Jindřich Nečas Center for Mathematical Modeling
Lecture notes

Volume 2

Topics on partial differential equations

Volume edited by P. Kaplický and Š. Nečasová

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VYDAVATELSTVÍ MATEMATICKO-FYZIKÁLNÍ FAKULTY
UNIVERZITY KARLOVY V PRAZE

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Jindřich Nečas Center for Mathematical Modeling
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Topics on partial differential equations

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Abstract. The volume provides a record of lectures given by visiting professors of the Jindřich Nečas Center for Mathematical Modeling during academic year 2006/2007. The volume contains both introductory as well as advanced level texts on various topics in theory of partial differential equations.

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Preface

This volume of Lecture Notes of the Jindřich Nečas Center for Mathematical Modeling contains contributions by Reinhard Farwig, Werner Varnhorn, Daniel Ševčovič, Pavol Quittner and Patrick J. Rabier who were among the first long time visiting professors of the Nečas Center. The contributions are based on the lectures delivered during their stays between November 2006 and May 2007.

One of the basic aims of the Nečas Center for Mathematical Modeling is to establish a scientific team studying mathematical properties of models in continuum mechanics and thermodynamics and to arrange collaboration between members of the team and world renowned scientists. To this end the experts, with research fields connected to the scientific program of the Nečas Center, are invited to present mini courses. How this particular aim is fulfilled can be demonstrated in this volume of Lecture Notes. Parts 1 and 3 are based on lectures of Reinhard Farwig and Werner Varnhorn devoted to the Navier–Stokes system, perhaps the most popular model in continuum mechanics. This model can also be treated by the methods presented in the lecture of Pavol Quittner which is base of Part 4 and by the methods developed in the lecture of Patrick J. Rabier described in Part 5, while the results from Part 2, which contains the lecture of Daniel Ševčovič, are applicable to the dynamics of phase boundaries in thermomechanics.

Another aim of the Nečas Center for Mathematical Modeling is to initiate Czech researchers to study new mathematical methods. This is also illustrated in this volume. In Part 1, Reinhard Farwig and coauthors present a new approach to the Navier–Stokes system through the theory of very weak solutions. These very weak solutions have *a priori* no differentiability neither in time nor in space, so they in general do not coincide with the weak solutions, they are however directly constructed in the Serrin’s class, and it is possible to show their uniqueness. Moreover, he describes new results of Serrin’s type such that the velocity field u is regular locally or globally in time or locally in space and time. In Part 2, Daniel Ševčovič presents the theory of curvature driven flow of planar curves, based on the direct approach. The evolution of the planar curve is described in the Lagrangian framework. A closed system of parabolic ordinary differential equations is constructed for relevant quantities, and properties of solutions of this system are studied. Efficient algorithms to calculate solutions numerically are also derived. Part 3 is again devoted to the Navier–Stokes system. Werner Varnhorn presents there a particle method to approximate it. Since the bad term in the Navier–Stokes system—the convective term—arises from the total material derivative, it is approximated by a kind of central total difference quotient, which does not invalidate the conservation of energy. In Part 4, Pavol Quittner studies qualitative properties of solutions to semilinear parabolic equations and systems. The main focus is on the question whether the solutions are global or whether they can blow up. Also the question how the blow up is created is studied. The last part presented by Patrick J. Rabier provides insight into the degree theory for Fredholm mappings of index 0 which is generalization of Leray–Schauder degree, however not derived from it. This theory is later applied to the stationary Navier–Stokes system considered on unbounded domains to obtain existence of solutions. Finally, also fredholmness of some evolutionary operators is studied.

It is a pleasure for us to present this volume of Lecture Notes of the Nečas Center for Mathematical Modeling to the reader. We think, it is a sample of excellent outputs of the top level research carried out at the Nečas Center, many of which will follow. We believe that this volume will be valuable and interesting not only for students still looking for their field of interest, but also for experts searching for new approaches and problems. At the end we hope that it will help to initiate new research in the framework of the Nečas Center for Mathematical Modeling and in the Czech science.

December 2007

P. Kaplický
Š. Nečasová

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Part 1

Very weak, weak and strong solutions to the instationary Navier–Stokes system

Reinhard Farwig, Hideo Kozono, Hermann Sohr

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Abstract. In this survey paper we discuss the theory of very weak solutions to the stationary and instationary (Navier–)Stokes system in a bounded domain of $\mathbb{R}^3$ and show how this new notion of solutions may be used to prove regularity locally or globally in space and time of a given weak solution in the sense of Leray–Hopf.

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CHAPTER 1

Introduction

We consider the instationary Navier–Stokes equations for a viscous incompressible fluid with density $\rho = 1$, i.e.,

$$\begin{aligned} u_t - \nu\Delta u + \operatorname{div}(uu) + \nabla p &= f \quad \text{in } \Omega \times (0,T) \\ \operatorname{div} u &= k \quad \text{in } \Omega \times (0,T) \\ u &= g \quad \text{on } \partial\Omega \times (0,T) \\ u &= u_0 \quad \text{at } t = 0 \end{aligned} \tag{1.1}$$

for the unknown velocity $u = (u_1, u_2, u_3)$ and pressure p in a domain $\Omega \subset \mathbb{R}^3$ and a time interval $(0,T)$, $0 < T \leq \infty$. Here f denotes the external force (force density), $u_0 = u_0(x)$ the initial value, and $\nu > 0$ is the given viscosity of the fluid. In the physical model the divergence $k = \operatorname{div} u$ is assumed to vanish. However, for mathematical reasons it will be convenient in particualar for linear problems to consider the more general case of a prescribed divergence $k \neq 0$; see also Remark 1.9(1) below. Moreover, the boundary data $g = u_{|\partial\Omega}$ is a generalization of the classical no-slip or adhesion condition $u_{|\partial\Omega} = 0$. Obviously, for a bounded domain, k and g must satisfy the necessary compatibility condition

$$\int_\Omega k\,dx = \int_{\partial\Omega} g \cdot N\,do\,; \tag{1.2}$$

here $N = N(x)$ is the external normal vector at $x \in \partial\Omega$, and do denotes the surface measure on $\partial\Omega$.

There are several notions of instationary solutions to (1.1) which are mainly considered for the case $k = 0$ and $g = 0$. In the following let us briefly discuss the notion and basic properties of weak solutions in the sense of J. Leray and E. Hopf and of strong solutions when $k = 0$, $g = 0$ before turning to the more recent concept of very weak solutions. For surveys on the instationary Navier–Stokes equations we refer to [34], [66].

1. Weak solutions in the sense of Leray–Hopf

Let us test the Navier–Stokes system (with $k = 0$, $g = 0$) formally with the solution u and use integration by parts in space. Then, since $\operatorname{div} u = 0$, $\operatorname{div}(uu) = u \cdot \nabla u$ and $u = 0$ on $\partial\Omega$,

$$\int_\Omega \nabla p \cdot u\,dx = 0 \quad \text{and} \quad \int_\Omega (u \cdot \nabla u) \cdot u\,dx = \int_\Omega u \cdot \nabla\big(\tfrac{1}{2}|u|^2\big)\,dx = 0$$

so that (1.1) yields the identity

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \nu\,\|\nabla u(t)\|_2^2 = (f,u)(t);$$

here $(\cdot,\cdot)$ denotes the L^2-scalar product on Ω. A further integration in time on the interval (s,t) leads to the *energy identity*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_s^t \|\nabla u\|_2^2\,d\tau = \frac{1}{2}\|u(s)\|_2^2 + \int_s^t (f,u)\,d\tau \tag{1.3}$$

for $0 \le s < t \le T$. Assume that the external force f has the form

$$f = f_0 + \operatorname{div} F, \quad f_0 \in L^1\big(0,T;L^2(\Omega)\big), \quad F \in L^2\big(0,T;L^2(\Omega)\big). \tag{1.4}$$

Then Young's inequality and Gronwall's Lemma yield the integrability properties

$$u \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T);H_0^1(\Omega)\big) \tag{1.5}$$

for every time interval $(0,T)$. Now (1.5) serves as starting point for the definition of a weak solution.

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a domain, let the initial value u_0 belong to the space

$$L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}, \quad C^\infty_{0,\sigma}(\Omega) = \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\},$$

and let f satisfy (1.4). Then a solenoidal vector field u satisfying (1.5) is called a *weak solution in the sense of Leray-Hopf* of the instationary Navier–Stokes system (1.1) with data f, u_0 (and with $k=0$, $g=0$) if

$$\begin{aligned} -\int_0^T (u,\varphi_t)d\tau + \nu\int_0^T (\nabla u,\nabla\varphi)d\tau + \int_0^T (u\cdot\nabla u,\varphi)d\tau \\ = \big(u_0,\varphi(0)\big) + \int_0^T \langle f,\varphi\rangle d\tau \end{aligned} \tag{1.6}$$

for all test functions $\varphi \in C_0^\infty\big([0,T);\,C^\infty_{0,\sigma}(\Omega)\big)$.

In (1.6) $\langle\cdot,\cdot\rangle$ denotes the duality product of $H^{-1}(\Omega) = H_0^1(\Omega)^*$ and $H_0^1(\Omega)$, and $(\cdot,\cdot)$ is used for measurable functions η, ψ on Ω in the sense $(\eta,\psi) = \int_\Omega \eta\cdot\psi\,dx$ provided $\eta\cdot\psi \in L^1(\Omega)$. Note that the same symbol, say $u \in C_0^\infty(\Omega)$, is used for a function as well as for vector fields or even matrix fields.

By the Galerkin approximation method or by the theory of analytic semigroups in the space $L^2_\sigma(\Omega)$ using Yosida approximation arguments it is shown that the Navier–Stokes system (1.6) has at least one weak solution in the sense of Leray-Hopf, see e.g. [24, §§2–3], [61, V.3]. Moreover, as a consequence of (1.6),

$$u : [0,T) \to L^2_\sigma(\Omega) \quad \text{is weakly continuous,} \tag{1.7}$$

and the initial value u_0 is attained in the sense: $\big(u(t),\varphi\big) \to (u_0,\varphi)$ as $t \to 0+$ for all $\varphi \in L^2_\sigma(\Omega)$ and even for all $\varphi \in L^2(\Omega)$.

However, due to the selection of a weakly convergent subsequence in the construction of the weak solution it cannot be guaranteed that u still satisfies the energy identity (1.3). The lower semicontinuity of norms with respect to weak convergences implies only that u satisfies the *energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u\rangle d\tau \tag{1.8}$$

rather than the energy identity (1.3). It is not clear whether any weak solution u according to Definition 1.1 does satisfy the energy inequality. However, each known construction method yields a weak solution satisfying (1.8).

If the domain $\Omega \subset \mathbb{R}^3$ is bounded, the compact embedding $H_0^1(\Omega) \subset L^2(\Omega)$ allows to construct a weak solution u satisfying also the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_s^t \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(s)\|_2^2 + \int_s^t \langle f, u\rangle \, d\tau \tag{1.9}$$

for almost all $s \in [0,T)$ including $s = 0$ and for all $t \in [s,T)$, see e.g. [61, Theorem V.3.6.2]. For unbounded domains the compactness argument is no longer available and more sophisticated tools based on maximal regularity, see Section 4 below, are needed to prove the existence of a weak solution satisfying the strong energy inequality; see [40], [62] for exterior domains and [16] for general unbounded domains with uniform C^2-regularity of the boundary.

Using (1.5) and the embedding $H_0^1(\Omega) \subset L^6(\Omega)$, we obtain for a weak solution u the space-time integrability $u \in L^s\big(0,T; L^q(\Omega)\big)$ for the pairs of exponents $s = \infty$, $q = 2$ and $s = 2$, $q = 6$, satisfying both the condition

$$\frac{2}{s} + \frac{3}{q} = \frac{3}{2}\,. \tag{1.10}$$

More generally, using the so-called *Serrin number*

$$\mathcal{S} = \mathcal{S}(s,q) = \frac{2}{s} + \frac{3}{q} \quad \text{for} \quad s,\, q \in [1,\infty],$$

Hölder's inequality yields

$$u \in L^s\big(0,T; L^q(\Omega)\big) \quad \text{when} \quad \mathcal{S} = \frac{3}{2}, \quad 2 < s,\, q < \infty, \tag{1.11}$$

see [61, Lemma V.1.2.1]. However, it is an open problem whether a weak solution with $\mathcal{S} = \frac{3}{2}$ is unique. But the uniqueness is known if $\mathcal{S} \le 1$.

THEOREM 1.2. *Let $\Omega \subseteq \mathbb{R}^3$ be any domain, and let u, v be weak solutions of the Navier–Stokes system (1.1) with the same data f, u_0 (and with $k = 0$, $g = 0$). Assume that u satisfies the energy inequality (1.8) and that*

$$v \in L^s\big(0,T; L^q(\Omega)\big) \quad \textit{where} \quad \mathcal{S}(s,q) \le 1,\ 2 < s < \infty,\ 3 < q < \infty.$$

Then $u = v$.

For a proof we refer to [58]. The same result holds in the limit case $s = \infty$, $q = 3$ when $\Omega \subset \mathbb{R}^3$ is a bounded or exterior domain with boundary of class C^2, see [35].

2. Regular solutions

One of the seven Millennium Problems of Clay Mathematics Institute in 2000 is the question whether a weak solution of the Navier–Stokes equations in a three-dimensional domain is smooth, i.e., whether $u \in C^\infty\big(\Omega \times (0,T)\big)$ when $f = 0$ or, more generally, $f \in C^\infty\big(\Omega \times (0,T)\big)$. The first step in this direction is the question whether u is a strong solution.

Definition 1.3. A weak solution u of the Navier–Stokes equations (with $k = 0$, $g = 0$) is called a *regular solution* if there exist exponents s, q such that

$$u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big), \quad \mathcal{S}(s,q) \le 1, \quad 3 < q < \infty,\ 2 < s < \infty. \tag{1.12}$$

For short, we say that u is *regular in the sense* $u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big)$. Moreover, u is called a *strong solution* if

$$u \in L^\infty_{\mathrm{loc}}\big([0,T); H^1_0(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); H^2(\Omega)\big). \tag{1.13}$$

Note that in (1.13), compared to (1.5), the regularity in space has been increased by one. Since $H^1_0(\Omega) \subset L^6(\Omega)$, we get $u \in L^\infty_{\mathrm{loc}}\big([0,T); L^6(\Omega)\big)$ with Serrin's number $\mathcal{S} = \frac{1}{2}$ so that u also satisfies (1.12).

The next two theorems state the local existence of a regular solution and the global regularity of a given weak solution under an additional assumption.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^3$ be any domain, $u_0 \in \mathcal{D}(A_2^{1/4})$, where A_2 denotes the Stokes operator on $L^2_\sigma(\Omega)$, see Section 4, and let $f = f_0 + \operatorname{div} F$ with $f_0 \in L^{4/3}\big(0,T; L^2(\Omega)\big)$, $F \in L^4\big(0,T; L^2(\Omega)\big)$. Then there exists $T' = T'(\nu, u_0, f_0, F) \in (0,T)$ such that the Navier–Stokes equations (1.1) with data u_0, f (and with $k = 0$, $g = 0$) have a uniquely determined regular solution*

$$u \in L^8\big(0,T'; L^4(\Omega)\big).$$

Proof. We refer to [24] for a proof of this result for a bounded domain Ω with $\partial\Omega \in C^2$ when $f = 0$ and $u_0 \in H^1_0(\Omega)$. In this case u even satisfies (1.13) in $(0,T')$. The more general result can be found in [61, Theorem V.4.2.2]. □

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^2$ and let u be a weak solution of (1.1) with data $f \in L^2\big(0,T; L^2(\Omega)\big)$, $u_0 \in L^2_\sigma(\Omega) \cap H^1_0(\Omega)$, $0 < T \le \infty$, (and with $k = 0$, $g = 0$) satisfying*

$$u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big), \quad \mathcal{S}(s,q) \le 1,\ 2 < s \le \infty,\ 3 \le q < \infty. \tag{1.14}$$

Then u is regular, uniquely determined by u_0, f, and a strong solution.

If $f \in C_0^\infty\big(\overline{\Omega} \times (0,T)\big)$ and $\partial\Omega \in C^\infty$, then $u \in C^\infty\big(\overline{\Omega} \times (0,T)\big)$.

Proof. The classical implication from (1.14) when $2 < s < \infty$, $3 < q < \infty$, i.e. from (1.12), to (1.13) can be found in [24], see also [61, Theorem V.1.8.1]. The limit case $s = \infty$, $q = 3$ was proved more recently in [11], [39] [52], [53], [54], [55] starting from a result [41] on the finite number of singular points in time and space for a weak solution $u \in L^\infty\big(0,T; L^3(\Omega)\big)$.

Interior regularity results in the sense $u \in C^\infty\big(\overline{\Omega'} \times (0,T)\big)$ for every subdomain $\Omega' \subset\subset \Omega$ are proved in [57], [58], [64]. Moreover, regularity up to the boundary $\partial\Omega$ of Ω is shown [29], [60]. □

At this point, several remarks are in order, for later use in Chapter 3 and for interest in its own. Concerning the energy identity and the energy inequality (1.8) which holds for every weak solution constructed so far in the literature, we note that every strong and every regular solution satisfies the energy identity, see the following Lemma 1.6.

LEMMA 1.6. *Let $\Omega \subseteq \mathbb{R}^3$ be any domain, and let u be a weak solution of (1.1) with data $u_0 \in L^2_\sigma(\Omega)$, $f = f_0 + \operatorname{div} F$, where $f_0 \in L^1\big(0,T;L^2(\Omega)\big)$, $F \in L^2\big(0,T;L^2(\Omega)\big)$ (and with $k=0$, $g=0$).*

(1) Suppose additionally that

$$u \in L^4\big(0,T;L^4(\Omega)\big)$$

or, more generally, that

$$u \in L^s\big(0,T;L^q(\Omega)\big), \quad \mathcal{S}(s,q) \le 1, \quad 2 \le s \le \infty, \quad 3 \le q \le \infty. \tag{1.15}$$

Then u satisfies the energy identity and is strongly continuous from $[0,T)$ to $L^2_\sigma(\Omega)$.

(2) If also v satisfies the integrability condition (1.5), then

$$u \cdot \nabla v \in L^s\big(0,T;L^q(\Omega)\big), \quad \mathcal{S}(s,q) = 4, \quad 1 \le s,q < 2.$$

PROOF. (1) The assumption $u \in L^4\big(0,T;L^4(\Omega)\big)$ implies that $uu \in L^2\big(0,T;L^2(\Omega)\big)$ so that $u\cdot\nabla u = \operatorname{div}(uu)$ may be written on the right-hand side of the equation as part of the external force $\operatorname{div} F$. Then u can be considered as the weak solution of a (linear) instationary Stokes system, and linear theory shows that u satisfies the energy identity.

Under the second assumption we may assume that $\frac{2}{s} + \frac{3}{q} = 1$. Since the given weak solution u also satisfies $u \in L^{s_1}\big(0,T;L^{q_1}(\Omega)\big)$ where $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$, and since $\frac{2}{4} + \frac{3}{4} = \frac{5}{4} \in (1, \frac{3}{2})$, Hölder's inequality easily implies that $u \in L^4\big(0,T;L^4(\Omega)\big)$, for details see [61, V.1.4].

