

As noted in [5], a trouble with $L^2_{ap}(\Pi)$ is that it is not complete. This is a known disadvantage of Besicovitch's spaces.

Next we study our boundary layer problem. For a given smooth almost-periodic function h it reads

$$\begin{aligned} \Delta\psi &= 0 && \text{in } \Pi, \\ \psi(x,0) &= h(x) && \text{on } \partial\Pi, \\ M[h] &= 0. \end{aligned} \tag{1.51}$$

It is well known that the unique smooth bounded solution for (1.51) is given by

$$\psi(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{yh(t)}{(x-t)^2+y^2}\,\mathrm{d}t. \tag{1.52}$$

Then we have the following result

THEOREM 1.7. *(see [5]) Let ψ be the unique bounded solution of (1.51). Then, for every fixed $y > 0$, the function $x \to \psi(x,y)$ is an almost-periodic function. Moreover, for any given $\gamma_0 > 0$, the following equivalence condition holds: $||\psi e^{\gamma y}|| < +\infty$ for every $0 < \gamma < \gamma_0$ if and only if $\tilde{h}(\lambda) = 0$ for every $|\lambda| < \gamma_0$.*

Further analysis in [5] lead to the conclusion that the necessary and sufficient condition for the exponential decay is that the frequencies λ of h are far from zero. It is worthwhile to point out that, in the purely periodic case, the frequencies are always far from zero and hence the exponential decay of the solution is in accordance with previous theorem. On the contrary, in the general almost-periodic case, the exponential decay property fails if the frequencies of h accumulate at zero. Difficulties are illustrated through the following explicit example from [5]:

Let $h(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2}\sin\left(\frac{x}{n^3}\right)$. Then the the series converges uniformly, the function h is well defined, almost-periodic and satisfies $M[h] = 0$. With this h, the problem (1.51) has a unique bounded solution

$$\psi(x,y) = \sum_{n=1}^{+\infty} \frac{1}{n^2}\sin\left(\frac{x}{n^3}\right)e^{-y/n^2}, \quad \text{with} \quad ||\psi|| = +\infty.$$

In this case not only that we do not have an exponential decay, but ψ is even not in the space $L^2_{ap}(\Pi)$.

We can only conclude that a reasonable theory would be possible in a correct setting and with well-prepared data.

Next difficulty is linked with the fact that in nature one has to handle *curved rough boundaries.* In the pioneering paper [1] the roughness was linked to a curved circular boundary. This work continued mainly with formal multiscale expansions and numerical simulations for flow problems (see [2], [3], [43] and references therein).

Nevertheless, there is a recent article [41] by Madureira and Valentin, with analysis of the curvature influence on 2D effective wall laws. Their geometry is essentially annular and it was possible to describe the rough surface using just angular variable. Their boundary layer problems are posed in an open angle and the connection with known results is to be established. Also their Laplace's operator in polar coordinates systematically misses a term. The paper gives ideas but not really the complete construction of the approximation. Furthermore, we note that the two-dimensional case is very special because it allows for a global isometric parametrization of the boundary, while in the multidimensional case even the correct formulation of the problem setting is not obvious.

Derivation of the approximations and effective boundary conditions for solutions of the Poisson equation on a domain in $\mathbb{R}^n$ whose boundary differs from the smooth boundary of a domain $\mathbb{R}^n$ by rapid oscillations of size ε, was considered in [45]. More precisely, the Poisson equation was supposed in a bounded or unbounded domain Ω of $\mathbb{R}^n$, $n \geq 2$, with smooth compact boundary $\Gamma = \partial\Omega$, being an $(n-1)$-dimensional Riemannian manifold. Using the unit outer normal ν to Γ, the tubular neighborhood of Γ was defined by the mapping $\mathcal{T} : (x,t) \to x + t\nu(x)$, defined on $\Gamma \times (-\delta, \delta)$. Then, using a function γ^ε from Γ to $\mathbb{R}$ such that $|\gamma^\varepsilon(x)| \leq \varepsilon M < \delta/2$ on Γ, and that γ^ε is locally ε-periodic through an atlas of charts, it was possible to define a rough boundary $\Gamma^\varepsilon = \mathcal{T}(x, \gamma^\varepsilon(x)); x \in \Gamma$. For this fairly general geometric situation it was possible to accomplish the steps 1 to 3 from the above construction, for the flat rough boundary. The wall law (1.45) was obtained again. Nevertheless, it was found that the coefficient C^{bl} depends on position. The position was present as a parameter in the boundary layer construction. The construction from [45] is to be extended to systems, most notably to the Stokes system.

3. Wall laws for the Stokes and Navier–Stokes equations

In the text which follows we will try to give a brief resume of the results concerning the wall laws for the incompressible Stokes and Navier–Stokes equations. Also we will recall the basic steps of the construction of the boundary layer corrections, following the approach from [31].

Flow problems over rough surfaces were considered by O. Pironneau and collaborators in [43], [2] and [3]. The paper [43] considers the flow over a rough surface and the flow over a wavy sea surface. It discusses a number of problems and announces a rigorous result for an approximation of the Stokes flow. Similarly, in the paper [2] numerical calculations are presented and rigorous results in [3] are announced. Finally, in the paper [3] the stationary incompressible flow at high Reynolds number $\mathbf{Re} \sim \frac{1}{\varepsilon}$ over a periodic rough boundary, with the roughness period ε, is considered. An asymptotic expansion is constructed and, with the help of boundary layer correctors defined in a semi-infinite cell, effective wall laws are obtained. A numerical validation is presented, but there are no mathematically rigorous convergence results. The error estimate for the approximation, announced in [2], was not proved in [3]. We mention also the article [14].

In this section we are going to present a sketch of the justification of the Navier slip law by the technique developed in [30] for Laplace's operator and then in

[31] for the Stokes system. The result for a 2D laminar stationary incompressible viscous flow over a rough boundary is in [35]. It presents a generalization of the analogous results on the justification of the law by Beavers and Joseph [16] for a tangential viscous flow over a porous bed, obtained in [32], [33], [34] and [36]. For a review we refer to [42] and [38]. In the subsections which follow we consider a 3D Couette flow over a rough boundary. In Subsection 3.1 we introduce the corresponding boundary layer problem and in Subsection 3.2 we present the main steps in obtaining the Navier slip condition from [37].

3.1. Navier's boundary layer. As observed in hydrodynamics, the phenomena relevant to the boundary occur in a thin layer surrounding it. We are not interested in the boundary layers corresponding to the inviscid limit of the Navier–Stokes equations, but we undertake to construct the viscous boundary layer describing effects of the roughness. There is a similarity with boundary layers describing effects of interfaces between a perforated and a non-perforated domain. The corresponding theory for the Stokes system is in [31] and, in a more pedagogical way, in [42]. In this subsection we are going to present a sketch of construction of the main boundary layer, used for determining the coefficient in Navier's condition. It is natural to call it the *Navier's boundary layer.* In [35] the 2D boundary layer was constructed and the 3D case was studied in [37] .

We suppose the layer geometry from the beginning of the subsection 2.2.

Following the construction from [35], the crucial role is played by an auxiliary problem. It reads as follows:

For a given constant vector $\lambda \in \mathbb{R}^2$, find $\{\beta^\lambda, \omega^\lambda\}$ that solve

$$-\Delta_y \beta^\lambda + \nabla_y \omega^\lambda = 0 \qquad \text{in } Z^+ \cup (Y - b_3\vec{e}_3) \tag{1.53}$$

$$\operatorname{div}_y \beta^\lambda = 0 \qquad \text{in } Z_{\text{bl}} \tag{1.54}$$

$$\left[\beta^\lambda\right]_S (\cdot, 0) = 0 \qquad \text{on } S \tag{1.55}$$

$$\left[\{\nabla_y \beta^\lambda - \omega^\lambda I\}\vec{e}_3\right]_S (\cdot, 0) = \lambda \qquad \text{on } S \tag{1.56}$$

$$\beta^\lambda = 0 \qquad \text{on } (\Upsilon - b_3\vec{e}_3) \tag{1.57}$$

$$\{\beta^\lambda, \omega^\lambda\} \qquad \text{is } y' = (y_1, y_2)\text{-periodic}, \tag{1.58}$$

where $S = (0, b_1) \times (0, b_2) \times \{0\}$, $Z^+ = (0, b_1) \times (0, b_2) \times (0, +\infty)$, and $Z_{\text{bl}} = Z^+ \cup S \cup (Y - b_3\vec{e}_3)$.

Let $V = \{z \in L^2_{loc}(Z_{\text{bl}})^3 : \nabla_y z \in L^2(Z_{\text{bl}})^9;\ z = 0 \text{ on } (\Upsilon - b_3\vec{e}_3);\ \operatorname{div}_y z = 0 \text{ in } Z_{\text{bl}} \text{ and } z \text{ is } y' = (y_1, y_2)\text{-periodic}\}$. Then, by the Lax–Milgram lemma, there is a unique $\beta^\lambda \in V$ satisfying

$$\int_{Z_{\text{bl}}} \nabla \beta^\lambda \nabla \varphi \, \mathrm{d}y = -\int_S \varphi \lambda \, \mathrm{d}y_1 \mathrm{d}y_2, \qquad \forall \varphi \in V. \tag{1.59}$$

Using De Rham's theorem we obtain a function $\omega^\lambda \in L^2_{loc}(Z_{\text{bl}})$, unique up to a constant and satisfying (1.53). By the elliptic theory, $\{\beta^\lambda, \omega^\lambda\} \in V \cap C^\infty(Z^+ \cup (Y - b_3\vec{e}_3))^3 \times C^\infty(Z^+ \cup (Y - b_3\vec{e}_3))$, for any solution to (1.53)–(1.58).

In the neighborhood of S we have $\beta^\lambda - (\lambda_1, \lambda_2, 0)(y_3 - y_3^2/2)e^{-y_3}H(y_3) \in W^{2,q}$ and $\omega^\lambda \in W^{1,q}$, $\forall q \in [1, \infty)$.

Then we have

LEMMA 1.8. *([31], [32], [42]). For any positive a, a_1 and a_2, $a_1 > a_2$, the solution $\{\beta^\lambda, \omega^\lambda\}$ satisfies*

$$\begin{aligned}
&\int_0^{b_1}\int_0^{b_2} \beta_2^\lambda(y_1, y_2, a)\, \mathrm{d}y_1\mathrm{d}y_2 = 0,\\
&\int_0^{b_1}\int_0^{b_2} \omega^\lambda(y_1, y_2, a_1)\, \mathrm{d}y_1\mathrm{d}y_2 = \int_0^{b_1}\int_0^{b_2} \omega^\lambda(y_1, y_2, a_2)\, \mathrm{d}y_1\mathrm{d}y_2,\\
&\int_0^{b_1}\int_0^{b_2} \beta_j^\lambda(y_1, y_2, a_1)\, \mathrm{d}y_1\mathrm{d}y_2 = \int_0^{b_1}\int_0^{b_2} \beta_j^\lambda(y_1, y_2, a_2)\, \mathrm{d}y_1\mathrm{d}y_2, \quad j = 1, 2,\\
C_\lambda^{\mathrm{bl}} &= \sum_{j=1}^{2} C_\lambda^{j,\mathrm{bl}}\lambda_j = \int_S \beta^\lambda \lambda\, \mathrm{d}y_1\mathrm{d}y_2 = -\int_{Z_{\mathrm{bl}}} |\nabla\beta^\lambda(y)|^2\, \mathrm{d}y < 0.
\end{aligned} \tag{1.60}$$

LEMMA 1.9. *Let $\lambda \in \mathbb{R}^2$ and let $\{\beta^\lambda, \omega^\lambda\}$ be the solution for (1.53)–(1.58) satisfying $\int_S \omega^\lambda\, \mathrm{d}y_1 dy_2 = 0$. Then $\beta^\lambda = \sum_{j=1}^2 \beta^j\lambda_j$ and $\omega^\lambda = \sum_{j=1}^2 \omega^j\lambda_j$, where $\{\beta^j, \omega^j\} \in V \times L^2_{loc}(Z_{\mathrm{bl}})$, $\int_S \omega^j\, \mathrm{d}y_1 dy_2 = 0$, is the solution for (1.53)–(1.58) with $\lambda = \vec{e}_j$, $j = 1, 2$.*

LEMMA 1.10. *Let $a > 0$ and let $\beta^{a,\lambda}$ be the solution for (1.53)–(1.58) with S replaced by $S_a = (0, b_1) \times (0, b_2) \times \{a\}$ and Z^+ by $Z_a^+ = (0, b_1) \times (0, b_2) \times (a, +\infty)$. Then we have*

$$C_\lambda^{a,\mathrm{bl}} = \int_0^{b_1}\int_0^{b_2} \beta^{a,\lambda}(y_1, y_2, a)\lambda\, \mathrm{d}y_1 = C_\lambda^{\mathrm{bl}} - a \mid \lambda \mid^2 b_1 b_2 \tag{1.61}$$

PROOF. It goes along the same lines as Lemma 2 from [35] and we omit it. □

LEMMA 1.11. *(see [37]) Let $\{\beta^j, \omega^j\}$ be as in Lemma 1.8 and let $M_{ij} = \frac{1}{b_1 b_2}\int_S \beta_i^j\, \mathrm{d}y_1\mathrm{d}y_2$ be the Navier matrix. Then the matrix M is symmetric negatively definite.*

LEMMA 1.12. *(see [37]) Let Y have the mirror symmetry with respect to y_j, where j is 1 or 2. The the matrix M is diagonal.*

LEMMA 1.13. *(see [37]) Let us suppose that the shape of the boundary doesn't depend on y_2 . Then for $\lambda = \vec{e}_2$ the system (1.53)–(1.58) has the solution $\beta^2 = (0, \beta_2^2(y_1, y_3), 0)$ and $\omega^2 = 0$, where β_2^2 is determined by*

$$-\frac{\partial^2\beta_2^2}{\partial y_1^2} - \frac{\partial^2\beta_2^2}{\partial y_3^2} = 0 \qquad \text{in } (0, b_1) \times (0, +\infty) \cup (Y \cap \{y_2 = 0\} - b_3\vec{e}_3) \tag{1.62}$$

$$\left[\beta_2^2\right](\cdot, 0) = 0 \qquad \text{on } (0, b_1) \times \{0\} \tag{1.63}$$

$$\left[\frac{\partial\beta_2^2}{\partial y_3}\right](\cdot, 0) = 1 \qquad \text{on } (0, b_1) \times \{0\} \tag{1.64}$$

$$\beta_2^2 = 0 \qquad \text{on } (\Upsilon \cap \{y_2 = 0\} - b_3\vec{e}_3), \tag{1.65}$$

$$\beta_2^2 \qquad \text{is } y_1\text{-periodic}, \tag{1.66}$$

Furthermore, for $\lambda = \vec{e}_1$*, the system (1.53)–(1.58) has the solution* $\beta^1 = (\beta_1^1(y_1, y_3), 0, \beta_3^1(y_1, y_3))$ *and* $\omega^1 = \omega(y_1, y_3)$ *satisfying*

$$-\frac{\partial \beta_j^1}{\partial y_1^2} - \frac{\partial \beta_j^1}{\partial y_3^2} + \frac{\partial \omega}{\partial y_j} = 0 \qquad \text{in } (0, b_1) \times (0, +\infty) \cup (Y \cap \{y_2 = 0\} - b_3 \vec{e}_3), \; j = 1 \text{ and } j = 3 \tag{1.67}$$

$$\frac{\partial \beta_1^1}{\partial y_1} + \frac{\partial \beta_3^1}{\partial y_3} = 0 \qquad \text{in } Z_{\rm bl} \cap \{y_2 = 0\} \tag{1.68}$$

$$\left[\beta_j^1\right](\cdot, 0) = 0 \qquad \text{on } (0, b_1) \times \{0\}, \; j = 1 \text{ and } j = 3 \tag{1.69}$$

$$\left.\begin{aligned} [\omega] &= 0 \\ \left[\frac{\partial \beta_1^1}{\partial y_3}\right](\cdot, 0) &= 1 \\ \left[\frac{\partial \beta_3^1}{\partial y_3}\right](\cdot, 0) &= 1 \end{aligned}\right\} \text{ on } (0, b_1) \times \{0\}, \tag{1.70}$$

$$\beta_1^1 = \beta_3^1 = 0 \qquad \text{on } (\Upsilon \cap \{y_2 = 0\} - b_3 \vec{e}_3), \tag{1.71}$$

$$\{\beta_1^1, \beta_3^1, \omega\} \qquad \text{is } y_1\text{-periodic.} \tag{1.72}$$

Finally,

$$\begin{aligned} M_{11} &= \frac{1}{b_1} \int_0^{b_1} \beta_1^1(y_1, 0)\, \mathrm{d}y_1 \\ M_{12} &= M_{21} = 0 \\ M_{22} &= \frac{1}{b_1} \int_0^{b_1} \beta_2^2(y_1, 0)\, \mathrm{d}y_1 \end{aligned} \tag{1.73}$$

and $| M_{11} | \leq | M_{22} |$.

LEMMA 1.14. *Let* $\{\beta^j, \omega^j\}, j = 1$ *and* $j = 3$*, be as in Lemma 1.8. Then we have*

$$\begin{aligned} | D^\alpha \operatorname{curl}_y \beta^j(y) | &\leq C e^{-2\pi y_3 \min\left\{\frac{1}{b_1}, \frac{1}{b_2}\right\}}, && y_3 > 0, \; \alpha \in \mathbb{N}^2 \cup (0,0), \\ | \beta^j(y) - (M_{1j}, M_{2j}, 0) | &\leq C(\delta) e^{-\delta y_3}, && \begin{cases} y_3 > 0, \\ \forall \delta < 2\pi \min\left\{\frac{1}{b_1}, \frac{1}{b_2}\right\} \end{cases} \\ | D^\alpha \beta^j(y) | &\leq C(\delta) e^{-\delta y_3}, && \begin{cases} y_3 > 0, \; \alpha \in \mathbb{N}^2, \\ \forall \delta < 2\pi \min\left\{\frac{1}{b_1}, \frac{1}{b_2}\right\} \end{cases} \\ | \omega^j(y) | &\leq C e^{-2\pi y_3 \min\left\{\frac{1}{b_1}, \frac{1}{b_2}\right\}}, && y_3 > 0. \end{aligned} \tag{1.74}$$

PROOF. As in [32] we take the curl of the equation (1.53) and obtain the following problem for $\xi_m^j = \left(\operatorname{curl} \beta^j\right)_m$, $m = 1, 2, 3$

$$\begin{aligned} &\Delta \xi_m^j = 0 && \text{in } Z^+ \\ &\xi_m^j \in W^{1-1/q, q}(S) && \forall q < +\infty \\ &\xi_m^j && \text{is periodic in } y' = (y_1, y_2) \end{aligned} \tag{1.75}$$

Now Tartar's lemma from [39] (see Lemma 1.5) implies an exponential decay of $\nabla \xi_m^j$ to zero and of ξ_m^j. Since $\xi_m^j \in L^2(Z^+)$, this constant equals to zero. Furthermore, having established an exponential decay, we are in situation to apply the separation of variables. Then explicit calculations, analogous to those in [36], give the first estimate in (1.74).

In the next step we use the following identity, holding for the divergence free fields:

$$-\Delta\beta^j = \operatorname{curl}\operatorname{curl}\beta^j = \operatorname{curl}\xi^j$$

and the same arguing as above leads to the second and the third estimate.

After taking the divergence of the equation (1.53) we find out that the pressure is harmonic in Z^+. Since the averages of the pressure over the sections $\{y_3 = a\}$ are zero, we obtain the last estimate in (1.75). □

COROLLARY 1.15. *The system (1.53)–(1.58) defines a boundary layer.*

3.2. Justification of the Navier slip condition for the laminar 3D Couette flow. A mathematically rigorous justification of the Navier slip condition for the 2D Poiseuille flow over a rough boundary is in [35]. Rough boundary was the periodic repetition of a basic cell of roughness, with characteristic heights and lengths of the impurities equal to a small parameter ε. Then the flow domain was decomposed to a rough layer and its complement.

The no-slip condition was imposed on the rough boundary and there were inflow and outflow boundaries, not interacting with the humps. The flow was governed by a given constant pressure drop. The mathematical model were the stationary Navier–Stokes equations. In [35] the flow under moderate Reynolds numbers was considered and the following results were proved:

a) A non-linear stability result with respect to small perturbations of the smooth boundary with a rough one;
b) An approximation result of order $\varepsilon^{3/2}$;
c) Navier's slip condition was justified.

In this review we are going to present analogous results for a 3D Couette flow from [37].

We consider a viscous incompressible fluid flow in a domain Ω^ε defined in Subsection 2.1.

Then, for a fixed $\varepsilon > 0$ and a given constant velocity $\vec{U} = (U_1, U_2, 0)$, the Couette flow is described by the following system

$$-\nu\Delta\mathbf{v}^\varepsilon + (\mathbf{v}^\varepsilon\nabla)\mathbf{v}^\varepsilon + \nabla p^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \tag{1.76}$$

$$\operatorname{div}\mathbf{v}^\varepsilon = 0 \quad \text{in } \Omega^\varepsilon, \tag{1.77}$$

$$\mathbf{v}^\varepsilon = 0 \quad \text{on } \mathcal{B}^\varepsilon, \tag{1.78}$$

$$\mathbf{v}^\varepsilon = \vec{U} \quad \text{on } \Sigma_2 = (0, L_1) \times (0, L_2) \times \{L_3\} \tag{1.79}$$

$$\{\mathbf{v}^\varepsilon, p^\varepsilon\} \quad \text{is periodic in } (x_1, x_2) \text{ with period } (L_1, L_2) \tag{1.80}$$

where $\nu > 0$ is the kinematic viscosity and $\int_\Omega^\varepsilon p^\varepsilon \, \mathrm{d}x = 0$.

Let us note that a similar problem was considered in [9], but in an infinite strip with a rough boundary. In [9] the authors were looking for solutions periodic in (x_1, x_2), with the period $\varepsilon(b_1, b_2)$.

Since we need not only existence for a given ε, but also the a priori estimates independent of ε, we give a non-linear stability result with respect to rough perturbations of the boundary, leading to uniform a priori estimates.

First, we observe that the Couette flow in P, satisfying the no-slip conditions at Σ, is given by

$$\mathbf{v}^0 = \frac{U_1 x_3}{L_3}\vec{e}_1 + \frac{U_2 x_3}{L_3}\vec{e}_2 = \vec{U}\frac{x_3}{L_3}, \qquad p^0 = 0. \tag{1.81}$$

Let $|U| = \sqrt{U_1^2 + U_2^2}$. Then it is easy to see that $\mathbf{v}^0$ is the unique solution for the Couette flow in P if $|U|L_3 < 2\nu$, i.e. if the Reynolds number is moderate.

We extend the velocity field to $\Omega^\varepsilon \setminus P$ by zero.

The idea is to construct the solution to (1.76)–(1.80) as a small perturbation to the Couette flow (1.81). Before the existence result, we prove an auxiliary lemma:

LEMMA 1.16. *([35]). Let $\varphi \in H^1(\Omega^\varepsilon \setminus P)$ be such that $\varphi = 0$ on $\mathcal{B}^\varepsilon$. Then we have*

$$\|\varphi\|_{L^2(\Omega^\varepsilon\setminus P)} \le C\varepsilon \|\nabla\varphi\|_{L^2(\Omega^\varepsilon\setminus P)^3}, \tag{1.82}$$

$$\|\varphi\|_{L^2(\Sigma)} \le C\varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon\setminus P)^3}. \tag{1.83}$$

Now we are in position to prove the desired non-linear stability result:

THEOREM 1.17. *([37]). Let $|U|L_3 \le \nu$. Then there exists a constant $C_0 = C_0(b_1, b_2, b_3,\ L_1, L_2)$ such that for $\varepsilon \le C_0 \left(\frac{L_3}{|U|}\right)^{3/4} \nu^{3/4}$ the problem (1.76)–(1.80) has a unique solution $\{\mathbf{v}^\varepsilon, p^\varepsilon\} \in H^2(\Omega^\varepsilon)^3 \times H^1(\Omega^\varepsilon)$, $\int_\Omega^\varepsilon p^\varepsilon \,\mathrm{d}x = 0$, satisfying*

$$\|\nabla(\mathbf{v}^\varepsilon - \mathbf{v}^0)\|_{L^2(\Omega^\varepsilon)^9} \le C\sqrt{\varepsilon}\frac{|U|}{L_3}. \tag{1.84}$$

Moreover,

$$\|\mathbf{v}^\varepsilon\|_{L^2(\Omega^\varepsilon\setminus P)^3} \le C\varepsilon\sqrt{\varepsilon}\frac{|U|}{L_3}, \tag{1.85}$$

$$\|\mathbf{v}^\varepsilon\|_{L^2(\Sigma)^3} + \|\mathbf{v}^\varepsilon - \mathbf{v}^0\|_{L^2(P)^3} \le C\varepsilon\frac{|U|}{L_3}, \tag{1.86}$$

$$\|p^\varepsilon - p^0\|_{L^2(P)} \le C\frac{|U|}{L_3}\sqrt{\varepsilon}, \tag{1.87}$$

where $C = C(b_1, b_2, b_3, L_1, L_2)$.

Therefore, we have obtained the uniform a priori estimates for $\{\mathbf{v}^\varepsilon, p^\varepsilon\}$. Moreover, we have found that Couette's flow in P is an $O(\varepsilon)$ L^2-approximation for $\mathbf{v}^\varepsilon$.

Following the approach from [35], the Navier slip condition should correspond to taking into the account the next order corrections for the velocity. Then formally we get

$$\mathbf{v}^\varepsilon = \mathbf{v}^0 - \frac{\varepsilon}{L_3}\sum_{j=1}^{2} U_j \left(\beta^j\left(\frac{x}{\varepsilon}\right) - (M_{j1}, M_{j2}, 0)H(x_3)\right) \\ - \frac{\varepsilon}{L_3}\sum_{j=1}^{2} U_j \left(1 - \frac{x_3}{L_3}\right)(M_{j1}, M_{j2}, 0)H(x_3) + O(\varepsilon^2)$$

where $\mathbf{v}^0$ is the Couette velocity in P and the last term corresponds to the counterflow generated by the motion of Σ. Then on the interface Σ

$$\frac{\partial {\mathbf{v}^\varepsilon}_j}{\partial x_3} = \frac{U_j}{L_3} - \frac{1}{L_3}\sum_{i=1}^{2} U_i \frac{\partial \beta^i_j}{\partial y_3} + O(\varepsilon) \quad \text{and} \quad \frac{1}{\varepsilon}{\mathbf{v}^\varepsilon}_j = -\frac{1}{L_3}\sum_{i=1}^{2} U_i \beta^i_j\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

After averaging we obtain the familiar form of the Navier slip condition

$$u^{\text{eff}}_j = -\varepsilon \sum_{i=1}^{2} M_{ji}\frac{\partial u^{\text{eff}}_i}{\partial x_3} \quad \text{on } \Sigma, \tag{1.88}$$

where u^{eff} is the average over the impurities and the matrix M is defined in Lemma 1.11. The higher order terms are neglected.

Now let us make this formal asymptotic expansion rigorous.

It is clear that in P the flow continues to be governed by the Navier–Stokes system. The presence of the irregularities would only contribute to the effective boundary conditions at the lateral boundary. The leading contribution for the estimate (1.84) were the interface integral terms $\int_\Sigma \varphi_j$. Following the approach from [35], we eliminate it by using the boundary layer-type functions

$$\beta^{j,\varepsilon}(x) = \varepsilon\beta^j\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \omega^{j,\varepsilon}(x) = \omega^j\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega^\varepsilon,\ j = 1, 2, \tag{1.89}$$

where $\{\beta^j, \omega^j\}$ is defined in Lemma 1.8. We have, for all $q \geq 1$ and $j = 1, 2$,

$$\frac{1}{\varepsilon}\|\beta^{j,\varepsilon} - \varepsilon(M_{1j}, M_{2j}, 0)\|_{L^q(P)^3} + \|\omega^{j,\varepsilon}\|_{L^q(P)} + \|\nabla\beta^{j,\varepsilon}\|_{L^q(\Omega)^9} = C\varepsilon^{1/q} \tag{1.90}$$

and

$$-\Delta\beta^{j,\varepsilon} + \nabla\omega^{j,\varepsilon} = 0 \qquad \text{in } \Omega^\varepsilon \setminus \Sigma, \tag{1.91}$$

$$\operatorname{div}\beta^{j,\varepsilon} = 0 \qquad \text{in } \Omega^\varepsilon, \tag{1.92}$$

$$\left[\beta^{j,\varepsilon}\right]_\Sigma(\cdot, 0) = 0 \qquad \text{on } \Sigma, \tag{1.93}$$

$$\left[\{\nabla\beta^{j,\varepsilon} - \omega^{j,\varepsilon} I\} e_3\right]_\Sigma(\cdot, 0) = e_j \qquad \text{on } \Sigma. \tag{1.94}$$

As in [35] stabilization of $\beta^{j,\varepsilon}$ towards a nonzero constant velocity $\varepsilon(M_{1j}, M_{2j}, 0)$, at the upper boundary, generates a counterflow. It is given by the 3D Couette flow $d^i = \left(1 - \frac{x_3}{L_3}\right)\vec{e}_i$ and $g^i = 0$.

Now, we would like to prove that the following quantities are $o(\varepsilon)$ for the velocity and $O(\varepsilon)$ for the pressure:

$$\mathcal{U}^{\varepsilon}(x) = \mathbf{v}^{\varepsilon} - \frac{1}{L_3}\left(x_3^+ \vec{U} - \varepsilon \sum_{j=1}^{2} U_j \beta^j (\frac{x}{\varepsilon}) + \varepsilon \frac{x_3^+}{L_3} M \vec{U} \right), \tag{1.95}$$

$$\mathcal{P}^{\varepsilon} = p^{\varepsilon} + \frac{\nu}{L_3} \sum_{j=1}^{2} U_j \omega^{j,\varepsilon}. \tag{1.96}$$

Then we have the following result:

THEOREM 1.18. *([37]). Let* $\mathcal{U}^{\varepsilon}$ *be given by (1.95) and* $\mathcal{P}^{\varepsilon}$ *by (1.96). Then* $\mathcal{U}^{\varepsilon} \in H^1(\Omega^{\varepsilon})^3$, $\mathcal{U}^{\varepsilon} = 0$ *on* Σ, *it is periodic in* (x_1, x_2), *exponentially small on* Σ_2 *and* $\operatorname{div} \mathcal{U}^{\varepsilon} = 0$ *in* Ω^{ε}. *Furthermore,* $\forall \varphi$ *satisfying the same boundary conditions, we have the following estimate*

$$\left| \nu \int_{\Omega^{\varepsilon}} \nabla \mathcal{U}^{\varepsilon} \nabla \varphi - \int_{\Omega^{\varepsilon}} \mathcal{P}^{\varepsilon} \operatorname{div} \varphi + \int_{\Omega^{\varepsilon}} \frac{x_3^+}{L_3} \sum_{j=1}^{2} U_j \frac{\partial \mathcal{U}^{\varepsilon}}{\partial x_j} \varphi + \int_{\Omega^{\varepsilon}} \mathcal{U}_3^{\varepsilon} \frac{\vec{U}}{L_3} \varphi \right.$$
$$\left. + \int_{\Omega^{\varepsilon}} \big((\mathbf{v}^{\varepsilon} - \mathbf{v}^0) \nabla \big) (\mathbf{v}^{\varepsilon} - \mathbf{v}^0) \varphi \right| \leq C \varepsilon^{3/2} \| \nabla \varphi \|_{L^2(\Omega^{\varepsilon})^9} \frac{|U|^2}{L_3}. \tag{1.97}$$

COROLLARY 1.19. *([37]). Let* $\mathcal{U}^{\varepsilon}(x)$ *and* $\mathcal{P}^{\varepsilon}$ *be defined by (1.95)–(1.96) and let*

$$\varepsilon \leq \frac{\nu^{6/7}}{|U|} \min \left\{ \frac{\nu^{1/7}}{4(|M| + \|\beta\|_{L^{\infty}})}, C(b_1, b_2, b_3, L_1, L_2) L_3^{3/7} |U|^{1/7} \right\}. \tag{1.98}$$

Then $\mathbf{v}^{\varepsilon}$, *constructed in Theorem 1.17, is a unique solution to (1.76)–(1.80) and*

$$\| \nabla \mathcal{U}^{\varepsilon} \|_{L^2(\Omega^{\varepsilon})^9} + \mathcal{P}^{\varepsilon} \|_{L^2(P)} \leq C \varepsilon^{3/2} \frac{|U|^2}{\nu L_3}, \tag{1.99}$$

$$\| \mathcal{U}^{\varepsilon} \|_{L^2(P)^3} + \| \mathcal{U}^{\varepsilon} \|_{L^2(\Sigma)^3} \leq C \varepsilon^2 \frac{|U|^2}{\nu L_3}. \tag{1.100}$$

The estimates (1.99)–(1.100) allow to justify Navier's slip condition.

REMARK 1.20. *It is possible to add further correctors and then our problem would contain an exponentially decreasing forcing term. This is in accordance with [9] for the Navier–Stokes system and with [7], [8] and [13] for the Stokes system. For the case of rough boundaries with different characteristic heights and lengths we refer to the doctoral dissertation of I. Cotoi [28]. The estimate (1.98) is of the same order in* ε *as the* H^1*-estimate in [4], obtained for the Laplace operator. The advantage of our approach is that we are going to obtain the Navier slip condition with a negatively definite matricial coefficient.*

Now we introduce the effective Couette–Navier flow through the following boundary value problem:

Find a velocity field $\mathbf{u}^{\text{eff}}$ and a pressure field p^{eff} such that

$$-\nu\Delta\mathbf{u}^{\text{eff}} + (\mathbf{u}^{\text{eff}}\nabla)\mathbf{u}^{\text{eff}} + \nabla p^{\text{eff}} = 0 \qquad \text{in } P, \tag{1.101}$$

$$\operatorname{div}\mathbf{u}^{\text{eff}} = 0 \qquad \text{in } P, \tag{1.102}$$

$$\mathbf{u}^{\text{eff}} = (U_1, U_2, 0) \qquad \text{on } \Sigma_2, \tag{1.103}$$

$$\mathbf{u}_3^{\text{eff}} = 0 \qquad \text{on } \Sigma, \tag{1.104}$$

$$\mathbf{u}_j^{\text{eff}} = -\varepsilon\sum_{i=1}^{2} M_{ji}\frac{\partial \mathbf{u}_i^{\text{eff}}}{\partial x_3} \qquad \text{on } \Sigma,\ j = 1, 2, \tag{1.105}$$

$$\{\mathbf{u}^{\text{eff}}, p^{\text{eff}}\} \qquad \begin{array}{l}\text{is periodic in } (x_1, x_2)\\ \text{with period } (L_1, L_2)\end{array} \tag{1.106}$$

If $|U|L_3 \le \nu$, the problem (1.101)–(1.106) has a unique solution

$$\begin{aligned} \mathbf{u}^{\text{eff}} = (\tilde{u}^{\text{eff}}, 0),\ \tilde{u}^{\text{eff}} &= \vec{U} + \left(\frac{x_3}{L_3} - 1\right)\left(I - \frac{\varepsilon}{L_3}M\right)^{-1}\vec{U}, \quad x \in P, \\ p^{\text{eff}} &= 0, \qquad\qquad x \in P. \end{aligned} \tag{1.107}$$

Let us estimate the error made when replacing $\{\mathbf{v}^{\varepsilon}, p^{\varepsilon}, \mathcal{M}^{\varepsilon}\}$ by $\{\mathbf{u}^{\text{eff}}, p^{\text{eff}}, \mathcal{M}^{\text{eff}}\}$.

THEOREM 1.21. *([37]). Under the assumptions of Theorem 1.17 we have*

$$\|\nabla(\mathbf{v}^{\varepsilon} - \mathbf{u}^{\text{eff}})\|_{L^1(P)^9} \le C\varepsilon, \tag{1.108}$$

$$\sqrt{\varepsilon}\|\mathbf{v}^{\varepsilon} - \mathbf{u}^{\text{eff}}\|_{L^2(P)^3} + \|\mathbf{v}^{\varepsilon} - \mathbf{u}^{\text{eff}}\|_{L^1(P)^3} \le C\varepsilon^2\frac{|U|}{L_3}. \tag{1.109}$$

Our next step is to calculate the *tangential drag force* or the *skin friction*

$$\mathcal{F}_{t,j}^{\varepsilon} = \frac{1}{L_1L_2}\int_{\Sigma}\nu\frac{\partial \mathbf{v}^{\varepsilon}{}_j}{\partial x_3}(x_1, x_2, 0)\,\mathrm{d}x_1\mathrm{d}x_2, \quad j = 1, 2. \tag{1.110}$$

THEOREM 1.22. *([37]). Let the skin friction $\mathcal{F}_t^{\varepsilon}$ be defined by (1.110). Then we have*

$$\left|\mathcal{F}_t^{\varepsilon} - \nu\frac{1}{L_3}\left(\vec{U} + \frac{\varepsilon}{L_3}M\vec{U}\right)\right| \le C\varepsilon^2\frac{|U|^2}{\nu L_3}\left(1 + \frac{\nu}{L_3|U|}\right). \tag{1.111}$$

COROLLARY 1.23. . *Let $\mathcal{F}_t^{\text{eff}} = \nu\frac{1}{L_3}\left(I - \frac{\varepsilon}{L_3}M\right)^{-1}\vec{U}$ be the tangential drag force corresponding to the effective velocity $\mathbf{u}^{\text{eff}}$. Then we have*

$$|\mathcal{F}_t^{\text{eff}} - \mathcal{F}_t^{\varepsilon}| \le C\varepsilon^2\frac{|U|^2}{\nu L_3}\left(1 + \frac{\nu}{L_3|U|}\right) \tag{1.112}$$

REMARK 1.24. *We see that the presence of the periodic roughness diminishes the tangential drag. The contribution is linear in ε, and consequently rather small. It coincides with the conclusion from [8] that for laminar flows there is no palpable drag reduction. Nevertheless, we are going to see in the next subsection that the calculations from the laminar case could be useful for turbulent Couette flow.*

3.3. Wall laws for fluids obeying Fourier's boundary conditions at the rough boundary. In number of situations, the adherence conditions, that are used to describe fluid behavior when moderate pressure and low surface stresses are involved, are no longer valid. Physical considerations lead to slip boundary conditions. These conditions are of particular interest in the study of polymers, blood flow, and flow through filters. We mention also the near wall models from turbulence theories.

These conditions are of Fourier's type and in number of recent publications, authors undertook the homogenization of Stokes and Navier–Stokes equations in such setting. An early reference is [10], but it was the work of Simon et al [26] which attracted lot of interest. This is a fast developing research area and we mention only the articles [22] and [23]. In most cases the effective boundary condition is the no-slip condition. Consequently, the boundary layers do not enter into the wall law and the effective models are valid for much larger class of the rough boundaries than the wall law derived in the previous section.

Finally, we mention that there is a work on roughness induced wall laws for geostrophic flows. For more information see the article [20] and references therein.

4. Rough boundaries and drag minimization

Drag reduction for planes, ships and cars reduces significantly the spending of the energy, and consequently the cost for all type of land, sea and air transportation.

Drag-reduction adaptations were important for the survival of Avians and Nektons, since their efficiency or speed, or both, have improved.

Essentially, there are three forms of drag. The largest drag component is pressure or form drag. It is particularly troublesome when flow separation occurs. The two remaining drag components are skin-friction drag and drag due to lift. Skin-friction drag is the result of the no-slip condition on the surface. Those components are present for both laminar (low Reynolds number) or turbulent (high Reynolds number) flows.

There are several drag-reduction methods and here we discuss only the use of drag-reducing surfaces. For an overview of other techniques we refer to Bushnell, Moore [24].

The inspiration comes from morphological observations. It is known that the skin of fast sharks is covered with tiny scales having little longitudinal ribs on their surface (shark dermal denticles). These are tiny ridges, closely spaced (less than 100 μm apart and still less in height). We note that the considered sharks have a length of approximately 2 m and swim at Reynolds numbers $\mathbf{Re} \approx 3 \cdot 10^7$ (see e.g. Vogel [50]). Such grooves are similar to ones used on the yacht "Stars and Strips" in America's Cup finals and seem to reduce the skin-friction for $\mathcal{O}(10\%)$ (see [24]).

In the applications, the main interest is in the turbulent case. Mathematical modeling of the turbulent flows in the presence of solid walls is still out of reach. However the turbulent boundary layers on surfaces with fine roughness contain a viscous sublayer. It was found that the viscous sublayer exhibits a streaky structure. Those "low-speed streaks" are believed to be produced by slowly rotating longitudinal vortices. For a streaky structure, with a preferred lateral wavelength, a turbulent shear stress reduction was observed.

The experimental facts were theoretically explained in the papers by Bechert and Bartenwerfer [17] and Luchini, Manzo and Pozzi [40] (see also [18] and references in mentioned articles).

In [37] the theory developed in the laminar situation was applied to the turbulent flow. It is known that the turbulent Couette flow has a 2-layer structure. There is a large core layer where the molecular momentum transfer can be neglected and a thin wall layer (or sublayer) where both turbulent and molecular momentum transfer are important. The flow in the wall layer is governed by the turbulent viscous shear stress τ_w, supposed to depend only on time. In connection with τ_w authors use the friction velocity $v = \sqrt{\frac{\tau_w}{\rho}}$, where ρ is the density. Then the wall layer thickness is $\delta_v = \frac{\nu}{v}$, we suppose that our riblets remain all the time in the pure viscous sublayer and try to apply the analysis from the Subsection 3.2.

The corresponding equations are (1.76)–(1.80) with $L_3 = \delta_v$ and velocity $v = \sqrt{\frac{\tau_w}{\rho}} = (v_1, v_2, 0)$ at $x_3 = \delta_v$. Since $\delta_v \sqrt{\frac{\tau_w}{\rho}} = \nu < 2\nu$, our results from Subsection 3.2 are applicable and we get

$$\left| \mathcal{F}_t^\varepsilon - \frac{\nu}{\delta_v}\left(v + \frac{\varepsilon}{\delta_v} M v\right)\right| \leq C \left(\frac{\varepsilon |U|}{\delta_v}\right)^2. \tag{1.113}$$

Since $\delta_v = \nu\sqrt{\frac{\rho}{\tau_w}}$, we see that the effects of roughness are significant.

For the shark skin $\varepsilon/\delta_v = 0.1, L_3 = \delta_v = 10^{-3} = \sqrt{\nu}$ and $|U| = \sqrt{\nu} = 10^{-3}$. The uniqueness condition from Corollary 1.19 applies if $\varepsilon \leq C\nu^{9/4}$. Since $\varepsilon \approx 10^{-4}$ and $\nu^{9/14} \approx 1.389 \cdot 10^{-4}$. We see that our theory is applicable to the swimming of Nektons. For more details we refer to [36].

Furthermore, let us suppose the geometry of the rough boundary from [17] and [40]. Then M is diagonal and the origins of the cross and longitudinal flows are at the characteristic walls coordinates (see [49]) $y^+ = \frac{\varepsilon}{\delta_v} M_{11}$ and $y^+ = \frac{\varepsilon}{\delta_v} M_{22}$, respectively. Hence the proposition is to model the flow in the viscous sublayer in the presence of the rough boundary by the Couette–Navier profile (1.106) instead of the simple Couette profile in the smooth case.

We note that these observations were implemented numerically into a shape optimization procedure in [29]. The numerically obtained drag reduction confirmed the theoretical predictions from [36].

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Part 5

Hyperbolic problems with characteristic boundary

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ABSTRACT. In this lecture notes we consider mixed initial-boundary value problems with characteristic boundary for symmetric hyperbolic systems.

First, we recall the main results of the regularity theory. Among the applications, we describe some free boundary problems for the equations of motion of inviscid compressible flows in Fluid Dynamics and ideal MHD. These are problems where the free boundary is a characteristic hypersurface and the Lopatinskiĭ condition for the associated linearized equations holds only in weak form. In particular, we describe the result obtained in some joint papers by J.F. Coulombel and P. Secchi about the stability and existence of 2D compressible vortex sheets.

Then we present a general result about the regularity of solutions to characteristic initial-boundary value problems for symmetric hyperbolic systems. We assume the existence of the strong L^2-solution, satisfying a suitable energy estimate, without assuming any structural assumption sufficient for existence, such as the fact that the boundary conditions are maximally dissipative or satisfy the Kreiss-Lopatinskiĭ condition. We show that this is enough in order to get the regularity of solutions, in the natural framework of weighted anisotropic Sobolev spaces H^m_*, provided the data are sufficiently smooth.

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Contents

CHAPTER 1

Introduction

1. Characteristic IBVP's of symmetric hyperbolic systems

For a given integer $n \geq 2$, let Ω be an open bounded connected subset of $\mathbb{R}^n$, and let $\partial\Omega$ denote its boundary. For $T > 0$ we set $Q_T = \Omega \times]0, T[$ and $\Sigma_T = \partial\Omega \times]0, T[$. We are interested in the following initial-boundary value problem (shortly written IBVP)

$$\begin{array}{ll} Lu = F & \text{in } Q_T, \\ Mu = G & \text{on } \Sigma_T, \\ u_{|t=0} = f & \text{in } \Omega, \end{array} \tag{1.1}$$

where L is a first order linear partial differential operator

$$L = A_0(x,t,u)\partial_t + \sum_{i=1}^{n} A_i(x,t,u)\partial_i + B(x,t,u), \tag{1.2}$$

$\partial_t := \frac{\partial}{\partial t}$ and $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$.

The coefficients A_i, B, for $i = 0, \dots, n$, are real $N \times N$ matrix-valued functions, defined on Q_T. The unknown $u = u(x,t)$, and the data $F = F(x,t)$, $G = G(x,t)$, $f = f(x)$ are vector-valued functions with N components, defined on $\overline{Q}_T$, $\overline{\Sigma}_T$ and $\overline{\Omega}$ respectively. $M = M(x,t)$ is a given real $d \times N$ matrix-valued function; M is supposed to have maximal constant rank d.

Let $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ be the unit outward normal to $\partial\Omega$ at a point x; then

$$A_\nu(x,t) = \sum_{i=1}^{n} A_i(x,t)\nu_i(x)$$

is called the *boundary matrix*.

Definition 1.1. L is *symmetric hyperbolic* if the matrix A_0 is definite positive and symmetric on $\overline{Q}_T$, and the matrices A_i, for $i = 1, \dots, n$, are also symmetric.

Definition 1.2. The boundary is said *characteristic* if the boundary matrix A_ν is singular on Σ_T.

The boundary is *characteristic of constant multiplicity* if the boundary matrix A_ν is singular on Σ_T and $\operatorname{rank} A_\nu(x,t)$ is constant for all $(x,t) \in \Sigma_T$.

The boundary is *uniformly characteristic* if the boundary matrix A_ν is singular on Σ_T and $\operatorname{rank} A_\nu(x,t)$ is constant in a neighborhood of Σ_T.

The assumption that the boundary is *characteristic of constant multiplicity* yields that the number of negative eigenvalues (counted with multiplicity) of A_ν is constant on the connected components of Σ_T.

The case when the boundary matrix A_ν is singular on Σ_T and $\operatorname{rank} A_\nu(x,t)$ is not constant on Σ_T is said *nonuniformly characteristic*. This case is also physically quite interesting, but only partial results are known, see [40, 47, 56, 57]. In the present notes we will not discuss this problem.

Example 1.3. *Let us consider the Euler equations for inviscid compressible fluids*

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho\, v) = 0\,, \\ \partial_t(\rho\, v) + \nabla\cdot(\rho\, v\otimes v) + \nabla p = 0\,, \\ \partial_t(\rho e + \frac{1}{2}\rho|v|^2) + \nabla\cdot\big(\rho v(e+\frac{1}{2}|v|^2) + vp\big) = 0\,. \end{cases} \tag{1.3}$$

Here ρ denotes the density, S the entropy, v the velocity field, $p = p(\rho, S)$ the pressure (such that $p'_\rho > 0$), and $e = e(\rho, S)$ the internal energy.

The "Gibbs relation"

$$T\,dS = de + p\,dV$$

(with T the absolute temperature, and $V = \dfrac{1}{\rho}$ the specific volume) yields

$$p = -\left(\frac{\partial e}{\partial V}\right)_S = \rho^2\left(\frac{\partial e}{\partial \rho}\right)_S\,, \qquad T = \left(\frac{\partial e}{\partial S}\right)_\rho\,.$$

Therefore (1.3) *is a closed system for the vector of unknowns (ρ, v, S). For smooth solutions, system* (1.3) *can be rewritten as*

$$\begin{cases} \frac{\rho_p}{\rho}(\partial_t p + v\cdot\nabla p) + \nabla\cdot v = 0, \\ \rho\{\partial_t v + (v\cdot\nabla)v\} + \nabla p = 0, \\ \partial_t S + v\cdot\nabla S = 0. \end{cases} \tag{1.4}$$

This is a quasi-linear symmetric hyperbolic system since it can be written in the form

$$\begin{pmatrix} (\rho_p/\rho)(\partial_t + v\cdot\nabla) & \nabla\cdot & 0 \\ \nabla & \rho(\partial_t + v\cdot\nabla)I_3 & \underline{0} \\ 0 & \underline{0}^T & \partial_t + v\cdot\nabla \end{pmatrix}\begin{pmatrix} p \\ v \\ S \end{pmatrix} = 0.$$

The boundary matrix is

$$A_\nu = \begin{pmatrix} (\rho_p/\rho)v\cdot\nu & \nu^T & 0 \\ \nu & \rho v\cdot\nu I_3 & \underline{0} \\ 0 & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

If $v\cdot\nu = 0$, then

$$\ker A_\nu = \{U' = (p', v', S')\ :\ p' = 0,\ v'\cdot\nu = 0\},$$

and $\operatorname{rank} A_\nu = 2$.

Example 1.4. *Let us consider the equations of ideal Magneto-Hydrodynamics (MHD) for the motion of an electrically conducting fluid, where "ideal" means that* *the effect of viscosity and electrical resistivity is neglected. The equations read*

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, v) = 0\,, \\ \partial_t (\rho\, v) + \nabla \cdot (\rho\, v \otimes v - H \otimes H) + \nabla(p + \frac{1}{2}|H|^2) = 0\,, \\ \partial_t H - \nabla \times (v \times H) = 0\,, \\ \partial_t \big(\rho e + \frac{1}{2}(\rho |v|^2 + |H|^2)\big) \\ \qquad\qquad + \nabla \cdot \big(\rho v (e + \frac{1}{2}|v|^2) + vp + H \times (v \times H)\big) = 0\,, \\ \nabla \cdot H = 0\,. \end{cases} \tag{1.5}$$

Here ρ denotes the density, S the entropy, v the velocity field, H the magnetic field, $p = p(\rho, S)$ the pressure (such that $p'_\rho > 0$), and $e = e(\rho, S)$ the internal energy.

The constraint $\nabla \cdot H = 0$ may be considered as a restriction on the initial data.

For smooth solutions, system (1.5) is written in equivalent form as a quasi-linear symmetric hyperbolic system:

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & \underline{0}^T & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ \underline{0} & 0_3 & I_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} p \\ v \\ H \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho) v \cdot \nabla & \nabla \cdot & \underline{0}^T & 0 \\ \nabla & \rho v \cdot \nabla I_3 & \nabla(\cdot) \cdot H - H \cdot \nabla I_3 & \underline{0} \\ \underline{0} & H \nabla \cdot - H \cdot \nabla I_3 & v \cdot \nabla I_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v \cdot \nabla \end{pmatrix} \begin{pmatrix} p \\ v \\ H \\ S \end{pmatrix} = 0\,.$$

A different symmetrization can be obtained by the introduction of the total pressure $q = p + |H|^2/2$:

$$\begin{cases} \frac{\rho_p}{\rho}\{(\partial_t + v \cdot \nabla) q - H \cdot (\partial_t + (v \cdot \nabla)) H\} + \nabla \cdot v = 0, \\ \rho(\partial_t + (v \cdot \nabla)) v + \nabla q - (H \cdot \nabla) H = 0, \\ (\partial_t + (v \cdot \nabla)) H - (H \cdot \nabla) v - \\ \qquad - \frac{\rho_p}{\rho} H \{(\partial_t + v \cdot \nabla) q - H \cdot (\partial_t + (v \cdot \nabla)) H\} = 0, \\ \partial_t S + v \cdot \nabla S = 0. \end{cases} \tag{1.6}$$

This system can be rewritten as

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & -(\rho_p/\rho) H^T & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ -(\rho_p/\rho) H & 0_3 & a_0 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho) v \cdot \nabla & \nabla \cdot & -(\rho_p/\rho) H^T v \cdot \nabla & 0 \\ \nabla & \rho v \cdot \nabla I_3 & -H \cdot \nabla I_3 & \underline{0} \\ -(\rho_p/\rho) H v \cdot \nabla & -H \cdot \nabla I_3 & a_0 v \cdot \nabla & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v \cdot \nabla \end{pmatrix} \begin{pmatrix} q \\ v \\ H \\ S \end{pmatrix} = 0,$$

where

$$a_0 = I_3 + (\rho_p/\rho) H \otimes H.$$

The boundary matrix is:

$$A_\nu = \begin{pmatrix} (\rho_p/\rho)v\cdot\nu & \nu^T & -(\rho_p/\rho)H^T v\cdot\nu & 0 \\ \nu & \rho v\cdot\nu I_3 & -H\cdot\nu I_3 & \underline{0} \\ -(\rho_p/\rho)Hv\cdot\nu & -H\cdot\nu I_3 & a_0 v\cdot\nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

The rank of the boundary matrix depends accordingly on the conditions satisfied by (ρ, v, H, S) at the boundary.
(i) If $v\cdot\nu = 0$, $H\cdot\nu = 0$, then

$$\ker A_\nu = \{U' = (q', v', H', S') \,:\, q' = 0,\, v'\cdot\nu = 0\},$$
$$\operatorname{rank} A_\nu = 2.$$

(ii) If $H\cdot\nu = 0$ and $v\cdot\nu \neq 0$, $v\cdot\nu \neq \frac{|H|}{\sqrt{\rho}} \pm c(\rho)$, then

$$\ker A_\nu = \{0\}.$$

This yields that the boundary matrix is invertible; in this case the boundary is non-characteristic.
(iii) If $v\cdot\nu = 0$ and $H\cdot\nu \neq 0$, then

$$\ker A_\nu = \{v' = 0,\, \nu q' - H\cdot\nu H' = 0\},$$
$$\operatorname{rank} A_\nu = 6.$$

2. Known results

It is well-known that full regularity (existence in usual Sobolev spaces $H^m(\Omega)$) of solutions to characteristic IBVP's for symmetric hyperbolic systems can't be expected, in general, because of the possible loss of normal regularity at $\partial\Omega$. This fact has been first noticed by Tsuji [71], see also Majda–Osher [30].

Ohno-Shirota [42] have proved that a mixed problem for the linearized MHD equations is ill-posed in $H^m(\Omega)$ for $m \geq 2$.

Generally speaking, one normal derivative (w.r.t. $\partial\Omega$) is controlled by two tangential derivatives, see [9]. The loss of normal regularity and the relation between normal and tangential derivatives will be shown in Chapter 3 with a very simple example.

The natural function space is the weighted anisotropic Sobolev space

$$H^m_*(\Omega) := \{u \in L^2(\Omega) :\; Z^\alpha \partial^k_{x_1} u \in L^2(\Omega),\; |\alpha| + 2k \leq m\},$$

where

$$Z_1 = x_1\partial_{x_1} \quad \text{and} \quad Z_j = \partial_{x_j} \quad \text{for } j = 2, \dots, n,$$

if $\Omega = \{x_1 > 0\}$ (a more rigorous definition will be given in Section 2). This function space has been first introduced by Chen Shuxing [9] and Yanagisawa - Matsumura [73] for the study of ideal MHD equations.

Most of the theory has been developed for symmetric hyperbolic systems and maximal non-negative boundary conditions:

DEFINITION 1.5. The boundary space $\ker M$ is said *maximally non-negative* for A_ν if, for every $(t, x) \in \Sigma_T$, $(A_\nu(t, x)u, u) \geq 0$ for all $u \in \ker M(t, x)$, and $\ker M(t, x)$ is not properly contained in any other subspace having this property.

Linear L^2 theory with maximal non-negative boundary conditions and characteristic boundaries with constant multiplicity has been developed by Rauch [46], where the tangential regularity of solutions is also proved.

Existence and regularity theory in $H^m_*(\Omega)$ has been treated by Guès [21], Ohno, Shizuta and Yanagisawa [44], Secchi [51, 53, 55, 58], Shizuta [62].

Resolution of the MHD equations in $H^m_*(\Omega)$ may be found in the already cited paper [73] and in Secchi [52, 59]. Applications to general relativity are in [20, 66], see also [49]. An extension to nonhomogeneous strictly dissipative boundary conditions has been considered by Casella, Secchi and Trebeschi in [7, 60]. For problems with a nonuniformly characteristic boundary we refer again to [40, 47, 56, 57].

Remark 1.6. *There is a very important exception to the phenomenon of the loss of normal regularity at $\partial\Omega$. This is given by the IBVP for the Euler compressible equations under the slip boundary condition $v \cdot \nu = 0$, see Example 1.3. The latter is a maximal non-negative boundary condition and the boundary matrix is singular at $\partial\Omega$ with constant* $\operatorname{rank} A_\nu = 2$.

That IBVP for the Euler equations can be solved in the usual Sobolev spaces $H^m(\Omega)$, i.e. solutions have full regularity with respect to the normal direction to the boundary, see [3, 50]. The reason is due to the vorticity equation, which represents an additional conservation law that can be used in order to estimate those normal derivatives that one cannot obtain by the inversion of the noncharacteristic part of the boundary matrix.

A similar remark holds for compressible vortex sheets, see Chapter 2, where solutions have full regularity with respect to the normal direction to the boundary, but a loss of regularity with respect to the initial data.

So far for characteristic boundaries of constant multiplicity and maximal non-negative boundary conditions.

For more general boundary conditions, some results have been proven for symmetrizable hyperbolic systems under suitable *structural assumptions*, that we briefly describe in Appendix C. Instead of maximal non-negative boundary conditions, the theory deals with *uniform Kreiss–Lopatinskiĭ conditions* (UKL) (that we introduce in Appendix B). Moreover the boundary is assumed to be uniformly characteristic.

The general theory has received major contributions by Majda and Osher [30], Ohkubo [41], Benzoni and Serre [4]. In the same framework we may also quote the papers about elasticity by Morando and Serre [35, 36].

3. Characteristic free boundary problems

In general, the global existence of regular solutions of quasi-linear hyperbolic systems can't be expected because the formation of singularities in finite time may occur. The breakdown of the smoothness property may come from the appearance of discontinuities in the solution, i.e. shock waves which develop no matter how smooth the initial data are, see [64].

In contrast to the 1D case, in higher space dimensions there is no general existence theorem for solutions which allows discontinuities. A fundamental part in the study of quasi-linear hyperbolic equations is the Riemann problem, i.e. the initial value problem where initial data are piecewise constant with a jump in between.

This initial discontinuity generates elementary waves of three kinds: centered rarefaction waves, shock waves and contact discontinuities. In general, the solution of the Riemann problem is expected to develop singularities or fronts of the above kind for all the characteristic fields.

Since the general case is too difficult, we will restrict the problem to the case with only one single wave front separating two smooth states. This is a free boundary problem because the single wave front separating the two smooth states on either sides is part of the unknowns.

The first attempt to extend the theory to several space variables is due to Majda [28, 29], who showed the short-time existence and stability of a single shock wave. See also Blokhin-Trakhinin [5] and the references therein for a different approach. A general presentation of Majda's result with some improvements may be found in Métivier [32]. See [19] for the uniform stability of weak shocks when the shock strength tends to zero. The existence of rarefaction waves was then showed by Alinhac [1], the existence of sound waves by Métivier [31].