(2) The proof is based on embedding theorems and Hölder's inequality, see [61, Lemma V.1.2.1]. □

REMARK 1.7. *The condition (1.15) for u to satisfy the energy identity may be relaxed to the condition that $u \in L^s\big(0,T;L^q(\Omega)\big)$ and*

$$\mathcal{S}(s,q) \le \min\Big(1+\frac{1}{q}, 1+\frac{1}{s}\Big), \quad 2 \le s \le \infty, \quad 3 \le q \le \infty. \tag{1.16}$$

The proof follows the lines of [61, V.1.4]; note that the region in the $\big(\frac{1}{q},\frac{1}{s}\big)$-plane described by (1.16) is the closed convex hull of the line $\mathcal{S}=1$ and the point $\big(\frac14,\frac14\big)$ in the first quadrant of the $\big(\frac{1}{q},\frac{1}{s}\big)$-plane. Hence the point $\big(\frac14,\frac14\big)$ can be written as a convex combination of any two points of this region and of the line $\mathcal{S}(s,q)=\frac32$, respectively; see also Figure 1 below.

For a further discussion of the energy inequality, energy identity and regularity of a weak solution we refer to the first paragraphs of Chapter 3 as well as Section 3.2 and to Section 3.2 in general.

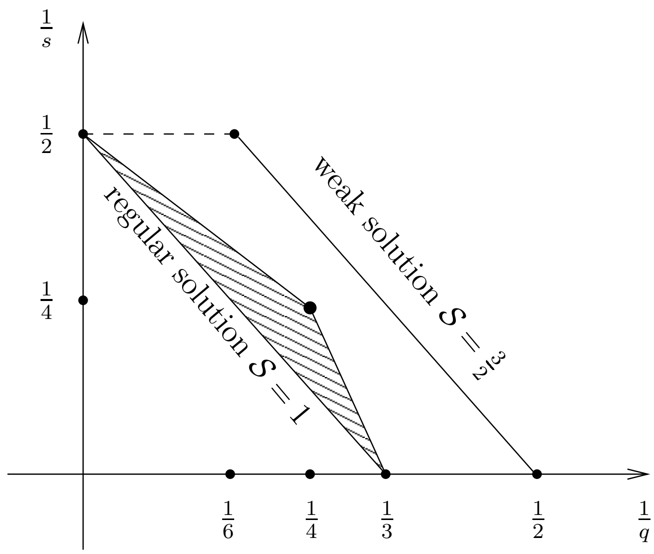

FIGURE 1. Weak and regular solutions represented by lines in the $\left(\frac{1}{q}, \frac{1}{s}\right)$-plane. The hatched region indicates the set described by (1.16) where the energy identity holds.

3. The concept of very weak solutions

In contrast to the definition of weak solutions, see Definition 1.1, where one integration by parts in space was used, the concept of very weak solutions allows all derivatives in space and time to be applied to the test functions. To give a precise definition we will use the spaces of test functions (vector fields)

$$C^2_{0,\sigma}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : \operatorname{div} v = 0,\ v_{|_{\partial\Omega}} = 0\}$$

such that in general ∇v does not vanish on $\partial\Omega$, and

$$C^1_0\big([0,T);\, C^2_{0,\sigma}(\overline{\Omega})\big)$$

of solenoidal vector fields w satisfying $supp\, w \subset \overline{\Omega} \times [0,T)$.

Given a sufficiently smooth solution u of the fully inhomogeneous Navier–Stokes system (1.1) and test functions $w \in C^1_0\big([0,T);\, C^2_{0,\sigma}(\Omega)\big)$ we are led to the identity

$$\int_0^T \big(-(u,w_t) - \nu(u,\Delta w) + \langle g, N\cdot\nabla w\rangle_{\partial\Omega} - (uu,\nabla w)\big)\, d\tau = \big(u_0, w(0)\big) + \int_0^T \langle f, w\rangle\, d\tau$$

where $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ are pairings between corresponding spaces on Ω and $\partial\Omega$, respectively, see Definition 1.8 below. The term $\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$ is due to the inhomogeneous boundary data $g = u_{|_{\partial\Omega}}$ and the fact that in general the normal derivative $N\cdot\nabla w$ of w on $\partial\Omega$ does not vanish. Since $\operatorname{div} w = 0$ for all $t \in [0,T)$, the term $N\cdot\nabla w$ is purely tangential on $\partial\Omega$; this fact is easily checked when $\partial\Omega$ is planar. Hence, the term $\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$ carries only the information of the tangential component of $g = u_{|_{\partial\Omega}}$. Secondly we test the equation $\operatorname{div} u = k$ in $\Omega\times(0,T)$ with test functions $\psi \in C_0^0\big((0,T); C^1(\overline{\Omega})\big)$ and get the identity

$$\int_0^T (k,\psi)d\tau = \int_0^T \big(-(u,\nabla\psi) + \langle g\cdot N,\psi\rangle_{\partial\Omega}\big)\,d\tau.$$

This identity may be rewritten in the pointwise form

$$\operatorname{div} u = k \quad \text{in } \Omega\times(0,T); \quad u\cdot N = g\cdot N \quad \text{on } \partial\Omega\times(0,T)$$

giving information on $\operatorname{div} u$ and the normal component of u on $\partial\Omega$. Summarizing the previous reasoning we are led to

Definition 1.8. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with $C^{1,1}$-boundary, let $f = \operatorname{div} F$ and

$$\begin{aligned} &F\in L^s\big(0,T;L^r(\Omega)\big), && k\in L^s\big(0,T;L^r(\Omega)\big)\\ &g\in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big), && u_0\in \mathcal{J}_\sigma^{q,s}(\Omega)\end{aligned} \tag{1.17}$$

where $\mathcal{J}_\sigma^{q,s}(\Omega)$ is a space of initial values to be defined below, see Definition 2.10, k,g satisfy the compatibility condition (1.2) in the sense

$$\int_\Omega k(t)dx = \langle g(t),N\rangle_{\partial\Omega} \quad \text{for a.a. } t\in(0,T), \tag{1.18}$$

and q,r,s satisfy the conditions

$$\mathcal{S} = \frac{2}{s}+\frac{3}{q} = 1, \quad \frac{1}{3}+\frac{1}{q} = \frac{1}{r}, \quad 2<s<\infty,\ 1<r<3<q<\infty. \tag{1.19}$$

Then a vector field

$$u\in L^s\big(0,T;L^q(\Omega)\big)$$

is called a *very weak solution* of the instationary Navier–Stokes system (1.1) if

$$\begin{aligned}\int_0^T &\big(-(u,w_t) - \nu(u,\Delta w) + \langle g, N\cdot\nabla w\rangle_{\partial\Omega} - (uu,\nabla w)\big)\,d\tau\\ &= \big(u_0,w(0)\big) - \int_0^T (F,\nabla w)\,d\tau\end{aligned} \tag{1.20}$$

for all test fields $w\in C_0^1\big([0,T);C_{0,\sigma}^2(\overline{\Omega})\big)$, and additionally

$$\operatorname{div} u = k \quad \text{in } \Omega\times(0,T), \quad u\cdot N = g\cdot N \quad \text{on } \partial\Omega\times(0,T). \tag{1.21}$$

Remark 1.9. *(1) Note that in [12], [14], [19], [21], [26] the authors considered the variational problem*

$$\begin{aligned}\int_0^T &\big(-(u,w_t) - \nu(u,\Delta w) + \langle g, N\cdot\nabla w\rangle_{\partial\Omega} - (uu,\nabla w) - (ku,w)\big)\,d\tau\\ &= \big(u_0,w(0)\big) - \int_0^T (F,\nabla w)\,d\tau\end{aligned} \tag{1.22}$$

instead of (1.20). The additional term (ku,w) in (1.22) or equivalently $-ku$ on the left-hand side of the first equation of (1.1) is due to the identity

$$u\cdot\nabla u = \operatorname{div}(uu) - ku, \quad \text{where} \quad k = \operatorname{div} u.$$

The difference of these variational problems originates from the derivation of the Navier–Stokes equations, see e.g. [48]. On the one hand, considering compressible fluids with density $\rho=\rho(x,t)$ the term $(\rho u)_t + \operatorname{div}(\rho uu)$ appears in the equation *for the balance of momentum; for constant* ρ *and in the time-independent case we are left with the term* $\operatorname{div}(uu)$ *as in (1.1). On the other hand, the term* $u_t + u \cdot \nabla u$ *denotes the acceleration of particles and leads to the additional term* $-ku$ *in (1.1). We note that both models are unphysical, since the equation for the conservation of mass* $\rho_t + \operatorname{div}(\rho u) = 0$ *leads to* $\operatorname{div} u = 0$ *when the density* ρ *is constant. For the model (1.1) the proofs of Theorems 2.9 and 2.18 below are shorter compared to the proofs in [12], [14], [19], [21], [26], although the assumptions on* $k = \operatorname{div} u$ *and the complexity of the proofs are the same.*

(2) The conditions (1.19) on q, r, s *are needed to give each term in (1.20) a well-defined meaning, particularly to define the nonlinear term* $(uu, \nabla w)$*. The exponents* q, r *are chosen such that the embeddings* $W^{1,r}(\Omega) \subset L^q$*,* $L^r(\Omega) \subset W^{-1,q}(\Omega) := W_0^{1,q'}(\Omega)^*$ *(= the dual space of* $W_0^{1,q'}(\Omega)^*$*,* $q' = \frac{q}{q-1}$*) and* $L^{q'}(\Omega) \subset W^{-1,r'}(\Omega)$ *hold.*

(3) The information on $\operatorname{div} u$ *can be recovered only from (1.21), but not from (1.20).*

(4) Analogous definitions of very weak solutions will be given also for the stationary Stokes and Navier–Stokes system, see Chapter 2. In these cases the conditions on q, r, s *in (1.19) are more general.*

Before turning to theorems on existence in Chapter 2 let us discuss the main features of this concept.

- The concept of very weak solutions was introduced in a series of papers by H. Amann [2], [3] in the setting of Besov spaces when $k = 0$.
- More recently this concept was modified by G.P. Galdi, C. Simader and the authors to a setting in classical L^q-spaces including the inhomogeneous data k, see [12], [13], [14], [19], [21], [26].
- By definition very weak solutions have no differentiability, neither in space nor in time, except for the existence of the divergence $k = \operatorname{div} u \in L^r(\Omega)$ for a.a. t.
- In general, a very weak solution does neither have a bounded kinetic energy in $L^\infty\big(0, T; L^2(\Omega)\big)$ nor a finite dissipation energy in $L^2\big(0, T; H^1(\Omega)\big)$. In particular, a very weak solution is not necessarily a weak solution.
- By definition, a very weak solution is contained in Serrin's uniqueness class $L^s\big(0, T; L^q(\Omega)\big)$ with $\mathcal{S} = 1$. Very weak solutions can be shown to be unique, see Chapter 2. However, in general, the regularity of the data is too low to guarantee any kind of regularity of the very weak solution.
- The concept of very weak solutions has been generalized by K. Schumacher to a setting in weighted Lebesgue and Bessel potential spaces using arbitrary Muckenhoupt weights, see [51].
- Although the data in Definition 1.8 imply no regularity for a very weak solution, the concept may be even further generalized so that neither boundary values nor initial values of a very weak solution can be defined, see [51] and Chapter 2.
- The concept of very weak solutions is strongly based on duality arguments concerning the theory of strong (or regular) solutions. Therefore, the boundary regularity required in this theory is the same as for strong solutions.

- The boundary is usually assumed to be of class $C^{2,1}$. Due to a new smoothing argument in the proof of an extension theorem, see [51], it suffices to require that $\partial\Omega \in C^{1,1}$.

4. Preliminaries

We summarize several auxiliary results on the Helmholtz projection and the Stokes operator introduced for later use only for bounded domains.

LEMMA 1.10. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1-boundary and let $1 < q < \infty$.*

(1) There exists a bounded projection

$$P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$$

from the space of all L^q-vector fields onto the subspace

$$L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$$

of all solenoidal vector fields u such that the normal component $u \cdot N$ of u vanishes on $\partial\Omega$ in the weak sense. In particular,

$$\mathcal{R}(P_q) = L^q_\sigma(\Omega), \quad \mathcal{N}(P_q) = G_q(\Omega) := \{\nabla p : \ p \in W^{1,q}(\Omega)\}.$$

Every vector field $u \in L^q(\Omega)$ has a unique decomposition

$$u = u_0 + \nabla p, \quad u_0 \in L^q_\sigma(\Omega), \quad \nabla p \in G_q(\Omega),$$

satisfying

$$\|u_0\|_q + \|\nabla p\|_q \le c\|u\|_q$$

with a constant $c = c(q,\Omega) > 0$.

(2) The adjoint operator $(P_q)^*$ of P_q equals $P_{q'}$, where $q' = \frac{q}{q-1}$, and the dual space $L^q(\Omega)^*$ is isomorphic to $L^{q'}(\Omega)$.

PROOF. See e.g. [59]. □

LEMMA 1.11. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$-boundary and let $1 < q < \infty$.*

(1) The Stokes operator, defined by

$$\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega), \quad A_q u = -P_q \Delta u,$$

is a closed bijective operator from $\mathcal{D}(A_q) \subset L^q_\sigma(\Omega)$ onto $L^q_\sigma(\Omega)$. If $u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_\rho)$ for $1 < \rho < \infty$, then $A_q u = A_\rho u$.

(2) For $0 \le \alpha \le 1$ the fractional powers

$$A^\alpha_q : \mathcal{D}(A^\alpha_q) \subset L^q_\sigma(\Omega \to L^q_\sigma(\Omega)$$

are well-defined, closed, bijective operators. In particular, the inverses $A^{-\alpha}_q := (A^\alpha_q)^{-1}$ are bounded operators on $L^q_\sigma(\Omega)$ with $\mathcal{R}(A^{-\alpha}_q) = \mathcal{D}(A^\alpha_q)$. The space $\mathcal{D}(A^\alpha_q)$ endowed with the graph norm $\|u\|_q + \|A^\alpha_q u\|_q$, equivalent to $\|A^\alpha_q u\|_q$ for bounded domains, is a Banach space. Moreover, for $1 > \alpha > \beta > 0$,

$$\mathcal{D}(A_q) \subset \mathcal{D}(A^\alpha_q) \subset \mathcal{D}(A^\beta_q) \subset L^q_\sigma(\Omega)$$

with strict dense inclusions, and $(A^\alpha_q)^* = A^\alpha_{q'}$ is the adjoint to A^α_q.

(3) The norms $\|u\|_{W^{2,q}}$ and $\|A_q u\|_q$ are equivalent for $u \in \mathcal{D}(A_q)$. Analogously, the norms $\|\nabla u\|_q$, $\|u\|_{W^{1,q}}$ and $\|A_q^{1/2} u\|_q$ are equivalent for $u \in \mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$. More generally, the embedding estimate

$$\|u\|_q \le c\|A_\gamma^\alpha u\|_\gamma \quad 1 < \gamma \le q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma} \tag{1.23}$$

holds for every $u \in \mathcal{D}(A_\gamma^\alpha)$; here $c = c(q,\gamma,\Omega) > 0$.

(4) The Stokes operator A_q generates a bounded analytic semigroup e^{-tA_q}, $t \ge 0$, on $L^q_\sigma(\Omega)$. Moreover, there exists a constant $\delta_0 = \delta_0(q,\Omega) > 0$ such that

$$\|A_q^\alpha\, e^{-tA_q} u\|_q \le c e^{-\delta_0 t} t^{-\alpha}\, \|u\|_q \quad \text{for } u \in L^q_\sigma(\Omega),\ t > 0, \tag{1.24}$$

with $c = c(q,\alpha,\Omega) > 0$.

Proof. See [1], [20], [27], [28], [30], [61]. Usually these results are proved for bounded domains with $\partial\Omega \in C^2$ or even $C^{2,\mu}$, $0 < \mu < 1$. However, a careful inspection of the proofs shows that $C^{1,1}$-regularity is sufficient. □

We note that most of the results of Lemma 1.11 also hold for exterior domains $\Omega \subset \mathbb{R}^3$. However, some results are more restrictive, since the Poincaré inequality on $W_0^{1,q}(\Omega)$ does not hold for an exterior domain.

The next auxiliary tool concerns the instationary Stokes system

$$\begin{aligned} u_t - \nu\Delta u + \nabla p &= f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega \times (0,T) \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T) \\ u(0) &= u_0 \quad \text{at } t = 0 \end{aligned} \tag{1.25}$$

for data $f \in L^s\big(0,T; L^q(\Omega)\big)$ and $u_0 \in L^q_\sigma(\Omega)$, $1 < s,\ q < \infty$.

Applying the Helmholtz projection P_q to (1.25) we get the abstract evolution equation

$$u_t + \nu A_q u = P_q f, \quad u(0) = u_0, \tag{1.26}$$

where we are looking for a solution u with $u(t) \in \mathcal{D}(A_q)$. The variation of constants formula yields the solution

$$u(t) = e^{-\nu t A_q} u_0 + \int_0^t e^{-\nu(t-\tau)A_q}\, P_q f(\tau)\, d\tau, \quad 0 \le t < T \le \infty. \tag{1.27}$$

Conversely, the solution of (1.26) yields $P_q(u_t - \nu\Delta u - f) = 0$ so that by Lemma 1.10 there exists a function p with $u_t - \nu\Delta u - f = -\nabla p$, i.e., (u,p) solves (1.25). To estimate u given by (1.27) (with $u_0 = 0$) and ∇p we introduce the notion of *maximal regularity.*

Lemma 1.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{1,1}$-boundary, let $1 < s$, $q < \infty$, $f \in L^s\big(0,T; L^q(\Omega)\big)$ and $u_0 = 0$. Then the Stokes equation (1.26) has a unique solution u satisfying the maximal regularity estimate*

$$\|u_t\|_{L^s(0,T;L^q(\Omega))} + \|\nu A_q u\|_{L^s(0,T;L^q(\Omega))} \le c\|f\|_{L^s(0,T;L^q(\Omega))} \tag{1.28}$$

where $c = c(q,s,\Omega) > 0$ is independent of ν and T. Moreover, there exists a function $p \in L^s\big(0,T; W^{1,q}(\Omega)\big)$ such that (u,p) satisfies (1.25) and the estimate

$$\|(u_t, \nabla p, \nu\nabla^2 u)\|_{L^s(0,T;L^q(\Omega))} \le c\|f\|_{L^s(0,T;L^q(\Omega))}. \tag{1.29}$$

Proof. The first proof of this result for $s = q \in (1, \infty)$ can be found in [63] and is based on potential theory, the generalization to arbitrary $s \in (1, \infty)$ is a consequence of abstract theory, see [1], [8], [30]. Different approaches are based on the theory of pseudodifferential operators [28], [31] and on the theory of weighted estimates, see A. Fröhlich [22], [23]. □

CHAPTER 2

Theory of very weak solutions

As already outlined in Section 1.3 the concept of very weak solutions introduces a new class of solutions to stationary and nonstationary Stokes and Navier–Stokes equations with data of very low regularity such that solutions may have (almost) no differentiability and no finite energy, but they are unique even in the nonlinear case.

1. The stationary Stokes system

First we consider the stationary Stokes problem

$$-\Delta u + \nabla p = f = \operatorname{div} F,\ \operatorname{div} u = k \text{ in } \Omega,\ u_{|_{\partial\Omega}} = g \tag{2.1}$$

for suitable data $f = \operatorname{div} F$, k and g in a bounded domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{1,1}$ and – for simplicity – with viscosity $\nu = 1$. Let

$$C^2_{0,\sigma}(\overline{\Omega}) = \{w \in C^2(\overline{\Omega}) : \operatorname{div} w = 0,\ w_{|_{\partial\Omega}} = 0\}$$

denote the corresponding space of test functions.

Definition 2.1. Let $1 < r \le q < \infty$ and $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$. Given data

$$F \in L^r(\Omega),\ k \in L^r(\Omega),\ g \in W^{-1/q,q}(\partial\Omega) \tag{2.2}$$

satisfying the compatibility condition

$$\int_\Omega k\,dx = \langle g, N\rangle_{\partial\Omega}\,, \tag{2.3}$$

a vector field $u \in L^q(\Omega)$ is called a *very weak solution* to (2.1) if

$$\begin{aligned} -(u, \Delta w) &= -\langle g, N\cdot\nabla w\rangle_{\partial\Omega} - (F, \nabla w) \quad \forall w \in C^2_{0,\sigma}(\overline{\Omega}) \\ \operatorname{div} u &= k \text{ in } \Omega,\ u\cdot N = g\cdot N \text{ on } \partial\Omega. \end{aligned} \tag{2.4}$$

Here $(\eta, \psi) := \int_\Omega \eta\psi\,dx$ for measurable functions η, ψ on Ω provided $\eta\cdot\psi \in L^1(\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the evaluation of the functional $g \in W^{-1/q,q}(\partial\Omega)$ at the admissible test function $N\cdot\nabla w = \frac{\partial w}{\partial N} \in W^{1-1/q',q'}(\partial\Omega)$; note that $N \in C^{0,1}(\partial\Omega) \subset W^{1-1/q',q'}(\partial\Omega)$ for every $q \in (1,\infty)$.

Since $N\cdot\nabla w$ is purely tangential on $\partial\Omega$ for $w \in C^2_{0,\sigma}(\overline{\Omega})$, the term $\langle g, N\cdot\nabla w\rangle_{\partial\Omega}$ concerns only the tangential component of $g = u_{|_{\partial\Omega}}$ on $\partial\Omega$. Testing the equation $\operatorname{div} u = k$ with an arbitrary scalar-valued test function $\psi \in C^1(\overline{\Omega})$, we get the second and third identity in (2.4) via the variational problem

$$-(u, \nabla\psi) = (k, \psi) - \langle g, \psi N\rangle_{\partial\Omega}. \tag{2.5}$$

Now let us define the functionals

$$\begin{aligned} \langle \mathcal{F}, w\rangle &= -(F,\nabla w) - \langle g, N\cdot\nabla w\rangle_{\partial\Omega}, \quad w\in Y^{2,q'}_\sigma(\Omega),\\ \langle \mathcal{K}, \psi\rangle &= (k,\psi) - \langle g, \psi N\rangle_{\partial\Omega}, \quad \psi\in W^{1,q'}(\Omega), \end{aligned} \tag{2.6}$$

where

$$Y^{2,q'}_\sigma(\Omega) := \mathcal{D}(A_{q'}) = W^{2,q'}(\Omega)\cap W^{1,q'}_0(\Omega)\cap L^{q'}_\sigma(\Omega).$$

Then the embeddings

$$W^{1,q'}(\Omega)\subset L^{r'}(\Omega),\ Y^{2,q'}_\sigma(\Omega)\subset W^{1,r'}(\Omega),$$

cf. Remark 1.9 (2), and the trace estimate

$$\|\psi\cdot N\|_{W^{1-1/q',q'}(\partial\Omega)} \le c\,\|\psi\|_{W^{1-1/q',q'}(\partial\Omega)} \le c\|\psi\|_{W^{1,q'}(\Omega)}$$

imply that

$$\begin{aligned} \mathcal{F}\in Y^{-2,q}_\sigma(\Omega) &:= Y^{2,q'}_\sigma(\Omega)^*\\ \mathcal{K}\in W^{-1,q}_0(\Omega) &:= W^{1,q'}(\Omega)^*. \end{aligned} \tag{2.7}$$

However, the functionals $\mathcal{F}$ and $\mathcal{K}$ are not distributions in the classical sense on their respective spaces of test functions, since in each case $C^\infty_0(\Omega)$ is *not* a dense subspace. Nevertheless, (2.6), (2.7) leads to a further useful generalization of the concept of very weak solutions, see [51].

Definition 2.2. Let $1<q<\infty$ and let $\mathcal{F}\in Y^{-2,q}_\sigma(\Omega)$, $\mathcal{K}\in W^{-1,q}_0(\Omega)$ be given. Then $u\in L^q(\Omega)$ is called a *very weak solution of the Stokes problem with data* $\mathcal{F},\mathcal{K}$ if

$$\begin{aligned} -(u,\Delta w) &= \langle\mathcal{F},w\rangle, \quad w\in Y^{2,q'}_\sigma(\Omega),\\ -(u,\nabla\psi) &= \langle\mathcal{K},\psi\rangle, \quad \psi\in W^{1,q'}(\Omega). \end{aligned} \tag{2.8}$$

The concept of Definition 2.2 has the drawback that *any* vector field $u\in L^q(\Omega)$ is the very weak solution of the Stokes problem for suitable data $\mathcal{F}\in Y^{-2,q}_\sigma(\Omega)$, $\mathcal{K}\in W^{-1,q}_0(\Omega)$, namely,

$$\langle\mathcal{F},w\rangle := -(u,\Delta w), \quad \langle\mathcal{K},\psi\rangle := -(u,\nabla\psi).$$

Hence there is no possibility to define boundary values of u in this very general setting. However, this concept immediately leads to the existence of a unique very weak solution using duality arguments.

Theorem 2.3. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega\in C^{1,1}$, let $1<q<\infty$ and $\mathcal{F}\in Y^{-2,q}_\sigma(\Omega)$, $\mathcal{K}\in W^{-1,q}_0(\Omega)$ be given. Then the Stokes problem (2.8) has a unique very weak solution $u\in L^q(\Omega)$; moreover, u satisfies the estimate*

$$\|u\|_q \le c\left(\|\mathcal{F}\|_{Y^{-2,q}_\sigma(\Omega)} + \|\mathcal{K}\|_{W^{-1,q}_0(\Omega)}\right) \tag{2.9}$$

with a constant $c=c(\Omega,q)>0$.

Proof. Consider an arbitrary vector field $v\in L^{q'}(\Omega)$. Then there exists a unique strong solution $w\in Y^{2,q'}_\sigma(\Omega)$, $\psi\in W^{1,q'}(\Omega)$ of the Stokes problem

$$-\Delta w - \nabla\psi = v,\ \operatorname{div} w = 0 \text{ in } \Omega,\ w\big|_{\partial\Omega} = 0,\ \int_\Omega \psi\,dx = 0; \tag{2.10}$$

moreover, w, ψ linearly depend on v and

$$\|w\|_{W^{2,q'}(\Omega)} + \|\psi\|_{W^{1,q'}(\Omega)} \le c\,\|v\|_{q'}$$

with a constant $c = c(\Omega, q) > 0$. Now, using the duality $L^q(\Omega) = L^{q'}(\Omega)^*$, define $u \in L^q(\Omega)$ by

$$(u, v) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle$$

such that

$$\begin{aligned} |(u,v)| &\le \|\mathcal{F}\|_{Y_\sigma^{-2,q}(\Omega)} \|w\|_{W^{2,q'}(\Omega)} + \|\mathcal{K}\|_{W_0^{-1,q}(\Omega)} \|\psi\|_{W^{1,q'}(\Omega)} \\ &\le c\big(\|\mathcal{F}\|_{Y_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{W_0^{-1,q}(\Omega)}\big) \|v\|_{q'}. \end{aligned}$$

Hence u satisfies the *a priori* estimate (2.9).

To show that u is a very weak solution to the data $\mathcal{F}, \mathcal{K}$, choose arbitrary test functions $w \in Y_\sigma^{2,q'}(\Omega)$ and $\psi \in W^{1,q'}(\Omega)$ and define $v = -\Delta w - \nabla \psi \in L^{q'}(\Omega)$. Then

$$(u, -\Delta w) - (u, \nabla\psi) = (u, v) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle,$$

i.e., (2.8) is satisfied.

To prove uniqueness, let $u \in L^q(\Omega)$ satisfy (2.8) with $\mathcal{F} = 0$, $\mathcal{K} = 0$. Then for all $v \in L^{q'}(\Omega)$ and corresponding solutions $w \in Y_\sigma^{2,q'}(\Omega)$, $\psi \in W^{1,q'}(\Omega)$ of (2.10) we get

$$(u, v) = (u, -\Delta w) - (u, \nabla\psi) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle = 0.$$

Thus $u = 0$. □

We note that the proof of Theorem 2.3 was based on duality arguments related to the (strong) Stokes operator

$$A_{q'} : Y_\sigma^{2,q'}(\Omega) \to L_\sigma^{q'}(\Omega)$$

where $A_{q'} = -P_{q'}\Delta$ is considered as a bounded bijective operator from $Y_\sigma^{2,q'}(\Omega) \subset W^{2,q'}(\Omega)$, endowed with the norm of $W^{2,q'}(\Omega)$, onto $L_\sigma^{q'}(\Omega)$, and to its adjoint

$$(A_{q'})^* : L_\sigma^q(\Omega) \to Y_\sigma^{-2,q}(\Omega),$$

which defines an isomorphism as well.

To return to Definition 2.1 of very weak solutions and to interpret their boundary values let us introduce the notion of normal and tangential components of ($\mathbb{R}^3$-valued) traces on $\partial\Omega$ and of functionals on $\partial\Omega$. Given $h = (h_1, h_2, h_3) \in W^{1-1/q',q'}(\partial\Omega)$ let

$$h_N = (h \cdot N)N \quad \text{and} \quad h_\tau = h - h_N \quad \text{for a.a. } x \in \partial\Omega$$

denote its normal and tangential component, respectively. Obviously

$$\begin{aligned} h_N &\in W_N^{1-1/q',q'}(\partial\Omega) := \{\varphi \in W^{1-1/q',q'}(\partial\Omega) : \varphi \| N \text{ on } \partial\Omega \text{ a.e.}\}, \\ h_\tau &\in W_\tau^{1-1/q',q'}(\partial\Omega) := \{\varphi \in W^{1-1/q',q'}(\partial\Omega) : \varphi \cdot N = 0 \text{ on } \partial\Omega \text{ a.e.}\}, \end{aligned}$$

and

$$\|h_N\|_{1-1/q',q',\partial\Omega} + \|h_\tau\|_{1-1/q',q',\partial\Omega} \le c\|h\|_{1-1/q',q',\partial\Omega}.$$

Actually,

$$W_N^{1-1/q',q'}(\partial\Omega) \oplus W_\tau^{1-1/q',q'}(\partial\Omega) = W^{1-1/q',q'}(\partial\Omega)$$

as a topological and algebraic direct decomposition.

For $g = (g_1, g_2, g_3) \in W^{-1/q,q}(\partial\Omega)$, we define the functionals

$$g_N \in W_N^{-1/q,q}(\partial\Omega) := W_N^{1-1/q',q'}(\partial\Omega)^*$$
$$g_\tau \in W_\tau^{-1/q,q}(\partial\Omega) := W_\tau^{1-1/q',q'}(\partial\Omega)^*$$

by

$$\langle g_N, h_N\rangle_{\partial\Omega} := \langle g, h_N\rangle_{\partial\Omega},\ \ h_N \in W_N^{1-1/q',q'}(\partial\Omega),$$

and

$$\langle g_\tau, h_\tau\rangle_{\partial\Omega} := \langle g, h_\tau\rangle_{\partial\Omega},\ \ h_\tau \in W_\tau^{1-1/q',q'}(\partial\Omega),$$

respectively. Hence

$$\|g_N\|_{W_N^{-1/q,q}(\partial\Omega)} + \|g_\tau\|_{W_\tau^{-1/q,q}(\partial\Omega)} \le c\|g\|_{-1/q,q,\partial\Omega}.$$

Since $g \in W^{-1/q,q}(\partial\Omega)$ is given, it is reasonable to extend g_N from $W_N^{-1/q,q}(\partial\Omega)$ to $W^{-1/q,q}(\partial\Omega)$ by defining $\langle g_N, h_\tau\rangle := 0$ for all tangential traces $h_\tau \in W_\tau^{1-1/q',q'}(\partial\Omega)$ and to extend g_τ from $W_\tau^{-1/q,q}(\partial\Omega)$ to $W^{-1/q,q}(\partial\Omega)$ by defining $\langle g_\tau, h_N\rangle := 0$ for all normal traces $h_N \in W_N^{1-1/q',q'}(\partial\Omega)$. That way, $W_N^{-1/q,q}(\partial\Omega)$ and $W_\tau^{-1/q,q}(\partial\Omega)$ may be considered as closed subspaces of $W^{-1/q,q}(\partial\Omega)$.

Hence

$$g = g_N + g_\tau \quad \text{on } W^{1-1/q',q'}(\partial\Omega), \tag{2.11}$$

and we get the topological and algebraic decomposition

$$W_N^{-1/q,q}(\partial\Omega) \oplus W_\tau^{-1/q,q}(\partial\Omega) = W^{-1/q,q}(\partial\Omega). \tag{2.12}$$

Finally, we define the functional $g \cdot N \in W^{-1/q,q}(\partial\Omega)$ by

$$\langle g \cdot N, \psi\rangle_{\partial\Omega} := \langle g, \psi N\rangle_{\partial\Omega}, \quad \psi \in W^{1,1/q',q'}(\partial\Omega),$$

satisfying $\|g\cdot N\|_{-1/q,q,\partial\Omega} \le c\,\|g\|_{-1/q,q,\partial\Omega}$. Obviously, $g\cdot N = g_N \cdot N$ and $g_\tau \cdot N = 0$. Moreover, $g_N = (g\cdot N)N$ formally and also in the pointwise sense when g is a vector field on $\partial\Omega$.

THEOREM 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{1,1}$, and let $1 < r \le q < \infty$ satisfy $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$.*

(1) *Given data F, k and g as in (2.2), (2.3) there exists a unique very weak solution $u \in L^q(\Omega)$ of (2.4). This solution satisfies the a priori estimate*

$$\|u\|_q \le c\Big(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\Big) \tag{2.13}$$

with a constant $c = c(q, r, \Omega) > 0$.

(2) *The very weak solution $u \in L^q(\Omega)$ in (1) has a* normal trace $u\cdot N = g\cdot N \in W^{-1/q,q}(\partial\Omega)$ *and a* tangential trace component $u_\tau = g_\tau \in W_\tau^{-1/q,q}(\partial\Omega)$ *in the following sense: The normal trace $u \cdot N = g \cdot N$ exists via the identity*

$$\langle u \cdot N, \psi\rangle_{\partial\Omega} = (k, \psi) + (u, \nabla\psi), \quad \psi \in W^{1,q'}(\Omega). \tag{2.14}$$

For the tangential component of the trace, u_τ, we use a bounded linear extension operator

$$E_\tau : W_\tau^{1-1/q',q'}(\partial\Omega) \to Y_\sigma^{2,q'}(\Omega)$$

such that

$$h = N\cdot\nabla E_\tau(h)\big|_{\partial\Omega} \quad \textit{for all } h\in W_\tau^{1-1/q',q'}(\partial\Omega).$$

Then

$$\langle u_\tau, h\rangle = \big(u,\Delta E_\tau(h)\big) - \big(F,\nabla E_\tau(h)\big),\ h\in W_\tau^{1-1/q',q'}(\partial\Omega), \tag{2.15}$$

is uniquely defined (not depending on the extension operator E_τ *with the above properties). Moreover,*

$$\begin{aligned} \|u\cdot N\|_{-1/q,q,\partial\Omega} &\le c\,\|g_N\|_{W_N^{-1/q,q}(\partial\Omega)},\\ \|u_\tau\|_{W_\tau^{-1/q,q}}(\partial\Omega) &\le c\,\|g_\tau\|_{W_\tau^{-1/q,q}(\partial\Omega)}. \end{aligned} \tag{2.16}$$

Defining the functional $u_N = (u\cdot N)N\in W_N^{-1/q,q}(\partial\Omega)$ *by* $\langle u_N, h_N\rangle_{\partial\Omega} := \langle u\cdot N, h_N\cdot N\rangle_{\partial\Omega}$ *for* $h_N\in W_N^{1-1/q',q'}(\partial\Omega)$*, it holds in view of (2.11), (2.12)*

$$u = u_N + u_\tau = g\in W^{-1/q,q}(\partial\Omega) \tag{2.17}$$

and

$$\|u\|_{-1/q,q,\partial\Omega}\le c\Big(\|u\cdot N\|_{-1/q,q,\partial\Omega} + \|u_\tau\|_{W_\tau^{-1/q,q}(\partial\Omega)}\Big). \tag{2.18}$$

(3) *Assume that* $\mathcal{F}\in Y_\sigma^{-2,q}(\Omega)$ *and* $\mathcal{K}\in W_0^{-1,q}(\Omega)$ *have the representations*

$$\begin{aligned} \langle\mathcal{F}, w\rangle &= -(F,\nabla w) - \langle g_\tau, N\cdot\nabla w\rangle_{\partial\Omega}, & w&\in Y_\sigma^{2,q'}(\Omega),\\ \langle\mathcal{K},\psi\rangle &= (k,\psi) - \langle\hat g,\psi\rangle_{\partial\Omega}, & \psi&\in W^{1,q'}(\Omega), \end{aligned} \tag{2.19}$$

respectively, with

$$F,k\in L^r(\Omega) \quad \textit{and} \quad g_\tau\in W_\tau^{-1/q,q}(\partial\Omega),\ \hat g\in W^{-1/q,q}(\partial\Omega).$$

Then F, g_τ *and* $k,\hat g$ *are uniquely determined by* $\mathcal{F}$ *and* $\mathcal{K}$*, respectively; see the proof below for details concerning the uniqueness of* F*.*

Proof. (1) Given F,k,g as in (2.2), (2.3) define $\mathcal{F},\mathcal{K}$ as in (2.6), and let $u\in L^q(\Omega)$ be the unique very weak solution of (2.8) due to Theorem 2.3. In view of (2.6), (2.9) u satisfies (2.13).

(2) Testing in $(2.8)_2$ with $\psi\in C_0^\infty(\Omega)$ we see from $(2.6)_2$ that $\operatorname{div} u = k\in L^r(\Omega)$ in the sense of distributions. Since $u\in L^q(\Omega)\subset L^r(\Omega)$, a classical result implies that u has a normal trace $u\cdot N\in W^{-1/r,r}(\partial\Omega)$ which by $(2.6)_2$, $(2.8)_2$ coincides with $g\cdot N\in W^{-1/q,q}(\partial\Omega)$.

Concerning the tangential trace we first construct the extension operator E_τ. Let $h\in W^{1-1/q',q'}(\partial\Omega)$. Then we find $w_h = E_1(h)\in W^{2,q'}(\Omega)\cap W_0^{1,q'}(\Omega)$ such that

$$w_h\big|_{\partial\Omega} = 0 \quad\text{and}\quad N\cdot\nabla w_h = h;$$

moreover, w_h depends linearly and continuously on h. The existence of an extension operator E_1 with these porperties is well-known in the case of bounded domains with boundary of class $C^{2,1}$, see [47], [65]. However, a mollification procedure, see [51], allows this extension even in the case when $\partial\Omega\in C^{1,1}$ only. Next, assume that $h\in W_\tau^{1-1/q',q'}(\partial\Omega)$. Then an easy calculation shows that $\operatorname{div} w_h\big|_{\partial\Omega} = 0$ so that $\operatorname{div} w_h\in W_0^{1,q'}(\Omega)$ and $\int_\Omega \operatorname{div} w_h\,dx = 0$. Next we need properties of Bogovskii's operator concerning the divergence problem ([6], [61]): There exists a bounded linear operator

$$B : \left\{ f \in W_0^{1,q'}(\Omega) : \int_\Omega f\,dx = 0 \right\} \to W_0^{2,q'}(\Omega)$$

such that $\operatorname{div} Bf = f$ for these f. Now we define the extension operator $E_\tau = E_1 - B \circ E_1$. Obviously, E_τ is a bounded operator from $W_\tau^{1-1/q',q'}(\partial\Omega)$ to $W^{2,q'}(\Omega)$ such that $E_\tau(h) = 0$ on $\partial\Omega$ and $\operatorname{div} E_\tau(h) = 0$ in Ω, i.e. $E_\tau(h) \in Y_\sigma^{2,q'}(\Omega)$. Moreover, $N \cdot \nabla E_\tau(h) = N \cdot \nabla w_h = h$ on $\partial\Omega$ due to the properties of B.