While rarefaction waves are continuous solutions with only a singularity at the initial time given by the initial jump, shocks and contact discontinuities are solutions with a discontinuity which persists in time; it is therefore useful to point out the differences between these two cases. Let us briefly recall the main definitions.

Consider a general $N \times N$ system of conservation laws in $\mathbb{R}^n$

$$\partial_t U + \sum_{j=1}^n \partial_{x_j} f_j(U) = 0,$$

where $f_j \in \mathcal{C}^\infty(\mathbb{R}^N;\mathbb{R}^N)$. Given $U \in \mathbb{R}^N$, denote

$$A_j(U) := f_j'(U), \quad A(U,\nu) := \sum_{j=1}^n \nu_j A_j(U) \quad \forall \nu \in \mathbb{R}^n;$$

let $\lambda_k(U,\nu)$ be the (real) eigenvalues (characteristic fields) of the matrix $A(U,\nu)$,

$$\lambda_1(U,\nu) \leq \cdots \leq \lambda_N(U,\nu).$$

Let us denote by $r_k(U,\nu)$ the right eigenvectors of $A(U,\nu)$.

Consider a planar discontinuity at $(\underline{t},\underline{x})$ with front

$$\Sigma := \{\nu \cdot (x - \underline{x}) = \sigma(t - \underline{t})\},$$

where σ is the velocity of the front. Denote by $U^\pm$ the values of U at $(\underline{t},\underline{x})$ from each side of Σ. We have the following definition introduced by Lax [26].

Definition. $U^\pm$ is a *shock* if there exists $k \in \{1,\dots,N\}$ such that $\nabla\lambda_k \cdot r_k \neq 0$ $\forall U$, $\forall \nu$ (λ_k is said *genuinely nonlinear*) and

$$\lambda_{k-1}(U^-,\nu) < \sigma < \lambda_k(U^-,\nu),$$
$$\lambda_k(U^+,\nu) < \sigma < \lambda_{k+1}(U^+,\nu).$$

The first inequality on the left (resp. the last on the right) is ignored when $k = 1$ (resp. $k = N$).

$U^\pm$ is a *contact discontinuity* if there exists $k \in \{1,\dots,N\}$ such that $\nabla\lambda_k \cdot r_k \equiv 0$ $\forall U$, $\forall \nu$ (λ_k is said *linearly degenerate*) and

$$\lambda_k(U^+,\nu) \leq \sigma = \lambda_k(U^-,\nu)$$

or

$$\lambda_k(U^+, \nu) = \sigma \leq \lambda_k(U^-, \nu).$$

In case of shocks, the definition shows that the velocity of the front is always different from the characteristic fields. It follows that the shock front is a *noncharacteristic* interface. On the contrary, the contact discontinuity is a *characteristic* interface, because of the possible equalities. Another crucial difference between shocks and contact discontinuities is that in the first case one has the *uniform stability*, which is the extension of the uniform Kreiss–Lopatinskiĭ condition for standard mixed problems. In case of contact discontinuities the Kreiss–Lopatinskiĭ condition holds only weakly, and not uniformly. This fact has consequences for the apriori energy estimate of solutions.

In the following we describe some characteristic free boundary value problems for piecewise smooth solutions: 2D vortex sheets for compressible Euler equations (which will be considered in more detail in Chapter 2) and the strong discontinuities of ideal MHD.

Other characteristic interfaces are the rarefaction waves and the sound waves, see [1, 31].

3.1. Compressible vortex sheets. 2D vortex sheets are piecewise smooth solutions for the compressible Euler equations for barotropic fluids:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, \mathbf{u}) = 0\,, \\ \partial_t(\rho\, \mathbf{u}) + \nabla_x \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla_x\, p(\rho) = 0\,, \end{cases} \tag{1.7}$$

where $t \geq 0$, $x \in \mathbb{R}^2$. At the unknown discontinuity front $\Sigma = \{x_1 = \varphi(x_2, t)\}$ it is required that

$$\partial_t \varphi = v^{\pm} \cdot \nu, \quad [p] = 0,$$

where $[p] = p^+ - p^-$ denotes the jump across Σ, and ν is a normal vector to Σ.

Stability and existence of solutions to the above problem have been proven by Coulombel-Secchi [16]. We will consider this problem in more detail in Chapter 2. For a comparison with the following analysis about strong discontinuities for ideal MHD, it may be useful to notice that here the mass flux $j = j^{\pm} := \rho^{\pm}(v^{\pm} \cdot \nu - \partial_t \varphi) = 0$ at Σ.

3.2. Strong discontinuities for ideal MHD. Consider a solution (ρ, v, H, S) of ideal MHD equations (1.5) in $\mathbb{R}^3$, with a single front of discontinuity $\Sigma = \{x_1 = \varphi(x_2, x_3, t)\}$. This is a piecewise smooth function which solves (1.5) on either side of the front and, in order to be a weak solution, satisfies the Rankine–Hugoniot jump conditions at Σ, taking the form

$$\begin{aligned} &[j] = 0, \quad [H_N] = 0, \quad j[v_N] + [q]|N|^2 = 0, \\ &j[v_\tau] = H_N^+[H_\tau], \quad j[H_\tau/\rho] = H_N^+[v_\tau] \\ &j[e + \tfrac{1}{2}|v|^2 + \tfrac{|H|^2}{2\rho}] + [qv_N - H_N(v \cdot H)] = 0, \end{aligned} \tag{1.8}$$

where $N = (1, -\partial_{x_2}\varphi, -\partial_{x_3}\varphi)$ is a normal vector to Σ, and we have set

$$\begin{aligned} &v_N = v \cdot N, \qquad H_N = H \cdot N,\\ &v_\tau = v - v_N N, \qquad H_\tau = H - H_N N,\\ &j := \rho(v_N - \partial_t\varphi) \quad \text{(mass flux)},\\ &q := p + \tfrac{1}{2}|H|^2 \quad \text{(total pressure)}. \end{aligned}$$

The Rankine–Hugoniot conditions (1.8) may be satisfied in different ways. This leads to different kinds of strong discontinuities classified as follows, see [25]:

DEFINITION 1.7. (i) The discontinuity front Σ is a *MHD shock* if

$$j^{\pm} \neq 0, \quad [\rho] \neq 0;$$

(ii) Σ is called an *Alfvén or rotational discontinuity (Alfvén shock)* if

$$j^{\pm} \neq 0, \quad [\rho] = 0;$$

(iii) Σ is a *contact discontinuity* if

$$j^{\pm} = 0, \quad H_N^{\pm} \neq 0;$$

(iv) Σ is a *current-vortex sheets* (also called *tangential discontinuities)* if

$$j^{\pm} = 0, \quad H_N^{\pm} = 0.$$

Except for MHD shocks, which are noncharacteristic interfaces, all the above free boundaries are characteristic surfaces.

Accordingly to the above classification, the Rankine–Hugoniot conditions (1.8) are satisfied as follows:

(1) If Σ is an *Alfvén discontinuities* then:

$$\begin{aligned} &[p] = 0, \quad [S] = 0, \quad [H_N] = 0, \quad [|H|^2] = 0, \quad [v - \frac{H}{\sqrt{\rho}}] = 0,\\ &j = j^{\pm} = \rho^{\pm}(v_N^{\pm} - \partial_t\varphi) = H_N^{+}\sqrt{\rho^{+}} \neq 0. \end{aligned} \tag{1.9}$$

As the modulus and the normal component of H are continuous across the front, in general H may only change its direction. For this reason Alfvén discontinuities are also called *rotational* discontinuities. Moreover, since the mass flux j is different from 0, Alfvén discontinuities are sometimes called *Alfvén shocks*.

Consider the problem obtained by linearizing equations (1.5) and (1.9) around a piecewise constant solution of (1.9). This problem may be formulated as a nonstandard boundary value problem, which is well-posed if the analogue of the Kreiss–Lopatinskiĭ condition is satisfied (see Appendix B for the definition).

It has been shown by Syrovatskii [65] for incompressible MHD, and Ilin - Trakhinin [24] for compressible MHD, that such planar Alfvén discontinuities are never uniformly stable, that is the uniform Lopatinskiĭ condition is always violated. In fact, planar Alfvén discontinuities are either violently unstable or weakly stable.

Violent instability means that the Kreiss–Lopatinskiĭ condition is violated, so that there exist exponentially exploding modes of instability. This instability corresponds to ill-posedness in the sense of Hadamard.

Weak stability means that the Kreiss- Lopatinskiĭ condition is satisfied (there are no growing modes) but not uniformly. In this case the solution can become unstable, but the instability is much slower to develop than in the case of violent instability.

Another remarkable fact about Alfvén discontinuities is that the symbol of the operator associated to the function φ describing the unknown front Σ, that is obtained from (1.9), is *not elliptic*. This leads to an additional loss of regularity of the front.

As already mentioned, Alfvén discontinuities are *characteristic* interfaces.

(2) If Σ is a *contact discontinuity* then:

$$[v] = 0, \quad [H] = 0, \quad [p] = 0. \tag{1.10}$$

(We may have $[\rho] \neq 0, \ [S] \neq 0$.)

The boundary conditions (1.10) are maximally non-negative (but not strictly dissipative). Using this fact, some a priori estimates for the solution of (1.5), (1.10) have been proven by Blokhin-Trakhinin [5], by the energy method.

As for Alfvén discontinuities, the symbol associated to the front in (1.10) is *not elliptic*. Again, the front of contact discontinuities is *characteristic*, Example 1.4 (iii).

(3) If Σ is a *current-vortex sheets* then:

$$\partial_t \varphi = v_N^{\pm}, \quad [q] = 0, \quad H_N^{\pm} = 0. \tag{1.11}$$

(We may have $[v_\tau] \neq 0, \ [H_\tau] \neq 0, \ [\rho] \neq 0, \ [S] \neq 0$.)

The analysis of the linearized equations around a piecewise constant solution of (1.11) shows that planar current-vortex sheets are never uniformly stable (i.e. the uniform Lopatinskiĭ condition is always violated). They are either weakly stable or violently unstable (Hadamard ill-posedness). The symbol associated to the front is *elliptic*. Again the front is *characteristic*, Example 1.4 (i).

The stability and existence of current-vortex sheets has been studied by Trakhinin. In [68] Trakhinin has considered the linearized equations with variable coefficients obtained from a new symmetrization of the MHD equations. Under the assumption $H^+ \times H^- \neq 0$ on Σ, and a smallness condition on $[v_\tau] \neq 0$, he has shown that the boundary conditions (1.11) are maximally non-negative (but not strictly dissipative). Using this fact he has proved by the energy method an a priori estimate in space $H^1_*(\Omega)$, without loss of regularity w.r.t. the initial data (but not w.r.t. the coefficients). For the stability in the incompressible case see [39, 69].

In [70] Trakhinin has proved the existence of current-vortex sheets. First he has extended the a priori estimate of [68] and proved a tame estimate in anisotropic Sobolev spaces $H^m_*(\Omega)$. Then the existence of the solution to the nonlinear problem has been shown by adapting a Nash–Moser iteration. This strategy is explained with more details in Chapter 2 on vortex sheets.

CHAPTER 2

Compressible vortex sheets

Let us consider Euler equations of isentropic gas dynamics in the whole space $\mathbb{R}^2$. Denoting by $\mathbf{u}$ the velocity of the fluid and ρ the density, the equations read:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\,, \\ \partial_t(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0\,, \end{cases} \tag{2.1}$$

where $p = p(\rho)$ is the pressure law, a $\mathcal{C}^\infty$ function of ρ, defined on $]0, +\infty[$, with $p'(\rho) > 0$ for all ρ. The speed of sound $c(\rho)$ in the fluid is then defined by the relation

$$c(\rho) := \sqrt{p'(\rho)}\,.$$

Let $\Sigma := \{x_2 = \varphi(t, x_1)\}$ be a smooth interface and $(\rho, \mathbf{u})$ a smooth function on either side of Σ:

$$(\rho, \mathbf{u}) := \begin{cases} (\rho^+, \mathbf{u}^+) & \text{if } x_2 > \varphi(t, x_1) \\ (\rho^-, \mathbf{u}^-) & \text{if } x_2 < \varphi(t, x_1). \end{cases}$$

DEFINITION 2.1. $(\rho, \mathbf{u})$ is a *weak solution* of (2.1) if and only if it is a classical solution on both sides of Σ and it satisfies the *Rankine–Hugoniot conditions* at Σ:

$$\begin{aligned} \partial_t \varphi\, [\rho] - [\rho \mathbf{u} \cdot \nu] = 0\,, \\ \partial_t \varphi\, [\rho \mathbf{u}] - [(\rho \mathbf{u} \cdot \nu)\mathbf{u}] - [p]\nu = 0\,, \end{aligned} \tag{2.2}$$

where $\nu := (-\partial_{x_1}\varphi, 1)$ is a (space) normal vector to Σ. As usual, $[q] = q^+ - q^-$ denotes the jump of a quantity q across the interface Σ.

Following Lax [26], we shall say that $(\rho, \mathbf{u})$ is a contact discontinuity if (2.2) is satisfied in the following way:

$$\begin{aligned} \partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu\,, \\ p^+ = p^-\,. \end{aligned}$$

Because p is monotone, the previous equalities read

$$\begin{aligned} \partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu\,, \\ \rho^+ = \rho^-\,. \end{aligned} \tag{2.3}$$

Since the density and the normal velocity are continuous across the interface Σ, the *only jump* experimented by the solution is on the *tangential velocity*. (Here, normal and tangential mean normal and tangential with respect to Σ). For this reason, a contact discontinuity is a *vortex sheet* and we shall make no distinction in the terminology we use.

Therefore the problem is to show the existence of a weak solution to (2.1), (2.3). Observe that the interface Σ, or equivalently the function φ, is part of the unknowns of the problem; we thus deal with a *free boundary problem.* Due to (2.3), the free boundary is characteristic with respect to both left and right sides.

Some information comes from the study of the *linearized* equations near a *piecewise constant* vortex sheet.

The stability of linearized equations for planar and rectilinear compressible vortex sheets around a piecewise constant solution has been analysed some time ago by Miles [18, 34], using tools of complex analysis. For a vortex sheet in $\mathbb{R}^n, n \geq 2$, the situation may be summarized as follows (see [61]):

$$
\begin{array}{l}
\text{for } n \geq 3, \text{ the problem is always } \textit{violently unstable};\\
\text{for } n = 2, \text{ there exists a critical value for the jump of the tangential velocity}\\
\text{such that :}\\
\quad \text{if } |[\mathbf{u}\cdot\tau]| < 2\sqrt{2}c(\rho) \text{ the problem is } \textit{violently unstable} \text{ (subsonic case),}\\
\quad \text{if } |[\mathbf{u}\cdot\tau]| > 2\sqrt{2}c(\rho) \text{ the problem is } \textit{weakly stable} \text{ (supersonic case),}
\end{array}
\tag{2.4}
$$

where $c(\rho) := \sqrt{p'(\rho)}$ is the sound speed and τ is a tangential unit vector to Σ.

As will be seen below, the problem given by the linearized equations obtained from (2.1) and the transmission boundary conditions at the interface (2.3) may be formulated as a nonstandard boundary value problem, which is well-posed if the analogue of the Kreiss–Lopatinskiĭ condition is satisfied (see Appendix B). In (2.4), violent instability means that the Kreiss–Lopatinskiĭ condition is violated, so that there exist exponentially exploding modes of instability. This instability corresponds to ill-posedness in the sense of Hadamard. Weak stability means that the Kreiss–Lopatinskiĭ condition is satisfied (there are no growing modes), but not uniformly. In this case the solution can become unstable, but the instability is much slower to develop than in the case of violent instability.

In the instability case no a priori energy estimate for the solution is possible, because of the ill-posedness. In the weak stability case, an $L^2 - L^2$ energy estimate (for the solution with respect to the data) is not expectable, because the Kreiss–Lopatinskiĭ condition doesn't hold uniformly. However, it is reasonable to look for an energy estimate with loss of derivatives with respect to the data. This is different from the case of shocks, where the Kreiss–Lopatinskiĭ condition holds uniformly, so that an energy estimate without loss of derivatives may be proved (see Majda [28, 29]).

The above result in 2D formally agrees with the theory of incompressible vortex sheets. In fact, in the incompressible limit the speed of sound tends to infinity, and the above result yields that two-dimensional incompressible vortex sheets are always unstable (the Kelvin–Helmhotz instability).

Recalling that in the theory for incompressible vortex sheets, solutions are shown to exist in the class of analytic functions, one may look for analytic solutions also in the compressible instability case; here the existence of a local in time analytic solution for the nonlinear problem may be obtained by applying Harabetian's result [22].

In the transition case $|[\mathbf{u} \cdot \tau]| = 2\sqrt{2}c(\rho)$ the problem is also weakly stable, as shown in [14], in a weaker sense than in the supersonic case. The complete analysis of linear stability of contact discontinuities for the nonisentropic Euler equations is carried out in [13], for both cases $n = 2$ and $n = 3$.

From now on we consider the 2D supersonic *weakly stable* regime.

In the paper [15], written with J.F. Coulombel, we show that supersonic constant vortex sheets are linearly stable, in the sense that the linearized system (around a piecewise constant solution) obeys an L^2-energy estimate. The failure of the uniform Kreiss–Lopatinskiĭ condition yields an energy estimate with the loss of one tangential derivative from the source terms to the solution. Moreover, since the problem is characteristic, the estimate we prove exhibits a loss of control on the trace of the solution. We also consider the linearized equations around a perturbation of a constant vortex sheet, and we show that these linearized equations with variable coefficients obey the same energy estimate with loss of one derivative w.r.t. the source terms.

In a second paper [16] written with J.F. Coulombel, we consider the nonlinear problem and prove the existence of supersonic compressible vortex sheets solutions. To prove our result we first extend the energy estimate of solutions to the linearized equations to Sobolev norms, by application of the L^2-estimate to tangential derivatives and combination with an a priori estimate for normal derivatives obtained by the energy method from a vorticity-type equation, see Remark 1.6.

Here solutions have full regularity with respect to the normal direction to the boundary; therefore they can be estimated in usual Sobolev spaces H^m instead of anisotropic Sobolev spaces H^m_*, in spite of the characteristic boundary. The failure of the uniform Kreiss–Lopatinskiĭ condition yields another type of loss of regularity, i.e. the loss of derivatives from the source terms to the solution.

The new estimate extended to Sobolev norms shows the loss of one derivative with respect to the source terms, and the loss of three derivatives with respect to the coefficients. The loss is fixed, and we can thus solve the nonlinear problem by a Nash–Moser iteration scheme. Recall that the Nash–Moser procedure was already used to construct other types of waves for multidimensional systems of conservation laws, see, e.g., [1, 19]. However, our Nash–Moser procedure is not completely standard, since the tame estimate for the linearized equations will be obtained under certain nonlinear constraints on the state about which we linearize. We thus need to make sure that these constraints are satisfied at each iteration step. The rest of the present chapter is devoted to the presentation of these results.

In [16] we also show how a similar analysis yields the existence of weakly stable shock waves in isentropic gas dynamics, and the existence of weakly stable liquid/vapor phase transitions.

In [17] we prove that sufficiently smooth 2-D compressible vortex sheets are unique.

Similar arguments to those of [15] have been considered by Morando and Trebeschi [38] in the analysis of the linearized stability of 2D vortex sheets for the nonisentropic Euler equations.

Adapting the proof of [16], Trakhinin [70] has shown the existence of current-vortex sheets in MHD, see Section 3.2.

1. The nonlinear equations in a fixed domain

The interface $\Sigma := \{x_2 = \varphi(t, x_1)\}$ is an unknown of the problem. We first straighten the unknown front in order to work in a fixed domain. Let us introduce the change of variables

$$\begin{aligned}&(\tau, y_1, y_2) \to (t, x_1, x_2),\\ &(t, x_1) = (\tau, y_1), \quad x_2 = \Phi(\tau, y_1, y_2),\end{aligned}$$

where

$$\Phi : \{(\tau, y_1, y_2) \, : y_2 > 0\} \to \mathbb{R},$$

is a smooth function such that

$$\partial_{y_2}\Phi(\tau, y_1, y_2) \geq \kappa > 0\,, \quad \Phi(\tau, y_1, 0) = \varphi(t, x_1)\,.$$

We define the new unknowns

$$\begin{aligned}&(\rho_\sharp^+, \mathbf{u}_\sharp^+)(\tau, y_1, y_2) := (\rho, \mathbf{u})(\tau, y_1, \Phi(\tau, y_1, y_2))\,,\\ &(\rho_\sharp^-, \mathbf{u}_\sharp^-)(\tau, y_1, y_2) := (\rho, \mathbf{u})(\tau, y_1, \Phi(\tau, y_1, -y_2))\,.\end{aligned}$$

The functions $\rho_\sharp^\pm, \mathbf{u}_\sharp^\pm$ are smooth on the fixed domain $\{y_2 > 0\}$. For convenience, we drop the $\sharp$ index and only keep the $+$ and $-$ exponents. Then, we again write (t, x_1, x_2) instead of (τ, y_1, y_2).

Let us denote $\mathbf{u}^\pm = (v^\pm, u^\pm)$. The existence of compressible vortex sheets amounts to proving the existence of smooth solutions to the following first order system:

$$\begin{aligned}&\partial_t\rho^\pm + v^\pm\partial_{x_1}\rho^\pm + (u^\pm - \partial_t\Phi^\pm - v^\pm\partial_{x_1}\Phi^\pm)\tfrac{\partial_{x_2}\rho^\pm}{\partial_{x_2}\Phi^\pm}\\ &+\rho^\pm\partial_{x_1}v^\pm + \rho^\pm\,\tfrac{\partial_{x_2}u^\pm}{\partial_{x_2}\Phi^\pm} - \rho^\pm\,\tfrac{\partial_{x_1}\Phi^\pm}{\partial_{x_2}\Phi^\pm}\,\partial_{x_2}v^\pm = 0\,,\\[1em] &\partial_t v^\pm + v^\pm\partial_{x_1}v^\pm + (u^\pm - \partial_t\Phi^\pm - v^\pm\partial_{x_1}\Phi^\pm)\tfrac{\partial_{x_2}v^\pm}{\partial_{x_2}\Phi^\pm}\\ &+\tfrac{p'(\rho^\pm)}{\rho^\pm}\,\partial_{x_1}\rho^\pm - \tfrac{p'(\rho^\pm)}{\rho^\pm}\,\tfrac{\partial_{x_1}\Phi^\pm}{\partial_{x_2}\Phi^\pm}\,\partial_{x_2}\rho^\pm = 0\,,\\[1em] &\partial_t u^\pm + v^\pm\partial_{x_1}u^\pm + (u^\pm - \partial_t\Phi^\pm - v^\pm\partial_{x_1}\Phi^\pm)\tfrac{\partial_{x_2}u^\pm}{\partial_{x_2}\Phi^\pm}\\ &+\tfrac{p'(\rho^\pm)}{\rho^\pm}\,\tfrac{\partial_{x_2}\rho^\pm}{\partial_{x_2}\Phi^\pm} = 0\,,\end{aligned} \tag{2.5}$$

in the fixed domain $\{x_2 > 0\}$, where

$$\Phi^\pm(t, x_1, x_2) := \Phi(t, x_1, \pm x_2)\,,$$

both defined on the half-space $\{x_2 > 0\}$.

The equations are not sufficient to determine the unknowns $U^\pm := (\rho^\pm, v^\pm, u^\pm)$ and $\Phi^\pm$. In fact, the change of variables is only requested to map Σ to $\{x_2 = 0\}$ and is arbitrary outside Σ. In order to simplify the transformed equations of motion we may prescribe that $\Phi^\pm$ solve the *eikonal* equations

$$\partial_t\Phi^\pm + v^\pm\partial_{x_1}\Phi^\pm - u^\pm = 0 \tag{2.6}$$

in the domain $\{x_2 > 0\}$.

This choice has the advantage that the boundary matrix of the system for $U^\pm$ has constant rank in the whole space domain $\{x_2 \geq 0\}$, and not only at the boundary.

The equations for $U^\pm$ are only coupled through the boundary conditions

$$\begin{aligned} &\Phi^+ = \Phi^- = \varphi\,,\\ &(v^+ - v^-)\,\partial_{x_1}\varphi - (u^+ - u^-) = 0\,,\\ &\partial_t\varphi + v^+\,\partial_{x_1}\varphi - u^+ = 0\,,\\ &\rho^+ - \rho^- = 0\,, \end{aligned} \tag{2.7}$$

on the fixed boundary $\{x_2 = 0\}$, which are obtained from (2.3). We will also consider the initial conditions

$$(\rho^\pm, v^\pm, u^\pm)_{|t=0} = (\rho_0^\pm, v_0^\pm, u_0^\pm)(x_1, x_2)\,, \quad \varphi_{|t=0} = \varphi_0(x_1)\,, \tag{2.8}$$

in the space domain $\mathbb{R}^2_+ = \{x_1 \in \rho\,, x_2 > 0\}$.

Thus, compressible vortex sheet solutions should solve (2.5), (2.6), (2.7), (2.8).

There exist many simple solutions of (2.5), (2.6), (2.7) that correspond (for the Euler equations (2.1) in the original variables) to stationary rectilinear vortex sheets:

$$(\rho, \mathbf{u}) = \begin{cases} (\overline{\rho}, \overline{v}, 0)\,, & \text{if } x_2 > 0,\\ (\overline{\rho}, -\overline{v}, 0)\,, & \text{if } x_2 < 0, \end{cases}$$

where $\overline{\rho}, \overline{v} \in \mathbb{R}$, $\overline{\rho} > 0$. Up to Galilean transformations, every rectilinear vortex sheet has this form. In the straightened variables, this stationary vortex sheet corresponds to the following smooth (stationary) solution to (2.5), (2.6), (2.7):

$$\overline{U}^\pm \equiv \begin{pmatrix} \overline{\rho} \\ \pm\overline{v} \\ 0 \end{pmatrix}, \quad \overline{\Phi}^\pm(t,x) \equiv \pm x_2\,, \quad \overline{\varphi} \equiv 0\,. \tag{2.9}$$

In this paper, we shall assume $\overline{v} > 0$, but the opposite case can be dealt with in the same way.

The following theorem is our main result: for the nonlinear problem (2.5), (2.6), (2.7), (2.8) of supersonic compressible vortex sheets we prove the existence of solutions close enough to the piecewise constant solution (2.9).

THEOREM 2.2. *[15, 16] Let $T > 0$, and let $\mu \in \mathbb{N}$, with $\mu \geq 6$. Assume that the stationary solution defined by* (2.9) *satisfies the "supersonic" condition:*

$$\overline{v} > \sqrt{2}\,c(\overline{\rho})\,. \tag{2.10}$$

Assume that the initial data $(U_0^\pm, \varphi_0)$ have the form

$$U_0^\pm = \overline{U}^\pm + \dot{U}_0^\pm\,,$$

with $\dot{U}_0^\pm \in H^{2\mu+3/2}(\mathbb{R}^2_+)$, $\varphi_0 \in H^{2\mu+2}(\mathbb{R})$, and that they satisfy sufficient compatibility conditions. Assume also that $(\dot{U}_0^\pm, \varphi_0)$ have a compact support. Then, there exists $\delta > 0$ such that, if $\|\dot{U}_0^\pm\|_{H^{2\mu+3/2}(\mathbb{R}^2_+)} + \|\varphi_0\|_{H^{2\mu+2}(\mathbb{R})} \leq \delta$, then there exists a solution $U^\pm = \overline{U}^\pm + \dot{U}^\pm, \Phi^\pm = \pm x_2 + \dot{\Phi}^\pm, \varphi$ of (2.5), (2.6), (2.7), (2.8), *on the time interval* $[0,T]$. *This solution satisfies $(\dot{U}^\pm, \dot{\Phi}^\pm) \in H^\mu(]0,T[\times\mathbb{R}^2_+)$, and $\varphi \in H^{\mu+1}(]0,T[\times\mathbb{R})$.*

For the compatibility conditions as for all the other details we refer the reader to [15, 16].

The rest of the chapter is organized as follows: in section 2 we introduce the linearized equations around a perturbation of the piecewise constant solution (2.9) and state the basic a priori L^2 estimate while in section 3 we describe the main steps of its proof. In 4 we give a tame a priori estimate in Sobolev spaces for the solution of the linearized problem.

In section 5, we reduce the nonlinear problem (2.5), (2.6), (2.7), (2.8), to another nonlinear system with zero initial data; then we describe the Nash–Moser iteration scheme that will be used to solve this reduced problem.

2. The L^2 energy estimate for the linearized problem

We introduce the linearized equations around a perturbation of the piecewise constant solution (2.9). More precisely, let us consider the functions

$$U_{r,l} = \overline{U}^{\pm} + \dot{U}_{r,l}(t, x_1, x_2),$$

$$\Phi_{r,l} = \pm x_2 + \dot{\Phi}_{r,l}(t, x_1, x_2),$$

where

$$\dot{U}_{r,l} \in W^{2,\infty}(\Omega)\,, \quad \dot{\Phi}_{r,l} \in W^{3,\infty}(\Omega)\,,$$

$(U_{r,l}, \Phi_{r,l})$ satisfy (2.6), (2.7), and the perturbations $\dot{U}_{r,l}$ and $\dot{\Phi}_{r,l}$ have compact support.

Let us consider the linearized equations around $U_{r,l}, \Phi_{r,l}$ with solutions denoted by $U_\pm, \Psi_\pm$. The equations take a simpler form by the introduction of the new unknowns (cfr. [1])

$$\dot{U}_+ := U_+ - \frac{\Psi_+}{\partial_{x_2}\Phi_r}\partial_{x_2}U_r, \quad \dot{U}_- := U_- - \frac{\Psi_-}{\partial_{x_2}\Phi_l}\partial_{x_2}U_l. \tag{2.11}$$

Then the equations are diagonalized and transformed to an equivalent form with constant (singular) boundary matrix.

Denote the new unknowns by $W^\pm$. The linearized equations are then equivalent to

$$\begin{aligned} \mathcal{N}_r W^+ &:= \mathbf{A}_0^r\,\partial_t W^+ + \mathbf{A}_1^r\,\partial_{x_1} W^+ + \mathbf{I}_2\,\partial_{x_2} W^+ + \mathbf{A}_0^r\,\mathbf{C}^r\,W^+ = F^+\,, \\ \mathcal{N}_l W^- &:= \mathbf{A}_0^l\,\partial_t W^- + \mathbf{A}_1^l\,\partial_{x_1} W^- + \mathbf{I}_2\,\partial_{x_2} W^- + \mathbf{A}_0^l\,\mathbf{C}^l\,W^- = F^-\,, \end{aligned} \tag{2.12}$$

with suitable matrices $\mathbf{A}_j^{r,l} = \mathbf{A}_j^{r,l}(U_{r,l}, \Phi_{r,l}), \mathbf{C}^{r,l} = \mathbf{C}^{r,l}(U_{r,l}, \Phi_{r,l})$, and boundary matrix

$$\mathbf{I}_2 := \text{ diag } (0, 1, 1).$$

We have

$$\mathbf{A}_j^{r,l} \in W^{2,\infty}(\Omega), \quad \mathbf{C}^{r,l} \in W^{1,\infty}(\Omega).$$

In view of the results in [1, 19], in (2.12) we have dropped the zero order terms in Ψ_+, Ψ_-. The linearized boundary conditions are

$$\begin{aligned} \Psi_{+|x_2=0} = \Psi_{-|x_2=0} = \psi\,, \\ \mathbf{b}\,\nabla\psi + \mathbf{M}\,U_{|x_2=0} = g\,, \end{aligned}$$

with suitable matrices $\mathbf{b} = \mathbf{b}(U_{r,l}), \mathbf{M} = \mathbf{M}(\nabla\varphi)$ and where $U = (U_+, U_-)^T$, $\nabla\psi = (\partial_t\psi, \partial_{x_1}\psi)^T$ and $g = (g_1, g_2, g_3)^T$. Introducing $W^\pm$ the linearized boundary conditions become equivalent to

$$\begin{aligned} &\Psi_+ = \Psi_- = \psi\,, \\ &\mathcal{B}(W^{nc},\psi) := \mathbf{b}\,\nabla\psi + \check{\mathbf{b}}\,\psi + \widetilde{\mathbf{M}}W_{|x_2=0} = g\,. \end{aligned} \tag{2.13}$$

Here $W = (W^+, W^-)^T$, and

$$\begin{aligned} &\mathbf{b}(U_{r,l}) \in W^{2,\infty}(\mathbb{R}^2)\,, \\ &\check{\mathbf{b}}(\partial_{x_2}U_{r,l}, \nabla\varphi, \partial_{x_2}\Phi_{r,l}) \in W^{1,\infty}(\mathbb{R}^2)\,, \\ &\widetilde{\mathbf{M}}(U_{r,l}, \nabla\varphi, \nabla\Phi_{r,l}) \in W^{2,\infty}(\mathbb{R}^2). \end{aligned}$$

Observe that the boundary conditions involve both ψ and W. Moreover, the matrix $\mathbf{M}$ only acts on the *noncharacteristic* part $W^{nc} := (W_2^+, W_3^+, W_2^-, W_3^-)$ of the vector W.

Our first goal is to obtain an L^2 a priori estimate of the solution to the linearized problem (2.12),(2.13). Let us define

$$\Omega := \{(t, x_1, x_2) \in \mathbb{R}^3 \text{ s.t. } x_2 > 0\} = \mathbb{R}^2 \times \mathbb{R}^+\,.$$

The boundary $\partial\Omega = \{x_2 = 0\}$ is identified to $\mathbb{R}^2$. Define also

$$H^s_\gamma = H^s_\gamma(\mathbb{R}^2) := \{u \in \mathcal{D}'(\mathbb{R}^2) \text{ s.t. } \exp(-\gamma t)u \in H^s(\mathbb{R}^2)\}\,,$$

equipped with the norm

$$\|u\|_{H^s_\gamma} := \|\exp(-\gamma t)u\|_{H^s(\mathbb{R}^2)}\,.$$

Define similarly the space $H^k_\gamma(\Omega)$. The space $L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^2))$ is equipped with the norm

$$|\!|\!|v|\!|\!|^2_{L^2(H^s_\gamma)} := \int_0^{+\infty} \|v(\cdot, x_2)\|^2_{H^s_\gamma(\mathbb{R}^2)}\,dx_2\,.$$

In the sequel, the variable in $\mathbb{R}^2$ is (t, x_1) while x_2 is the variable in $\mathbb{R}^+$.

Our first result is the following (here we denote $\mathcal{N} := (\mathcal{N}_r, \mathcal{N}_l)$).

THEOREM 2.3. *[15] Assume that the particular solution defined by* (2.9) *satisfies* (2.10)*, that* $(U_{r,l}, \Phi_{r,l})$ *satisfy* (2.6), (2.7)*, and that the perturbation* $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ *is sufficiently small in* $W^{2,\infty}(\Omega) \times W^{3,\infty}(\Omega)$ *and has compact support. Then, for all* $\gamma \geq 1$ *large enough and for all* $(W, \psi) \in H^2_\gamma(\Omega) \times H^2_\gamma(\mathbb{R}^2)$*, the following estimate holds:*

$$\begin{aligned} &\gamma\,|\!|\!|W|\!|\!|^2_{L^2_\gamma(\Omega)} + \|W^{nc}{}_{|x_2=0}\|^2_{L^2_\gamma} + \|\psi\|^2_{H^1_\gamma} \\ &\leq C\left(\frac{1}{\gamma^3}\,|\!|\!|\mathcal{N}W|\!|\!|^2_{L^2(H^1_\gamma)} + \frac{1}{\gamma^2}\,\|\mathcal{B}(W^{nc},\psi)\|^2_{H^1_\gamma}\right)\,. \end{aligned} \tag{2.14}$$

Observe that there is the loss of one (tangential) derivative for W with respect to the source terms, but no loss of derivatives for the front function ψ (as in Majda's work [28, 29] on shock waves). Since the problem is characteristic, only the trace of the noncharacteristic part of the solution may be controlled at the boundary. The loss of control regards the tangential velocity.

3. Proof of the L^2-energy estimate

We describe the main steps of the proof of the above Theorem 2.3.

(1) Paralinearization of the equations.

Using the paradifferential calculus of Bony [6] and Meyer [33], we substitute in the equations the paradifferential operators (w.r.to the tangential variables (t, x_1)) and obtain a system of ordinary differential equations with derivatives in x_2 and symbols instead of derivatives in (t, x_1). This step essentially reduces to the constant coefficient case.

(2) Elimination of the front.

The projected boundary condition onto a suitable subspace of the frequency space gives an elliptic equation of order one for the front ψ. This property is a key point in our work since it allows to *eliminate* the unknown front and to consider a standard boundary value problem with a symbolic boundary condition (this ellipticity property is also crucial in Majda's analysis on shock waves [28, 29]). One obtains an estimate of the form

$$\begin{aligned}\|\psi\|^2_{H^1_\gamma} \leq C\left(\tfrac{1}{\gamma^2}\,\|\mathcal{B}(W^{nc},\psi)\|^2_{H^1_\gamma} + \|W^{nc}{}_{|x_2=0}\|^2_{L^2_\gamma}\right)\\ +\textit{error terms}\,,\end{aligned}$$

with no loss of regularity with respect to the source terms. In view of (2.14), it is enough to estimate W.

(3) Problem with reduced boundary conditions.

The projection of the boundary condition onto the orthogonal subspace gives a boundary condition involving only W^{nc}, i.e. without involving ψ. Thus we are left with the (paradifferential version of the) linear problem for W

$$\begin{aligned}&\mathcal{N}_r W^+ = \mathbf{A}_0^r\,\partial_t W^+ + \mathbf{A}_1^r\,\partial_{x_1} W^+ + \mathbf{I}_2\,\partial_{x_2} W^+ + \mathbf{A}_0^r\,\mathbf{C}^r\,W^+ = F^+\,, \qquad x_2>0\,,\\ &\mathcal{N}_l W^- = \mathbf{A}_0^l\,\partial_t W^- + \mathbf{A}_1^l\,\partial_{x_1} W^- + \mathbf{I}_2\,\partial_{x_2} W^- + \mathbf{A}_0^l\,\mathbf{C}^l\,W^- = F^-\,, \qquad x_2>0\,,\\ &\mathbf{\Pi}\,\widetilde{\mathbf{M}}\,W_{|x_2=0} = \mathbf{\Pi}\,g\,, \qquad x_2=0\,,\end{aligned} \tag{2.15}$$

where $\mathbf{\Pi}$ denotes the suitable projection operator.

For this problem the boundary is characteristic with constant multiplicity, as in the analysis of Majda and Osher [30]. Differently from [30], our problem satisfies a Kreiss–Lopatinskiĭ condition in the weak sense and not uniformly. In fact, the Lopatinskiĭ determinant associated to the boundary condition vanishes at some points. Recalling that the uniform Kreiss–Lopatinskiĭ condition is a necessary and sufficient condition for the L^2 estimate with no loss of derivatives, the failure of the uniform Kreiss–Lopatinskiĭ condition yields necessarily a loss of derivatives with respect to the source terms.

The proof of the main energy estimate is based on the construction of a *degenerate Kreiss' symmetrizer*. We add the techniques of Majda and Osher [30] for the analysis of characteristic boundaries to Coulombel's technique [10, 11] for the analysis of the singularities near the frequencies where the Lopatinskiĭ condition fails.

In order to explain the main ideas, let us consider for simplicity the linearization around the piecewise constant solution (2.9).

Then, instead of (2.15), we have a problem of the form ($\widehat{W} = \widehat{W}(\delta, \eta)$ is the Fourier transform in (t, x_1))

$$\begin{array}{ll} (\tau \mathcal{A}_0 + i\eta \mathcal{A}_1)\widehat{W} + \mathcal{A}_2 \frac{d\widehat{W}}{dx_2} & = 0\,, \quad x_2 > 0\,, \\ \beta(\tau, \eta)\widehat{W^{nc}} & = \widehat{h}\,, \quad x_2 = 0\,, \end{array} \tag{2.16}$$

where $\tau = \delta + i\eta$ and where $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are matrices with constant coefficients. Because of the characteristic boundary, the two first equations do not involve differentiation with respect to the normal variable x_2:

$$\begin{array}{ll} (\tau + iv_r\eta)\,\widehat{W}_1^+ - ic^2\eta\,\widehat{W}_2^+ + ic^2\eta\,\widehat{W}_3^+ & = 0\,, \\ (\tau + iv_l\eta)\,\widehat{W}_1^- - ic^2\eta\,\widehat{W}_2^- + ic^2\eta\,\widehat{W}_3^- & = 0\,. \end{array}$$

For Re $\tau > 0$, we obtain an expression for $\widehat{W}_1^+$ and $\widehat{W}_1^-$ that we plug into the other equations in (2.16). This operation yields a system of O.D.E. of the form:

$$\begin{array}{ll} \frac{d\widehat{W^{nc}}}{dx_2} = \mathcal{A}(\tau, \eta)\,\widehat{W^{nc}}\,, & x_2 > 0, \\ \beta(\tau, \eta)\widehat{W^{nc}}(0) = \widehat{h}\,, & x_2 = 0. \end{array}$$

By microlocalization, the analysis is performed locally in the neighborhood of points (τ, η) with $\mathrm{Re}\,\tau \geq 0$. In points with $\mathrm{Re}\,\tau > 0$ the matrix $\mathcal{A}(\tau, \eta)$ is regular and the Lopatinskiĭ determinant doesn't vanish; therefore in the neighborhood of those points we can construct a classical Kreiss' symmetrizer. This symmetrizer would yield an L^2 estimate with no loss of derivatives. When $\mathrm{Re}\,\tau = 0$ we find points of the following type:

1) Points where $\mathcal{A}(\tau, \eta)$ is diagonalizable and the Lopatinskiĭ condition is satisfied.

In these points the analysis is the same as for the interior points with $\mathrm{Re}\,\tau > 0$. Therefore we can construct a classical Kreiss' symmetrizer. This symmetrizer would yield an L^2 estimate with no loss of derivatives.

2) Points where $\mathcal{A}(\tau, \eta)$ is diagonalizable and the Lopatinskiĭ condition breaks down.

The points where the Lopatinskiĭ determinant vanishes correspond to critical speeds which are exactly the speeds of the kink modes in [2]. Since the Lopatinskiĭ determinant has simple roots, it behaves like $\gamma = \mathrm{Re}\,\tau$ uniformly in a neighborhood of the points. Using this fact and the diagonalizability of $A(\tau, \eta)$ we construct a degenerate Kreiss' symmetrizer; this yields an L^2 estimate with loss of one derivative.

3) Points where $\mathcal{A}(\tau, \eta)$ is not diagonalizable. In those points, the Lopatinskiĭ condition is satisfied.

Differently from the other cases we construct a suitable non-diagonal symmetrizer. This case doesn't yield a loss of derivatives.

4) Poles of $\mathcal{A}(\tau, \eta)$. At those points, the Lopatinskiĭ condition is satisfied.

The matrix $\mathcal{A}(\tau, \eta)$ is not smoothly diagonalizable. Consequently, Majda and Osher [30] construction of a symmetrizer in this case involves a singularity in the symmetrizer. We avoid this singularity and construct a smooth symmetrizer by working on the original system (2.15).

In the end, we consider a partition of unity to patch things together and we get the degenerate Kreiss' symmetrizer used in order to derive the energy estimate.

4. Tame estimate in Sobolev norms

Our second result concerns the well-posedness in Sobolev norm. In view of the future application to the initial-boundary value problem, we consider functions defined up to time T. Let us set

$$\begin{aligned} \Omega_T &:= \left\{(t,x_1,x_2) \in \mathbb{R}^3 \text{ s.t. } -\infty < t < T,\, x_2 > 0\right\}, \\ \omega_T &:= \left\{(t,x_1,x_2) \in \mathbb{R}^3 \text{ s.t. } -\infty < t < T,\, x_2 = 0\right\}. \end{aligned} \tag{2.17}$$

THEOREM 2.4. *[16] Let $T > 0$ and $m \in \mathbb{N}$. Assume that (i) the particular solution $\overline{U}^{\pm}$ defined by* (2.9) *satisfies* (2.10)*, (ii) $(\overline{U}^{\pm} + \dot{U}_{r,l}, \pm x_2 + \dot{\Phi}_{r,l})$ satisfies* (2.6) *and* (2.7)*, (iii) the perturbation $(\dot{U}_{r,l}, \dot{\Phi}_{r,l}) \in H^{m+3}_\gamma(\Omega_T)$ has compact support and is sufficiently small in $H^6(\Omega_T)$.*

Then there exist some constants $C > 0$ and $\gamma \geq 1$ such that, if $(F_\pm, g) \in H^{m+1}(\Omega_T) \times H^{m+1}(\omega_T)$ vanish in the past (i.e. for $t < 0$), then there exists a unique solution $(W^\pm, \psi) \in H^m(\Omega_T) \times H^{m+1}(\omega_T)$ to (2.12), (2.13) *that vanishes in the past. Moreover the following estimate holds:*

$$\begin{aligned} &\|W\|_{H^m_\gamma(\Omega_T)} + \|W^{nc}_{|x_2=0}\|_{H^m_\gamma(\omega_T)} + \|\psi\|_{H^{m+1}_\gamma(\omega_T)} \leq C\Big\{\|F\|_{H^{m+1}_\gamma(\Omega_T)} \\ &+\|g\|_{H^{m+1}_\gamma(\omega_T)} + \Big(\|F\|_{H^4_\gamma(\Omega_T)} + \|g\|_{H^4_\gamma(\omega_T)}\Big)\|(\dot{U}_{r,l}, \dot{\Phi}_{r,l})\|_{H^{m+3}_\gamma(\Omega_T)}\Big\}. \end{aligned} \tag{2.18}$$

Observe that there is the loss of *one* derivative for W with respect to the source terms, and the loss of *three* derivatives with respect to the coefficients. Again we have no loss of derivatives for the front function ψ (as in Majda's work [28, 29] on shock waves).

For the forthcoming analysis of the nonlinear problem by a Nash–Moser procedure it's important to observe that (2.18) is a *"tame estimate"* (roughly speaking: linear in high norms which are multiplied by low norms).

PROOF. We describe the main steps of the proof of the above theorem.

(1) Estimate of tangential derivatives. The tangential derivatives $\partial_t^h \partial_{x_1}^k W$ and the front function ψ are estimated by differentiation along tangential directions of the equations and application of the L^2 energy estimate given in Theorem 2.3. We

obtain

$$\sqrt{\gamma}\, \| W \|_{L^2(H^m_\gamma)} + \| W^{nc}{}_{|x_2=0} \|_{H^m_\gamma(\omega_T)} + \|\psi\|_{H^{m+1}_\gamma(\omega_T)}$$

$$\le \frac{C}{\gamma}\Big\{ \| F \|_{L^2(H^{m+1}_\gamma)} + \|g\|_{H^{m+1}_\gamma(\omega_T)} + \|W\|_{W^{1,\infty}(\Omega_T)} \|(\dot U_{r,l}, \nabla\dot\Phi_{r,l}\|_{H^{m+2}_\gamma(\Omega_T)} +$$

$$+ \Big(\|W^{nc}_{|x_2=0}\|_{L^\infty(\omega_T)} + \|\psi\|_{W^{1,\infty}(\omega_T)}\Big) \|(\dot U_{r,l}, \partial_{x_2}\dot U_{r,l}, \nabla\dot\Phi_{r,l})_{|x_2=0}\|_{H^{m+1}(\omega_T)} \Big\}, \tag{2.19}$$

where $\| \cdot \|_{L^2(H^m_\gamma)}$ denotes the norm of $L^2(\mathbb{R}^+; H^m_\gamma(\omega_T))$.

2) *Estimate of the linearized vorticity.*

Consider the original non linear equations. On both sides of the interface the solution is smooth and satisfies

$$\rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) + \nabla\, p(\rho) = 0\,.$$

Hence the vorticity $\xi := \partial_{x_1} u - \partial_{x_2} v$ satisfies on both sides

$$\partial_t \xi + \mathbf{u}\cdot\nabla\xi + \xi\nabla\cdot\mathbf{u} = 0\,.$$

Recalling that the interface is a streamline and that there is continuity of the normal velocity across the interface, this suggests the possibility of estimates of the vorticity on either part of the front. This leads to introduce the "linearized vorticity"

$$\dot\xi_\pm := \partial_{x_1}\dot u_\pm - \frac{1}{\partial_{x_2}\Phi_{r,l}}\big(\partial_{x_1}\Phi_{r,l}\,\partial_{x_2}\dot u_\pm + \partial_{x_2}\dot v_\pm\big).$$

Then

$$\partial_t\dot\xi_+ + v_r\partial_{x_1}\dot\xi_+ = \partial_{x_1}\mathcal{F}_2^+ - \tfrac{1}{\partial_{x_2}\Phi_r}(\partial_{x_1}\Phi_r\partial_{x_2}\mathcal{F}_2^+ + \partial_{x_2}\mathcal{F}_1^+) + \Lambda_1^r\cdot\partial_{x_1}\dot U_+ + \Lambda_2^r\cdot\partial_{x_2}\dot U_+\,,$$

$$\partial_t\dot\xi_- + v_l\partial_{x_1}\dot\xi_- = \partial_{x_1}\mathcal{F}_2^- - \tfrac{1}{\partial_{x_2}\Phi_l}(\partial_{x_1}\Phi_l\partial_{x_2}\mathcal{F}_2^- + \partial_{x_2}\mathcal{F}_1^-) + \Lambda_1^l\cdot\partial_{x_1}\dot U_- + \Lambda_2^l\cdot\partial_{x_2}\dot U_-\,, \tag{2.20}$$

where $\Lambda^{r,l}_{1,2}$ are $\mathcal{C}^\infty$ functions of $(\dot U_{r,l}, \nabla\dot U_{r,l}, \nabla\dot\Phi_{r,l}, \nabla^2\dot\Phi_{r,l})$ and where $\mathcal{F}^\pm_{1,2}$ are $\mathcal{C}^\infty$ functions of $U_{r,l}, \nabla\Phi_{r,l}$ and depend linearly on $F^\pm$ and $W^\pm$.

A standard energy argument may be applied to (2.20). In fact we may observe that, if we take any derivative ∂^α of (2.20), multiply by $\partial^\alpha\xi^\pm$ and integrate over ω_T, then the usual integrations by parts give no boundary terms. We obtain the estimate

$$\gamma\|\dot\xi_\pm\|_{H^{m-1}_\gamma(\Omega_T)} \le C\Big\{ \|F\|_{H^m_\gamma(\Omega_T)} + \|F\|_{L^\infty(\Omega_T)}\|\nabla\dot\Phi_{r,l}\|_{H^m_\gamma(\Omega_T)}$$

$$+ \|W\|_{H^m_\gamma(\Omega_T)} + \|W\|_{W^{1,\infty}(\Omega_T)}\big(\|\dot U_{r,l}\|_{H^{m+1}(\Omega_T)} + \|\nabla\dot\Phi_{r,l}\|_{H^m(\Omega_T)}\big)\Big\}. \tag{2.21}$$

3) *Estimate of normal derivatives.* We have

$$\partial_{x_2} W_1^\pm =$$

$$= \frac{1}{\langle\partial_{x_1}\Phi_{r,l}\rangle^2}\Big\{\partial_{x_2}\Phi_{r,l}\,(\partial_{x_1}\dot u_\pm - \dot\xi_\pm) - \partial_{x_1}\Phi_{r,l}\,(\partial_{x_2}T_{r,l}\,W^\pm)_3 - (\partial_{x_2}T_{r,l}\,W^\pm)_2\Big\},$$

where $T_{r,l} = T(U_{r,l}, \Phi_{r,l})$ denotes a suitable invertible matrix such that $W^\pm = T_{r,l}^{-1}\dot{U}_\pm$ (recall that $\dot{U}_\pm$ is the unknown defined in (2.11)). The above equality shows that we may estimate $\partial_{x_2} W_1^\pm$ by the previous steps. The estimate of normal derivatives $\partial_{x_2} W^{nc}$ of the noncharacteristic part of the solution follows directly from the equations:

$$\mathbf{I}_2\, \partial_{x_2} W^\pm = F^\pm - \mathbf{A}_0^{r,l}\, \partial_t W^\pm - \mathbf{A}_1^{r,l}\, \partial_{x_1} W^\pm - \mathbf{A}_0^{r,l}\, \mathbf{C}^{r,l}\, W^\pm\,,$$

since

$$\mathbf{I}_2 := \text{ diag } (0,1,1), \quad W^{nc} := (W_2^+, W_3^+, W_2^-, W_3^-).$$

We obtain for $k = 1, \dots, m$

$$\begin{aligned} &|\!|\!|\partial_{x_2}^k W|\!|\!|_{L^2(H_\gamma^{m-k})} \\ &\leq C\Big\{ \|F\|_{H_\gamma^{m-1}(\Omega_T)} + \|\dot{\xi}_\pm\|_{H_\gamma^{m-1}(\Omega_T)} + \|\dot{\xi}_\pm\|_{L^\infty(\Omega_T)} \|\nabla\dot{\Phi}_{r,l}\|_{H_\gamma^{m-1}(\Omega_T)} \qquad (2.22) \\ &+ \|W\|_{L^\infty(\Omega_T)} \|(\dot{U}_{r,l}, \nabla\dot{\Phi}_{r,l})\|_{H_\gamma^m(\Omega_T)} + |\!|\!|W|\!|\!|_{L^2(H_\gamma^m)} + \|W\|_{H_\gamma^{m-1}(\Omega_T)} \Big\}. \end{aligned}$$

By a combination of (2.19), (2.21) and (2.22) we finally obtain (2.18). The existence of the solution of the linear problem (2.12), (2.13) is a consequence of the well-posedness result of [12]. □

5. The Nash–Moser iterative scheme

5.1. Preliminary steps. We reduce the nonlinear problem (2.5), (2.6), (2.7), (2.8), to a new problem with solution vanishing in the past. We proceed as follows.

1) Given initial data $U_0^\pm = \overline{U}^\pm + \dot{U}_0^\pm$, $\dot{U}_0^\pm \in H^{2\mu+3/2}(\mathbb{R}_+^2)$, and $\varphi_0 \in H^{2\mu+2}(\mathbb{R})$, $\dot{U}_0^\pm$ and φ_0 with compact support and small enough, there exist an approximate ”solution” U^a, Φ^a, φ^a, such that $U^a - \overline{U} = \dot{U}^a \in H^{2\mu+2}(\Omega)$, $\Phi^{a\pm} \mp x_2 = \dot{\Phi}^{a\pm} \in H^{2\mu+3}(\Omega)$, $\varphi^a \in H^{2\mu+5/2}(\omega)$, and such that

$$\partial_t^j\, \mathbb{L}(U^a, \Phi^a)_{|t=0} = 0\,, \qquad \text{for } j = 0, \dots, 2\mu\,, \tag{2.23}$$

$$\partial_t \Phi^a + v^a\, \partial_{x_1} \Phi^a - u^a = 0\,, \tag{2.24}$$

$$\varphi^a = \Phi^{a+}_{|x_2=0} = \Phi^{a-}_{|x_2=0}\,, \tag{2.25}$$

$$\mathbb{B}(U^a_{|x_2=0}, \varphi^a) = 0\,. \tag{2.26}$$

The functions $\dot{U}^a, \dot{\Phi}^{a\pm}, \varphi^a$ satisfy a suitable a priori estimate and may be taken with compact supports.

2) We write the equations (2.5), (2.7) for $U = (U^+, U^-), \Phi = (\Phi^+, \Phi^-)$ in the form

$$\mathbb{L}(U, \Phi) = 0\,,$$
$$\mathbb{B}(U_{|x_2=0}, \varphi) = 0\,,$$

and introduce

$$\begin{cases} f^a := -\mathbb{L}(U^a, \Phi^a)\,, & t > 0\,, \\ f^a := 0\,, & t < 0\,. \end{cases}$$

Because $\dot{U}^a \in H^{2\mu+2}(\Omega)$ and $\dot{\Phi}^a \in H^{2\mu+3}(\Omega)$, (2.23) yields $f^a \in H^{2\mu+1}(\Omega)$.

3) For all real number $T > 0$, we let Ω_T^+, and ω_T^+ denote the sets

$$\omega_T^+ :=]0, T[\times \rho\,, \quad \Omega_T^+ :=]0, T[\times \rho \times]0, +\infty[= \omega_T^+ \times \mathbb{R}^+\,.$$

Given the approximate solution (U^a, Φ^a) and the function f^a, then $(U, \Phi) = (U^a, \Phi^a) + (V, \Psi)$ is a solution on Ω_T^+ of (2.5), (2.6), (2.7), (2.8), if $V = (V^+, V^-), \Psi = (\Psi^+, \Psi^-)$ satisfy the following system:

$$\begin{aligned}
&\mathcal{L}(V, \Psi) = f^a\,, && \text{in } \Omega_T\,,\\
&\mathcal{E}(V, \Psi) := \partial_t \Psi + (v^a + v)\,\partial_{x_1}\Psi - u + v\,\partial_{x_1}\Phi^a = 0\,, && \text{in } \Omega_T\,,\\
&\Psi^+_{|x_2=0} = \Psi^-_{|x_2=0} =: \psi\,, && \text{on } \omega_T\,,\\
&\mathcal{B}(V_{|x_2=0}, \psi) = 0\,, && \text{on } \omega_T\,,\\
&(V, \Psi) = 0\,, && \text{for } t < 0\,,
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
\mathcal{L}(V, \Psi) &:= \mathbb{L}(U^a + V, \Phi^a + \Psi) - \mathbb{L}(U^a, \Phi^a)\,,\\
\mathcal{B}(V_{|x_2=0}, \psi) &:= \mathbb{B}(U^a_{|x_2=0} + V_{|x_2=0}, \varphi^a + \psi)\,.
\end{aligned} \tag{2.28}$$

We note that $(V, \Psi) = 0$ satisfy (2.27) for $t < 0$, because $f^a = 0$ for $t < 0$, and $\mathbb{B}(U^a_{|x_2=0}, \varphi^a) = 0$ for all $t \in \mathbb{R}$. Therefore the initial nonlinear problem on Ω_T^+ is now substituted by a problem on Ω_T. The initial data (2.8) are absorbed into the equations by the introduction of the approximate solution (U^a, Φ^a, φ^a), and the problem has to be solved in the class of functions vanishing in the past (i.e., for $t < 0$), which is exactly the class of functions in which we have a well-posedness result for the linearized problem, see Theorem 2.4.

4) We solve problem (2.27) by a Nash–Moser type iteration. This method requires a family of smoothing operators. For $T > 0$, $s \geq 0$, and $\gamma \geq 1$, we let

$$\mathcal{F}^s_\gamma(\Omega_T) := \left\{ u \in H^s_\gamma(\Omega_T)\,, \quad u = 0 \text{ for } t < 0 \right\}.$$

The definition of $\mathcal{F}^s_\gamma(\omega_T)$ is entirely similar.