Let $h \in W_\tau^{1-1/q',q'}(\partial\Omega)$. Then we use $w = E_\tau(h) \in Y_\sigma^{2,q'}(\Omega)$ as a test function in $(2.8)_1$ to see that

$$\begin{aligned} -\big(u, \Delta E_\tau(h)\big) &= \langle \mathcal{F}, E_\tau(h) \rangle \\ &= -\big(F, \nabla E_\tau(h)\big) - \langle g, N \cdot \nabla E_\tau(h) \rangle_{\partial\Omega} \\ &= -\big(F, \nabla E_\tau(h)\big) - \langle g_\tau, h \rangle_{\partial\Omega}. \end{aligned}$$

With $u_\tau := g_\tau$ the former identity coincides with (2.15) and does not depend on the particular choice of the extension operator E_τ.

(3) It suffices to consider F, g_τ or $k, \hat{g}$ such that $\mathcal{F} = 0$ or $\mathcal{K} = 0$, respectively. If $\mathcal{K} = 0$ so that $0 = (k, \psi) - \langle \hat{g}, \psi \rangle_{\partial\Omega}$ for all $\psi \in W^{1,q'}(\Omega)$, then $k = 0$ since we may consider the dense subset $C_0^\infty(\Omega)$ of $L^{r'}(\Omega)$ for the test functions ψ. Hence $0 = \langle \hat{g}, \psi \rangle_{\partial\Omega}$ for all $\psi \in W^{1,q'}(\Omega)$ and consequently $\hat{g} = 0$.

Now let $\mathcal{F} = 0$ so that, using the notation $f = \operatorname{div} F$,

$$0 = \langle f, w \rangle - \langle g_\tau, N \cdot \nabla w \rangle_{\partial\Omega} \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega). \tag{2.20}$$

Hence

$$\langle f, w \rangle = 0 \quad \text{for all } w \in C_{0,\sigma}^\infty(\Omega),$$

and a classical theorem on weak solutions of the Stokes problem proves that $f = \nabla p$ with $p \in L^r(\Omega)$. Therefore,

$$-(F, \nabla w) = \langle f, w \rangle = \langle \nabla p, w \rangle = -\int_\Omega p \operatorname{div} w\,dx = 0$$

for all $w \in Y_\sigma^{2,q'}(\Omega)$ and even for all $w \in W_{0,\sigma}^{1,r'}(\Omega) := W_0^{1,r'}(\Omega) \cap L_\sigma^{r'}(\Omega)$. In this sense $F = 0$ and $f = 0$, and (2.20) implies that

$$\langle g_\tau, N \cdot \nabla w \rangle_{\partial\Omega} = 0 \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega).$$

Using the operator E_τ we get that $\langle g_\tau, h \rangle_{\partial\Omega} = 0$ for all $h \in W_\tau^{1-1/q',q'}(\Omega)$ and hence $g_\tau = 0$. □

Let us introduce a further notation for very weak solutions of the Stokes system which will be helpful in the analysis of nonstationary problems, see Section 3 and Section 4.

Definition 2.5. For $f \in Y_\sigma^{-2,q}(\Omega)$ let $A_q^{-1} P_q f$ denote the unique vector field in $L_\sigma^q(\Omega)$ satisfying

$$(A_q^{-1} P_q f, v) = \langle f, A_{q'}^{-1} v \rangle \quad \text{for all } v \in L_\sigma^{q'}(\Omega),$$

or, equivalently, with $v = A_{q'}w$,

$$(A_q^{-1}P_q f, A_{q'}w) = \langle f, w\rangle \quad \text{for all } w \in Y_\sigma^{2,q'}(\Omega). \tag{2.21}$$

REMARK 2.6. *(1) Formally, every gradient field ∇p, $p \in L^{q'}(\Omega)$, vanishes when being considered as an element of $Y_\sigma^{-2,q}(\Omega)$. In this sense we have to identify two elements $f, f' \in Y_\sigma^{-2,q}(\Omega)$ when $f - f'$ is a gradient field, or, formally, when $P_q f = P_q f'$. The notation $P_q f$ and $A_q^{-1}P_q f$ in Definition 2.5 is formal and indicates that only solenoidal test functions v are used.*

(2) Since $A_q^{-1}P_q f \in L^q_\sigma(\Omega)$ for $f \in Y_\sigma^{-2,q}(\Omega)$, (2.21) also reads

$$-(A_q^{-1}P_q f, \Delta w) = \langle f, w\rangle \quad \textit{for all } w \in Y_\sigma^{2,q'}(\Omega).$$

Hence $A_q^{-1}P_q f$ is the unique very weak solution of (2.8) with $\mathcal{F} = f$ and $\mathcal{K} = 0$, i.e.,

$$A_q^{-1}P_q : Y_\sigma^{-2,q}(\Omega) \to L^q_\sigma(\Omega)$$

is the corresponding bounded solution operator. In particular,

$$\|A_q^{-1}P_q \mathrm{div}\, F\|_q \le c\,\|F\|_r, \quad F \in L^r(\Omega), \tag{2.22}$$

by (2.6), (2.7), (2.13) when using $\mathcal{F} = f = \mathrm{div}\, F$.

(3) Let us discuss the relation of Definition 2.5 to the weak Stokes problem. Given $F \in L^\rho(\Omega)$, $1 < \rho < \infty$, there exists a unique weak solution $u \in W^{1,\rho}_{0,\sigma}(\Omega) = \mathcal{D}(A_\rho^{1/2})$ such that

$$(\nabla u, \nabla v) = \langle \mathrm{div}\, F, v\rangle = -(F, \nabla v) \quad \textit{for all } W^{1,\rho'}_{0,\sigma}(\Omega)$$
$$\|\nabla u\|_\rho \le c\,\|F\|_\rho \tag{2.23}$$

where $c = c(\rho, \Omega) > 0$. Using as a test function $v \in Y_\sigma^{2,\rho'}(\Omega)$ we get that

$$\langle \mathrm{div}\, F, v\rangle = -(u, \Delta v\rangle = (u, A_{\rho'}v).$$

Hence u coincides with the unique very weak solution $A_\rho^{-1}P_\rho\ \mathrm{div}\, F \in L^\rho_\sigma(\Omega)$, and we conclude that $A_\rho^{-1}P_q\ \mathrm{div}\, F \in \mathcal{D}(A_\rho^{1/2})$, and, from (2.23), that

$$\|A_\rho^{1/2}A_\rho^{-1}P_\rho\ \mathrm{div}\, F\|_\rho \le c\,\|F\|_\rho \tag{2.24}$$

where $c = c(\rho, \Omega) > 0$. For short, we will write $A_\rho^{-1/2}P_\rho \,\mathrm{div}\, F = A_\rho^{1/2}A_\rho^{-1}P_\rho \,\mathrm{div}\, F$ so that (2.24) reads

$$\|A_\rho^{-1/2}P_\rho \,\mathrm{div}\, F\|_\rho \le c\,\|F\|_\rho$$

2. The stationary Navier–Stokes system

DEFINITION 2.7. Let $1 < r$, $q < \infty$ satisfy $\frac{2}{q} \le \frac{1}{r} \le \frac{1}{3} + \frac{1}{q}$ and let the data F, k, g be given as in (2.2), (2.3). Then $u \in L^q(\Omega)$ is called a *very weak solution* of the stationary Navier–Stokes system

$$-\nu\Delta u + \mathrm{div}\,(uu) + \nabla p = f = \mathrm{div}\, F, \quad \mathrm{div}\, u = k \text{ in } \Omega,\ u|_{\partial\Omega} = g \tag{2.25}$$

if for all $w \in C^2_{0,\sigma}(\overline{\Omega})$

$$-\nu(u, \Delta w) - (uu, \nabla w) = -(F, \nabla w) - \nu\langle g,\ N\cdot\nabla w\rangle_{\partial\Omega} \tag{2.26}$$

and

$$\operatorname{div} u = k \text{ in } \Omega,\ u \cdot N_{|_{\partial\Omega}} = g \cdot N. \tag{2.27}$$

REMARK 2.8. *As already noted in Remark 1.9, the variational problem (2.26) is missing the term* (ku, w) *compared to the approach in [12], [14], [19], [21], [26] where the authors considered the equation*

$$-\nu(u, \Delta w) - (uu, \nabla w) - (ku, w) = -(F, \nabla w) - \nu\langle g, N \cdot \nabla w\rangle_{\partial\Omega},$$

$w \in C^2_{0,\sigma}(\overline{\Omega})$. *The only reason for this change is to keep the proofs shorter than for the model including the term* ku.

THEOREM 2.9. *There exists a constant* $\varepsilon_* = \varepsilon_*(q, r, \Omega)$ *independent of the data* F, k, g *and the viscosity* $\nu > 0$ *with the following property:*

(1) *If*

$$\|F\|_r + \nu\|k\|_r + \nu\|g\|_{-1/q,q,\partial\Omega} \le \varepsilon_* \nu^2, \tag{2.28}$$

then there exists a very weak solution $u \in L^q(\Omega)$ *to the stationary Navier–Stokes system (2.25). This solution satisfies the a priori estimate*

$$\nu\|u\|_q \le c\big(\|F\|_r + \nu\|k\|_r + \nu\|g\|_{-1/q,q,\partial\Omega}\big) \tag{2.29}$$

where $c = c(q, r, \Omega) > 0$.

(2) *A very weak solution* u *to data* F, k, g *is unique in* $L^q(\Omega)$ *under the smallness condition* $\|u\|_q \le \varepsilon_* \nu$.

We note that in Definition 2.7 and Theorem 2.9 we need the restrictions $2r \le q$ and $q \ge 3$ in contrast to the linear case. The proof of existence (and hence of local uniqueness) is based on Banach's Fixed Point Theorem, whereas the proof of uniqueness in all of $L^q(\Omega)$ requires a bootstrapping argument; the case $q = 3$ needs a further approximation step and will be omitted.

PROOF. (1) Since $2r \le q$, every vector field $u \in L^q(\Omega)$ satisfies the estimate

$$\|uu\|_r \le c\,\|u\|^2_{2r} \le c\,\|u\|^2_q. \tag{2.30}$$

Now, for arbitrary data F, k, g as in (2.2), (2.3), let $u = S(F, k, g) \in L^q(\Omega)$ denote the very weak solution of the Stokes problem (2.1) with $\nu = 1$. Then, in view of (2.30) a very weak solution $u \in L^q(\Omega)$ of the Navier–Stokes system (2.25) is a fixed point of the nonlinear map

$$\mathcal{N}(u) = S\big(\frac{1}{\nu}(F - uu), k, g\big) = S(\frac{1}{\nu}F, k, g) - \frac{1}{\nu}S(uu, 0, 0).$$

To apply Banach's fixed Point Theorem we estimate $\mathcal{N}(u)$ by using (2.30) and the *a priori* estimate (2.13) for the operator S as follows:

$$\begin{aligned} \|\mathcal{N}(u)\|_q &\le c\big(\tfrac{1}{\nu}(\|F\|_r + \|u\|^2_q) + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big) \\ &= a\|u\|^2_q + b \end{aligned} \tag{2.31}$$

where $a = \frac{c}{\nu}$ and $b = c\big(\frac{1}{\nu}\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\big)$. Moreover, for $u, u' \in L^q(\Omega)$ we get the estimate

$$\begin{aligned} \|\mathcal{N}(u) - \mathcal{N}(u')\|_q &= \|\tfrac{1}{\nu}S(uu - u'u', 0, 0)\|_q \\ &\le \tfrac{c}{\nu}\|u - u'\|_q(\|u\|_q + \|u'\|_q) \end{aligned} \tag{2.32}$$

with the same constant $c > 0$ as above. Now consider the closed ball $\mathcal{B}_\rho \subset L^q(\Omega)$ of radius $\rho > 0$ and center 0 where ρ is the smallest positive root of the quadratic equation $y = ay^2 + b$; for the existence of $\rho > 0$ we need the smallness condition

$$4\,ab < 1$$

which is equivalent to (2.28) with a suitable constant $\varepsilon_* = \varepsilon_*(q, r, \Omega) > 0$. Furthermore note that $\rho < \frac{1}{2a}$ so that by (2.32)

$$\|\mathcal{N}(u) - \mathcal{N}(u')\|_q \le \kappa \|u - u'\|_q,\ u, u' \in \mathcal{B}_\rho,$$

with $\kappa = 2a\rho < 1$. Since $\mathcal{N}$ maps $\mathcal{B}_\rho$ into $\mathcal{B}_\rho$ by (2.31) and is a strict contraction on $\mathcal{B}_\rho$, Banach's Fixed Point Theorem yields a unique fixed point $u \in \mathcal{B}_\rho$ of $\mathcal{N}$. Finally the trivial bound $\rho \le 2b$ yields the *a priori* estimate (2.29).

(2) To prove uniqueness of a very weak solution u in $L^q(\Omega)$ we start with the case when $q > 3$. Let $u, v \in L^q(\Omega)$ be fixed points of $\mathcal{N}$. Then $w = u - v$ is the unique very weak solution of the linear Stokes system

$$-\nu\Delta w + \nabla p = -\mathrm{div}\,(wu + vw),\ \mathrm{div}\, w = 0 \quad \text{in } \Omega,\ w|_{\partial\Omega} = 0 \tag{2.33}$$

with "known" right-hand side $-\mathrm{div}\,(wu+vw)$. Since $u, v \in L^q(\Omega)$ and consequently $w \in L^{q_1}_\sigma(\Omega)$ where $q_1 = q$, we get that

$$wu + vw \in L^{\rho_1}(\Omega),\ \frac{1}{\rho_1} = \frac{1}{q} + \frac{1}{q_1}.$$

Hence w coincides with the unique *weak* solution of the Stokes problem (2.33) and satisfies

$$w \in \mathcal{D}(A^{1/2}_{\rho_1}) = W^{1,\rho_1}_{0,\sigma}(\Omega) \subset L^{q_1}_\sigma(\Omega),\ \frac{1}{q_1} = \frac{1}{\rho_1} - \frac{1}{3} = \frac{1}{q} + (\frac{1}{q} - \frac{1}{3}).$$

If $\rho_1 < 2$, i.e., $q < 4$, we repeat this argument finitely many times to get in the m-th step, $m = 1, 2, 3, \ldots$, that

$$w \in L^{q_m}_\sigma(\Omega),\ \frac{1}{q_m} = \frac{1}{q} + m(\frac{1}{q} - \frac{1}{3}).$$

Since $q > 3$, we will arrive at the property

$$wu + vw \in L^{\rho_m}(\Omega),\ \frac{1}{\rho_m} = \frac{1}{q} + \frac{1}{q_m} = \frac{2}{q} + m(\frac{1}{q} - \frac{1}{3}) \le \frac{1}{2}$$

for sufficiently large $m \in \mathbb{N}$. Now we see that $wu + vw \in L^2(\Omega)$, consequently $w \in \mathcal{D}(A^{1/2}_2) = W^{1,2}_{0,\sigma}(\Omega)$, and that we may test in (2.33) with w. By these means we get that

$$\begin{aligned}
\nu\|\nabla w\|_2^2 &= \int_\Omega u(w \cdot \nabla w)\,dx + \int_\Omega w(v \cdot \nabla w)\,dx = \int_\Omega u(w \cdot \nabla w)\,dx \\
&\le \|u\|_3\,\|w\|_6\,\|\nabla w\|_2 \\
&\le c\,\|u\|_q\,\|\nabla w\|_2^2.
\end{aligned}$$

Hence, under the smallness condition $\|u\|_q \le \varepsilon_*\nu$ we may conclude that $\nabla w = 0$ and $u = v$.

The limit case $q = 3$, in which the above iteration is stationary ($q_m = q$ for all $m \in \mathbb{N}$), requires a complicated approximation and smoothing argument. For details we refer to [21]. □

3. The instationary Stokes system

Looking at very weak solutions $u \in L^s\big(0,T;L^q(\Omega)\big)$, $1 < s,q < \infty$, of the initial-boundary value problem of the Stokes system we carefully introduce the set of admissible initial values, $\mathcal{J}^{q,s}_\sigma(\Omega)$, as a subset of $Y^{-2,q}_\sigma(\Omega)$. In this subsection we set $\nu = 1$ for simplicity.

Definition 2.10. Given $1 < s,q < \infty$ let

$$\mathcal{J}^{q,s}_\sigma(\Omega) = \Big\{u_0 \in Y^{-2,q}_\sigma(\Omega) : \int_0^\infty \|A_q e^{-\tau A_q}(A_q^{-1}P_q u_0)\|_q^s \, d\tau < \infty\Big\},$$

endowed with the norm

$$\|u_0\|_{\mathcal{J}^{q,s}_\sigma} := \Big(\int_0^\infty \|A_q e^{-\tau A_q}(A_q^{-1}P_q u_0)\|_q^s \, d\tau\Big)^{1/s}.$$

Remark 2.11. *(1) The term $\|\cdot\|_{\mathcal{J}^{q,s}_\sigma}$ defines a norm on $\mathcal{J}^{q,s}_\sigma(\Omega)$: If $\|u_0\|_{\mathcal{J}^{q,s}_\sigma} = 0$, then $A_q e^{-tA_q}(A_q^{-1}P_q u_0) = 0$ and consequently $e^{-tA_q}A_q^{-1}P_q u_0 = 0$ for a.a. $t > 0$; as $t \to 0+$, we conclude that $A_q^{-1}P_q u_0 = 0$, i.e., $u_0 = 0$ as an element of $Y^{-2,q}_\sigma(\Omega)$. Note that $\|u_0\|_{\mathcal{J}^{q,s}_\sigma(\Omega)}$ equals the $L^s\big(0,T;L^q(\Omega)\big)$-norm of $Au(t)$ where $u(t)$ denotes the strong solution of the homogeneous instationary Stokes problem with initial value $A_q^{-1}P_q u_0 \in L^q_\sigma(\Omega)$.*

(2) The spaces $\mathcal{J}^{q,s}_\sigma(\Omega)$ can be considered as real interpolation spaces and identified with solenoidal subspaces of Besov spaces. Actually,

$$u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega) \Leftrightarrow A_q^{-1}P_q u_0 \in \big(\mathcal{D}(A_q), L^q_\sigma(\Omega)\big)_{1/s,s}$$

and

$$\|u_0\|_{\mathcal{J}^{q,s}_\sigma} + \|A_q^{-1}P_q u_0\|_q \sim \|A_q^{-1}P_q u_0\|_{(\mathcal{D}(A_q),L^q_\sigma(\Omega))_{1/s,s}}$$

in the sense of norm equivalence, see [30, (2.5)], [65]. Moreover, consider the solenoidal Besov spaces $\mathbb{B}^{2-2/s}_{q,s}(\Omega)$ introduced in [3, (0.6)], with the property

$$\mathbb{B}^{2-2/s}_{q,s}(\Omega) = \begin{cases} \{u \in B^{2-2/s}_{q,s}(\Omega) : \operatorname{div} u = 0,\ u|_{\partial\Omega} = 0\}, & \frac{1}{q} < 2 - \frac{2}{s}, \\ \{u \in B^{2-2/s}_{q,s}(\Omega) : \operatorname{div} u = 0,\ u \cdot N|_{\partial\Omega} = 0\}, & \frac{1}{q} > 2 - \frac{2}{s}, \end{cases}$$

cf. [65], where $B^{2-2/s}_{q,s}(\Omega)$ are the usual Besov spaces. By [3, Proposition 3.4]

$$u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega) \Leftrightarrow A_q^{-1}P_q u_0 \in \big(\mathcal{D}(A_q), L^q_\sigma(\Omega)\big)_{1/s,s} = \mathbb{B}^{2-2/s}_{q,s}(\Omega).$$

(3) Consider $u_0 \in Y^{-2,q}_\sigma(\Omega)$ such that

$$|\langle u_0, w\rangle| \le c\,\|A_{q'}^{-1/s+\varepsilon} w\|_{q'}, \quad w \in Y^{2,q'}_\sigma(\Omega),$$

where $0 < \varepsilon < \frac{1}{s}$. Then by (1.24) $u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega)$.