PROPOSITION 2.5. *Let $T > 0$, $\gamma \geq 1$, and let $M \in \mathbb{N}$, with $M \geq 4$. There exists a family $\{S_\theta\}_{\theta \geq 1}$ of operators*

$$S_\theta : \quad \mathcal{F}^3_\gamma(\Omega_T) \times \mathcal{F}^3_\gamma(\Omega_T) \longrightarrow \bigcap_{\beta \geq 3} \mathcal{F}^\beta_\gamma(\Omega_T) \times \mathcal{F}^\beta_\gamma(\Omega_T)\,,$$

and a constant $C > 0$ (depending on M), such that

$$\begin{aligned}
&\|S_\theta U\|_{H^\beta_\gamma(\Omega_T)} \leq C\,\theta^{(\beta-\alpha)_+}\,\|U\|_{H^\alpha_\gamma(\Omega_T)}\,, && \forall\,\alpha, \beta \in \{1, \ldots, M\}\,,\\
&\|S_\theta U - U\|_{H^\beta_\gamma(\Omega_T)} \leq C\,\theta^{\beta-\alpha}\,\|U\|_{H^\alpha_\gamma(\Omega_T)}\,, && 1 \leq \beta \leq \alpha \leq M\,,\\
&\|\frac{d}{d\theta} S_\theta U\|_{H^\beta_\gamma(\Omega_T)} \leq C\,\theta^{\beta-\alpha-1}\,\|U\|_{H^\alpha_\gamma(\Omega_T)}\,, && \forall\,\alpha, \beta \in \{1, \ldots, M\}\,.
\end{aligned}$$

Moreover, (i) if $U = (u^+, u^-)$ satisfies $u^+ = u^-$ on ω_T, then $S_\theta u^+ = S_\theta u^-$ on ω_T, (ii) the following estimate holds:

$$\begin{aligned}
&\|(S_\theta u^+ - S_\theta u^-)_{|x_2=0}\|_{H^\beta_\gamma(\omega_T)} \leq C\,\theta^{(\beta+1-\alpha)_+}\,\|(u^+ - u^-)_{|x_2=0}\|_{H^\alpha_\gamma(\omega_T)}\,,\\
&\qquad \forall\,\alpha, \beta \in \{1, \ldots, M\}\,.
\end{aligned}$$

There is another family of operators, still denoted S_θ, that acts on functions that are defined on the boundary ω_T, and that enjoy the above properties with the norms $\|\cdot\|_{H^\alpha_\gamma(\omega_T)}$.

In our case it appears to be convenient the choice $M := 2\mu + 3$.

5.2. Description of the iterative scheme. The iterative scheme starts from $V_0 = 0, \Psi_0 = 0, \psi_0 = 0$. Assume that V_k, Ψ_k, ψ_k are already given for $k = 0, \dots, n$ and verify

$$\begin{aligned} &(V_k, \Psi_k, \psi_k) = 0\,, && \text{for } t < 0, \\ &(\Psi_k^+)_{|x_2=0} = (\Psi_k^-)_{|x_2=0} = \psi_k\,, && \text{on } \omega_T. \end{aligned}$$

Given $\theta_0 \geq 1$, let us set $\theta_n := (\theta_0^2 + n)^{1/2}$, and consider the smoothing operators S_{θ_n}. Let us set

$$V_{n+1} = V_n + \delta V_n\,, \qquad \Psi_{n+1} = \Psi_n + \delta\Psi_n\,, \qquad \psi_{n+1} = \psi_n + \delta\psi_n\,. \tag{2.29}$$

We introduce the decomposition ($\mathcal{L}$ is defined in (2.28))

$$\begin{aligned} \mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) &= \mathbb{L}(U^a + V_{n+1}, \Phi^a + \Psi_{n+1}) - \mathbb{L}(U^a + V_n, \Phi^a + \Psi_n) \\ &= \mathbb{L}'(U^a + V_n, \Phi^a + \Psi_n)(\delta V_n, \delta\Psi_n) + e'_n \\ &= \mathbb{L}'(U^a + S_{\theta_n} V_n, \Phi^a + S_{\theta_n}\Psi_n)(\delta V_n, \delta\Psi_n) + e'_n + e''_n\,, \end{aligned}$$

where e'_n denotes the usual "quadratic"error of Newton's scheme, and e''_n the "substitution"error, due to the regularization of the state where the operator is calculated.

Thanks to the properties of the smoothing operators, we have $(S_{\theta_n}\Psi_n^+)_{|x_2=0} = (S_{\theta_n}\Psi_n^-)_{|x_2=0}$ and we denote $\psi_n^\sharp$ the common trace of these two functions. With this notation, we have

$$\begin{aligned} &\mathcal{B}((V_{n+1})_{|x_2=0}, \psi_{n+1}) - \mathcal{B}((V_n)_{|x_2=0}, \psi_n) \\ &\qquad = \mathbb{B}'\big((U^a + V_n)_{|x_2=0}, \varphi^a + \psi_n\big)((\delta V_n)_{|x_2=0}, \delta\psi_n) + \tilde{e}'_n \\ &\qquad = \mathbb{B}'\big((U^a + S_{\theta_n} V_n)_{|x_2=0}, \varphi^a + \psi_n^\sharp\big)((\delta V_n)_{|x_2=0}, \delta\psi_n) + \tilde{e}'_n + \tilde{e}''_n\,, \end{aligned}$$

where $\tilde{e}'_n$ denotes the "quadratic"error, and $\tilde{e}''_n$ the "substitution"error.

The inversion of the operator $(\mathbb{L}', \mathbb{B}')$ requires the linearization around a state satisfying the constraints (2.6), (2.7). We thus need to introduce a smooth modified state, denoted $V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2}$, that satisfies the above mentioned constraints, see [16] for details of the construction. Accordingly, we introduce the decompositions

$$\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = \mathbb{L}'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2})(\delta V_n, \delta\Psi_n) + e'_n + e''_n + e'''_n\,,$$

and

$$\begin{aligned} &\mathcal{B}((V_{n+1})_{|x_2=0}, \psi_{n+1}) - \mathcal{B}((V_n)_{|x_2=0}, \psi_n) \\ &\qquad = \mathbb{B}'\big((U^a + V_{n+1/2})_{|x_2=0}, \varphi^a + \psi_{n+1/2}\big)((\delta V_n)_{|x_2=0}, \delta\psi_n) + \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n\,, \end{aligned}$$

where $e'''_n, \tilde{e}'''_n$ denote the second "substitution"errors. The final step is the introduction of the "good unknown":

$$\delta\dot{V}_n := \delta V_n - \delta\Psi_n \, \frac{\partial_{x_2}(U^a + V_{n+1/2})}{\partial_{x_2}(\Phi^a + \Psi_{n+1/2})}\,. \tag{2.30}$$

This leads to

$$\mathcal{L}(V_{n+1},\Psi_{n+1})-\mathcal{L}(V_n,\Psi_n)=\mathbb{L}'_e(U^a+V_{n+1/2},\Phi^a+\Psi_{n+1/2})\delta\dot{V}_n$$
$$+e'_n+e''_n+e'''_n+\frac{\delta\Psi_n}{\partial_{x_2}(\Phi^a+\Psi_{n+1/2})}\,\partial_{x_2}\Big\{\mathbb{L}(U^a+V_{n+1/2},\Phi^a+\Psi_{n+1/2})\Big\}\,, \quad (2.31)$$

and

$$\mathcal{B}((V_{n+1})_{|x_2=0},\psi_{n+1})-\mathcal{B}((V_n)_{|x_2=0},\psi_n)$$
$$=\mathbb{B}'_e((U^a+V_{n+1/2})_{|x_2=0},\varphi^a+\psi_{n+1/2})((\delta\dot{V}_n)_{|x_2=0},\delta\psi_n)+\tilde{e}'_n+\tilde{e}''_n+\tilde{e}'''_n\,, \quad (2.32)$$

Here $\mathbb{L}'_e\delta\dot{V}$ denotes the “effective” linear operator obtained by linearizing $\mathbb{L}\delta V$, substituting the good unknown $\delta\dot{V}$ in place of the unknown δV and neglecting the zero order term in $\delta\Psi$, see (2.12). Similarly, $\mathbb{B}'_e$ is the operator obtained from linearization of the boundary conditions and the introduction of the good unknown.

For the sake of brevity we set

$$D_{n+1/2}:=\frac{1}{\partial_{x_2}(\Phi^a+\Psi_{n+1/2})}\,\partial_{x_2}\Big\{\mathbb{L}(U^a+V_{n+1/2},\Phi^a+\Psi_{n+1/2})\Big\}\,,$$

$$\mathbb{B}'_{n+1/2}:=\mathbb{B}'_e\big((U^a+V_{n+1/2})_{|x_2=0},\varphi^a+\psi_{n+1/2}\big)\,.$$

Let us also set

$$e_n:=e'_n+e''_n+e'''_n+D_{n+1/2}\,\delta\Psi_n\,,$$

$$\tilde{e}_n:=\tilde{e}'_n+\tilde{e}''_n+\tilde{e}'''_n.$$

The iteration proceeds as follows. Given

$$\begin{array}{llll} V_0:=0\,, & \Psi_0:=0\,, & \psi_0:=0\,, & \\ f_0:=S_{\theta_0}f^a\,, & g_0:=0\,, & E_0:=0\,, & \tilde{E}_0:=0\,,\\ V_1,\dots,V_n\,, & \Psi_1,\dots,\Psi_n\,, & \psi_1,\dots,\psi_n\,, & \\ f_1,\dots,f_{n-1}\,, & g_1,\dots,g_{n-1}\,, & & \\ e_0,\dots,e_{n-1}, & \tilde{e}_0,\dots,\tilde{e}_{n-1}, & & \end{array}$$

we first compute for $n\ge 1$

$$E_n:=\sum_{k=0}^{n-1}e_k\,,\qquad \tilde{E}_n:=\sum_{k=0}^{n-1}\tilde{e}_k\,.$$

These are the accumulated errors at the step n. Then we compute f_n, and g_n from the equations:

$$\sum_{k=0}^{n}f_k+S_{\theta_n}E_n=S_{\theta_n}f^a\,,\qquad \sum_{k=0}^{n}g_k+S_{\theta_n}\tilde{E}_n=0\,, \quad (2.33)$$

and we solve the linear problem

$$\begin{array}{ll} \mathbb{L}'_e(U^a+V_{n+1/2},\Phi^a+\Psi_{n+1/2})\,\delta\dot{V}_n=f_n & \text{in } \Omega_T\,,\\ \mathbb{B}'_{n+1/2}((\delta\dot{V}_n)_{|x_2=0},\delta\psi_n)=g_n & \text{on } \omega_T\,,\\ \delta\dot{V}_n=0,\quad \delta\psi_n=0 & \text{for } t<0\,, \end{array} \quad (2.34)$$

finding $(\delta\dot{V}_n, \delta\psi_n)$. Now we need to construct $\delta\Psi_n = (\delta\Psi_n^+, \delta\Psi_n^-)$ that satisfies $(\delta\Psi_n^\pm)_{|x_2=0} = \delta\psi_n$. Using the explicit expression of the boundary conditions in (2.34), we first note that $\delta\psi_n$ solves the equation:

$$\begin{aligned}\partial_t\delta\psi_n &+ (v^{a+} + v^+_{n+1/2})_{|x_2=0}\,\partial_{x_1}\delta\psi_n\\ &+ \left\{\partial_{x_1}(\varphi^a + \psi_{n+1/2})\,\frac{\partial_{x_2}(v^{a+} + v^+_{n+1/2})_{|x_2=0}}{\partial_{x_2}(\Phi^{a+} + \Psi^+_{n+1/2})_{|x_2=0}} - \frac{\partial_{x_2}(u^{a+} + u^+_{n+1/2})_{|x_2=0}}{\partial_{x_2}(\Phi^{a+} + \Psi^+_{n+1/2})_{|x_2=0}}\right\}\delta\psi_n\\ &+ \partial_{x_1}(\varphi^a + \psi_{n+1/2})\,(\delta\dot{v}_n^+)_{|x_2=0} - (\delta\dot{u}_n^+)_{|x_2=0} = g_{n,2}\,, \end{aligned} \tag{2.35}$$

and the equation

$$\begin{aligned}\partial_t\delta\psi_n &+ (v^{a-} + v^-_{n+1/2})_{|x_2=0}\,\partial_{x_1}\delta\psi_n\\ &+ \left\{\partial_{x_1}(\varphi^a + \psi_{n+1/2})\,\frac{\partial_{x_2}(v^{a-} + v^-_{n+1/2})_{|x_2=0}}{\partial_{x_2}(\Phi^{a-} + \Psi^-_{n+1/2})_{|x_2=0}} - \frac{\partial_{x_2}(u^{a-} + u^-_{n+1/2})_{|x_2=0}}{\partial_{x_2}(\Phi^{a-} + \Psi^-_{n+1/2})_{|x_2=0}}\right\}\delta\psi_n\\ &+ \partial_{x_1}(\varphi^a + \psi_{n+1/2})\,(\delta\dot{v}_n^-)_{|x_2=0} - (\delta\dot{u}_n^-)_{|x_2=0} = g_{n,2} - g_{n,1}\,. \end{aligned} \tag{2.36}$$

We shall thus define $\delta\Psi_n^+, \delta\Psi_n^-$ as the solutions to the following equations:

$$\begin{aligned}\partial_t\delta\Psi_n^+ &+ (v^{a+} + v^+_{n+1/2})\,\partial_{x_1}\delta\Psi_n^+\\ &+ \left\{\partial_{x_1}(\Phi^{a+} + \Psi^+_{n+1/2})\,\frac{\partial_{x_2}(v^{a+} + v^+_{n+1/2})}{\partial_{x_2}(\Phi^{a+} + \Psi^+_{n+1/2})} - \frac{\partial_{x_2}(u^{a+} + u^+_{n+1/2})}{\partial_{x_2}(\Phi^{a+} + \Psi^+_{n+1/2})}\right\}\delta\Psi_n^+\\ &+ \partial_{x_1}(\Phi^{a+} + \Psi^+_{n+1/2})\,\delta\dot{v}_n^+ - \delta\dot{u}_n^+ = \mathcal{R}_T g_{n,2} + h_n^+\,, \end{aligned} \tag{2.37}$$

and

$$\begin{aligned}\partial_t\delta\Psi_n^- &+ (v^{a-} + v^-_{n+1/2})\,\partial_{x_1}\delta\Psi_n^-\\ &+ \left\{\partial_{x_1}(\Phi^{a-} + \Psi^-_{n+1/2})\,\frac{\partial_{x_2}(v^{a-} + v^-_{n+1/2})}{\partial_{x_2}(\Phi^{a-} + \Psi^-_{n+1/2})} - \frac{\partial_{x_2}(u^{a-} + u^-_{n+1/2})}{\partial_{x_2}(\Phi^{a-} + \Psi^-_{n+1/2})}\right\}\delta\Psi_n^-\\ &+ \partial_{x_1}(\Phi^{a-} + \Psi^-_{n+1/2})\,\delta\dot{v}_n^- - \delta\dot{u}_n^- = \mathcal{R}_T(g_{n,2} - g_{n,1}) + h_n^-\,. \end{aligned} \tag{2.38}$$

In (2.37), and (2.38), the source terms $h_n^\pm$ have to be chosen suitably. First we require that $h_n^\pm$ vanish on the boundary ω_T, and in the past, so that the unique smooth solutions to (2.37) and (2.38) will vanish in the past, and will satisfy the continuity condition $(\delta\Psi_n^\pm)_{|x_2=0} = \delta\psi_n$. In order to compute the source terms $h_n^\pm$, we use a decomposition that is similar to (2.31) for the operator $\mathcal{E}$ (defined in (2.27)). We have:

$$\mathcal{E}(V_{n+1}, \Psi_{n+1}) - \mathcal{E}(V_n, \Psi_n) = \mathcal{E}'(V_{n+1/2}, \Psi_{n+1/2})(\delta V_n, \delta\Psi_n) + \hat{e}'_n + \hat{e}''_n + \hat{e}'''_n\,, \tag{2.39}$$

where $\hat{e}'_n$ is the "quadratic"error, $\hat{e}''_n$ is the first "substitution"error, and $\hat{e}''_n$ is the second "substitution"error. We denote

$$\hat{e}_n := \hat{e}'_n + \hat{e}''_n + \hat{e}'''_n\,, \quad \hat{E}_n := \sum_{k=0}^{n-1} \hat{e}_k\,.$$

Using the good unknown (2.30), and omitting the ± superscripts, we compute

$$\mathcal{E}'(V_{n+1/2},\Psi_{n+1/2})(\delta V_n,\delta\Psi_n)=\partial_t\delta\Psi_n+(v^a+v_{n+1/2})\,\partial_{x_1}\delta\Psi_n$$
$$+\left\{\partial_{x_1}(\Phi^a+\Psi_{n+1/2})\,\frac{\partial_{x_2}(v^a+v_{n+1/2})}{\partial_{x_2}(\Phi^a+\Psi_{n+1/2})}-\frac{\partial_{x_2}(u^a+u_{n+1/2})}{\partial_{x_2}(\Phi^a+\Psi_{n+1/2})}\right\}\delta\Psi_n$$
$$+\partial_{x_1}(\Phi^a+\Psi_{n+1/2})\,\delta\dot{v}_n-\delta\dot{u}_n\,.$$

Consequently, (2.37) and (2.39) yield

$$\mathcal{E}(V^+_{n+1},\Psi^+_{n+1})-\mathcal{E}(V^+_n,\Psi^+_n)=\mathcal{R}_T g_{n,2}+h^+_n+\hat{e}^+_n\,.$$

Summing these relations, and using $\mathcal{E}(V^+_0,\Psi^+_0)=0$, we get

$$\mathcal{E}(V^+_{n+1},\Psi^+_{n+1})=\mathcal{R}_T\Big(\sum_{k=0}^n g_{k,2}\Big)+\sum_{k=0}^n h^+_k+\hat{E}^+_{n+1}$$
$$=\mathcal{R}_T\Big(E((V^+_{n+1})_{|x_2=0},\psi_{n+1})-\tilde{E}_{n+1,2}\Big)+\sum_{k=0}^n h^+_k+\hat{E}^+_{n+1}\,,$$

where in the last equality, we have summed (2.32) and used the relation

$$\Big(\mathcal{B}((V_{n+1})_{|x_2=0},\psi_{n+1})\Big)_2=\mathcal{E}((V^+_{n+1})_{|x_2=0},\psi_{n+1})\,,$$

which simply shows that the second line of the boundary operator $\mathcal{B}$ coincides with $\mathcal{E}$ at the boundary, see the definitions in (2.27), (2.28). The previous relations lead to the following definition of the source term h^+_n:

$$\sum_{k=0}^n h^+_k+S_{\theta_n}\big(\hat{E}^+_n-\mathcal{R}_T\tilde{E}_{n,2}\big)=0\,.$$

The definition of h^-_n is entirely similar:

$$\sum_{k=0}^n h^-_k+S_{\theta_n}\big(\hat{E}^-_n-\mathcal{R}_T\tilde{E}_{n,2}+\mathcal{R}_T\tilde{E}_{n,1}\big)=0\,.$$

Once $\delta\Psi_n$ is computed, the function δV_n is obtained from (2.30), and the functions V_{n+1}, Ψ_{n+1}, ψ_{n+1} are obtained from (2.29).

Finally, we compute $e_n,\hat{e}_n,\tilde{e}_n$ from

$$\begin{aligned}
&\mathcal{L}(V_{n+1},\Psi_{n+1})-\mathcal{L}(V_n,\Psi_n)=f_n+e_n\,,\\
&\mathcal{E}(V^+_{n+1},\Psi^+_{n+1})-\mathcal{E}(V^+_n,\Psi^+_n)=\mathcal{R}_T g_{n,2}+h^+_n+\hat{e}^+_n\,,\\
&\mathcal{E}(V^-_{n+1},\Psi^-_{n+1})-\mathcal{E}(V^-_n,\Psi^-_n)=\mathcal{R}_T(g_{n,2}-g_{n,1})+h^-_n+\hat{e}^-_n\,,\\
&\mathcal{B}((V_{n+1})_{|x_2=0},\psi_{n+1})-\mathcal{B}((V_n)_{|x_2=0},\psi_n)=g_n+\tilde{e}_n\,.
\end{aligned}\tag{2.40}$$

To compute V_1,Ψ_1,ψ_1 we only consider steps (2.34), (2.37), (2.38), (2.40) for $n=0$.

Adding (2.40) from 0 to N, and combining with (2.33) gives

$$\mathcal{L}(V_{N+1}, \Psi_{N+1}) - f^a = (S_{\theta_N} - I) f^a + (I - S_{\theta_N}) E_N + e_N \,,$$

$$\mathcal{E}(V^+_{N+1}, \Psi^+_{N+1}) = \mathcal{R}_T\Big(\mathcal{E}((V^+_{N+1})_{|x_2=0}, \psi_{N+1})\Big) + (I - S_{\theta_N})(\hat{E}^+_N - \mathcal{R}_T \tilde{E}_{N,2}) \\ + \hat{e}^+_N - \mathcal{R}_T \tilde{e}_{N,2} \,,$$

$$\mathcal{E}(V^-_{N+1}, \Psi^-_{N+1}) = \mathcal{R}_T\Big(\mathcal{E}((V^-_{N+1})_{|x_2=0}, \psi_{N+1})\Big) + (I - S_{\theta_N})(\hat{E}^-_N - \mathcal{R}_T(\tilde{E}_{N,2} - \tilde{E}_{N,1})) \\ + \hat{e}^-_N - \mathcal{R}_T(\tilde{e}_{N,2} - \tilde{e}_{N,1}) \,,$$

$$\mathcal{B}\big((V_{N+1})_{|x_2=0}, \psi_{N+1}\big) = (I - S_{\theta_N}) \tilde{E}_N + \tilde{e}_N \,.$$

Because $S_{\theta_N} \to I$ as $N \to +\infty$, and since we expect $(e_N, \hat{e}_n, \tilde{e}_N) \to 0$, we will formally obtain the solution of the problem (2.27) from $\mathcal{L}(V_{N+1}, \Psi_{N+1}) \to f^a$, $\mathcal{B}((V_{N+1})_{|x_2=0}, \psi_{N+1}) \to 0$, and $\mathcal{E}(V_{N+1}, \Psi_{N+1}) \to 0$.

The rigorous proof of convergence follows from a priori estimates of V_k, Ψ_k, ψ_k proved by induction for every k. In the limit we obtain a solution (V, Ψ) on Ω_T of (2.27), vanishing in the past, which yields that $(U, \Phi) = (U^a, \Phi^a) + (V, \Psi)$ is a solution on Ω^+_T of (2.5), (2.6), (2.7), (2.8). This concludes the proof of Theorem 2.2.

CHAPTER 3

An example of loss of normal regularity

1. A toy model

In $\Omega = \mathbb{R}^2_+ = \{x > 0\}$ let us consider the linear IBVP

$$\begin{cases} u_t + u_x + v_y = 0 \\ v_t + u_y = 0 \\ u_{|x=0} = 0 \\ (u, v)_{|t=0} = (u_0, v_0), \end{cases} \tag{3.1}$$

In matrix form the differential equations can be written as

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

Clearly the system is symmetric hyperbolic and the boundary is (uniformly) characteristic. It is also immediate to verify that the boundary condition is maximally nonnegative.

We look for a priori estimates of the solution. Assume that

$$(u_0, v_0) \in H^1(\Omega) \quad \text{with} \quad ||(u_0, v_0)||_{H^1(\Omega)} \le K.$$

(I) We multiply the first equation by u, the second one by v, integrate over $(0, t) \times \Omega$ and obtain ($|| \cdot ||$ stands for $|| \cdot ||_{L^2(\Omega)}$)

$$||u(t,)||^2 + ||v(t,)||^2 = ||u_0||^2 + ||v_0||^2 \qquad \forall t > 0.$$

It follows that

$$||u(t,)|| + ||v(t,)|| \le C(K) \qquad \forall t > 0.$$

(II) Consider the tangential derivatives (u_y, v_y). By taking the $y-$derivative of the problem we see that (u_y, v_y) solves the same problem as (u, v), with initial data (u_{0y}, v_{0y}). In particular it satisfies the same boundary condition as (u, v). It follows that

$$||u_y(t,)||^2 + ||v_y(t,)||^2 = ||u_{0y}||^2 + ||v_{0y}||^2 \qquad \forall t > 0.$$

Thus

$$||u_y(t,)|| + ||v_y(t,)|| \le C(K) \qquad \forall t > 0.$$

(III) By taking the $t-$derivative of the equations we see that (u_t, v_t) is also a solution. This yields

$$||u_t(t,)||^2 + ||v_t(t,)||^2 = ||u_t(0,)||^2 + ||v_t(0,)||^2 = ||u_{0x} + v_{0y}||^2 + ||u_{0y}||^2,$$

$$||u_t(t,)|| + ||v_t(t,)|| \le C(K) \qquad \forall t > 0.$$

(IV) From

$$u_x = -u_t - v_y$$

we may estimate the normal derivative u_x:

$$||u_x(t,)|| \leq ||u_t(t,)|| + ||v_y(t,)|| \leq C(K) \qquad \forall t > 0.$$

Let P be the orthogonal projection onto $ker\, A_\nu(x,t)^\perp$. Then

$$P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \tag{3.2}$$

(this is called the *noncharacteristic component* of $(u, v)^T$).

(V) Now we want to estimate the normal derivative v_x. We have

$$(I - P)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{3.3}$$

(called the *characteristic component* of $(u, v)^T$). Take the $x-$derivative of the second equation in (3.1):

$$v_{tx} + u_{xy} = 0. \tag{3.4}$$

Take also the $y-$derivative of the first equation in (3.1):

$$u_{ty} + u_{xy} + v_{yy} = 0.$$

Multiply (3.4) by v_x and integrate over Ω. Then ($\int = \int_\Omega dxdy$)

$$\frac{1}{2}\frac{d}{dt}||v_x||^2 = -\int u_{xy}v_x = \int (u_{ty} + v_{yy})v_x$$

$$= \frac{d}{dt}\int u_y v_x - \int u_y v_{tx} + \int v_{yy} v_x$$

$$= \frac{d}{dt}\int u_y v_x + \int u_y u_{xy} - \int v_y v_{xy}$$

$$= \frac{d}{dt}\int u_y v_x + \frac{1}{2}\int (u_y^2)_x - \frac{1}{2}\int (v_y^2)_x$$

$$= \frac{d}{dt}\int u_y v_x - \underbrace{\frac{1}{2}\int_{|x=0} u_y^2(t,0,y)dy}_{=0} + \frac{1}{2}\int_{|x=0} v_y^2(t,0,y)dy.$$

From $v_{ty} = -u_{yy}$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{|x=0} v_y^2 dy = \int_{|x=0} v_y v_{ty} dy = -\int_{|x=0} v_y u_{yy} dy = 0.$$

Then

$$\int_{|x=0} v_y^2(t,0,y)dy = \int_{|x=0} v_{0y}^2(y)dy = \text{ constant in time.}$$

We then obtain

$$\frac{d}{dt}||v_x||^2 = 2\frac{d}{dt}\int u_y v_x + \int_{|x=0} v_{0y}^2(y)dy.$$

Integration in time between 0 and $t > 0$ gives

$$||v_x(t,)||^2 = ||v_{0x}||^2 + 2\int u_y v_x - 2\int u_{0y} v_{0x} + t\int_{|x=0} v_{0y}^2(y)dy.$$

By the Young's inequality we finally obtain

$$\frac{1}{2}\,t\int_{|x=0} v_{0y}^2(y)dy - C_1(K) \quad \le ||v_x(t,)||^2 \le$$

$$\le 2\,t\int_{|x=0} v_{0y}^2(y)dy + C_2(K), \quad t>0.$$

This shows that

$$v_x(t,) \in L^2(\Omega) \text{ for } t>0 \textit{ if and only if } v_0 \in H^1(\partial\Omega).$$

By the trace theorem, $v_0 \in H^1(\Omega)$ only gives $v_{0|\partial\Omega} \in H^{1/2}(\partial\Omega)$. Therefore

$$\boxed{(u_0,v_0) \in H^1(\Omega) \not\Rightarrow (u(t,),v(t,)) \in H^1(\Omega) \text{ for } t>0.}$$

2. Two for one

We consider the problem of determining a function space X characterized by the property of persistence of regularity, that is such that

$$(u_0,v_0) \in X \Rightarrow (u(t,),v(t,)) \in X, \quad \forall t>0\,.$$

We assume

$$(u_0,v_0) \in H^2(\Omega) \quad \text{with} \quad ||(u_0,v_0)||_{H^2(\Omega)} \le K_2.$$

After the above analysis, we don't expect to obtain $(u(t,),v(t,)) \in H^2(\Omega)$. Calculations as above give

$$\partial_t^h\partial_y^k u(t,),\ \partial_t^h\partial_y^k v(t,) \in L^2(\Omega), \quad t>0,\ h+k\le 2,$$

with norms bounded by $C(K_2)$. By the t and y differentiation of the first equation in (3.1) we readily obtain

$$u_{tx} = -u_{tt} - v_{ty} \in L^2(\Omega), \quad ||u_{tx}(t,)|| \le C(K_2), \quad t>0,$$

$$u_{xy} = -u_{ty} - v_{yy} \in L^2(\Omega), \quad ||u_{xy}(t,)|| \le C(K_2), \quad t>0.$$

$v_0 \in H^2(\Omega)$ yields $v_{0|\partial\Omega} \in H^1(\partial\Omega)$, so that by the above analysis

$$v_x(t,) \in L^2(\Omega), \quad ||v_x(t,)|| \le C(K_2), \quad 0<t<T,$$

for any $T < +\infty$. We look for an estimate of the mixed derivative v_{xy}. Here we start from

$$\begin{cases} u_{tyy} + u_{xyy} + v_{yyy} = 0, \\ v_{txy} + u_{xyy} = 0. \end{cases}$$

Multiply the second equation by v_{xy} and integrate over Ω. Then

$$\frac{1}{2}\frac{d}{dt}||v_{xy}||^2 = -\int u_{xyy}v_{xy} = \int (u_{tyy}+v_{yyy})v_{xy}$$

$$= \frac{d}{dt}\int u_{yy}v_{xy} - \int u_{yy}v_{txy} - \int v_{yy}v_{xyy}$$

$$= \frac{d}{dt}\int u_{yy}v_{xy} + \int u_{yy}u_{xyy} - \frac{1}{2}\int (v_{yy}^2)_x$$

$$= \frac{d}{dt}\int u_{yy}v_{xy} \underbrace{- \frac{1}{2}\int_{|x=0} u_{yy}^2(t,0,y)dy}_{=0} + \frac{1}{2}\int_{|x=0} v_{yy}^2(t,0,y)dy.$$

Since $v_{tyy} = -u_{yyy}$, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\int_{|x=0} v_{yy}^2 dy = \int_{|x=0} v_{yy}v_{tyy}dy = -\int_{|x=0} v_{yy}u_{yyy}dy = 0,$$

again by the boundary condition on u. It follows that

$$\int_{|x=0} v_{yy}^2(t,0,y)dy = \int_{|x=0} v_{0yy}^2(y)dy = \text{ constant in time.}$$

We then obtain

$$\tfrac{d}{dt}||v_{xy}||^2 = 2\tfrac{d}{dt}\int u_{yy}v_{xy} + \int_{|x=0} v_{0yy}^2(y)dy.$$

Integrating in time between 0 and $t > 0$ yields

$$\begin{aligned} \tfrac{1}{2}\,t\int_{|x=0} v_{0yy}^2(y)dy - C_1(K_2) \quad &\leq ||v_{xy}(t,)||^2 \leq \\ &\leq 2\,t\int_{|x=0} v_{0yy}^2(y)dy + C_2(K_2), \quad t > 0. \end{aligned}$$

It follows that, if $v_0 \in H^2(\Omega)$, but $v_{0|\partial\Omega} \notin H^2(\partial\Omega)$, then $v_{xy}(t,) \notin L^2(\Omega)$. Since $u_{xx} = -u_{tx} - v_{xy}$ and $u_{tx}(t,) \in L^2(\Omega)$, then $u_{xx}(t,) \notin L^2(\Omega)$. A fortiori we also have $v_{xx}(t,) \notin L^2(\Omega)$.

The first two cases that we have considered suggest to define the following functions spaces.

Given $m \geq 1$, let us define the anisotropic Sobolev spaces

$$K_*^m(\Omega) = \{u \in L^2(\Omega)|\partial_x^k\partial_y^h u \in L^2(\Omega) \text{ for } 2k + h \leq m\},$$

$$K_{**}^m(\Omega) = \{u \in L^2(\Omega)|\partial_x^k\partial_y^h u \in L^2(\Omega) \text{ for } 2k + h \leq m+1, h \leq m\}.$$

Observe that $K_{**}^1(\Omega) = H^1(\Omega)$. When $m = 0$ we set $K_*^0 = K_{**}^0 = L^2$. Deriving the above a priori estimates we had assumed $(u_0, v_0) \in H^m(\Omega), m = 1, 2$, but not all the derivatives had been used. We go back and check which particular derivatives have to be L^2 in order to get the estimates. We summarize as follows

(I) If $u_0 \in K_{**}^1(\Omega)$, $v_0 \in K_*^1(\Omega)$, then $u(t,) \in K_{**}^1(\Omega)$, $v(t,) \in K_*^1(\Omega)$. If $v_{0|\partial\Omega} \notin H^1(\partial\Omega)$ then $v(t,) \notin K_{**}^1(\Omega)$.

(II) If $\partial_t^h u(0,) \in K_{**}^{2-h}(\Omega)$, $\partial_t^h v(0,) \in K_*^{2-h}(\Omega)$, then $\partial_t^h u(t,) \in K_{**}^{2-h}(\Omega)$, $\partial_t^h v(t,) \in K_*^{2-h}(\Omega)$, $h = 0, 1, 2$. If $v_{0|\partial\Omega} \notin H^2(\partial\Omega)$ then $v(t,) \notin K_{**}^2(\Omega)$.

(III) In order to check if this is the correct choice when m is odd we also prove: if $\partial_t^h u(0,) \in K_{**}^{3-h}(\Omega)$, $\partial_t^h v(0,) \in K_*^{3-h}(\Omega)$, then $\partial_t^h u(t,) \in K_{**}^{3-h}(\Omega)$, $\partial_t^h v(t,) \in K_*^{3-h}(\Omega)$, $h = 0, \ldots, 3$. If $v_{0x|\partial\Omega} \in H^2(\partial\Omega)$, but $v_{0|\partial\Omega} \notin H^3(\partial\Omega)$, then $v_{xx}(t,) \notin L^2(\Omega)$ and thus v may loose two normal derivatives even if the data are in $H^3(\Omega)$. Under the same assumption one also shows $v_{xyy}(t,) \notin L^2(\Omega), u_{xxy}(t,) \notin L^2(\Omega)$; it follows that $v(t,) \notin K_{**}^3(\Omega)$.

These are not yet the best choices for the function spaces appropriate for the general problem. A better insight is obtained with a little modification of the model problem.

3. Modified toy model

Let $\sigma \in \mathcal{C}^\infty(\overline{R_+})$ be a monotone increasing function such that $\sigma(x) = x$ in a neighborhood of the origin and $\sigma(x) = 1$ for any x large enough. In $\Omega = \mathbb{R}^2_+ = \{x > 0\}$ we consider the linear initial-boundary value problem

$$\begin{cases} u_t + u_x + \sigma v_x + v_y = 0 \\ v_t + \sigma u_x + \sigma v_x + u_y = 0 \\ u_{|x=0} = 0 \\ (u, v)_{|t=0} = (u_0, v_0). \end{cases} \tag{3.5}$$

Dropping the terms with σ we get (3.1). This is a symmetric hyperbolic system with variable coefficients. The boundary matrix is singular at the boundary with constant rank 1, thus the boundary is characteristic; again the boundary condition is maximally nonnegative. Let us assume

$$(u_0, v_0) \in H^2(\Omega).$$

It is understood that not all the derivatives of the initial data will be used and we will have to take care of that. We multiply the first equation by u, the second one by v, and integrate over Ω. After integrating by parts, using the boundary condition and $\sigma(0) = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(||u||^2 + ||v||^2) = \int \sigma' uv + \frac{1}{2}\int \sigma' v^2. \tag{3.6}$$

By the Gronwall lemma it follows that

$$||u(t,)||^2 + ||v(t,)||^2 \leq e^{Ct}(||u_0||^2 + ||v_0||^2), \qquad t > 0. \tag{3.7}$$

By taking the $y-$ and $t-$derivative of the problem we see that (u_y, v_y) and (u_t, v_t) are solution of the differential equations and boundary condition in (3.5) (with suitable initial data, of course). From the previous calculation it follows that

$$\begin{aligned} &\tfrac{d}{dt}(||u_y||^2 + ||v_y||^2) \leq C(||u_y||^2 + ||v_y||^2), \\ &\tfrac{d}{dt}(||u_t||^2 + ||v_t||^2) \leq C(||u_t||^2 + ||v_t||^2). \end{aligned} \tag{3.8}$$

Now we consider the normal derivatives. From the first equation we have $u_x = -u_t - \sigma v_x - v_y$, which shows the necessity to estimate at first σv_x. We apply the weighted differential operator $\sigma\partial_x$ to both equations in (3.5) and obtain the system

$$\begin{cases} (\sigma u_x)_t + (\sigma u_x)_x + \sigma(\sigma v_x)_x + (\sigma v_x)_y = \sigma' u_x \\ (\sigma v_x)_t + \sigma(\sigma u_x)_x + \sigma(\sigma v_x)_x + (\sigma u_x)_y = 0 \\ \sigma u_{x|x=0} = 0 \\ (\sigma u_x, \sigma v_x)_{|t=0} = (\sigma u_{x0}, \sigma v_{x0}). \end{cases} \tag{3.9}$$

The boundary condition follows from $\sigma(0) = 0$. We multiply the two equations respectively by $\sigma u_x, \sigma v_x$ and integrate over Ω. The result is similar to (3.6); we only have to write σu_x instead of u, σv_x instead of v, and take care of the forcing term in (3.9). We obtain

$$\frac{1}{2}\frac{d}{dt}(||\sigma u_x||^2 + ||\sigma v_x||^2) = \int \sigma'(\sigma u_x)(\sigma v_x) + \frac{1}{2}\int \sigma'(\sigma v_x)^2 + \int \sigma' u_x(\sigma u_x).$$

This yields

$$\frac{1}{2}\frac{d}{dt}(||\sigma u_x||^2 + ||\sigma v_x||^2) \le C(||\sigma u_x||^2 + ||\sigma v_x||^2 + ||u_x||\,||\sigma u_x||). \tag{3.10}$$

The estimate may be closed by using the first equation of (3.5) which gives

$$||u_x|| \le ||u_t|| + ||\sigma v_x|| + ||v_y||. \tag{3.11}$$

Let us denote by $\partial_\star$ any of the derivatives ∂_t, $\sigma\partial_x$, ∂_y, and set also $\partial_\star^\alpha = \partial_t^{\alpha_0}(\sigma\partial_x)^{\alpha_1}\partial_y^{\alpha_2}$ for $\alpha = (\alpha_0, \alpha_1, \alpha_2)$.

We substitute (3.11) in (3.10), add (3.8), apply the Gronwall lemma and finally obtain

$$\begin{aligned} &||\partial_\star u(t,)|| + ||\partial_\star v(t,)|| + ||u_x(t,)|| \\ &\le Ce^{Ct}(||\partial_\star u(0,)|| + ||\partial_\star v(0,)||), \quad t > 0. \end{aligned} \tag{3.12}$$

Let us consider the second order derivatives. As (u_y, v_y) and (u_t, v_t) solve problem (3.5) with suitable initial data, the estimate (3.12) holds also with (u_y, v_y) and (u_t, v_t) replacing (u, v).

Substituting $u_x = -(u_t + \sigma v_x + v_y)$ in the right-hand side of (3.9) we can also obtain an estimate for $((\sigma\partial_x)^2 u, (\sigma\partial_x)^2 v)$. Hence we find

$$\begin{aligned} &\textstyle\sum_{|\alpha|=2}(||\partial_\star^\alpha u(t,)|| + ||\partial_\star^\alpha v(t,)||) + ||\partial_\star u_x(t,)|| \\ &\le Ce^{Ct}\textstyle\sum_{|\alpha|=2}(||\partial_\star^\alpha u(0,)|| + ||\partial_\star^\alpha v(0,)||), \quad t > 0. \end{aligned} \tag{3.13}$$

Let us denote $\partial_*^\alpha = (\sigma\partial_x)^{\alpha_1}\partial_y^{\alpha_2}$ for $\alpha = (\alpha_1, \alpha_2)$. Given $m \ge 1$, we define the anisotropic Sobolev spaces

$$H_*^m(\Omega) = \{u \in L^2(\Omega) | \partial_*^\alpha \partial_x^k u \in L^2(\Omega) \text{ for } |\alpha| + 2k \le m\},$$

$$H_{**}^m(\Omega) = \{u \in L^2(\Omega) | \partial_*^\alpha \partial_x^k u \in L^2(\Omega) \text{ for } |\alpha| + 2k \le m+1, |\alpha| \le m\}.$$

Observe that $H_{**}^1(\Omega) = H^1(\Omega)$. When $m = 0$ we set $H_*^0 = H_{**}^0 = L^2$. Notice that in $H_*^m(\Omega)$ there is one normal derivative ∂_x every two tangential derivatives ∂_*. In the space $H_{**}^m(\Omega)$ every normal derivative admits one more tangential derivative than in $H_*^m(\Omega)$.

From (3.13) we see that $H_{**}^m(\Omega)$ is a good space for u in the sense that we have $u(t,) \in H_{**}^2(\Omega)$ with $\partial_t u(t,) \in H_*^1(\Omega)$.

Notice that the normal regularity of u follows from the first equation in (3.5), that we can write as $u_x = -(u_t + \sigma v_x + v_y)$. More generally, recalling that P denotes the orthogonal projection onto $ker\, A_\nu(x,t)^\perp$, the normal regularity of the noncharacteristic component $P\begin{pmatrix} u \\ v \end{pmatrix}$ follows by inverting in the equations the nonsingular part of the boundary matrix $A_\nu(x,t)$, in a neighborhood of the boundary.

On the other hand, the good choice for v is the space $H_*^m(\Omega)$. In fact, the second order tangential derivatives are already estimated in (3.13). It rests to estimate the first order normal derivative v_x.

We differentiate the second equation in (3.5) w.r.t. x and obtain the transport-type equation

$$\partial_t v_x + \sigma\partial_x v_x + \sigma' v_x = -(\sigma' u_x + \sigma\partial_x u_x + u_{xy}). \tag{3.14}$$

We notice that no boundary condition is needed for (3.14) because $\sigma(0) = 0$. We also observe that the right-hand side has already been estimated. By multiplying (3.14) by v_x and integrating over Ω, plus an integration by parts, we get an estimate for v_x in $L^2(\Omega)$. Thus we have obtained $v(t,) \in H^2_*(\Omega)$ with $\partial_t v(t,) \in H^1_*(\Omega)$.

More generally, applying the projection $I - P$ to the differential equations in (3.5) gives a transport-type equation for the normal derivatives of the characteristic component $(I - P)\begin{pmatrix} u \\ v \end{pmatrix}$, with vanishing boundary matrix (no need of a boundary condition) and right-hand side estimated at previous step. Then, an energy argument gives the a priori estimate. A similar strategy will be employed in Section 3.3.

The above analysis gives an answer to the problem set at the beginning of Section 2 of determining a function space X characterized by the property of persistence of regularity, that is such that

$$(u_0, v_0) \in X \Rightarrow (u(t,), v(t,)) \in X, \quad \forall t > 0\,.$$

The function space X is characterized as follows:

(i) If $(u_0, v_0) \in H^m_*(\Omega) \times H^m_*(\Omega)$ are such that $(\partial^k_t u(0,), \partial^k_t v(0,)) \in H^{m-k}_*(\Omega) \times H^{m-k}_*(\Omega)$, for $k = 1, \ldots, m$, then $(u(t,), v(t,)) \in H^m_*(\Omega) \times H^m_*(\Omega)$ with $(\partial^k_t u(t,), \partial^k_t v(t,)) \in H^{m-k}_*(\Omega) \times H^{m-k}_*(\Omega)$, for $k = 1, \ldots, m$, $\forall t > 0$.

Alternatively one may require (using the equations one can prove that (i) and (ii) are equivalent):

(ii) If $u_0 \in H^m_{**}(\Omega), v_0 \in H^m_*(\Omega)$ are such that $\partial^k_t u(0,) \in H^{m-k}_{**}(\Omega), \partial^k_t v(0,) \in H^{m-k}_*(\Omega)$, for $k = 1, \ldots, m$, then $u(t,) \in H^m_{**}(\Omega), v(t,) \in H^m_*(\Omega)$ with $\partial^k_t u(t,) \in H^{m-k}_{**}(\Omega), \partial^k_t v(t,) \in H^{m-k}_*(\Omega)$, for $k = 1, \ldots, m$, $\forall t > 0$.

CHAPTER 4

Regularity for characteristic symmetric IBVP's

1. Problem of regularity and main result

We consider an initial-boundary value problem for a linear Friedrichs symmetrizable system, with characteristic boundary of constant multiplicity. It is well-known that for solutions of symmetric or symmetrizable hyperbolic systems with characteristic boundary the full regularity (i.e. solvability in the usual Sobolev spaces H^m) cannot be expected generally because of the possible loss of derivatives in the normal direction to the boundary, see [42, 71] and Chapter 3.
The natural space is the anisotropic Sobolev space H^m_*, which comes from the observation that the one order gain of normal differentiation should be compensated by two order loss of tangential differentiation (cf. [9]). The theory has been developed mostly for characteristic boundaries of *constant multiplicity* (see Definition 1.2 or the definition in assumption (B)) and *maximally non-negative* boundary conditions, see Definition 1.5 and [9, 21, 44, 51, 53, 54, 55, 62].
However, there are important characteristic problems of physical interest where boundary conditions are not maximally non-negative. Under the more general *Kreiss–Lopatinskiĭ condition* (KL), see Appendix B, the theory has been developed for problems satisfying the *uniform* KL condition with *uniformly* characteristic boundaries (when the boundary matrix has constant rank in a neighborhood of the boundary), see [4, 30] and references therein.
In this chapter we are interested in the problem of the regularity. We assume the existence of the strong L^2-solution, satisfying a suitable energy estimate, without assuming any structural assumption sufficient for existence, such as the fact that the boundary conditions are maximally dissipative or satisfy the Kreiss–Lopatinskiĭ condition. We show that this is enough in order to get the regularity of solutions, in the natural framework of weighted anisotropic Sobolev spaces H^m_*, provided the data are sufficiently smooth. Obviously, the present results contain in particular what has been previously obtained for maximally nonnegative boundary conditions.

Let Ω be an open bounded subset of $\mathbb{R}^n$ (for a fixed integer $n \geq 2$), lying locally on one side of its smooth, connected boundary $\Gamma := \partial\Omega$. For any real $T > 0$, we set $Q_T := \Omega\times]0,T[$ and $\Sigma_T := \Gamma\times]0,T[$; we also define $Q_\infty := \Omega \times [0,+\infty[$, $\Sigma_\infty := \partial\Omega \times [0,+\infty[$, $Q := \Omega \times \mathbb{R}$ and $\Sigma := \partial\Omega \times \mathbb{R}$. We are interested in the following IBVP

$$Lu = F, \quad \text{in } Q_T \tag{4.1}$$
$$Mu = G, \quad \text{on } \Sigma_T \tag{4.2}$$
$$u_{|t=0} = f, \quad \text{in } \Omega, \tag{4.3}$$

where L is the first order linear partial differential operator

$$L = \partial_t + \sum_{i=1}^{n} A_i(x,t)\partial_i + B(x,t), \tag{4.4}$$

$\partial_t := \frac{\partial}{\partial t}$, $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$ and $A_i(x,t), B(x,t)$ are $N \times N$ real matrix-valued functions of (x,t), for a given integer size $N \geq 1$, defined over Q_∞. The unknown $u = u(x,t)$ and the data $F = F(x,t)$, $f = f(x)$ are real vector-valued functions with N components, defined on $\overline{Q}_T$ and $\overline{\Omega}$ respectively. In the boundary conditions (4.2), M is a smooth $d \times N$ matrix-valued function of (x,t), defined on Σ_∞, with maximal constant rank d. The boundary datum $G = G(x,t)$ is a $d-$vector valued function, defined on $\overline{\Sigma}_T$.
Let us denote by $\nu(x) := (\nu_1(x), \dots, \nu_n(x))$ the unit outward normal to Γ at the point $x \in \Gamma$; then

$$A_\nu(x,t) = \sum_{i=1}^{n} A_i(x,t)\nu_i(x)\,, \quad (x,t) \in \Sigma_\infty\,, \tag{4.5}$$

is the *boundary matrix*. Let $P(x,t)$ be the orthogonal projection onto the orthogonal complement of $\ker A_\nu(x,t)$, denoted $\ker A_\nu(x,t)^\perp$; it is defined by

$$P(x,t) = \frac{1}{2\pi i}\int_{C(x,t)} (\lambda - A_\nu(x,t))^{-1}\, d\lambda\,, \quad (x,t) \in \Sigma_\infty\,, \tag{4.6}$$

where $C(x,t)$ is a closed rectifiable Jordan curve with positive orientation in the complex plane, enclosing all and only all non-zero eigenvalues of $A_\nu(x,t)$. Denoting again by P an arbitrary smooth extension on $\overline{Q}_\infty$ of the above projection, Pu and $(I-P)u$ are called respectively the *noncharacteristic* and the *characteristic* components of the vector field $u = u(x,t)$. Examples of projector P for problems of physical interest are given in Appendix A. See also (3.2), (3.3) in Section 1.
We study the problem (4.1)-(4.3) under the following assumptions:

(A) The operator L is *Friedrichs symmetrizable*, namely for all $(x,t) \in \overline{Q}_\infty$ there exists a symmetric positive definite matrix $S_0(x,t)$ such that the matrices $S_0(x,t)A_i(x,t)$, for $i = 1, \cdots, n$, are also real symmetric; this implies, in particular, that the *symbol* $A(x,t,\xi) = \sum_{i=1}^{n} A_i(x,t)\xi_i$ is diagonalizable with real eigenvalues, whenever $(x,t,\xi) \in \overline{Q}_\infty \times \mathbb{R}^n$.

(B) The boundary is *characteristic, with constant multiplicity*, namely the boundary matrix A_ν is singular on Σ_∞ and has constant rank $0 < r := \operatorname{rank} A_\nu(x,t) < N$ for all $(x,t) \in \Sigma_\infty$; this assumption, together with the symmetrizability of L and that Γ is connected, yields that the number of negative eigenvalues of A_ν (the so-called *incoming modes*) remains constant on Σ_∞.

(C) $\ker A_\nu(x,t) \subseteq \ker M(x,t)$, for all $(x,t) \in \Sigma_\infty$; moreover $d = \operatorname{rank} M(x,t)$ must equal the number of negative eigenvalues of $A_\nu(x,t)$.

(D) The orthogonal projection $P(x,t)$ onto $\ker A_\nu(x,t)^\perp$, $(x,t) \in \Sigma_\infty$, can be extended as a matrix-valued $\mathcal{C}^\infty$ function over $\overline{Q}_\infty$.

Concerning the solvability of the IBVP (4.1)-(4.3), we state the following well-posedness assumption:

(E) *Existence of the* L^2-*weak solution.* Assume that $S_0, A_i \in Lip(\overline{Q}_\infty)$, for $i = 1, \ldots, n$. For all $T > 0$ and all matrices $B \in L^\infty(\overline{Q}_T)$, there exist constants $\gamma_0 \geq 1$ and $C_0 > 0$ such that for all $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$, $f \in L^2(\Omega)$ there exists a unique solution $u \in L^2(Q_T)$ of (4.1)-(4.3), with data (F, G, f), satisfying the following properties:
   i. $u \in C([0,T]; L^2(\Omega))$;
   ii. $Pu_{|\,\Sigma_T} \in L^2(\Sigma_T)$;
   iii. for all $\gamma \geq \gamma_0$ and $0 < \tau \leq T$ the solution u enjoys the following a priory estimate

$$\begin{aligned} & e^{-2\gamma\tau}||u(\tau)||^2_{L^2(\Omega)} + \gamma \textstyle\int_0^\tau e^{-2\gamma t}||u(t)||^2_{L^2(\Omega)}\, dt \\ & + \textstyle\int_0^\tau e^{-2\gamma t}||Pu_{|\partial\Omega}(t)||^2_{L^2(\partial\Omega)}\, dt \\ & \leq C_0 \left(||f||^2_{L^2(\Omega)} + \textstyle\int_0^\tau e^{-2\gamma t}(\frac{1}{\gamma}||F(t)||^2_{L^2(\Omega)} + ||G(t)||^2_{L^2(\partial\Omega)})\, dt \right) . \end{aligned} \tag{4.7}$$

When the IBVP (4.1)-(4.3) admits an a priori estimate of type (4.7), with $F = Lu$, $G = Mu$, for all $\tau > 0$ and all sufficiently smooth functions u, one says that the problem is *strongly* L^2 *well posed*, see e.g. [4]. A necessary condition for (4.7) is the validity of the *uniform Kreiss–Lopatinskiĭ condition* (UKL) (see Appendix B, an estimate of type (4.7) has been obtained by Rauch [45]). On the other hand, UKL is not sufficient for the well posedness and other structural assumptions have to be taken into account, see Appendix C and [4].
Finally, we require the following technical assumption that for $\mathcal{C}^\infty$ approximations of problem (4.1)-(4.3) one still has the existence of L^2-solutions. This stability property holds true for maximally nonnegative boundary conditions and for uniform KL conditions.

(F) Given matrices $(S_0, A_i, B) \in \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$, where $\sigma \geq [(n+1)/2] + 4$, enjoying properties (A)-(E), let $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$ be $\mathcal{C}^\infty$ matrix-valued functions converging to (S_0, A_i, B) in $\mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$ as $k \to \infty$, and satisfying properties (A)-(D). Then, for k sufficiently large, property (E) holds also for the approximating problems with coefficients $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$.

The solution of (4.1)-(4.3), considered in the statements (E), (F), must be intended in the sense of Rauch [46]. This means that for all $v \in H^1(Q_T)$ such that $v_{|\,\Sigma_T} \in (A_\nu(\ker M))^\perp$ and $v(T, \cdot) = 0$ in Ω, there holds:

$$\int_0^T \langle u(t), L^* v(t)\rangle\, dt = \int_0^T \langle F(t), v(t)\rangle\, dt - \int_{\Sigma_T} \langle A_\nu g, v\rangle\, d\sigma_x\, dt + \int_\Omega \langle f, v(0)\rangle\, dx,$$

where L^* is the adjoint operator of L and g is a function defined on Σ_T such that $Mg = G$. Notice also that for such a weak solution to (4.1)-(4.3), the boundary condition (4.2) makes sense. Indeed, in [46, Theorem 1] it is shown that for any $u \in L^2(Q_T)$, with $Lu \in L^2(Q_T)$, the trace of $A_\nu u$ on Σ_T exists in $H^{-1/2}(\Sigma_T)$. Moreover, for a given boundary matrix $M(x,t)$ satisfying assumption (C), there exists another matrix $M_0(x,t)$ such that $M(x,t) = M_0(x,t)A_\nu(x,t)$ for all $(x,t) \in$

Σ_∞. Therefore, for L^2-solutions of (4.1) one has

$$Mu = G \quad \text{on } \Sigma_T \iff M_0 A_\nu u_{|\Sigma_T} = G \quad \text{on } \Sigma_T. \tag{4.8}$$

In order to study the regularity of solutions to the IBVP (4.1)-(4.3), the data F, G, f need to satisfy some compatibility conditions. The compatibility conditions are defined in the usual way (see [48]). Given the IBVP (4.1)-(4.3), we recursively define $f^{(h)}$ by formally taking $h-1$ time derivatives of $Lu = F$, solving for $\partial_t^h u$ and evaluating it at $t = 0$; for $h = 0$ we set $f^{(0)} := f$. The *compatibility condition* of order $k \geq 0$ for the IBVP reads as

$$\sum_{h=0}^{p} \binom{p}{h} (\partial_t^{p-h} M)_{|\,t=0} f^{(h)} = \partial_t^h G_{|\,t=0}\,, \quad \text{on } \Gamma\,,\ p = 0, \dots, k\,. \tag{4.9}$$

In the framework of the preceding assumptions, we are able to prove the following theorem.

THEOREM 4.1. *[37] Let $m \in \mathbb{N}$ and $s = \max\{m, \big[(n+1)/2\big] + 5\}$. Assume that $S_0, A_i \in \mathcal{C}_T(H^s_*)$, for $i = 1, \dots, n$, and $B \in \mathcal{C}_T(H^{s-1}_*)$ (or $B \in \mathcal{C}_T(H^s_*)$ if $m = s$). Assume also that problem* (4.1)-(4.3) *obeys the assumptions* (A)-(F). *Then for all $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, $f \in H^m_*(\Omega)$, with $f^{(h)} \in H^{m-h}_*(\Omega)$ for $h = 1, \dots, m$, satisfying the compatibility condition* (4.9) *of order $m-1$, the unique solution u to* (4.1)–(4.3)*, with data (F, G, f), belongs to $\mathcal{C}_T(H^m_*)$ and $Pu_{|\,\Sigma_T} \in H^m(\Sigma_T)$. Moreover u satisfies the a priori estimate*

$$\begin{aligned} &||u||_{\mathcal{C}_T(H^m_*)} + ||Pu_{|\,\Sigma_T}||_{H^m(\Sigma_T)} \\ &\leq C_m \left(|||f|||_{m,*} + ||F||_{H^m_*(Q_T)} + ||G||_{H^m(\Sigma_T)} \right), \end{aligned} \tag{4.10}$$

with a constant $C_m > 0$ depending only on A_i, B.

The function spaces involved in the statement above (cf. also the assumption (F)), and the norms appearing in the energy estimate (4.10) are introduced in the next section.

2. Function spaces

For every integer $m \geq 1$, $H^m(\Omega)$, $H^m(Q_T)$ denote the usual Sobolev spaces of order m, over Ω and Q_T respectively.
In order to define the anisotropic Sobolev spaces, first we need to introduce the differential operators in *tangential direction*. Throughout the paper, for every $j = 1, 2, \dots, n$ the differential operator Z_j is defined by

$$Z_1 := x_1 \partial_1\,, \quad Z_j := \partial_j\,, \text{ for } j = 2, \dots, n\,.$$

Then, for every multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the tangential differential operator Z^α of order $|\alpha| = \alpha_1 + \dots + \alpha_n$ is defined by setting

$$Z^\alpha := Z_1^{\alpha_1} \dots Z_n^{\alpha_n}$$

(we also write, with the standard multi-index notation, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$).
We denote by $\mathbb{R}^n_+$ the $n-$dimensional positive half-space $\mathbb{R}^n_+ := \{x = (x_1, x') \in \mathbb{R}^n : x_1 > 0 \ \ x' := (x_2, \dots, x_n) \in \mathbb{R}^{n-1}\}$. For every positive integer m, the *tangential* (or

conormal) Sobolev space $H^m_{tan}(\mathbb{R}^n_+)$ and the *anisotropic* Sobolev space $H^m_*(\mathbb{R}^n_+)$ are defined respectively by:

$$H^m_{tan}(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) \,:\, Z^\alpha w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| \leq m\}\,, \tag{4.11}$$

$$H^m_*(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) \,:\quad Z^\alpha \partial_1^k w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| + 2k \leq m\}\,, \tag{4.12}$$

and equipped respectively with norms

$$||w||^2_{H^m_{tan}(\mathbb{R}^n_+)} := \sum_{|\alpha|\leq m} ||Z^\alpha w||^2_{L^2(\mathbb{R}^n_+)}\,, \tag{4.13}$$

$$||w||^2_{H^m_*(\mathbb{R}^n_+)} := \sum_{|\alpha|+2k\leq m} ||Z^\alpha \partial_1^k w||^2_{L^2(\mathbb{R}^n_+)}\,. \tag{4.14}$$

To extend the definition of the above spaces to an open bounded subset Ω of $\mathbb{R}^n$ (fulfilling the assumptions made at the beginning of the previous section), we proceed as follows. First, we take an open covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ such that $U_j \cap \overline{\Omega}$, $j = 1, \ldots, l$, are diffeomorphic to $\overline{\mathbb{B}}_+ := \{x_1 \geq 0, |x| < 1\}$, with Γ corresponding to $\partial\mathbb{B}_+ := \{x_1 = 0, |x| < 1\}$, and $U_0 \subset\subset \Omega$. Next we choose a smooth partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to the covering $\{U_j\}_{j=0}^l$. We say that a distribution u belongs to $H^m_{tan}(\Omega)$, if and only if $\psi_0 u \in H^m(\mathbb{R}^n)$ and, for all $j = 1, \ldots, l$, $\psi_j u \in H^m_{tan}(\mathbb{R}^n_+)$, in local coordinates in U_j. The space $H^m_{tan}(\Omega)$ is provided with the norm

$$||u||^2_{H^m_{tan}(\Omega)} := ||\psi_0 u||^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^l ||\psi_j u||^2_{H^m_{tan}(\mathbb{R}^n_+)}. \tag{4.15}$$

The anisotropic Sobolev space $H^m_*(\Omega)$ is defined in a completely similar way as the set of distributions u in Ω such that $\psi_0 u \in H^m(\mathbb{R}^n)$ and $\psi_j u \in H^m_*(\mathbb{R}^n_+)$, in local coordinates in U_j, for all $j = 1, \ldots, l$; it is provided with the norm

$$||u||^2_{H^m_*(\Omega)} := ||\psi_0 u||^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^l ||\psi_j u||^2_{H^m_*(\mathbb{R}^n_+)}\,. \tag{4.16}$$

The definitions of $H^m_{tan}(\Omega)$ and $H^m_*(\Omega)$ do not depend on the choice of the coordinate patches $\{U_j\}_{j=0}^l$ and the corresponding partition of unity $\{\psi_j\}_{j=0}^l$, and the norms arising from different choices of U_j, ψ_j are equivalent.