Definition 2.12. Let $1 < s,q < \infty$, $1 < r \le q$, $\frac{1}{3} + \frac{1}{q} \ge \frac{1}{r}$, $0 < T \le \infty$, let the data F, k, g satisfy

$$F \in L^s\big(0,T;L^r(\Omega)\big),\ k \in L^s\big(0,T;L^r(\Omega)\big),\ g \in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big) \tag{2.34}$$

$$\textstyle\int_\Omega k(t)\,dx = \langle g(t), N\rangle_{\partial\Omega} \quad \text{for a.a. } t \in (0,T), \tag{2.35}$$

and let $u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega)$. Then $u \in L^s\big(0,T;L^q(\Omega)\big)$ is called a *very weak solution* of the instationary Stokes system

$$\begin{aligned} u_t - \Delta u + \nabla p &= \operatorname{div} F, & \operatorname{div} u &= k \text{ in } \Omega \times (0,T) \\ u(0) &= u_0 \text{ at } t = 0, & u &= g \text{ on } \partial\Omega \times (0,T) \end{aligned} \tag{2.36}$$

if

$$\begin{aligned} -(u, w_t)_{\Omega,T} - (u, \Delta w)_{\Omega,T} &= \langle u_0, w(0)\rangle - (F, \nabla w)_{\Omega,T} - \langle g, N \cdot \nabla w\rangle_{\partial\Omega,T} \\ \operatorname{div} u = k \text{ in } \Omega \times (0,T), &\quad u \cdot N = g \cdot N \text{ on } \partial\Omega \times (0,T) \end{aligned} \tag{2.37}$$

for all test functions $w \in C^1_0\big([0,T);C^2_{0,\sigma}(\overline{\Omega})\big)$.

REMARK 2.13. *(1) As shown in Theorem 2.14 below the very weak solution* $u \in L^s\big(0,T;L^q(\Omega)\big)$ *of (2.36), (2.37) has the property* $A_q^{-1}P_q u(\cdot) \in C^0\big([0,T);L^q(\Omega)\big)$ *or equivalently,* $u \in C^0\big([0,T);Y^{-2,q}_\sigma(\Omega)\big)$. *Hence the initial value* $u(0) = u_0$ *in (2.36)*$_2$ *is attained in* $Y^{-2,q}_\sigma(\Omega)$, *i.e.*

$$\langle u(0), w\rangle = \langle u_0, w\rangle \textit{ for all } w \in Y^{2,q'}_\sigma(\Omega),$$

or equivalently $(A_q^{-1}P_q u)(0) = A_q^{-1}P_q u_0$.

(2) Definition 2.12 may be extended, correspondingly to Definition 2.2, to the problem

$$\begin{aligned} (u, w_t)_{\Omega,T} - (u, \Delta w)_{\Omega,T} &= \langle \mathcal{F}, w\rangle \\ -(u, \nabla\psi)_{\Omega,T} &= \langle \mathcal{K}, \psi\rangle \end{aligned} \tag{2.38}$$

with data $\mathcal{F} \in L^s\big(0,T;Y^{-2,q}_\sigma(\Omega)\big)$ *and* $\mathcal{K} \in L^s\big(0,T;W^{-1,q}_0(\Omega)\big)$ *and for suitable test function* w *and* ψ, *cf. [51]. Then existence and uniqueness of a very weak solution* $u \in L^s\big(0,T;L^q(\Omega)\big)$ *to (2.38) is a direct consequence of duality arguments and results on the strong instationary Stokes system in* $L^{s'}\big(0,T;L^{q'}(\Omega)\big)$. *As in Section 1, in this very general setting neither initial values nor boundary values of* u *are well-defined. Actually, every* $u \in L^s\big(0,T;L^q(\Omega)\big)$ *is the very weak solution of (2.38) for certain data* $\mathcal{F}$ *and* $\mathcal{K}$. *However, in contrast to our approach in Section 1, we will follow a different idea to solve (2.37).*

THEOREM 2.14. *Suppose that the data* F, k, g *satisfy the conditions (2.34), (2.35) and that* $u_0 \in \mathcal{J}^{q,s}_\sigma(\Omega)$, *where* $1 < s,\ q < \infty$, $1 < r \le q$, $\frac{1}{q} + \frac{1}{3} \ge \frac{1}{r}$. *Then there exists a unique very weak solution* $u \in L^s\big(0,T;L^q(\Omega)\big)$ *of (2.36), satisfying*

$$u_t \in L^s\big(0,T;Y^{-2,q}_\sigma(\Omega)\big),\ u \in C^0\big([0,T);Y^{-2,q}_\sigma(\Omega)\big).$$

Moreover, there exists a constant $c = c(q,r,s,\Omega) > 0$ *independent of* $T > 0$ *such that*

$$\begin{aligned} &\|u\|_{L^s(L^q)} + \|u_t\|_{L^s(Y^{-2,q}_\sigma)} \\ &\qquad \le c\big(\|F\|_{L^s(L^r)} + \|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))} + \|u_0\|_{\mathcal{J}^{q,s}_\sigma}\big). \end{aligned} \tag{2.39}$$

PROOF. For almost all $t \in (0,T)$ let $H(t)$ denote the solution of the weak Neumann problem

$$\Delta H = k \text{ in } \Omega,\ N \cdot (\nabla H - g) = 0 \text{ on } \partial\Omega.$$

Since $k(t) \in L^r(\Omega) \subset W_0^{-1,q}(\Omega)$, we find a unique solution $\nabla H(t) \in L^q(\Omega)$ satisfying

$$\nabla H(t) \in L^s\big(0,T;L^q(\Omega)\big),\ \|\nabla H\|_{L^s(L^q)} \le c\left(\|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}\right). \tag{2.40}$$

Moreover, for almost all $t \in (0,T)$ let $\gamma(t) = \gamma_{F(t),k(t),g(t)} \in L^q(\Omega)$ denote the very weak solution of the inhomogeneous Stokes problem

$$-\Delta\gamma + \nabla p = \operatorname{div} F,\ \operatorname{div}\gamma = k \text{ in } \Omega,\ \gamma\big|_{\partial\Omega} = g, \tag{2.41}$$

satisfying the estimate

$$\|\gamma\|_{L^s(L^q)} \le c\big(\|F\|_{L^s(L^r)} + \|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}\big). \tag{2.42}$$

Assume that $u \in L^s\big(0,T;L^q(\Omega)\big)$ is a very weak solution of (2.36). Obviously

$$P_q u = u - \nabla H \quad \text{and} \quad P_q\gamma = \gamma - \nabla H \quad \text{for a.a. } t \in (0,T),$$

where P_q denotes the usual Helmholtz projection on $L^q(\Omega)$. Thus

$$\hat{u} := P_q u = u - \nabla H = u - \gamma + P_q\gamma \in L^s\big(0,T;L^q_\sigma(\Omega)\big).$$

Next let us prove that $U = A_q^{-1}\hat{u} \in L^s\big(0,T;\mathcal{D}(A_q)\big)$ is a strong solution of the Stokes system

$$U_t + A_q U = P_q\gamma \text{ on } (0,T),\ U(0) = A_q^{-1}P_q u_0. \tag{2.43}$$

For this reason consider any test function $v \in C_0^1\big([0,T);L^{q'}_\sigma(\Omega)\big)$ and also $w = A_{q'}^{-1}v \in C_0^1\big([0,T);Y^{2,q'}_\sigma(\Omega)\big)$. Then

$$\begin{aligned} &-(U,v_t)_{\Omega,T} + (A_qU,v)_{\Omega,T} - (P_q\gamma,v)_{\Omega,T} \\ &\quad= -(\hat{u},w_t)_{\Omega,T} + (\hat{u},A_{q'}w)_{\Omega,T} - (P_q\gamma,A_{q'}w)_{\Omega,T} \\ &\quad= -(u,w_t)_{\Omega,T} - (u-\gamma,\Delta w)_{\Omega,T}, \end{aligned}$$

since $(\nabla H,w_t)_{\Omega,T} = 0$ and $\operatorname{div}(u-\gamma) = 0$. Due to (2.41) we know that

$$-(\gamma,\Delta w)_{\Omega,T} = -(F,\nabla w)_{\Omega,T} - \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T},$$

so that we may proceed as follows:

$$\begin{aligned} &-(U,v_t)_{\Omega,T} + (A_qU,v)_{\Omega,T} - (P_q\gamma,v)_{\Omega,T} \\ &\quad= -(u,w_t)_{\Omega,T} - (u,\Delta w)_{\Omega,T} + (F,\nabla w)_{\Omega,T} + \langle g,N\cdot\nabla w\rangle_{\partial\Omega,T} \\ &\quad= \langle u_0, w(0)\rangle \\ &\quad= \big(A_q^{-1}P_q u_0, v(0)\big). \end{aligned}$$

This identity, valid for all $v \in C^1\big([0,T);L^{q'}_\sigma(\Omega)\big)$, proves that U satisfies (2.43) and that $U(0) = A_q^{-1}P_q u_0$. Moreover, by Lemma 1.12 on maximal regularity, the estimates (1.28), (2.42) and the variation of constants formula (1.27) we know that $U_t \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$, in particular, $U \in C^0\big([0,T);L^q_\sigma(\Omega)\big)$,

$$U(t) = e^{-A_q t}(A_q^{-1}P_q u_0) + \int_0^t e^{-A_q(t-\tau)}P_q\gamma(\tau)\,d\tau \tag{2.44}$$

and

$$\begin{aligned}
&\|U_t\|_{L^s(L^q)} + \|A_qU\|_{L^s(L^q)} \\
&\quad\le\ c\Big(\int_0^T \|A_q\,e^{-A_qt}(A_q^{-1}P_qu_0)\|_q^s\,dt\Big)^{1/s} + \|P_q\gamma\|_{L^s(L^q)} \qquad (2.45)\\
&\quad\le\ c\big(\|u_0\|_{\mathcal{J}_\sigma^{q,s}} + \|F\|_{L^s(L^r)} + \|k\|_{L^s(L^r)} + \|g\|_{L^s(W^{-1/q,q}(\partial\Omega))}\big).
\end{aligned}$$

Since $u = \hat u + \nabla H = A_qU + \nabla H$, we proved so far that u necessarily has the representation

$$u = \nabla H + A_q\,e^{-A_qt}(A_q^{-1}P_qu_0) + \int_0^t A_qe^{-A_q(t-\tau)}P_q\gamma(\tau)\,d\tau. \qquad (2.46)$$

Hence u is uniquely defined by the data F, k, g and u_0 and satisfies (2.36) in the very weak sense, since we may pass through the previous computations in reverse order. Finally, (2.45) and (2.46) imply (2.39). □

REMARK 2.15. *The very weak solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ constructed in Theorem 2.14 has a trace $u|_{\partial\Omega} \in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big)$. Actually, since $k = \operatorname{div} u \in L^s\big(0,T;L^r(\Omega)\big)$, we get that $u\cdot N|_{\partial\Omega} \in L^s\big(0,T;W^{-1/r,r}(\partial\Omega)\big)$ and even*

$$u\cdot N|_{\partial\Omega} = g\cdot N \in L^s\big(0,T;W^{-1/q,q}(\partial\Omega)\big).$$

Concerning the tangential component of u on $\partial\Omega$ we consider $h \in C_0^1\big((0,T);W_\tau^{1-1/q',q'}(\partial\Omega)\big)$ and $w = E_\tau(h) \in C_0^1\big((0,T);Y_\sigma^{2,q'}(\Omega)\big)$ satisfying $h = N\cdot\nabla w|_{\partial\Omega}$, cf. Theorem 2.4. Inserting w in (2.37) we obtain the formula

$$\langle g,h\rangle_{\partial\Omega,T} = (u,w_t)_{\Omega,T} + (u,\Delta w)_{\Omega,T} - (F,\nabla w)_{\Omega,T}.$$

This formula yields a well-defined expression for the tangential component $g_\tau = g-(g\cdot N)N$ of the boundary values. Obviously, if u is sufficiently smooth, integration by parts shows that $u_\tau|_{\partial\Omega} = g_\tau$.

4. The instationary Navier–Stokes system

Let us consider the instationary Navier–Stokes system

$$\begin{aligned}
u_t - \nu\Delta u + \operatorname{div}(uu) + \nabla p = f, \quad \operatorname{div} u = k \text{ in } \Omega\times(0,T)\\
u(0) = u_0 \text{ at } t = 0, \quad u = g \text{ on } \partial\Omega\times(0,T).
\end{aligned} \qquad (2.47)$$

DEFINITION 2.16. Let the data F, k, g satisfy (2.34), (2.35) and let $u_0 \in \mathcal{J}_\sigma^{q,s}(\Omega)$ where

$$2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1 \quad\text{and}\quad \frac{1}{3} + \frac{1}{q} \ge \frac{1}{r} \ge \frac{2}{q}\,. \qquad (2.48)$$

Then $u \in L^s\big(0,T;L^q(\Omega)\big)$ is called a very weak solution of (2.47) if for all test functions $w \in C_0^1\big([0,T);C_{0,\sigma}^2(\overline{\Omega})\big)$

$$\begin{aligned}
&-(u,w_t)_{\Omega,T} - \nu(u,\Delta w)_{\Omega,T} - (uu,\nabla w)_{\Omega,T}\\
&\qquad = -(F,\nabla w)_{\Omega,T} - \nu\langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} + \langle u_0, w(0)\rangle, \qquad (2.49)\\
&\operatorname{div} u = k \text{ in } \Omega\times(0,T),\quad u\cdot N|_{\partial\Omega} = g\cdot N \text{ on } \partial\Omega\times(0,T).
\end{aligned}$$

REMARK 2.17. *(1) In (2.48) we added the condition* $\mathcal{S}(s,q) = \frac{2}{s} + \frac{3}{q} = 1$ *in order to allow an estimate of the nonlinear term* $(uu, \nabla w)_{\Omega,T}$. *Compared to (1.19) in Definition 1.8 the assumptions on* q, r, s *are a little bit weaker in (2.48).*

(2) Looking at [12], [19] we omitted the term $(-k, uw)_{\Omega,T}$ *on the left-hand side of* $(2.49)_1$ *leading to some simplifications in the proof, cf. Remarks 1.9 and 2.8.*

THEOREM 2.18. *Given data* F, k, g, u_0 *as in Definition 2.16 there exists some* $T' = T'(\nu, F, k, g, u_0) \in (0,T]$ *and a unique very weak solution* $u \in L^s\big(0,T'; L^q(\Omega)\big)$ *of the Navier–Stokes system (2.47). Moreover,* u *satisfies*

$$u_t \in L^{s/2}_{\text{loc}}\big([0,T'); Y^{-2,q}_\sigma(\Omega)\big),$$

and the interval of existence, $[0,T')$, *is determined by the condition*

$$\begin{aligned}\Big(\int_0^{T'} \|\nu A_q e^{-\nu t A_q}(A_q^{-1}P_q u_0)\|_q^s\,dt\Big)^{1/s} + \|F\|_{L^s(0,T';L^r)} \\ +\|\nu k\|_{L^s(0,T';L^r)} + \|\nu g\|_{L^s(0,T';W^{-1/q,q}(\partial\Omega))} \le \varepsilon_* \nu^{2-1/s}.\end{aligned} \tag{2.50}$$

We note that the first term in (2.50) coincides with $\|u_0\|_{\mathcal{J}^{s,q}_\sigma}$ except for the interval of integration $(0,T')$ and the viscosity $\nu > 0$. If $T = \infty$, the case $T' = \infty$ is possible provided the data F, k, g, u_0 are sufficiently small. Formally, (2.50) contains the smallness condition (2.28) in the case $s = \infty$ which, however, is excluded by (2.48).

PROOF OF THEOREM 2.18. Let $\gamma(t) = \gamma_{F(t),k(t),g(t),u_0}$ denote the unique very weak solution in $L^s\big(0,T;L^q(\Omega)\big)$ of the linear system

$$\begin{aligned}\tfrac{\partial\gamma}{\partial t} - \nu\Delta\gamma + \nabla p &= \operatorname{div} F, & \operatorname{div}\gamma &= k \text{ in } \Omega\times(0,T),\\ \gamma(0) &= u_0, & \gamma &= g \text{ on } \partial\Omega\times(0,T),\end{aligned}$$

as constructed in Section 3 when $\nu = 1$. Obviously Theorem 2.14 extends to the case of a general viscosity $\nu > 0$, and the *a priori* estimate (2.39) reads as follows:

$$\begin{aligned}\|\nu\gamma\|_{L^s(0,T',L^q)} \le c\Big(\Big(\int_0^{T'} \|\nu A_q e^{-\nu\tau A_q}(A_q^{-1}P_q u_0)\|_q^s\,d\tau\Big)^{1/s} \\ +\|F\|_{L^s(0,T';L^r)} + \|\nu k\|_{L^s(0,T';L^r)} + \|\nu g\|_{L^s(0,T';W^{-1/q,q}(\partial\Omega))}\Big)\end{aligned} \tag{2.51}$$

for every $T' \in (0,T]$ with a constant $c = c(q,r,s,\Omega) > 0$ independent of $\nu > 0$ and T'.

Assume that $u \in L^s\big(0,T';L^q(\Omega)\big)$ is a very weak solution of (2.47). Then $\tilde u = u - \gamma$ is a very weak solution of the system

$$\begin{aligned}\tilde u_t - \nu\Delta\tilde u + \nabla p &= -\operatorname{div}(uu), & \operatorname{div}\tilde u &= 0 \text{ in } \Omega\times(0,T')\\ \tilde u &= 0 \text{ at } t=0, & \tilde u &= 0 \text{ on } \partial\Omega\times(0,T')\end{aligned} \tag{2.52}$$

with the right-hand side $-\operatorname{div}(uu) = -\operatorname{div}\big((\tilde u+\gamma)(\tilde u+\gamma)\big)$. Since $2r \le q$, we get $\|(\tilde u+\gamma)(\tilde u+\gamma)(t)\|_r \le c\,\|\tilde u+\gamma\|_q^2$ for a.a $t \in (0,T')$ and consequently $(\tilde u+\gamma)(\tilde u+\gamma) \in L^{s/2}\big(0,T';L^r(\Omega)\big)$, cf. (2.30). Hence by Theorem 2.14, $\tilde u$ in (2.52) is the unique very weak solution in $L^{s/2}\big(0,T';L^q(\Omega)\big)$ and

$$\tilde u(t) = \mathcal{N}(\tilde u)(t) := -\int_0^t A_q e^{-\nu A_q(t-\tau)} A_q^{-1} P_q \operatorname{div}(\tilde u+\gamma)(\tilde u+\gamma)(\tau)\,d\tau \tag{2.53}$$

for a.a. $t \in (0,T')$, cf. (2.46).