For an extensive study of the anisotropic Sobolev spaces, we refer the reader to [37, 43, 44, 53, 58, 63]; here we just remark that the continuous imbeddings

$$\begin{aligned}
&H^m_{tan}(\Omega) \hookrightarrow H^p_{tan}(\Omega)\,, \quad H^m_*(\Omega) \hookrightarrow H^p_*(\Omega)\,, \quad \forall m \geq p \geq 1\,,\\
&H^m(\Omega) \hookrightarrow H^m_*(\Omega) \hookrightarrow H^m_{tan}(\Omega)\,, \quad \forall m \geq 1\,,\\
&H^m_*(\Omega) \hookrightarrow H^{[m/2]}(\Omega)\,, \quad H^1_*(\Omega) = H^1_{tan}(\Omega)
\end{aligned} \tag{4.17}$$

hold true. For the sake of convenience, we also set $H^0_*(\Omega) = H^0_{tan}(\Omega) = L^2(\Omega)$. The spaces $H^m_{tan}(\Omega)$, $H^m_*(\Omega)$, endowed with their norms (4.15), (4.16), become Hilbert spaces. Analogously, we define the spaces $H^m_{tan}(Q_T)$ and $H^m_*(Q_T)$.

Let $C^m([0,T];X)$ denote the set of all m-times continuously differentiable functions over $[0,T]$, taking values in a Banach space X. We define the spaces

$$\mathcal{C}_T(H^m_{tan}) := \bigcap_{j=0}^m C^j([0,T]; H^{m-j}_{tan}(\Omega))\,, \quad \mathcal{C}_T(H^m_*) := \bigcap_{j=0}^m C^j([0,T]; H^{m-j}_*(\Omega))\,,$$

equipped respectively with the norms

$$\begin{aligned}||u||^2_{\mathcal{C}_T(H^m_{tan})} &:= \sum_{j=0}^{m} \sup_{t\in[0,T]} ||\partial_t^j u(t)||^2_{H^{m-j}_{tan}(\Omega)}\,,\\ ||u||^2_{\mathcal{C}_T(H^m_*)} &:= \sum_{j=0}^{m} \sup_{t\in[0,T]} ||\partial_t^j u(t)||^2_{H^{m-j}_*(\Omega)}\,.\end{aligned} \tag{4.18}$$

For the initial datum f we set

$$|||f|||^2_{m,*} := \sum_{j=0}^{m} ||f^{(j)}||^2_{H^{m-j}_*(\Omega)}\,.$$

3. The scheme of the proof of Theorem 4.1

The proof of Theorem 4.1 is made of several steps.
In order to simplify the forthcoming analysis, hereafter we only consider the case when the operator L has smooth coefficients. For the general case of coefficients with the finite regularity prescribed in Theorem 4.1, we refer the reader to [37]; this case is treated by a reduction to the smooth coefficients case, based upon the stability assumption (F). Thus, from now on, we assume that S_0, A_i, B are given functions in $\mathcal{C}^\infty(\overline{Q}_\infty)$. Just for simplicity, we even assume that the coefficients A_i of L are symmetric matrices (in this case the matrix S_0 reduces to I_N, the identity matrix of size N); the case of a symmetrizable operator can be easily reduced to this one, just by the application of the symmetrizer S_0 to system (4.1) (see [37] for details).
Below, we introduce the new unknown $u_\gamma(x,t) := e^{-\gamma t}u(x,t)$ and the new data $F_\gamma(x,t) := e^{-\gamma t}F(x,t)$, $G_\gamma(x,t) = e^{-\gamma t}G(x,t)$. Then problem (4.1)-(4.3) becomes equivalent to

$$\begin{aligned}&(\gamma+L)u_\gamma = F_\gamma \quad \text{in } Q_T\,,\\ &Mu_\gamma = G_\gamma\,, \quad \text{on } \Sigma_T\,,\\ &u_{\gamma\,|\,t=0} = f\,, \quad \text{in } \Omega\,.\end{aligned} \tag{4.19}$$

Let us now summarize the main steps of the proof of Theorem 4.1.

1. We firstly consider the *homogeneous* IBVP

$$\begin{aligned}&(\gamma+L)u_\gamma = F_\gamma \quad \text{in}\, Q_T\,,\\ &Mu_\gamma = G_\gamma \quad \text{on}\, \Sigma_T\,,\\ &u_{\gamma\ |t=0} = 0 \quad \text{in}\, \Omega\,.\end{aligned} \tag{4.20}$$

   We study (4.20), by reducing it to a *stationary* boundary value problem (see (4.26)), for which we deduce the *tangential* regularity. From the tangential regularity of this stationary problem, we deduce the tangential regularity of the homogeneous problem (4.20) (see the next Theorem 4.2).
2. We study the general problem (4.19). The anisotropic regularity, stated in Theorem 4.1, is obtained in two steps.
   - 2.i Firstly, from the tangential regularity of problem (4.20) we deduce the *anisotropic* regularity of (4.19) at the order $m = 1$.
   - 2.ii Eventually, we obtain the anisotropic regularity of (4.19), at any order $m > 1$, by an induction argument.

3.1. The homogeneous IBVP, tangential regularity. In this section, we concentrate on the study of the tangential regularity of solutions to the IBVP (4.19), where the initial datum f is identically zero, and the compatibility conditions are fulfilled in a more restrictive form than the one given in (4.9). More precisely, we consider the *homogeneous* IBVP (4.20) where, for a given integer $m \geq 1$, we assume that the data F_γ, G_γ satisfy the following conditions:

$$\partial_t^h F_{\gamma\,|\,t=0} = 0\,, \quad \partial_t^h G_{\gamma\,|\,t=0} = 0\,, \quad h = 0, \dots, m-1\,. \tag{4.21}$$

One can prove that conditions (4.21) imply the compatibility conditions (4.9) of order $m-1$, in the case $f = 0$.

THEOREM 4.2. *Assume that A_i, B, for $i = 1, \dots, n$, are in $\mathcal{C}^\infty(\overline{Q}_\infty)$, and that problem* (4.20) *satisfies assumptions* (A)-(E)*; then for all $T > 0$ and $m \in \mathbb{N}$ there exist constants $C_m > 0$ and γ_m, with $\gamma_m \geq \gamma_{m-1}$, such that for all $\gamma \geq \gamma_m$, for all $F_\gamma \in H^m_{tan}(Q_T)$ and all $G_\gamma \in H^m(\Sigma_T)$ satisfying* (4.21) *the unique solution u_γ to* (4.20) *belongs to $H^m_{tan}(Q_T)$, the trace of Pu_γ on Σ_T belongs to $H^m(\Sigma_T)$ and the a priori estimate*

$$\gamma||u_\gamma||^2_{H^m_{tan}(Q_T)} + ||Pu_{\gamma|\,\Sigma_T}||^2_{H^m(\Sigma_T)} \leq C_m\Big(\frac{1}{\gamma}||F_\gamma||^2_{H^m_{tan}(Q_T)} + ||G_\gamma||^2_{H^m(\Sigma_T)}\Big) \tag{4.22}$$

is fulfilled.

The first step to prove Theorem 4.2 is reducing the original mixed *evolution* problem (4.20) to a *stationary* boundary value problem, where the time is allowed to span the whole real line and it is treated then as an additional tangential variable. To make this reduction, we extend the data F_γ, G_γ and the unknown u_γ of (4.20) to all positive and negative times, by following methods similar to those of [4, Ch.9]. In the sequel, for the sake of simplicity, we remove the subscript γ from the unknown u_γ and the data F_γ, G_γ.
Because of (4.21), we extend F and G through $]-\infty, 0]$, by setting them equal to zero for all negative times; then for times $t > T$, we extend them by "reflection", following Lions–Magenes [27, Theorem 2.2]. Let us denote by $\breve{F}$ and $\breve{G}$ the resulting extensions of F and G respectively; by construction, $\breve{F} \in H^m_{tan}(Q)$ and $\breve{G} \in H^m(\Sigma)$. As we did for the data, the solution u to (4.20) is extended to all negative times, by setting it equal to zero. To extend u also for times $t > T$, we exploit the assumption (E). More precisely, for every $T' > T$ we consider the mixed problem

$$\begin{aligned} &(\gamma + L)u = \breve{F}_{|\,]0,T'[} \quad \text{in } Q_{T'}\,,\\ &Mu = \breve{G}_{|\,]0,T'[}\,, \quad \text{on } \Sigma_{T'}\,,\\ &u_{|\,t=0} = 0\,, \quad \text{in } \Omega\,. \end{aligned} \tag{4.23}$$

Assumption (E) yields that (4.23) admits a unique solution $u_{T'} \in C([0,T']; L^2(\Omega))$, such that $Pu_{T'} \in L^2(\Sigma_{T'})$ and the energy estimate

$$\begin{aligned} &||u_{T'}(T')||^2_{L^2(\Omega)} + \gamma||u_{T'}||^2_{L^2(Q_{T'})} + ||Pu_{T'\,|\,\Sigma_{T'}}||^2_{L^2(\Sigma_{T'})}\\ &\leq C'\left(\tfrac{1}{\gamma}||\breve{F}_{|\,]0,T'[}||^2_{L^2(Q_{T'})}) + ||\breve{G}_{|\,]0,T'[}||^2_{L^2(\Sigma_{T'})}\right) \end{aligned} \tag{4.24}$$

is satisfied for all $\gamma \geq \gamma'$ and some constants $\gamma' \geq 1$ and $C' > 0$ depending only on T' (and the norms $||A_i||_{Lip(Q_{T'})}$, $||B||_{L^\infty(Q_{T'})}$).
From the uniqueness of the L^2-solution, we infer that for arbitrary $T'' > T' \geq T$

we have $u_{T''} = u_{T'}$ ($u_T := u$) over $]0, T'[$. Therefore, we may extend u beyond T, by setting it equal to the unique solution of (4.23) over $]0, T'[$ for all $T' > T$. Thus we define

$$\breve{u}(t) := \begin{cases} u_{T'}(t)\,, & \forall\, t \in]0, T'[\,,\ \forall T' > T\,, \\ 0\,, & \forall\, t < 0\,. \end{cases} \tag{4.25}$$

Since $\breve{u}$, $\breve{F}$, $\breve{G}$ are all identically zero for negative times, we can take arbitrary smooth extensions of the coefficients of the differential operator L and the boundary operator M (originally defined on Q_∞ and Σ_∞) on Q and Σ respectively, with the only care to preserve $\text{rank}A_\nu = r$ and $\text{rank}M = d$ and $\ker A_\nu \subset \ker M$ for all $t < 0$. These extensions, that we fix once and for all, are denoted again by A_i, B, M. Moreover, we denote by L the corresponding extension on Q of the differential operator (4.4).
By construction, we have that $\breve{u}$ solves the boundary value problem (BVP)

$$\begin{aligned} &(\gamma + L)u = \breve{F} \quad \text{in } Q\,, \\ &Mu = \breve{G}\,, \quad \text{on } \Sigma\,. \end{aligned} \tag{4.26}$$

Using the estimate (4.24), for all $T' > T$, and noticing that the extended data $\breve{F}$, $\breve{G}$, as well as the solution $\breve{u}$, vanish identically for large $t > 0$, we derive that $\breve{u}$ enjoys the following estimate

$$\gamma||\breve{u}||^2_{L^2(Q)} + ||P\breve{u}_{|\,\Sigma}||^2_{L^2(\Sigma)} \le \breve{C}\left(\frac{1}{\gamma}||\breve{F}||^2_{L^2(Q)} + ||\breve{G}||^2_{L^2(\Sigma)}\right)\,, \tag{4.27}$$

for all $\gamma \ge \breve{\gamma}$, and suitable constants $\breve{\gamma} \ge 1$, $\breve{C} > 0$.
For the sake of simplicity, in the sequel we remove the superscript from the unknown $\breve{u}$ and the data $\breve{F}$, $\breve{G}$ of (4.26).
The next step is to move from the BVP (4.26) to a similar BVP posed in the $(n+1)-$dimensional positive half-space $\mathbb{R}^{n+1}_+ := \{(x_1, x', t) : x_1 > 0,\ (x', t) \in \mathbb{R}^n\}$. To make this reduction into a problem in $\mathbb{R}^{n+1}_+$, we follow a standard localization procedure of the problem (4.26) near the boundary of the spatial domain Ω; this is done by taking a covering $\{U_j\}^l_{j=0}$ of $\overline{\Omega}$ and a partition of unity $\{\psi_j\}^l_{j=0}$ subordinate to this covering, as in Section 2. Assuming that each patch $U_j, j = 1, \dots, l$, is sufficiently small, we can write the resulting localized problem in the form

$$\begin{aligned} &(\gamma + L)u = F \quad \text{in } \mathbb{R}^{n+1}_+\,, \\ &Mu = G\,, \quad \text{on } \mathbb{R}^n\,. \end{aligned} \tag{4.28}$$

As a consequence of the localization, the data F and G of the problem (4.28) are functions in $H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $H^m(\mathbb{R}^n)$ respectively; without loss of generality, we may also assume that the forcing term F and the solution u are supported in the set $\overline{\mathbb{B}}_+ \times [0, +\infty[$, and the boundary datum G is supported in $\partial\mathbb{B}_+ \times [0, +\infty[$. In $(4.28)_1$, L is now a differential operator in $\mathbb{R}^{n+1}$ of the form

$$L = \partial_t + \sum_{i=1}^n A_i(x, t)\partial_i + B(x, t)\,, \tag{4.29}$$

where the coefficients A_i, B are matrix-valued functions of (x, t) belonging to $\mathcal{C}^\infty_{(0)}(\mathbb{R}^{n+1}_+)$, namely the space of the restrictions onto $\mathbb{R}^{n+1}_+$ of (matrix-valued) functions in $\mathcal{C}^\infty_0(\mathbb{R}^{n+1})$. Let us remark that the boundary matrix of (4.28) is now

$-A_{1\,|\,\{x_1=0\}}$, because the outward unit vector to the boundary is $\nu = (-1, 0, \ldots, 0)$. It is a crucial step that the previously described localization process can be performed in such a way that A_1 has the following block structure

$$A_1(x,t) = \begin{pmatrix} A_1^{I,I} & A_1^{I,II} \\ A_1^{II,I} & A_1^{II,II} \end{pmatrix}, \quad (x,t) \in \mathbb{R}^{n+1}_+ , \tag{4.30}$$

where $A_1^{I,I}, A_1^{I,II}, A_1^{II,I}, A_1^{II,II}$ are respectively $r \times r$, $r \times (N-r)$, $(N-r) \times r$, $(N-r) \times (N-r)$ sub-matrices. Moreover, $A_1^{I,I}(x,t)$ is invertible over the support of $u(x,t)$ and we have

$$A_1^{I,II} = 0, \quad A_1^{II,I} = 0, \quad A_1^{II,II} = 0, \quad \text{in } \{x_1 = 0\} \times \mathbb{R}^n_{x',t}. \tag{4.31}$$

In view of assumption (C), we may even assume that the matrix M in the boundary condition $(4.28)_2$ is just $M = (I_d, 0)$, where I_d is the identity matrix of size d. According to (4.30), let us decompose the unknown u as $u = (u^I, u^{II})$; then we have $Pu = (u^I, 0)$.
Following the arguments of [8], one can prove that a local counterpart of the global estimate (4.27), associated to the stationary problem (4.26), can be attached to the local problem (4.28). More precisely, there exist constants $C_0 > 0$ and $\gamma_0 \geq 1$ such that for all $\varphi \in L^2(\mathbb{R}^{n+1}_+)$, supported in $\overline{\mathbb{B}}_+ \times [0, +\infty[$, such that $L\varphi \in L^2(\mathbb{R}^{n+1}_+)$ and $\gamma \geq \gamma_0$ there holds

$$\begin{aligned} &\gamma ||\varphi||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||\varphi^I_{|\,\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)} \\ &\leq C_0 \big(\tfrac{1}{\gamma} ||(\gamma + L)\varphi||^2_{L^2(\mathbb{R}^{n+1}_+)} + ||M\varphi_{|\,\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)}\big) . \end{aligned} \tag{4.32}$$

3.1.1. *Regularity of the stationary problem* (4.28). The analysis performed in the previous section shows that the tangential regularity of the homogeneous IBVP (4.20) can be deduced from the study of the regularity of the stationary BVP (4.28). For this stationary problem, we are able to show that if the data F and G belong to $H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $H^m(\mathbb{R}^n)$ respectively, and the L^2 a priori estimate (4.32) is fulfilled, then the L^2-solution of the problem (4.28) belongs to $H^m_{tan}(\mathbb{R}^{n+1}_+)$, the trace of its noncharacteristic part u^I belongs to $H^m(\mathbb{R}^n)$ and the estimate of order m

$$\gamma ||u||^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)} + ||u^I_{|\{x_1=0\}}||^2_{H^m(\mathbb{R}^n)} \leq C_m \Big(\frac{1}{\gamma} ||F||^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)} + ||G||^2_{H^m(\mathbb{R}^n)}\Big) \tag{4.33}$$

is satisfied with some constants $C_m > 0$, $\gamma_m \geq 1$ and for all $\gamma \geq \gamma_m$.
Then we recover the tangential regularity of the solution u to problem (4.26), posed on $Q = \Omega \times \mathbb{R}$, and we find an associated estimate of order m analogous to (4.33). Recalling that the solution u to (4.26) is the extension of the solution u_γ of the homogeneous IBVP (4.20), from the tangential regularity of u we can now derive the tangential regularity of u_γ, namely that $u_\gamma \in H^m_{tan}(Q_T)$ and $Pu_{\gamma\,|\,\Sigma_T} \in H^m(\Sigma_T)$. To get the energy estimate (4.22), we observe that the extended data $\breve{F}$ and $\breve{G}$ are defined in such a way that

$$||\breve{F}||_{H^m_{tan}(Q)} \leq C ||F_\gamma||_{H^m_{tan}(Q_T)}, \quad ||\breve{G}||_{H^m(\Sigma)} \leq C ||G_\gamma||_{H^m(\Sigma_T)} ,$$

with positive constant C independent of F_γ, G_γ and γ.

In order to prove the announced tangential regularity of the BVP (4.28), we adapt the classical technique of Friedrichs' mollifiers to our setting. More precisely, following Nishitani and Takayama [40], we introduce a “tangential” mollifier J_ε well suited to the tangential Sobolev spaces. Let χ be a function in $\mathcal{C}_0^\infty(\mathbb{R}^{n+1})$. For all $0<\varepsilon<1$, we set $\chi_\varepsilon(y):=\varepsilon^{-(n+1)}\chi(y/\varepsilon)$. We define $J_\varepsilon : L^2(\mathbb{R}^{n+1}_+)\to L^2(\mathbb{R}^{n+1}_+)$ by

$$J_\varepsilon w(x):=\int_{\mathbb{R}^{n+1}} w(x_1e^{-y_1},x'-y')e^{-y_1/2}\chi_\varepsilon(y)dy\,, \tag{4.34}$$

which differs from the one introduced in Rauch [46] by the factor $e^{-y_1/2}$. Using J_ε we follow the same lines in Tartakoff [67], Nishitani and Takayama [40] to get regularity of the weak solution u.
Starting from a classical characterization of the ordinary Sobolev spaces given in [23, Theorem 2.4.1], the following characterization of tangential Sobolev spaces $H^m_{tan}(\mathbb{R}^{n+1}_+)$ by means of J_ε can be proved.

Proposition 4.3. *Assume that $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^{n+1}_+)$ satisfies the following conditions:*

$$\widehat{\chi}(\xi)=O(|\xi|^p)\quad as\ \xi\to 0,\quad for\ some\ p\in\mathbb{N}; \tag{4.35}$$

$$\widehat{\chi}(t\xi)=0\,,\quad for\ all\ t\in\mathbb{R}\,,\quad implies\ \xi=0. \tag{4.36}$$

Then for all $m\in\mathbb{N}$ with $m<p$, we have that $u\in H^m_{tan}(\mathbb{R}^{n+1}_+)$ if and only if

a. $u\in H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)$;

b. $\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}$ is uniformly bounded for $0<\delta\le 1$.

In view of Proposition 4.3, showing that the solution $u\in H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)$ of (4.28) actually belongs to $H^m_{tan}(\mathbb{R}^{n+1}_+)$ amounts to provide a uniform bound, with respect to δ, for the integral quantity appearing in [b.], computed for the *mollified* solution $J_\varepsilon u$. To get this bound, the scheme is the following:

1. We notice that $J_\varepsilon u$ solves the following BVP

$$\begin{aligned}&(\gamma+L)J_\varepsilon u=J_\varepsilon F+[L,J_\varepsilon]u\,,\ \text{ in }\mathbb{R}^{n+1}_+\,,\\&MJ_\varepsilon u=G_\varepsilon\,,\quad\text{on }\mathbb{R}^n\,,\end{aligned} \tag{4.37}$$

   where $[L,J_\varepsilon]$ is the commutator between the operators L and J_ε, and G_ε is a suitable boundary datum that can be computed from the original datum G and the function χ_ϵ involved in (4.34) (see [37]).
2. Since the BVP (4.37) is the same as (4.28), with data $J_\varepsilon F+[L,J_\varepsilon]u$ and G_ε, the L^2 estimate (4.32) applied to (4.37) gives that the L^2-norm of $J_\varepsilon u$ can be estimated by

$$\begin{aligned}&\gamma||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}+||J_\varepsilon u^I_{|\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)}\\&\le C_0\left(\frac{1}{\gamma}||J_\varepsilon F+[L,J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}+||G_\varepsilon||^2_{L^2(\mathbb{R}^n)}\right)\,.\end{aligned} \tag{4.38}$$

3. From the preceding estimate, we immediately derive, for the integral quantity in [b.] and the analogous integral quantity associated to the trace of

noncharacteristic part of the solution, the following bound

$$\begin{aligned}
&\gamma \int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&+\int_0^1 ||J_\varepsilon u^I_{|\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\le C_0(\frac{1}{\gamma}\int_0^1 ||J_\varepsilon F||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&+\frac{1}{\gamma}\int_0^1 ||[L,J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&+\int_0^1 ||G_\varepsilon||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}).
\end{aligned}\tag{4.39}$$

Since $F\in H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $G\in H^m(\mathbb{R}^n)$, the first and the last integrals in the right-hand side of (4.39) can be estimated by $||F||^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}$ and $||G||^2_{H^m(\mathbb{R}^n)}$ respectively.
It remains to provide a uniform estimate for the middle integral involving the commutator $[L,J_\varepsilon]u$. For this term we get the following estimate

$$\begin{aligned}
&\int_0^1 ||[L,J_\varepsilon]u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\le C\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&+C\gamma^2||u||^2_{H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)}+C||F||^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}\,.
\end{aligned}\tag{4.40}$$

The estimate (4.40) is obtained by treating separately the different contributions to the commutator $[L,J_\varepsilon]$ associated to the different terms in the expression (4.29) of L (see [37] for details). The terms of the commutator involving the tangential derivatives $[A_i\partial_i,J_\varepsilon]$, for $i=2,\dots,n$ (note that $[\partial_t,J_\varepsilon]=0$), and the zero-th order term $[B,J_\varepsilon]$ are estimated by using [40, Lemma 9.2]. The term $[A_1\partial_1,J_\varepsilon]$, involving the normal derivative ∂_1, needs a more careful analysis; to estimate it, it is essential to make use of the structure (4.30), (4.31) of the boundary matrix in (4.28). Actually, by inverting $A_1^{I,I}$ in $(4.28)_1$, we can write $\partial_1 u^I$ as the sum of *space-time* tangential derivatives by

$$\partial_1 u^I=\Lambda Zu+R\,,$$

where

$$\begin{aligned}
\Lambda Zu&=-(A_1^{I,I})^{-1}\big[(\partial_t u^I+\sum_{j=2}^n A_jZ_ju)^I+A_1^{I,II}\partial_1u^{II}\big],\\
R&=(A_1^{I,I})^{-1}(F-\gamma u-Bu)^I.
\end{aligned}$$

Here, we use the fact that, if a matrix A vanishes on $\{x_1=0\}$, we can write $A\partial_1 u=HZ_1u$, where H is a suitable matrix; this trick transforms some normal derivatives into tangential derivatives.

Combining the inequalities (4.39) and (4.40), and arguing by finite induction on m to estimate $||u||_{H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)}$ in the right-hand side of (4.40), we get the desired uniform bounds of the integrals

$$\int_0^1 ||J_\varepsilon u||^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\,,$$
$$\int_0^1 ||J_\varepsilon u^I_{|\{x_1=0\}}||^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\,,$$

appearing in the left-hand side of (4.39). From this, in view of Proposition 4.3 and [23, Theorem 2.4.1], we conclude that $u\in H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $u^I\in H^m(\mathbb{R}^n)$. The a priori estimate (4.33) is deduced from (4.39), by following the same arguments.

3.2. The nonhomogeneous IBVP, case $m=1$. For *nonhomogeneous* IBVP, we mean the problem (4.1)-(4.3) where the initial datum f is different from zero. As announced before, we firstly prove the statement of Theorem 4.1 for $m=1$, namely we prove that, under the assumptions (A)-(F), for all $F\in H^1_*(Q_T)$, $G\in H^1(\Sigma_T)$ and $f\in H^1_*(\Omega)$, with $f^{(1)}\in L^2(\Omega)$, satisfying the compatibility condition $M_{|t=0}f_{|\partial\Omega}=G_{|t=0}$, the unique solution u to (4.1)–(4.3), with data (F,G,f), belongs to $\mathcal{C}_T(H^1_*)$ and $Pu_{|\,\Sigma_T}\in H^1(\Sigma_T)$; moreover, there exists a constant $C_1>0$ such that u satisfies the a priori estimate

$$||u||_{\mathcal{C}_T(H^1_*)}+||Pu_{|\,\Sigma_T}||_{H^1(\Sigma_T)}\leq C_1\big(|||f|||_{1,*}+||F||_{H^1_*(Q_T)}+||G||_{H^1(\Sigma_T)}\big)\,. \qquad (4.41)$$

To this end, we approximate the data with regularized functions satisfying one more compatibility condition. In this regard we get the following result, for the proof of which we refer to [37] and the references therein.

LEMMA 4.4. *Assume that problem* (4.1)-(4.3) *obeys the assumptions* (A)-(E). *Let $F\in H^1_*(Q_T)$, $G\in H^1(\Sigma_T)$, $f\in H^1_*(\Omega)$, with $f^{(1)}\in L^2(\Omega)$, such that $M_{|t=0}f_{|\partial\Omega}=G_{|t=0}$. Then there exist $F_k\in H^3(Q_T)$, $G_k\in H^3(\Sigma_T)$, $f_k\in H^3(\Omega)$, such that $M_{|t=0}f_k=G_{k|t=0}$, $\partial_tM_{|t=0}f_k+M_{|t=0}f^{(1)}_k=\partial_tG_{k|t=0}$ on $\partial\Omega$, and such that $F_k\to F$ in $H^1_*(Q_T)$, $G_k\to G$ in $H^1(\Sigma_T)$, $f_k\to f$ in $H^1_*(\Omega)$, $f^{(1)}_k\to f^{(1)}$ in $L^2(\Omega)$, as $k\to+\infty$.*

Given the functions F_k,G_k,f_k as in Lemma 4.4, we first calculate through equation $Lu=F_k, u_{|t=0}=f_k$, the initial time derivatives $f^{(1)}_k\in H^2(\Omega)$, $f^{(2)}_k\in H^1(\Omega)$. Then we take a function $w_k\in H^3(Q_T)$ such that

$$w_{k|t=0}=f_k\,,\quad \partial_tw_{k|t=0}=f^{(1)}_k\,,\quad \partial^2_{tt}w_{k|t=0}=f^{(2)}_k\,.$$

Notice that this yields

$$(Lw_k)_{|t=0}=F_{k|t=0}\,,\quad \partial_t(Lw_k)_{|t=0}=\partial_tF_{k|t=0}\,. \qquad (4.42)$$

Now we look for a solution u_k of problem (4.1)-(4.3), with data F_k,G_k,f_k, of the form $u_k=v_k+w_k$, where v_k is solution to

$$\begin{array}{ll} Lv_k=F_k-Lw_k, & \text{in } Q_T\\ Mv_k=G_k-Mw_k, & \text{on } \Sigma_T\\ v_{k|t=0}=0, & \text{in } \Omega. \end{array} \qquad (4.43)$$

Let us denote again $u_{k\gamma} = e^{-\gamma t}u_k$, $v_{k\gamma} = e^{-\gamma t}v_k$ and so on. Then (4.43) is equivalent to

$$\begin{array}{ll} (\gamma + L)v_{k\gamma} = F_{k\gamma} - (\gamma + L)w_{k\gamma}, & \text{in } Q_T \\ Mv_{k\gamma} = G_{k\gamma} - Mw_{k\gamma}, & \text{on } \Sigma_T \\ v_{k\gamma|t=0} = 0, & \text{in } \Omega. \end{array} \tag{4.44}$$

We easily verify that (4.42) yields

$$(F_{k\gamma} - (\gamma + L)w_{k\gamma})_{|t=0} = 0\,, \quad \partial_t\left(F_{k\gamma} - (\gamma + L)w_{k\gamma}\right)_{|t=0} = 0$$

and $M_{|t=0}f_{k|\partial\Omega} = G_{k|t=0}$, $\partial_t M_{|t=0}f_{k|\partial\Omega} + M_{|t=0}f^{(1)}_{k|\partial\Omega} = \partial_t G_{k|t=0}$ yield

$$(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0, \quad \partial_t(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0.$$

Thus the data of problem (4.44) obey conditions (4.21) for $h = 0, 1$; then we may apply to (4.44) Theorem 4.2 for γ large enough and find $v_k \in H^2_{tan}(Q_T)$, with $Pv_{k|\Sigma_T} \in H^2(\Sigma_T)$. Accordingly, we infer that $u_k \in H^2_{tan}(Q_T) \hookrightarrow \mathcal{C}_T(H^1_*)$ and $Pu_{k|\Sigma_T} \in H^2(\Sigma_T)$. Moreover $u_k \in L^2(Q_T)$ solves

$$\begin{array}{ll} Lu_k = F_k, & \text{in } Q_T \\ Mu_k = G_k, & \text{on } \Sigma_T \\ u_{k|t=0} = f_k, & \text{in } \Omega. \end{array} \tag{4.45}$$

Arguing as in the previous section, we take a covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ and a related partition of unity $\{\psi_j\}_{j=0}^l$, and we reduce problem (4.45) into a corresponding problem posed in the positive half-space $\mathbb{R}^n_+$, with new data $F_k \in H^3(\mathbb{R}^n_+ \times]0,T[)$, $G_k \in H^3(\mathbb{R}^{n-1} \times]0,T[)$, $f_k \in H^3(\mathbb{R}^n_+)$, and boundary matrix $M = (I_d, 0)$.

We also write the similar problem solved by the first order derivatives $Zu_k = (Z_1u_k, \ldots, Z_{n+1}u_k) \in H^1_{tan}(Q_T) = H^1_*(Q_T)$ (where $Z_{n+1} = \partial_t$). Here a crucial remark regards the commutators of L and the tangential operators Z_i, see [37, 46]: there exist matrices $\Gamma_\beta, \Gamma_0, \Psi$ such that

$$[L, Z_i] = -\sum_{|\beta|=1}\Gamma_\beta Z^\beta + \Gamma_0 + \Psi L, \qquad i = 1, \ldots, n+1. \tag{4.46}$$

Therefore the commutators contain only tangential derivatives, and no normal derivative.

Since assumption (E) is satisfied, applying the a priori estimate (4.7) to a difference of solutions $u_h - u_k$ of those problems satisfied by the first order derivatives readily gives

$$\begin{aligned} &||u_k - u_h||_{\mathcal{C}_T(H^1_*)} + ||P(u_k - u_h)_{|\Sigma_T}||_{H^1(\Sigma_T)} \\ &\leq C\left(|||f_k - f_h|||_{1,*} + ||F_k - F_h||_{H^1_*(Q_T)} + ||G_k - G_h||_{H^1(\Sigma_T)}\right). \end{aligned}$$

From Lemma 4.4, we infer that $\{u_k\}$ is a Cauchy sequence in $\mathcal{C}_T(H^1_*)$ and $\{Pu_{k|\,\Sigma_T}\}$ is a Cauchy sequence in $H^1(\Sigma_T)$. Therefore there exists a function in $\mathcal{C}_T(H^1_*)$ which is the limit of $\{u_k\}$. Passing to the limit in (4.45) as $k \to \infty$, we see that this function is a solution to (4.1)-(4.3). The uniqueness of the L^2-solution yields $u \in \mathcal{C}_T(H^1_*)$ and $Pu_{|\,\Sigma_T} \in H^1(\Sigma_T)$. Applying the a priori estimate (4.7) to the solution u_k of (4.45) and its first order derivatives, and passing to the limit finally gives (4.41). This completes the proof of Theorem 4.1 for $m = 1$ in the case of $\mathcal{C}^\infty$ coefficients. As we already say, here we do not deal with the case of less regular coefficients, for which the reader is referred to [37, Sect. 5].

3.3. The nonhomogeneous IBVP, proof for $m \geq 2$. The proof proceeds by finite induction on m. Assume that Theorem 4.1 holds up to $m-1$. Let $f \in H^m_*(\Omega)$, $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, with $f^{(k)} \in H^{m-k}_*(\Omega)$, $k = 1, \cdots, m$, and assume also that the compatibility conditions (4.9) hold up to order $m-1$. By the inductive hypothesis there exists a unique solution u of the problem (4.1)-(4.3) such that $u \in \mathcal{C}_T(H^{m-1}_*)$.

In order to show that $u \in \mathcal{C}_T(H^m_*)$, we have to increase the regularity of u by order one, that is by one more tangential derivative and, if m is even, also by one more normal derivative. This can be done as in [51, 53], with the small change of the elimination of the auxiliary system (introduced in [51, 53]) as in [7, 55]. At every step we can estimate some derivatives of u through equations where in the right-hand side we can put other derivatives of u that have already been estimated at previous steps. The reason why the main idea in [51] works, even though here we do not have maximally nonnegative boundary conditions, is that for the increase of regularity we consider the system (4.50) of equations for purely tangential derivatives of the type of (4.1)-(4.3), where we can use the inductive assumption, and other systems (4.52), (4.53) of equations for mixed tangential and normal derivatives where the boundary matrix vanishes identically, so that no boundary condition is needed and we can apply an energy method, under the assumption of the symmetrizable system. Without entering in too many details we briefly describe the different steps of the proof, for the reader's convenience. It can be useful to compare this strategy with Section 3.

As before, we take a covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ and a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to this covering. Assuming that each patch $U_j, j = 1, \ldots, l$, is sufficiently small we can write the resulting localized problem in the form

$$\begin{array}{ll} Lu = F, & \text{in } \mathbb{R}^n_+ \times]0,T[, \\ Mu = G, & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\ u_{|t=0} = f, & \text{in } \mathbb{R}^n_+. \end{array} \tag{4.47}$$

with L as in (4.29), and $M = (I_d, 0)$. The boundary matrix $-A_1^j$ has the block form as in (4.30), (4.31). According to (4.30), let us decompose the unknown u as $u = (u^I, u^{II})$; then we have $Pu = (u^I, 0)$. Hereafter we will denote $Z = (Z_1, \ldots, Z_{n+1}), Z_{n+1} = \partial_t$.

3.4. Purely tangential regularity. Let us start by considering all the tangential derivatives $Z^\alpha u$, $|\alpha| = m-1$. We decompose $\partial_1 u = \begin{pmatrix} \partial_1 u^I \\ \partial_1 u^{II} \end{pmatrix}$. By inverting $A_1^{I,I}$ in $(4.47)_1$, we can write $\partial_1 u^I$ as the sum of tangential derivatives by

$$\partial_1 u^I = \Lambda Z u + R \tag{4.48}$$

where

$$\begin{aligned} \Lambda Z u &= (A_1^{I,I})^{-1} \big[(A_{n+1} Z_{n+1} u + \sum_{j=2}^n A_j Z_j u)^I + A_1^{I,II} \partial_1 u^{II} \big], \\ R &= (A_1^{I,I})^{-1} (Bu - F)^I. \end{aligned}$$

Here and below, everywhere it is needed, we use the fact that, if a matrix A vanishes on $\{x_1 = 0\}$, we can write $A\partial_1 u = H Z_1 u$, where H is a suitable matrix such that $||H||_{H^{s-2}_*(\Omega)} \le c||A||_{H^s_*(\Omega)}$, see [37, App. B]; this trick transforms some normal derivatives into tangential derivatives. We obtain $\Lambda \in \mathcal{C}_T(H^{s-2}_*)$.

Applying the operator Z^α to (4.47), with $\alpha = (\alpha', \alpha_{n+1})$, $\alpha' = (\alpha_1, \cdots, \alpha_n)$, and substituting (4.48) gives equation (5.3) in [51], that is

$$\begin{aligned} & L(Z^\alpha u) + \sum_{|\gamma|=|\alpha|-1} (ZA_{n+1}Z_{n+1} + \sum_{j=2}^n ZA_jZ_j)Z^\gamma u + \sum_{|\gamma|=|\alpha|-1} ZA_1 \begin{pmatrix} \Lambda Z(Z^\gamma u) \\ 0 \end{pmatrix} \\ & \quad -\alpha_1 A_1 \begin{pmatrix} \Lambda Z(Z_1^{\alpha_1-1}Z_2^{\alpha_2}\cdots Z_{n+1}^{\alpha_{n+1}}u) \\ 0 \end{pmatrix} \\ & +\Big(\sum_{|\gamma|=|\alpha|-1} ZA_1Z^\gamma - \alpha_1 A_1 Z_1^{\alpha_1-1}Z_2^{\alpha_2}\cdots Z_{n+1}^{\alpha_{n+1}} \Big) \begin{pmatrix} 0 \\ \partial_1 u^{II} \end{pmatrix} = F_\alpha, \end{aligned} \tag{4.49}$$

with $F_\alpha \in H^1_*(Q_T)$, see [51] for its explicit expression. Equation (4.49) takes the form $(L + \overline{B})Z^\alpha u = F_\alpha$ with $\overline{B} \in \mathcal{C}_T(H^{s-3}_*)$.

Then we consider the problem satisfied by the vector of all tangential derivatives $Z^\alpha u$ of order $|\alpha| = m - 1$. From (4.49) this problem takes the form

$$\begin{array}{ll} (\mathcal{L} + \mathcal{B})Z^\alpha u = \mathcal{F} & \text{in } \mathbb{R}^n_+ \times]0,T[, \\ \mathcal{M}Z^\alpha u = Z^\alpha G & \text{on } \{x_1 = 0\} \times \mathbb{R}^{n-1}_{x'} \times]0,T[, \\ Z^\alpha u|_{t=0} = \tilde{f} & \text{in } \mathbb{R}^n_+, \end{array} \tag{4.50}$$

where

$$\mathcal{L} = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix},$$

$\mathcal{B} \in \mathcal{C}_T(H^{s-3}_*)$ is a suitable linear operator and $\mathcal{F}$ is the vector of all right-hand sides F_α. The initial datum $\tilde{f}$ is the vector of functions $Z^{\alpha'} f^{(\alpha_{n+1})}$.

We have $\mathcal{F} \in H^1_*(Q_T)$, $\tilde{f} \in H^1_*$, $Z^\alpha G \in H^1(\Sigma_T)$. Moreover the data satisfy the compatibility conditions of order 0. We infer that the solution of (4.50) satisfies $Z^\alpha u \in \mathcal{C}_T(H^1_*)$, for all $|\alpha| = m - 1$.

3.5. Tangential and one normal derivatives. We apply to the part II of $(4.47)_1$ the operator $Z^\beta \partial_1$, with $|\beta| = m - 2$. We obtain equation (28) in [7], that is

$$\begin{aligned} & \big[(L + \partial_1 A_1)Z^\beta + \sum_{|\gamma|=|\beta|-1} (ZA_0\partial_t + \sum_{j=1}^n ZA_j\partial_j)Z^\gamma \\ & -\beta_1 A_1 \partial_1 Z_1^{\beta_1-1}Z_2^{\beta_2}\cdots Z_{n+1}^{\beta_{n+1}} \big]^{II,II} \partial_1 u^{II} = \mathcal{G}, \end{aligned} \tag{4.51}$$

where the exact expression of $\mathcal{G}$ may be found in [7]. It is important to observe that $\mathcal{G}$ contains only tangential derivatives of order at most m. Hence, we can estimate it by using the previous step and infer $\mathcal{G} \in L^2(Q_T)$. Using (4.48) again, we write (4.51) as

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}})Z^\beta \partial_1 u^{II} = \mathcal{G}, \tag{4.52}$$

where

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{L} & & \\ & \ddots & \\ & & \tilde{L} \end{pmatrix}$$

with $\tilde{L} = A_0^{II,II}\partial_t + \sum_{j=1}^n A_j^{II,II}\partial_j$ and where $\tilde{\mathcal{C}} \in \mathcal{C}_T(H_*^{s-2})$ is a suitable linear operator. Here a crucial point is that (4.52) is a transport-type equation, because the boundary matrix of $\tilde{\mathcal{L}}$ vanishes at $\{x_1 = 0\}$. Thus we do not need any boundary condition. We infer that equation (4.52) has a unique solution $Z^\beta \partial_1 u^{II} \in C_T(L^2) := C([0,T]; L^2(\mathbb{R}^n_+))$, for all $|\beta| = m-2$. Using (4.48) again, we deduce $Z^\beta \partial_1 u \in C_T(L^2)$, for all $|\beta| = m-2$.

3.6. Normal derivatives. The last step is again by induction, as in [51], page 867, (ii). For convenience of the reader, we provide a brief sketch of the proof.

Suppose that for some fixed k, with $1 \le k < [m/2]$, it has already been shown that $Z^\alpha \partial_1^h u$ belongs to $C_T(L^2)$, for any h and α such that $h = 1, \cdots, k$, $|\alpha| + 2h \le m$. From (4.48) it immediately follows that $Z^\alpha \partial_1^{k+1} u^I \in C_T(L^2)$. It rests to prove that $Z^\alpha \partial_1^{k+1} u^{II} \in C_T(L^2)$.

We apply operator $Z^\alpha \partial_1^{k+1}$, $|\alpha| + 2k = m-2$, to the part II of $(4.47)_1$ and obtain an equation similar to (4.52) of the form

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}}_k) Z^\alpha \partial_1^{k+1} u^{II} = \mathcal{G}_k, \tag{4.53}$$

where $\tilde{\mathcal{C}}_k \in \mathcal{C}_T(H_*^{s-3})$ is a suitable linear operator. The right-hand side $\mathcal{G}_k$ contains derivatives of u of order m (in H_*^m, i.e. counting 1 for each tangential derivative and 2 for normal derivatives), but contains only normal derivatives that have already been estimated. We infer $\mathcal{G}_k \in L^2(Q_T)$. Again it is crucial that the boundary matrix of $\tilde{\mathcal{L}}$ vanishes at $\{x_1 = 0\}$. We infer that the solution $Z^\alpha \partial_1^{k+1} u^{II}$ is in $C_T(L^2)$ for all α, k with $|\alpha| + 2k = m-2$. By repeating this procedure we obtain the result for any $k \le [m/2]$, hence $u \in \mathcal{C}_T(H_*^m)$.

The a priori estimate (4.10) follows from (4.41) (namely estimate (4.10) in case $m = 1$) applied to the solution of (4.50), plus standard L^2 energy estimates for equations (4.52) and (4.53), and the direct estimate of the normal derivative of u by tangential derivatives via (4.48). All products of functions are estimated in spaces H_*^m by using the properties of these spaces given in [37, App. B]. We refer the reader to [7, 37, 51, 53] for all details.

This concludes the proof of Theorem 4.1.

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APPENDIX A

The Projector P

In this Appendix we see in various examples of physical interest what is the form of the orthogonal projection P onto $(\ker A_\nu)^\perp$, and which is the price of the hypothesis $P \in \mathcal{C}^\infty$, taken in assumption (D) of Theorem 4.1, Section 1.

EXAMPLE A.1. *Consider the Euler equations* (1.4). *The boundary matrix is:*

$$A_\nu = \begin{pmatrix} (\rho_p/\rho) v\cdot\nu & \nu^T & 0 \\ \nu & \rho v\cdot\nu I_3 & \underline{0} \\ 0 & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

If $v\cdot\nu = 0$, *then*

$$\ker A_\nu = \{U' = (p', v', S') \,:\, p' = 0,\, v'\cdot\nu = 0\},$$

The projection onto $(\ker A_\nu)^\perp$ *is:*

$$P = \begin{pmatrix} 1 & \underline{0}^T & 0 \\ \underline{0} & \nu\otimes\nu & \underline{0} \\ 0 & \underline{0}^T & 0 \end{pmatrix},$$

and P *has the regularity of* ν: *if* $\partial\Omega \in \mathcal{C}^\infty$ *then* $P \in \mathcal{C}^\infty$.

EXAMPLE A.2. *Consider the ideal MHD equations* (1.6). *The boundary matrix is:*

$$A_\nu = \begin{pmatrix} (\rho_p/\rho) v\cdot\nu & \nu^T & -(\rho_p/\rho) H^T v\cdot\nu & 0 \\ \nu & \rho v\cdot\nu I_3 & -H\cdot\nu I_3 & \underline{0} \\ -(\rho_p/\rho) H v\cdot\nu & -H\cdot\nu I_3 & a_0 v\cdot\nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & v\cdot\nu \end{pmatrix}.$$

(i) If $v\cdot\nu = 0$, $H\cdot\nu = 0$, *then the projection* P *onto* $(\ker A_\nu)^\perp$ *is:*

$$P = \begin{pmatrix} 1 & \underline{0}^T & \underline{0}^T & 0 \\ \underline{0} & \nu\otimes\nu & 0_3 & \underline{0} \\ \underline{0} & 0_3 & 0_3 & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 0 \end{pmatrix},$$

and P *has the regularity of* ν: *if* $\partial\Omega \in \mathcal{C}^\infty$ *then* $P \in \mathcal{C}^\infty$.
(ii) If $H\cdot\nu = 0$ *and* $v\cdot\nu \neq 0$, $v\cdot\nu \neq \frac{|H|}{\sqrt{\rho}} \pm c(\rho)$, *then*

$$\ker A_\nu = \{0\}, \quad P = Id \in \mathcal{C}^\infty.$$

(Noncharacteristic boundary)
(iii) If $v\cdot\nu = 0$ *and* $H\cdot\nu \neq 0$, *then*

$$\ker A_\nu = \{v' = 0,\, \nu q' - H\cdot\nu H' = 0\},$$
$$(\ker A_\nu)^\perp = \{H\cdot\nu\, q' + H'\cdot\nu = 0,\, S' = 0\}$$

The projection onto $(\ker A_\nu)^\perp$ *is:*

$$P = \begin{pmatrix} \Lambda & \underline{0}^T & -\Lambda(H\cdot\nu)\nu^T & 0 \\ \underline{0} & I_3 & 0_3 & \underline{0} \\ -\Lambda(H\cdot\nu)\nu & 0_3 & I_3 - \Lambda\nu\otimes\nu & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 0 \end{pmatrix}.$$

where $\Lambda := [1 + (H\cdot\nu)^2]^{-1}$. *P has the (finite) regularity of* $H\cdot\nu$ *(for* $\partial\Omega \in \mathcal{C}^\infty$*), while we would need at least* $P \in \mathcal{C}^{m+1}$ *(this is probably the least that we can ask instead of assumption* (D) *of Theorem 4.1, i.e. in place of* $P \in \mathcal{C}^\infty$*). Therefore our method does not seem to be applicable in this case.*

It is interesting to notice that, in spite of that, this problem may have full regularity (solvability in H^m*), see Yanagisawa [72].*

APPENDIX B

Kreiss-Lopatinskiĭ condition

For the sake of simplicity, instead of an initial-boundary value problem, we consider the boundary value problem (BVP)

$$\begin{cases} Lu = F\,, & \text{in}\,\{x_1 > 0\}\,, \\ Mu = G\,, & \text{on}\,\{x_1 = 0\}\,. \end{cases} \tag{B.1}$$

where

$$L := \partial_t + \sum_{j=1}^{n} A_j \partial_{x_j}$$

is a hyperbolic operator (with eigenvalues of constant multiplicity); moreover $A_j \in \mathbf{M}_{N\times N}$, $j = 1, \dots, n$, with constant entries. For simplicity we assume that the boundary is noncharacteristic, i.e. $\det A_1 \neq 0$. We also assume that $M \in \mathbf{M}_{d\times N}$ with constant entries, $\operatorname{rank}(M) = d$ where d denotes the number of positive eigenvalues of the matrix A_1.

Let $u = u(x_1, x', t)$ ($x' = (x_2, \dots, x_n)$) be a solution to (B.1) for $F = 0$ and $G = 0$. Let $\widehat{u} = \widehat{u}(x_1, \eta, \tau)$ be Fourier-Laplace transform of u w.r.t. x' and t respectively (η and τ dual variables of x' and t respectively).

Then $\widehat{u}$ solves the ODE problem

$$\begin{cases} \frac{d\widehat{u}}{dx_1} = \mathcal{A}(\eta,\tau)\widehat{u}\,, & x_1 > 0\,, \\ M\widehat{u}(0) = 0\,, \end{cases} \tag{B.2}$$

where $\mathcal{A}(\eta,\tau) := -(A_1)^{-1}\left(\tau I_n + i\sum_{j=2}^{n} A_j\eta_j\right)$. Let $\mathcal{E}^-(\eta,\tau)$ denote the stable subspace of (B.2).

Definition B.1. Problem (B.1) satisfies the *Kreiss-Lopatinskiĭ condition* (KL) if:

$$\ker M \cap \mathcal{E}^-(\eta,\tau) = \{0\}, \quad \forall(\eta,\tau) \in \mathbb{R}^{n-1}\times\mathbb{C},\ \Re\tau > 0.$$

An equivalent formulation is the following.

Proposition B.2. *The Kreiss-Lopatinskiĭ condition holds if and only if*

$$\begin{aligned} &\forall(\eta,\tau) \in \mathbb{R}^{n-1}\times\mathbb{C},\ \Re\tau > 0,\ \exists C = C(\eta,\tau) > 0\ : \\ &|A_1 V| \le C|MV| \quad \forall V \in \mathcal{E}^-(\eta,\tau). \end{aligned}$$

When the constant in the above estimate is independent of (η,τ) we have the so-called uniform Kreiss-Lopatinskiĭ condition:

DEFINITION B.3. Problem (B.1) satisfies the *Uniform Kreiss-Lopatinskiĭ condition* (UKL) if:

$$\exists C > 0 : \ \forall (\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau > 0 :$$
$$|A_1 V| \leq C|MV| \quad \forall V \in \mathcal{E}^-(\eta,\tau).$$

An useful tool for checking whether (KL) or (UKL) holds is given by the *Lopatinskiĭ determinant*.

For all $(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C}$, $\Re\tau > 0$, let $\{X_1(\eta,\tau),\dots,X_d(\eta,\tau)\}$ be an orthonormal basis of $\mathcal{E}^-(\eta,\tau)$ ($\dim \mathcal{E}^-(\eta,\tau) = \operatorname{rank} M = d$).

The assumption that the eigenvalues have constant multiplicity yields that $X_j(\eta,\tau)$, $j = 1,\dots,d$, and $\mathcal{E}^-(\eta,\tau)$ can be extended to all $(\eta,\tau) \neq (0,0)$ with $\Re\tau = 0$.

DEFINITION B.4. The *Lopatinskiĭ determinant* is the determinant defined by

$$\Delta(\eta,\tau) := \det\left[M\left(X_1(\eta,\tau),\dots,X_d(\eta,\tau)\right)\right]$$
$$\forall(\eta,\tau) \in \mathbb{R}^{n-1} \times \mathbb{C},\ \Re\tau \geq 0.$$

PROPOSITION B.5. *The Kreiss-Lopatinskiĭ condition holds if and only if*

$$\Delta(\eta,\tau) \neq 0\,, \quad \forall \Re\tau > 0, \forall \eta \in \mathbb{R}^{n-1}\,.$$

The Uniform Kreiss-Lopatinskiĭ condition holds if and only if

$$\Delta(\eta,\tau) \neq 0\,, \quad \forall \Re\tau \geq 0, \forall \eta \in \mathbb{R}^{n-1}\,.$$

Below we summarize the relation between the Kreiss-Lopatinskiĭ condition and the well posedness of (B.1).

1. $\boxed{\det A_1 \neq 0}$ (i.e. noncharacteristic boundary)

- (KL) does NOT hold ⇒ (B.1) is ill posed in Hadamard's sense;
- (UKL) ⇔ L^2-strong well posedness of (B.1);
- (KL) holds but NOT (UKL) ⇒ Weak well posedness of (B.1) (energy estimate with possible loss of regularity?).

2. $\boxed{\det A_1 = 0}$ (i.e. characteristic boundary)

- (KL) does NOT hold ⇒ (B.1) is ill posed in Hadamard's sense;
- (UKL) + *structural assumptions* (see Appendix C) on L ⇒ L^2-strong well posedness of (B.1).

APPENDIX C

Structural assumptions for well-posedness

For more general boundary conditions than those maximally non-negative, the well posedness has been proven for symmetrizable hyperbolic systems under suitable structural assumptions. Instead of maximally non-negative boundary conditions, the theory deals with *uniform Kreiss-Lopatinskiĭ conditions* (UKL), see Appendix B. Moreover the boundary is assumed to be *uniformly characteristic*, see Definition 1.2.

The general theory has received major contributions by Majda and Osher [30], Ohkubo [41], Benzoni and Serre [4]. In the same framework we may also quote the papers about elasticity by Morando and Serre [35, 36]. We briefly recall these results.

(I) Majda and Osher [30] prove the well-posedness of (1.1) under the following assumptions:

- the operator L is symmetric hyperbolic, with variable coefficients,
- the boundary is uniformly characteristic,
- the uniform Kreiss-Lopatinskiĭ condition (UKL) holds,
- several technical assumptions on L and M, among which the symbol of L is such that:

$$A(\eta) := \sum_{j=2}^{n} A_j \eta_j = \begin{pmatrix} a_1(\eta) & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & a_2(\eta) \end{pmatrix}, \tag{C.1}$$

where $a_2(\eta)$ has *only simple* eigenvalues for $|\eta| = 1$. In (C.1) the block decomposition is as in (4.30), where $a_1(\eta)$ takes the place of the invertible part $A_1^{I,I}(x,t)$.

The above assumptions are satisfied in many interesting cases: strictly hyperbolic systems, MHD equations, Maxwell's equations, linearized shallow water equations. They are not satisfied by the 3D isotropic elasticity, where $a_2(\eta) = 0_3$.

(II) Benzoni-Gavage and Serre [4] prove the well-posedness of (1.1) under the following assumptions:

- L is symmetric hyperbolic, with constant coefficients, M is constant,
- the boundary is uniformly characteristic, and $\ker A_\nu \subset \ker M$,
- the uniform Kreiss-Lopatinskiĭ condition (UKL) holds,
- Instead of (C.1), one has

$$A(\eta) = \begin{pmatrix} a_1(\eta) & a_{2,1}(\eta)^T \\ a_{2,1}(\eta) & 0 \end{pmatrix} \tag{C.2}$$

with $a_1(\eta) = 0$. This is the case of Maxwell's equations and linearized acoustics. Unfortunately the above assumption (C.2) is not satisfied by isotropic elasticity, where $a_1(\eta) \neq 0$.

(III) The well posedness of linear isotropic elasticity in $2D$ and $3D$ has been shown by Morando and Serre [35, 36] by the construction *ad hoc* of a symbolic Kreiss symmetrizer.

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Jindřich Nečas

Jindřich Nečas was born in Prague on December 14th, 1929. He studied mathematics at the Faculty of Natural Sciences at the Charles University from 1948 to 1952. After a brief stint as a member of the Faculty of Civil Engineering at the Czech Technical University, he joined the Czechoslovak Academy of Sciences where he served as the Head of the Department of Partial Differential Equations. He held joint appointments at the Czechoslovak Academy of Sciences and the Charles University from 1967 and became a full time member of the Faculty of Mathematics and Physics at the Charles University in 1977. He spent the rest of his life there, a significant portion of it as the Head of the Department of Mathematical Analysis and the Department of Mathematical Modeling.

His initial interest in continuum mechanics led naturally to his abiding passion to various aspects of the applications of mathematics. He can be rightfully considered as the father of modern methods in partial differential equations in the Czech Republic, both through his contributions and through those of his numerous students. He has made significant contributions to both linear and non-linear theories of partial differential equations. That which immediately strikes a person conversant with his contributions is their breadth without the depth being compromised in the least bit. He made seminal contributions to the study of Rellich identities and inequalities, proved an infinite dimensional version of Sard's Theorem for analytic functionals, established important results of the type of Fredholm alternative, and most importantly established a significant body of work concerning the regularity of partial differential equations that had a bearing on both elliptic and parabolic equations. At the same time, Nečas also made important contributions to rigorous studies in mechanics. Notice must be made of his work, with his collaborators, on the linearized elastic and inelastic response of solids, the challenging field of contact mechanics, a variety of aspects of the Navier–Stokes theory that includes regularity issues as well as important results concerning transonic flows, and finally non-linear fluid theories that include fluids with shear-rate dependent viscosities, multi-polar fluids, and finally incompressible fluids with pressure dependent viscosities.

Nečas was a prolific writer. He authored or co-authored eight books. Special mention must be made of his book "Les méthodes directes en théorie des équations elliptiques" which has already had tremendous impact on the progress of the subject and will have a lasting influence in the field. He has written a hundred and forty seven papers in archival journals as well as numerous papers in the proceedings of conferences all of which have had a significant impact in various areas of applications of mathematics and mechanics.

Jindřich Nečas passed away on December 5th, 2002. However, the legacy that Nečas has left behind will be cherished by generations of mathematicians in the Czech Republic in particular, and the world of mathematical analysts in general.

Jindřich Nečas Center for Mathematical Modeling

The Nečas Center for Mathematical Modeling is a collaborative effort between the Faculty of Mathematics and Physics of the Charles University, the Institute of Mathematics of the Academy of Sciences of the Czech Republic and the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University.

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