To find $\tilde u$ as the fixed point of the nonlinear map $\mathcal N$ in $L^s\big(0,T';L^q(\Omega)\big)$ we estimate $\mathcal N(\tilde u)$. Let $\alpha = \frac12 - \frac1s$ so that $2\alpha + \frac3q = \frac{3}{q/2}$ since $\frac2s + \frac3q = 1$. Then by Lemma 1.11 (4), (3) and (2.24)

$$\begin{aligned} \|\mathcal N(\tilde u)(t)\|_q &\le c\int_0^t \frac{1}{\big(\nu(t-\tau)\big)^{1/2+\alpha}} \|A_q^{1/2-\alpha}A_q^{-1}P_q \operatorname{div}(uu)(\tau)\|_q\,d\tau \\ &\le c\int_0^t \frac{1}{\big(\nu(t-\tau)\big)^{1-1/s}} \|A_{q/2}^{1/2}A_{q/2}^{-1}P_{q/2} \operatorname{div}(uu)(\tau)\|_{q/2}\,d\tau \\ &\le c\int_0^t \frac{1}{\big(\nu(t-\tau)\big)^{1-1/s}} \|u(\tau)\|_q^2\,d\tau. \end{aligned}$$

Next we use the Hardy–Littlewood inequality, see [61, p. 103],

$$\left(\int_0^T \Big|\int_0^t \frac{1}{(t-\tau)^{1-1/s}} h(\tau)\,d\tau\Big|^s dt\right)^{1/s} \le c\,\|h\|_{L^{s/2}(0,T)},$$

where $c = c(s) > 0$ is independent of T. Hence there exists a constant $c = c(q,r,s,\Omega) > 0$ independent of T' such that

$$\begin{aligned} \|\mathcal N(\tilde u)\|_{L^s(0,T';L^q)} &\le \frac{c}{\nu^{1-1/s}}\|u\|^2_{L^s(0,T';L^q)} \\ &\le \frac{c}{\nu^{1-1/s}}\left(\|\tilde u\|^2_{L^s(0,T';L^q)} + \|\gamma\|^2_{L^s(0,T';L^q)}\right). \end{aligned}$$

By analogy, we prove for $u' \in L^s\big(0,T';L^q(\Omega)\big)$ and $\tilde u' = u' - \gamma$ that

$$\begin{aligned} &\|\mathcal N(\tilde u) - \mathcal N(\tilde u')\|_{L^s(0,T';L^q)} \\ &\le \frac{c}{\nu^{1-1/s}}\|\tilde u - \tilde u'\|_{L^s(0,T';L^q)}\big(\|u\|_{L^s(0,T';L^q)} + \|u'\|_{L^s(0,T';L^q)}\big). \end{aligned} \tag{2.54}$$

Now we may proceed as in the proof of Theorem 2.9. Let $a = \frac{c}{\nu^{1-1/s}}$ and $b = \frac{c}{\nu^{1-1/s}}\|\gamma\|^2_{L^2(0,T';L^q)}$. The smallness condition $4ab < 1$ is equivalent to the estimate $\|\nu\gamma\|_{L^s(0,T':L^q)} \le \varepsilon_* \nu^{2-1/s}$, so that in view of (2.51) the condition (2.50) is sufficient to guarantee that $4ab<1$. Since (2.51) holds for $T' \in (0,T)$ sufficiently small (or even for $T' = T = \infty$), Banach's Fixed Point Theorem proves the existence of a unique solution to the equation $\tilde u = \mathcal N(\tilde u)$ in a sufficiently small closed ball of $L^s\big(0,T';L^q(\Omega)\big)$.

Let us write (2.53) in the form

$$A_q^{-1}\tilde u(t) = -\int_0^t e^{-\nu(t-\tau)A_q} A_q^{-1}P_q \operatorname{div}(uu)(\tau)\,d\tau,\ 0 \le t \le T'.$$

Then by the maximal regularity estimate (1.28) and (2.22)

$$\begin{aligned} \|\big(A_q^{-1}\tilde u(\cdot)\big)_t\|_{L^{s/2}(0,T',L^q)} &\le c\,\|A_q^{-1}P_q \operatorname{div}(uu)\|_{L^{s/2}(0,T';L^q)} \\ &\le c\,\|uu\|_{L^{s/2}(0,T';L^r)} \\ &\le c\left(\|\tilde u\|^2_{L^{s/2}(0,T';L^q)} + \|\gamma\|^2_{L^{s/2}(0,T';L^q)}\right) \end{aligned}$$

so that $\tilde{u}_t \in L^{s/2}\big(0,T';Y_\sigma^{-2,q}(\Omega)\big)$. Since by Theorem 2.14 $\gamma_t \in L^s\big(0,T;Y_\sigma^{-2,q}(\Omega)\big)$, we conclude that $u_t \in L^{s/2}\big(0,T';Y_\sigma^{-2,q}(\Omega)\big)$. Moreover, it is easily seen that $u = \tilde{u} + \gamma$ is a very weak solution of the Navier–Stokes system (2.47).

Finally we prove that u is the unique very weak solution of (2.47) in all of $L^s\big(0,T';L^q(\Omega)\big)$. Assume that $v \in L^s\big(0,T';L^q(\Omega)\big)$ is also a very weak solution to (2.47). Then $U = u - v \in L^s\big(0,T';L^q(\Omega)\big)$ is a very weak solution to the system

$$\begin{aligned} U_t - \nu\Delta U + \nabla P &= -\operatorname{div}(Uu + vU), \quad \operatorname{div} U = 0 \text{ in } \Omega \times (0,T') \\ U &= 0 \text{ at } t = 0, \qquad\qquad U = 0 \text{ on } \partial\Omega \times (0,T'). \end{aligned}$$

Using similar estimates as in the derivation of (2.54) we get that for all $T'' \in (0,T')$

$$\|U\|_{L^s(0,T'';L^q)} \le \frac{c}{\nu^{1-1/s}} \|U\|_{L^s(0,T'';L^q)} \big(\|u\|_{L^s(0,T'';L^q)} + \|v\|_{L^s(0,T'';L^q)}\big) \tag{2.55}$$

with a constant $c > 0$ independent of T''. Hence there exists some $T'' \in (0,T')$ depending on u, v such that (2.55) is reduced to the inequality $\|U\|_{L^s(0,T'';L^q)} \le \frac{1}{2}\|U\|_{L^s(0,T'';L^q)}$ and that consequently $U = 0$, $u = v$ holds on $[0,T'']$. This argument may be repeated finitely many times with the same T'' on the intervals $(T'', 2T'')$, $(2T'', 3T'')$ etc. and finally leads to $u = v$ on $[0,T')$. Now the proof of Theorem 2.18 is complete. □

CHAPTER 3

Regularity of weak solutions

Let u be a weak solution of the instationary Navier–Stokes system

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p &= f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega\times(0,T) \\ u\big|_{\partial\Omega} &= 0, \quad u(0) = u_0 \quad \text{at } t=0, \end{aligned} \tag{3.1}$$

in the bounded domain $\Omega \subset \mathbb{R}^3$. Besides the classical Serrin condition

$$u \in L^s\big(0,T;L^q(\Omega)\big), \quad \mathcal{S}(s,q) \le 1,\ 2 < s \le \infty,\ 3 \le q < \infty, \tag{3.2}$$

cf. (1.14) in Theorem 1.5, there are numerous other assumptions of *conditional regularity* imposed on specific components of u, ∇u or $\omega = \mathrm{r}ot\, u$ to imply regularity of u. Most of these conditions are related to (3.2) with a different upper bound for $\mathcal{S}$, cf. [9], [42], [43], [49], [50]; other conditions have a more geometric character, see [4], [10], [44], [45], [46], or are related to the pressure [5], [56], [68]. In the following we describe new results of Serrin's type, i.e., we assume

$$u \in L^r\big(0,T;L^q(\Omega)\big)$$

where $\frac{2}{r}+\frac{3}{q}$ is allowed to be larger than 1 such that u is regular locally or globally in time or locally in space and time. The proofs are based on a local or global identification of the weak solution u with a very weak solution v having the same initial value at $t_0 \ge 0$ and the same boundary value as u.

1. Local in time regularity

In addition to the definition of the global regularity in $(0,T)$, see (1.12), we say that u *is regular at* $t \in (0,T)$ if there exists $0 < \delta' < \min(t, T-t)$, such that

$$u \in L^{s_*}\big(t-\delta', t+\delta'; L^{q_*}(\Omega)\big), \quad \mathcal{S}(s_*,q_*) = 1,\ 2 < s_* < \infty,\ 3 < q_* < \infty. \tag{3.3}$$

By analogy, u *is regular in* $(a,b) \subset (0,T)$, if u is regular at every $t \in (a,b)$. Note that in Sections 1–3 we will use the notation s_*, q_* for exponents satisfying $\mathcal{S}(s_*,q_*) = 1$, but s, q if $\mathcal{S}(s,q) \ge 1$ is allowed.

Now our first result, see also [17], [18], reads as follows:

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$, and let*

$$2 < s_* < \infty, \quad 3 < q_* < \infty, \quad \mathcal{S}(s_*,q_*) = 1, \quad \frac{1}{3} + \frac{1}{q_*} = \frac{1}{\rho}, \quad 1 \le s \le s_*\,. \tag{3.4}$$

Given data

$$f = \operatorname{div} F,\ F \in L^2\big(0,T;L^2(\Omega)\big) \cap L^{s_*}\big(0,T;L^\rho(\Omega)\big) \quad \text{and} \quad u_0 \in L^2_\sigma(\Omega), \tag{3.5}$$

let u be a weak solution of the Navier–Stokes system (3.1) satisfying the strong energy inequality (1.9) on $[0,T)$, where $0 < T \le \infty$.

(1) Left-side $L^{s_*}(L^{q_*})$-condition: *If for $t \in (0,T)$*

$$u \in L^{s_*}\big(t-\delta, t; L^{q_*}(\Omega)\big) \quad \textit{for some } 0 < \delta = \delta(t) < t, \tag{3.6}$$

then u is regular at t.

(2) Left-side $L^{s}(L^{q_*})$-condition: *If at $t \in (0,T)$*

$$\liminf_{\delta \to 0+} \frac{1}{\delta} \int_{t-\delta}^{t} \|u(\tau)\|_{q_*}^{s}\, d\tau < \infty, \tag{3.7}$$

then u is regular at t. Assumption (3.7) may be replaced by the essentially weaker condition

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{1-s/s_*}} \int_{t-\delta}^{t} \|u(\tau)\|_{q_*}^{s}\, d\tau = 0, \tag{3.8}$$

which includes (3.6) when $s = s_*$. Moreover, (3.8) is even a necessary condition for regularity of u at t.

(3) Global $L^{s}(L^{q_*})$-condition. *There exists a constant $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$ independent of u, u_0, f and ν with the following property: If $u_0 \in L^{q_*}_\sigma(\Omega)$, $u \in L^s\big(0,T; L^{q_*}(\Omega)\big)$,*

$$\int_0^T \|F(\tau)\|_\rho^{s_*}\, d\tau \le \varepsilon_* \nu^{2s_*-1} \quad \textit{and} \quad \int_0^T \|u(\tau)\|_{q_*}^{s}\, d\tau < \varepsilon_* \frac{\nu^{s_*-1}}{\|u_0\|_{q_*}^{s_*-s}}, \tag{3.9}$$

then u is regular in the sense $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$.

The proof of Theorem 3.1 is based on a key lemma, see Lemma 3.2, combining the notions of weak and very weak solutions, and on a technical lemma, see Lemma 3.4, from which the results of Theorem 3.1 and also of Section 2 will follow easily.

Lemma 3.2. *In addition to the assumptions of Theorem 3.1 assume $u_0 \in L^{q_*}_\sigma(\Omega)$. Then there exists a constant $\varepsilon_* = \varepsilon_*(q_*, \Omega) > 0$ independent of u_0, f and ν with the following property: If*

$$\int_0^T \|F\|_\rho^{s_*}\, d\tau \le \varepsilon_* \nu^{2s_*-1} \quad \textit{and} \quad \int_0^T \|e^{-\nu\tau A_{q_*}} u_0\|_{q_*}^{s_*}\, d\tau \le \varepsilon_* \nu^{s_*-1}, \tag{3.10}$$

then the Navier–Stokes system (3.1) has a unique weak solution u in the sense of Leray and Hopf satisfying Serrin's condition $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ and moreover the energy inequality (1.8).

We note that the weak solution $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ constructed in Lemma 3.2 even satisfies the energy identity (1.3), see Lemma 1.6 (1).

Proof of Lemma 3.2. Given the smallness condition (3.10) Theorem 2.18 yields a unique very weak solution $u \in L^{s_*}\big(0,T; L^{q_*}(\Omega)\big)$ of (3.1). Moreover,

$$u(t) = \gamma(t) + \tilde{u}(t)$$

where γ solves the instationary Stokes system with data u_0, f in $\Omega \times (0,T)$, i.e.

$$\gamma(t) = e^{-\nu t\, A_{q_*}} u_0 + \int_0^t A_{q_*} e^{-\nu(t-\tau)A_{q_*}}\, A_{q_*}^{-1} P_{q_*} \operatorname{div} F(\tau)\, d\tau\ , \tag{3.11}$$

and where $\tilde{u}$ solves the nonlinear equation

$$\tilde{u}(t) = -\int_0^t A_{q_*/2}^{1/2} e^{-\nu(t-\tau)A_{q_*/2}} A_{q_*/2}^{-1/2} P_{q_*/2} \mathrm{div}\,(uu)\,d\tau. \tag{3.12}$$

Since $F \in L^2\big(0,T;L^2(\Omega)\big)$ and $u_0 \in L^2_\sigma(\Omega)$, we see that γ is the weak solution of the instationary Stokes system; in particular,

$$\gamma \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big).$$

The major part of the proof concerns the property

$$\tilde{u} \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H_0^1(\Omega)\big) \tag{3.13}$$

so that $u = \gamma + \tilde{u} \in L^{s_*}\big(0,T;L^{q_*}(\Omega)\big)$ is a weak solution in the sense of Leray and Hopf. Hence u satisfies the energy (in-)equality, and Serrin's Uniqueness Theorem 1.2 shows that u is the unique weak solution with these properties.

To prove (3.13) we recall from (2.24) that

$$\|A_{q_*/2}^{-1/2} P_{q_*} \mathrm{div}\,(uu)\|_{q_*/2} \le c\|uu\|_{q_*/2} \le c\|u\|_{q_*}^2 \quad \text{for a.a. } t \in (0,T). \tag{3.14}$$

Consequently, (3.12) implies the identity

$$A_{q_*/2}^{1/2}\tilde{u}(t) = -A_{q_*/2}\Big(\int_0^t e^{-\nu(t-\tau)A_{q_*/2}} A_{q_*/2}^{-1/2} P_{q_*/2}\,\mathrm{div}\,(uu)\,d\tau\Big). \tag{3.15}$$

Now the maximal regularity estimate (1.28), Lemma 1.11 (3) and (3.14) yield the estimate

$$\begin{aligned} \nu\|\nabla\tilde{u}\|_{L^{s_*/2}(L^{q_*/2})} &\le c\nu\|A_{q_*/2}^{1/2}\tilde{u}\|_{L^{s_*/2}(L^{q_*/2})} \\ &\le c\|uu\|_{L^{s_*/2}(L^{q_*/2})} \le c\|u\|_{L^{s_*}(L^{q_*})}^2 \end{aligned} \tag{3.16}$$

and particularly the result

$$\nabla\tilde{u} \in L^{s_*/2}\big(0,T;L^{q_*/2}(\Omega)\big). \tag{3.17}$$

We will consider four cases concerning the exponent s_*, starting with the case $2 < s_* < 4$ (and $q_* > 6$). Let $s_1 = s_*$, $q_1 = q_*$. Then (3.12) and (1.24) (with $\alpha = \frac{1}{2}$) imply that

$$\|\tilde{u}(t)\|_{q_1/2} \le \frac{c}{\sqrt{\nu}} \int_0^t \frac{1}{(t-\tau)^{1/2}}\,\|uu\|_{q_1/2}\,d\tau,$$

where $\|uu(\tau)\|_{q_1/2} \in L^{s_1/2}(0,T)$. Hence the Hardy-Littlewood inequality proves with

$$\frac{1}{s_2} = \frac{1}{s_1/2} - \frac{1}{2}, \quad q_2 = \frac{q_1}{2}$$

that

$$\tilde{u} \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big).$$

Here $\frac{2}{s_2} + \frac{3}{q_2} = 1$ since $\frac{2}{s_1} + \frac{3}{q_1} = 1$, and $s_2 > s_1$, $q_2 < q_1$. To get the same result for γ, note that

$$\gamma_1(t) := e^{-\nu t A_{q_*}} u_0 \in L^\infty\big(0,T;L^{q_*}(\Omega)\big) \subset L^{s_2}\big(0,T;L^{q_2}(\Omega)\big).$$

Concerning $\gamma_2(t) = \gamma(t) - \gamma_1(t)$, the second term on the right-hand side of (3.11), we use (1.23) with $\alpha = \frac{1}{s_1}$ and conclude, since $A_\rho^{-1/2} P_\rho \operatorname{div} F \in L^\rho(\Omega)$, see (2.24), that

$$v := A_\rho^{-1/s_1} A_\rho^{-1/2} P_\rho \operatorname{div} F \in L^{s_1}\big(0, T; L^{q_2}(\Omega)\big).$$

Hence $\gamma_2(t)$ satisfies the estimate

$$\|\gamma_2(t)\|_{q_2} \leq c_\nu \int_0^t \frac{1}{(t-\tau)^{1/2+1/s_1}} \|v(\tau)\|_{q_2} \, d\tau,$$

from which we deduce by the Hardy-Littlewood inequality that $\gamma_2 \in L^{s_2}\big(0, T; L^{q_2}(\Omega)\big)$; here we used that $\frac{1}{2} + \frac{1}{s_1} = 1 - \big(\frac{1}{s_1} - \frac{1}{s_2}\big)$.

Summarizing the results for γ_1 and γ_2 we get that $\gamma \in L^{s_2}\big(0, T; L^{q_2}(\Omega)\big)$ so that also $u \in L^{s_2}\big(0, T; L^{q_2}(\Omega)\big)$ and

$$\nabla \tilde{u} \in L^{s_2/2}\big(0, T; L^{q_2/2}(\Omega)\big),$$

cf. (3.17). Repeating this step finitely many times, we finally arrive at exponents $s_k \in [4, \infty)$, $q_k \in (3, 6]$. The problem of exponents $s \geq 4$, $q \leq 6$ will be considered in the following three cases.

Now let $s_* = 4$, $q_* = 6$. In this special case (3.16) yields $\nabla \tilde{u} \in L^2\big(0, T; L^2(\Omega)\big)$. Since by (3.14)

$$A_{q_*/2}^{-1/2} P_{q_*/2} \operatorname{div}(uu) \in L^{s_*/2}\big(0, T; L^{q_*/2}(\Omega)\big) \subset L^2\big(0, T; L^2(\Omega)\big),$$

we may consider $A_{q_*/2}^{-1/2} \tilde{u}$ as the strong solution of the instationary Stokes system with an external force in $L^2\big(0, T; L^2(\Omega)\big)$ and vanishing initial value. Hence

$$\tilde{u} = A_{q_*/2}^{1/2} A_{q_*/2}^{-1/2} \tilde{u} \in L^\infty\big(0, T; L^2(\Omega)\big)$$

and $\nabla \tilde{u} \in L^2\big(0, T; L^2(\Omega)\big)$ so that $u = \gamma + \tilde{u}$ satisfies

$$u \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2\big(0, T; H_0^1(\Omega)\big).$$

Moreover, since $u \in L^{s_*}\big(0, T; L^{q_*}(\Omega)\big)$, we see that $uu \in L^2\big(0, T; L^2(\Omega)\big)$. An elementary calculation shows that u is not only a very weak solution, but also a weak one in the sense of Leray and Hopf. Hence u is even a regular solution by Theorem 1.5 and satisfies the energy (in-)equality. Furthermore, the uniqueness assertion follows from Theorem 1.2.

Next let $4 < s_* \leq 8$ (and $4 \leq q_* < 6$) so that (3.17) immediately yields $\nabla \tilde{u} \in L^2\big(0, T; L^2(\Omega)\big)$ and $\tilde{u} \in L^2\big(0, T; H_0^1(\Omega)\big)$. Applying (1.24) and (3.14) to (3.12), Hölder's inequality implies the estimate

$$\begin{aligned}
\|\tilde{u}(t)\|_2 &\leq \frac{c}{\sqrt{\nu}} \int_0^t \frac{1}{(t-\tau)^{1/2}} e^{-\nu\delta(t-\tau)} \|uu\|_2 \, d\tau \\
&\leq \frac{c}{\sqrt{\nu}} \int_0^t \frac{1}{(t-\tau)^{1/2}} e^{-\nu\delta(t-\tau)} \|uu\|_{q_*/2} \, d\tau \\
&\leq c\nu^{-1+2/s_*} \|uu\|_{L^{s_*/2}(0,T;L^{q_*/2}(\Omega))} \\
&\leq c\nu^{-1+2/s_*} \|u\|^2_{L^{s_*}(0,T;L^{q_*}(\Omega))}.
\end{aligned}$$

Consequently, $\tilde{u}$ and even u belong to $L^\infty\big(0,T;L^2(\Omega)\big)$. Now we complete the proof as in the previous case.

Finally assume that $8 < s_* < \infty$ (and $3 < q_* < 4$). Now we need finitely many steps to reduce this case to the former one. Let $s_1 = s_*$ and $q_1 = q_*$. Then $\nabla\tilde{u} \in L^{s_1/2}\big(0,T;L^{q_1/2}(\Omega)\big)$ by (3.17). Defining $s_2 < s_1$, $q_2 > q_1$ by

$$s_2 = \frac{s_1}{2}\,, \quad \frac{1}{3} + \frac{1}{q_2} = \frac{2}{q_1}$$

we get by Sobolev's embedding theorem that $\tilde{u} \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big)$. By Lemma 1.11 we conclude that also $\gamma \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big)$ so that

$$u \in L^{s_2}\big(0,T;L^{q_2}(\Omega)\big),$$

where again $\frac{2}{s_2} + \frac{3}{q_2} = 1$. Repeating this step finitely many times, if necessary, we arrive at exponents $s_k \in (4,8]$, $q_k \in [4,6)$, i.e. in the previous case.

Now Lemma 3.2 is completely proved. □

COROLLARY 3.3. *In the situation of Lemma 3.2 assume that $T = \infty$. Then there exists a constant $\varepsilon_* = \varepsilon_*(q_*,\Omega) > 0$ with the following property: If*

$$\int_0^\infty \|F\|_\rho^{s_*}\,d\tau \le \varepsilon_*\nu^{2s_*-1} \quad \textit{and} \quad \|u_0\|_{q_*} \le \varepsilon_*\nu,$$

then the Navier–Stokes system (3.1) has a unique weak solution u in $\Omega \times (0,\infty)$ satisfying $u \in L^{s_*}\big(0,\infty;L^{q_*}(\Omega)\big)$ and the energy inequality.

PROOF. From (1.24) with $\alpha = 0$ we obtain that

$$\int_0^\infty \|e^{-\nu t\,A_{q_*}}u_0\|_{q_*}^{s_*}\,dt \le c\|u_0\|_{q_*}^{s_*}\int_0^\infty e^{-\nu s_*\delta_0 t}\,dt \le \frac{c}{\nu}\,\|u_0\|_{q_*}^{s_*}\,.$$

Now the result follows from Lemma 3.2 when using a different constant $\varepsilon_* = \varepsilon_*(q_*,\Omega) > 0$. □

The next lemma has a technical character, but will immediately imply the assertions of Theorem 3.1. We will use the notation

$$⨍_a^b h(\tau)\,d\tau = \frac{1}{b-a}\int_a^b h(\tau)\,d\tau$$

for the mean value of an integral.

LEMMA 3.4. *Under the assumptions of Theorem 3.1 there exists a constant $\varepsilon_* = \varepsilon_*(q_*,s,\Omega) > 0$ with the following property:*

If $0 < t_0 < t \le t_1 < T$, $0 \le \beta \le \frac{s}{s_*}$ and if

$$\int_{t_0}^{t_1} \|F\|_\rho^{s_*}\,d\tau \le \varepsilon_*\nu^{2s_*-1} \quad \textit{and} \quad ⨍_{t_0}^{t} (t_1-\tau)^\beta\,\|u\|_{q_*}^{s}\,d\tau \le \varepsilon_*\nu^{s-\beta}, \tag{3.18}$$

then u is regular in the interval $(t-\delta,t_1)$ for some $\delta > 0$ in the sense that $u \in L^{s_*}\big(t-\delta,t_1;L^{q_*}(\Omega)\big)$. In particular, if $t_1 > t$, then t is a regular point of u. If $\beta = 0$, then $t_1 = T \le \infty$ is allowed.

PROOF. From the second condition in (3.18) and the fact that u satisfies the strong energy inequality we find a null set $N \subset (t_0, t)$ such that for $\tau_0 \in (t_0, t)\backslash N$

$$\frac{1}{2}\|u(\tau_1)\|_2^2 + \nu \int_{\tau_0}^{\tau_1} \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(\tau_0)\|_2^2 + \int_{\tau_0}^{\tau_1} \langle f, u\rangle d\tau, \quad \tau_0 < \tau_1 < T, \tag{3.19}$$

and $u(\tau_0) \in L^{q_*}_\sigma(\Omega)$. Now, if we find $\tau_0 \in (t_0, t)\backslash N$ such that

$$\int_0^{t_1-\tau_0} \|e^{-\nu\tau A_{q_*}} u(\tau_0)\|_{q_*}^{s_*} \, d\tau \le \varepsilon_* \nu^{s_*-1}, \tag{3.20}$$

Lemma 3.2 will yield a unique weak solution $v \in L^{s_*}([\tau_0, t_1); L^{q_*}_\sigma(\Omega))$ to the Navier–Stokes system (3.1) with initial value $v(\tau_0) = u(\tau_0)$ at τ_0. Then (3.19) and Serrin's Uniqueness Theorem 1.2 show that

$$u = v \in L^{s_*}\big(\tau_0, t_1; L^{q_*}_\sigma(\Omega)\big)$$

and complete the proof.

To prove (3.20) note that the second condition in (3.18) yields the existence of $\tau_0 \in (t_0, t)\backslash N$ such that

$$(t_1 - \tau_0)^\beta \, \|u(\tau_0)\|_{q_*}^s \le \fint_{t_0}^t (t_1 - \tau)^\beta \, \|u(\tau)\|_{q_*}^s \, d\tau \le \varepsilon_* \nu^{s-\beta}\,; \tag{3.21}$$

otherwise $(t_1 - \tau)^\beta \, \|u(\tau)\|_{q_*}^s$ is strictly larger than $\fint_{t_0}^t (t_1 - \tau)^\beta \, \|u\|_{q_*}^s \, d\tau$ for every $\tau \in (t_0, T)\backslash N$, and we are led to a contradiction. Now, by Lemma 1.11, Hölder's inequality and (3.21),

$$\begin{aligned} \int_0^{t_1-\tau_0} \|e^{-\nu\tau A_{q_*}} u(\tau_0)\|_{q_*}^{s_*} \, d\tau \quad &\le \quad \int_0^{t_1-\tau_0} e^{-\delta_0 \nu s_* \tau} \, d\tau \, \|u(\tau_0)\|_{q_*}^{s_*} \\ &\le \quad c(t_1 - \tau_0)^{\beta s_*/s} \, \nu^{-1+\beta s_*/s} \|u(\tau_0)\|_{q_*}^{s_*} \\ &\le \quad c\, \varepsilon_*^{s_*/s} \, \nu^{s_*-1}. \end{aligned}$$

Hence, with a new constant $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$, (3.20) is proved. If $\beta = 0$, then $t_1 = T \le \infty$ is admitted. □

PROOF OF THEOREM 3.1. (1) Assuming (3.6) we choose $s = s_*$, $\beta = \frac{s}{s_*} = 1$. Furthermore, let $t_0 = t - \delta$, $t_1 = t + \delta$, where $\delta > 0$ is chosen so small that

$$\fint_{t-\delta}^t (t_1 - \tau)\|u\|_{q_*}^s \, d\tau \le 2 \int_{t-\delta}^t \|u\|_{q_*}^s \, d\tau \le \varepsilon_* \nu^{s-\beta}$$

and

$$\int_{t-\delta}^t \|F\|_\rho^{s_*} \, d\tau \le \varepsilon_* \nu^{2s_*-1}.$$

Then Lemma 3.4 implies that u is regular at t.

(2) Given (3.8) let $t_0 = t - \delta$, $t_1 = t + \delta$ such that with $\beta = \frac{s}{s_*}$

$$\fint_{t-\delta}^t (t_1 - \tau)^\beta \|u\|_{q_*}^s \, d\tau \le 2^\beta \frac{1}{\delta^{1-\beta}} \int_{t-\delta}^t \|u\|_{q_*}^s \, d\tau.$$

By (3.8) we find $\delta > 0$ such that the second condition of (3.18) is satisfied. Obviously, the condition on F in (3.18) can be fulfilled as well. Then Lemma 3.4 proves the sufficiency of (3.8) to imply regularity of u at t. The necessity of (3.8) is a simple consequence of Hölder's inequality.

(3) Given the initial value $u_0 \in L^{q_*}_\sigma(\Omega)$, Lemma 3.2 yields a unique weak solution $v \in L^{s_*}\big(0,\delta_1; L^{q_*}_\sigma(\Omega)\big)$ for some $\delta_1 > 0$ which coincides with u on $[0,\delta_1)$ by Theorem 1.2. Moreover, the elementary estimate

$$\int_0^{\delta_1} \|e^{-\nu\tau A_{q_*}} u_0\|^{s_*}_{q_*}\, d\tau \le c\,\delta_1 \|u_0\|^{s_*}_{q_*}$$

and (3.10) imply that we may choose

$$\delta_1 = \frac{\varepsilon_* \nu^{s_*-1}}{c\|u_0\|^{s_*}_{q_*}}.$$

In Lemma 3.4 let $\beta = \frac{s}{s_*}$, $t_0 = t - \frac{\delta_1}{2}$ and $t_1 = t + \frac{\delta_1}{2}$ where $t \ge \delta_1$ is arbitrary. Then

$$⨍_{t_0}^{t} (t_1-\tau)^\beta \|u\|^s_{q_*}\, d\tau \le \frac{2}{\delta_1^{1-\beta}} \int_0^T \|u\|^s_{q_*}\, d\tau$$

which by (3.9) is smaller than

$$2\left(\frac{\varepsilon_*\nu^{s_*-1}}{c\|u_0\|^{s_*}_{q_*}}\right)^{\frac{s}{s_*}-1} \cdot \varepsilon_* \frac{\nu^{s_*-1}}{\|u_0\|^{s_*-s}_{q_*}} \;=\; c\,\varepsilon_*^{s/s_*} \nu^{s-s/s_*}.$$

Redefining ε_*, we see that (3.18) is fulfilled. Hence u is regular at every $t \in [\delta_1, T)$ by Lemma 3.4; more precisely, u is regular in $\big(t-\delta(t), t+\frac{\delta_1}{2}\big)$. This argument completes the proof when $T < \infty$.

If $T = \infty$, applying the previous result for each finite interval we obtain that $u \in L^{s_*}_{\mathrm{loc}}\big([0,\infty); L^{q_*}_\sigma(\Omega)\big)$. Due to (3.9) we find a sufficiently large τ_0 satisfying $\|u(\tau_0)\|_{q*} \le \varepsilon_*\nu$ and the energy inequality (3.19). Then Corollary 3.3 yields the existence of a unique weak solution $v \in L^{s_*}\big(\tau_0,\infty; L^{q_*}_\sigma(\Omega)\big)$ with $v(\tau_0) = u(\tau_0)$ which must coincide with u on $[\tau_0,\infty)$. This argument proves (3). □

COROLLARY 3.5. *Under the assumptions of Theorem 3.1 we have the following results:*

(1) There exists $\varepsilon_* = \varepsilon_*(q_*, s, \Omega) > 0$ such that u is regular for all $t \ge T_1$ where

$$T_1 > \frac{1}{\varepsilon_*\nu^s} \|u\|^s_{L^s(0,\infty;L^{q_*}(\Omega))} \tag{3.22}$$

provided that $u \in L^s\big(0,\infty; L^{q_*}_\sigma(\Omega)\big)$ and $\int_0^\infty \|F\|^{s_*}_\rho\, d\tau \le \varepsilon_*\nu^{2s_*-1}$.

(2) Assume that $t \in (0,T)$ is a singular point of the weak solution u in the sense that $u \notin L^{s_*}\big(t-\delta, t+\delta; L^{q_*}(\Omega)\big)$ for any $\delta > 0$. Then

$$\liminf_{\delta\to 0+} \frac{1}{\delta^{1-\beta}} \int_{t-\delta}^t \|u\|^s_{q_*} d\tau > 0 \quad \textit{for all } \beta \in \Big[0, \frac{s}{s_*}\Big] \tag{3.23}$$

and even

$$\lim_{\delta\to 0+} ⨍_{t-\delta}^t \|u\|^s_{q_*}\, d\tau = \infty. \tag{3.24}$$

The set of singular points of u is either empty or at least a set of Lebesgue measure zero, if $u \in L^s\big(0,T; L^{q^*}(\Omega)\big)$.

PROOF. (1) Let $\beta = 0$ in Lemma 3.4. Then by assumption

$$\lim_{t_0 \to 0+} \fint_{t_0}^{T_1} \|u\|_{q_*}^s \, d\tau < \varepsilon_* \nu^s,$$

and Lemma 3.4 yields the regularity of u for $t \geq T_1$.

(2) Let $t \in (0, T)$ be a singular point of u and assume that the left hand side of (3.23) is zero. Then, setting $t_0 = t - \delta$, $t_1 = t + \delta$ we conclude that there exists some sufficiently small $\delta > 0$ such that (3.18) is satisfied. Hence we get the contradiction that u is regular at t. If (3.24) does not hold, then $\liminf_{\delta \to 0+} \fint_{t-\delta}^{t} \|u\|_{q_*}^s \, d\tau < \infty$ and consequently $\liminf_{\delta \to 0+} \frac{1}{\delta^{1-\beta}} \int_{t-\delta}^{t} \|u\|_{q_*}^s \, d\tau = 0$ for $\beta \in \big(0, \frac{s}{s_*}\big]$ which is a contradiction to (3.23).

It is a simple consequence of Leray's Structure Theorem, see [24], that the Lebesgue measure of the set of singular points in time vanishes. Here we may also argue as follows if $u \in L^s\big(0, T; L^{q_*}_\sigma(\Omega)\big)$. By Lebesgue's Differentiation Theorem

$$\lim_{\delta \to 0+} \fint_{t-\delta}^{t} \|u\|_{q_*}^s \, d\tau = \|u(t)\|_{q_*}^s \quad \text{for almost all} \quad t \in (0, T).$$

Hence (3.24) can hold only on a Lebesgue null set. □

2. Energy–based criteria for regularity

Let u be a weak solution in the sense of Leray and Hopf satisfying the energy inequality. Assume that $f = 0$ and $0 \neq u_0 \in H^1_0(\Omega) \cap L^2_\sigma(\Omega)$ so that there exists an interval $[0, T)$ on which u is a strong solution and satisfies even the energy identity (1.3). Then the *kinetic energy*

$$E(t) = \frac{1}{2} \|u(t)\|_2^2$$

is a strictly decreasing continuous function of $t \in [0, T)$. However, at $t = T$ the energy identity could loose its validity; either the kinetic energy has a jump discontinuity downward at $t = T$ or $E(t)$ will be strictly less than the continuously decreasing function

$$-\nu \int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau + E(0)$$

for certain $t > T$ close to T. In the first case the jump must be downward since $\|u(t)\|_2$ is lower semicontinuous by (1.7). Assuming that $\|u(t)\|_2$ is continuous and decreasing in an open interval to the right of T, there are three possibilities: $E(T+) := \lim_{t \to T+} E(t)$ equals either $E(T)$ or

$$E(T) < E(T+) < E(T-),$$

where $E(T-) := \lim_{t \to T-} E(t)$, or $E(T+) = E(T-)$. The fourth possibility $E(T+) > E(T-)$ is excluded since u satisfies the energy inequality for $t \geq T$ as well; if we want to exclude this possibility at a further jump discontinuity $\tilde{T} > T$, we have to use the strong energy inequality. If u satisfies the strong energy inequality and T is an initial point in time where the energy inequality holds ($T = s$ in (1.9)), then necessarily $E(T+) = E(T)$; otherwise the other two possibilities cannot be ruled out.

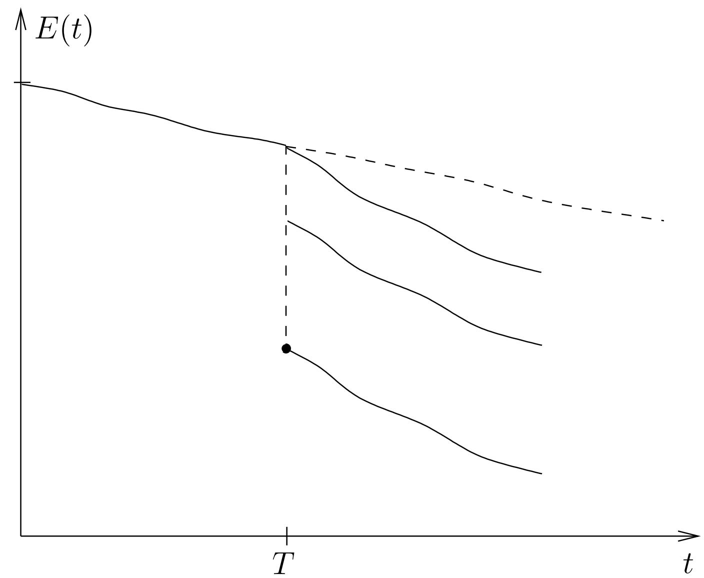

FIGURE 1. The kinetic energy $E(t)$ in the neighborhood of a jump discontinuity T.

In the following assume that $E(\cdot)$ is continuous in time, so that (1.7) implies $u \in C^0\big([0,T); L^2_\sigma(\Omega)\big)$ rather than only $u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big)$. Nevertheless we are not allowed to conclude that u is a regular solution. Actually, this conclusion is related to the modulus of continuity of the function $E(t)$ (or to that of the function $t \mapsto \|u(t)\|_2$ since $u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big)$).

THEOREM 3.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$ and let u be a weak solution of the instationary Navier–Stokes system (3.1) satisfying the strong energy inequality on $(0,T)$. The data u_0, f satisfy $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^{s_*/s}\big(0,T; L^2(\Omega)\big)$, $f = \operatorname{div} F$, $F \in L^2\big(0,T; L^2(\Omega)\big) \cap L^{s_*}\big((0,T; L^\rho(\Omega)\big)$ where ρ, s, s_* will be given in (3.29) below.*

(1) *Let $\alpha \in (\frac{1}{2}, 1)$ and let u satisfy at $t \in (0,T)$ the condition*

$$\sup_{t' \neq t} \frac{|E(t) - E(t')|}{|t-t'|^\alpha} < \infty$$

or only

$$\liminf_{\delta \to 0+} \frac{1}{\delta^\alpha} \, |E(t) - E(t-\delta)| < \infty, \tag{3.25}$$

where $E(\cdot)$ denotes the kinetic energy. Then u is regular at t.

(2) (*The case $\alpha = \frac{1}{2}$*) *There exists a constant $\varepsilon_* = \varepsilon_*(\Omega) > 0$ such that if*

$$\sup_{t' \neq t} \frac{|E(t) - E(t')|}{|t-t'|^{1/2}} \leq \varepsilon_* \, \nu^{5/2}$$

or only

$$\liminf_{\delta\to 0+} \frac{1}{\delta^{1/2}} \, |E(t) - E(t-\delta)| \le \varepsilon_* \, \nu^{5/2}, \tag{3.26}$$

then u is regular at $t \in (0,T)$.

REMARK 3.7. *(1) By Theorem 3.6 (1), Hölder continuity of the kinetic energy $E(\tau)$ from the left at t implies regularity at t if the Hölder exponent α is larger than $\frac{1}{2}$. In the case $\alpha = \frac{1}{2}$ the corresponding Hölder seminorm (from the left) is assumed to be sufficiently small. In both cases the function $E(\tau)$ may be replaced by the function $\|u(\tau)\|_2$.*

(2) The proof of Theorem 3.6, see (3.30), (3.31) below, will yield the following regularity criterion using $\|\nabla u\|_2$ instead of $\|u\|_2$. If

$$\alpha \in (\frac{1}{2}, 1) \quad \text{and} \quad \liminf_{\delta\to 0+} \frac{1}{\delta^{\alpha}} \int_{t-\delta}^{t} \|\nabla u(\tau)\|_2^2 \, d\tau < \infty \tag{3.27}$$

or

$$\alpha = \frac{1}{2} \quad \text{and} \quad \liminf_{\delta\to 0+} \frac{1}{\delta^{1/2}} \int_{t-\delta}^{t} \|\nabla u(\tau)\|_2^2 \, d\tau \le \varepsilon_* \, \nu^{5/2}, \tag{3.28}$$

then u is regular at t.

(3) In the case $\alpha = \frac{1}{2}$ a smallness condition as in (3.26) or (3.28) is necessary. Indeed, if $f = 0$ and $(0,t)$, $0 < t < \infty$, is a maximal regularity interval of u, then

$$\|\nabla u(\tau)\|_2 \ge \frac{c_0}{(t-\tau)^{1/4}} \ \ , 0 < \tau < t,$$

where $c_0 = c_0(\Omega) > 0$, see [24]. Hence

$$\liminf_{\delta\to 0+} \frac{1}{\delta^{1/2}} \int_{t-\delta}^{t} \|\nabla u\|_2^2 \, d\tau \ge 2\, c_0^2 > 0,$$

and due to the strong energy inequality,

$$\liminf_{\delta\to 0+} \frac{E(t-\delta) - E(t)}{\delta^{1/2}} \ge 2\,\nu\, c_0^2 > 0.$$

PROOF OF THEOREM 3.6. (see also [15] for the proof of (1)). The proof is based on Lemma 3.4 with $t_0 = t - \delta$, $t_1 = t + \delta$ and the exponents

$$\begin{cases} \text{if } \alpha > \frac{1}{2}: \ s = 4\alpha - \varepsilon > 2, \ \frac{2}{s} + \frac{3}{q_*} = \frac{3}{2}, \ \frac{2}{s_*} + \frac{3}{q_*} = 1, \ \beta = \frac{s}{s_*}, \\ \text{if } \alpha = \frac{1}{2}: \ s = 2, \ \varepsilon = 0, \ q_* = 6, \ s_* = 4, \ \beta = \frac{1}{2}. \end{cases} \tag{3.29}$$

In both cases the weak solution u satisfies $u \in L^s\big(0,T; L^{q_*}(\Omega)\big)$, cf. (1.11), and $1 - \frac{s}{s_*} = \frac{s}{4}$. To control the second term in (3.18) we will use the interpolation inequality

$$\|u\|_{q_*} \le c\, \|u\|_2^{1-2/s} \, \|\nabla u\|_2^{2/s}, \ c = c\,(q_*, \Omega) > 0,$$

and get that

$$\begin{aligned} I(\delta) := \int_{t_0}^{t} (t_1-\tau)^{\beta} \|u\|_{q_*}^{s}\, d\tau &\le 2^{\beta}\, \delta^{\beta-1} \int_{t-\delta}^{t} \|u\|_{q_*}^{s}\, d\tau \\ &\le c\, \delta^{-s/4} \int_{t-\delta}^{t} \|\nabla u\|_2^2\, \|u\|_2^{s-2}\, d\tau \qquad (3.30) \\ &\le c\, \|u\|_{L^\infty(L^2)}^{s-2}\, \delta^{-s/4} \int_{t-\delta}^{t} \|\nabla u\|_2^2\, d\tau\,. \end{aligned}$$

Since u is supposed to satisfy the strong energy inequality, we may proceed for almost all $\delta > 0$ as follows:

$$\begin{aligned} I(\delta) &\le \frac{c}{\nu}\, \delta^{-s/4} \Big(|E(t-\delta) - E(t)| + \Big| \int_{t-\delta}^{\delta} (f,u)\, d\tau \Big| \Big) \\ &= \frac{c}{\nu}\, \delta^{\varepsilon/4} \left(\frac{|E(t-\delta)-E(t)|}{\delta^{\alpha}} + \Big| \frac{1}{\delta^{\alpha}} \int_{t-\delta}^{t} (f,u)\, d\tau \Big| \right), \qquad (3.31) \end{aligned}$$

where the constant c depends on $\|u_0\|_2$ when $\alpha > \frac{1}{2}$.

First consider the case $\alpha > \frac{1}{2}$ in which $\varepsilon > 0$. Then

$$\Big| \frac{1}{\delta^{s/4}} \int_{t-\delta}^{t} (f,u)\, d\tau \Big| \le \frac{c}{\delta^{s/4}} \int_{t-\delta}^{t} \|f\|_2\, d\tau \le c \left(\int_{t-\delta}^{t} \|f\|_2^{4/(4-s)}\, d\tau \right)^{(4-s)/4}.$$

Hence, if $f \in L^{4/(4-s)}\big(0,T;L^2(\Omega)\big)$, the left-hand term in the previous inequality converges to 0 as $\delta \to 0+$. Moreover, due to the assumption (3.25), the term

$$\frac{c}{\nu}\, \delta^{\varepsilon/4} \cdot \frac{|E(t-\delta)-E(t)|}{\delta^{\alpha}} \qquad (3.32)$$

in (3.31) converges to 0 as $\delta \to 0+$. Hence the right-hand side in (3.31) converges to 0 as $\delta \to 0+$, and the continuity of $I(\delta)$ for $\delta > 0$ implies that the condition $(3.18)_2$ can be fulfilled for some $\delta' > 0$. Finally, the assumption $F \in L^{s_*}\big(0,T;L^{\rho}(\Omega)\big)$ shows that also $(3.18)_1$ can be satisfied.

Secondly, in the case $\alpha = \frac{1}{2}$ (and $\varepsilon = 0$), the assumption $f \in L^2\big(0,T;L^2(\Omega)\big)$ implies as above that

$$\frac{1}{\delta^{\alpha}} \int_{t-\delta}^{t} (f,u)\, d\tau \to 0 \text{ as } \delta \to 0+\,.$$

Moreover, the term (3.32) is bounded by $2\, c\, \varepsilon_* \nu^{3/2}$ for a sequence (δ_j), $0 < \delta_j \to 0$ as $j \to \infty$, due to the assumption (3.26). Hence the continuity of $I(\delta)$, $\delta > 0$, proves that $(3.18)_2$ can be satisfied. Concerning $(3.18)_1$ we proceed as before.

Now Theorem 3.6 is completely proved. □

3. Local in space–time regularity

Consider a weak solution u of the Navier–Stokes system (3.1) in a general domain $\Omega \subset \mathbb{R}^3$. In this subsection we are looking for conditions on u locally in space and time to guarantee that u is regular locally in space and time. The fundamental result in this direction is due to L. Caffarelli, R. Kohn and L. Nirenberg [7] and requires the definition of a suitable weak solution.

DEFINITION 3.8. A weak solution u to (3.1) is called a *suitable weak solution* if the associated pressure term satisfies

$$\nabla p \in L^q_{\text{loc}}\big(0,\infty; L^q_{\text{loc}}(\overline{\Omega})\big) \quad \text{with } q = \frac{5}{4} \tag{3.33}$$

and the *localized energy inequality*

$$\begin{aligned}\frac{1}{2}\|\varphi u(t)\|_2^2 + \nu \int_{t_0}^t \|\varphi \nabla u\|_2^2\, d\tau \;\; &\le \;\; \frac{1}{2}\|\varphi u(t_0)\|_2^2 + \int_{t_0}^t (\varphi f, \varphi u)\, d\tau \\ &- \frac{1}{2}\int_{t_0}^t (\nabla |u|^2, \nabla\varphi^2)\, d\tau + \int_{t_0}^t \Big(\frac{1}{2}|u|^2 + p,\, u\cdot\nabla\varphi^2\Big)\, d\tau \end{aligned} \tag{3.34}$$

holds for almost all $t_0 \ge 0$, all $t \ge t_0$ and all $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Using a standard mollification procedure we obtain from (3.34) the inequality

$$\begin{aligned}\int_{\Omega\times(0,T)} |\nabla u|^2\, \phi\, dx\, dt \;\; &\le \;\; \int_{\Omega\times(0,T)} u\cdot f\phi\, dx\, d\tau \\ &+ \frac{1}{2}\int_{\Omega\times(0,T)} |u|^2(\phi_t + \Delta\phi)\, dx\, dt + \int_{\Omega\times(0,T)} \Big(\frac{1}{2}|u|^2 + p,\, u\cdot\nabla\phi\Big)\, dx\, dt \end{aligned} \tag{3.35}$$

for all non-negative test functions $\phi \in C_0^\infty\big(\Omega\times(0,T)\big)$. This version of the localized energy inequality was used in [7]. However, note that (3.34) is a stronger condition than (3.35) in the sense that the test functions in (3.34) are not assumed to vanish in a neighborhood of $\partial\Omega$. The existence of a suitable weak solution satisfying (3.35) has been proved, under certain smoothness assumptions on the boundary $\partial\Omega$, for a bounded domain in [7], for an exterior domain in [25], and for a general uniform C^2-domain in [16], with (3.34) instead of (3.35).

To describe the local regularity result from [7] we introduce the space-time cylinder

$$Q_r = Q_r(x_0,t_0) = B_r(x_0)\times(t_0 - r^2, t_0), \quad B_r(x_0) = \{x\in\mathbb{R}^3 : |x - x_0| < r\},$$

for $(x_0,t_0) \in \Omega\times(0,T)$ such that $Q_r \subset \Omega\times(0,T)$. The following result is a simplified version of the local results in [7], [36], [37].

THEOREM 3.9. *Let u be a suitable weak solution of (3.1) and let $Q_r = Q_r(x_0,t_0) \subset \Omega\times(0,T)$, $r > 0$. There exists an absolute constant $\varepsilon_* > 0$ with the following property:*

(1) If

$$\|u\|^3_{L^3(Q_r)} + \|p\|^{3/2}_{L^{3/2}(Q_r)} \le \varepsilon_* r^2, \tag{3.36}$$

then $u \in L^\infty(Q_{r/2})$.

(2) If

$$\limsup_{\rho\to 0} \frac{1}{\rho}\, \|\nabla u\|^2_{L^2(Q_\rho)} \le \varepsilon_* \tag{3.37}$$

then there exists $r_0 > 0$ with $Q_{r_0} \subset \Omega\times(0,T)$ such that $u \in L^\infty(Q_{r_0})$.

REMARK 3.10. *(1) The condition (3.36) requires the existence of a suitable radius $r > 0$ and information on u as well as on the pressure p. However, (3.37) needs information for ∇u only, but on all parabolic cylinders Q_r, $r > 0$ sufficiently small.*

(2) The main condition on u in (3.36), i.e. $\|u\|^3_{L^3(Q_r)} \le \varepsilon_* r^2$, may be rewritten in the integral mean form

$$\fint_{t_0-r^2}^{t_0} \fint_{B_r(x_0)} |ru|^3 \, dx \, d\tau \le \varepsilon_* \, .$$

Obviously this condition is satisfied when $|u(x,t)| \le \frac{\varepsilon_*}{r}$ in Q_r. By analogy, the other terms in (3.36) and (3.37) may be treated. Conversely, if u is not regular at (x_0, t_0), then we are heuristically led to the blow-up rate

$$|u(x,t)| \ge \frac{c_0}{\big(|x-x_0|^2 + |t-t_0|\big)^{1/2}},$$

$c_0 > 0$, in a neighborhood of (x_0, t_0), see [7].

(3) The conclusion $u \in L^\infty(Q_{r/2})$ in Theorem 3.9 does not imply that $u \in C^\infty(Q_{r/2})$ even if $f \in C^\infty$ or $f = 0$. However, u is of class C^∞ in x, but not necessarily in t, see [57], [64]. In [36] it is proved that a suitable weak solution satisfying (3.37) is Hölder continuous in space and time locally.

(4) In (3.37) the term ∇u may be replaced by its symmetric part $\frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$ or its skew-symmetric part $\frac{1}{2}\big(\nabla u - (\nabla u)^T\big)$, i.e. by the vorticity $\omega = \mathrm{curl}\ u$, see [38], [67].

(5) More general results concerning regularity criteria for suitable weak solutions using local smallness conditions on $u, \nabla u, \mathrm{curl}\, u$ or $\nabla^2 u$ without any condition on the pressure can be found in [33]. If e.g. $1 \le \frac{2}{s} + \frac{3}{q} \le 2$ and

$$\limsup_{r \to 0} r^{-(\frac{2}{s}+\frac{3}{q}-1)} \|u\|_{L^s(t_0-r^2,t_0;\, L^q(B_r(x_0)))} \le \varepsilon_* \tag{3.38}$$

for some smallness constant $\varepsilon_* > 0$, then u is regular at (x_0, t_0) in the sense that u is essentially bounded in a space time cylinder $Q_{r'}(x_0, t_0) \subset \Omega \times (0,T)$, $0 < r' < r$. For similar results near the boundary of Ω see [32].

To describe our main result on local space-time regularity of suitable weak (or only weak) solutions we use the short notation

$$\|u\|_{L^s L^q(Q_r)} = \|u\|_{L^s(t_0-r^2,t_0;L^q(B_r(x_0)))}$$

when $Q_r = Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0)$. Note that the condition (3.39) in Theorem 3.11 below does not use the $\limsup_{r\to 0}$, but requires the existence of a single sufficiently small radius $r > 0$, and only norms of u, but not of ∇u or the pressure.

THEOREM 3.11. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary domain and let u be a suitable weak solution of the Navier–Stokes system in $\Omega \times (0,T)$ where for simplicity $f = 0$. Let $2 < s < \infty$, $3 < q < \infty$ satisfy the conditions*

$$\frac{2}{s} + \frac{3}{q} \le 1 + \frac{1}{q} \quad \textit{and} \quad q \ge 4.$$

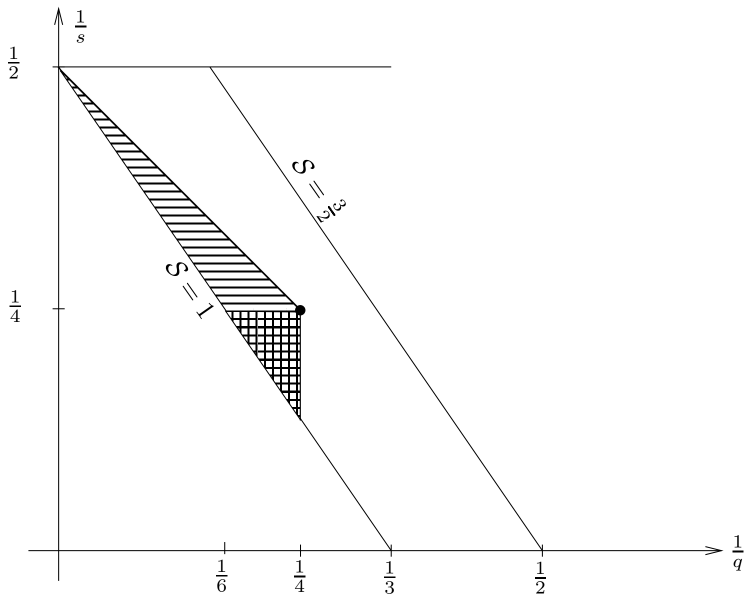

FIGURE 2. In the hatched region ($s < 4$) the localized energy inequality is needed to prove local regularity, in the doubly hatched region ($s \geq 4, q \geq 4$) no local version of an energy inequality is needed.

Then there exists an absolute constant $\varepsilon_* = \varepsilon_*(s,q) > 0$ independent of $\nu > 0$, $x_0 \in \Omega$, $t_0 \in (0,T)$ and $r > 0$ with $Q_r(x_0,t_0) \subset \Omega \times (0,T)$ and of u with the following property: If

$$\|u\|_{L^sL^q(Q_r)} \leq \varepsilon_* \min\left(\nu, \nu^{1-\frac{1}{s}}\right) r^{\frac{2}{s}+\frac{3}{q}-1}, \tag{3.39}$$

then u is regular in $Q_{r/2}$ in the sense

$$u \in L^{s_*}\big(t_0-(r/2)^2, t_0; L^{q_*}\big(B_{r/2}(x_0)\big)\big), \quad \frac{2}{s_*}+\frac{3}{q_*}=1.$$

Here, $s_* = 4$, $q_* = 6$ if $s \geq 4$; in this case, it suffices to assume that u is a weak solution only. If $2 < s < 4$, then s_*, q_* are defined by $\frac{2}{s_*}+\frac{3}{q} = 1+\frac{1}{q}$ and $\frac{2}{s_*}+\frac{3}{q_*} = 1$.

PROOF. Rewriting (3.39) in the integral mean form

$$\left(⨍_{t_0-r^2}^{t_0}\left(⨍_{B_r(x_0)} |ru|^q\,dx\right)^{s/q} ds\right)^{1/s} \leq \varepsilon_* \min\left(\nu, \nu^{1-\frac{1}{s}}\right)$$

where ε_* from (3.39) must be replaced by $\frac{\varepsilon_*}{|B_1(0)|^{1/q}}$, Hölder's inequality shows that we may replace s, q in (3.39) by any smaller s and smaller q, respectively. In particular, when $s \geq 4$ and $q \geq 4$, we may assume that $s = s_* = 4, q = 4$. When $2 < s < 4$, then let $s = s_*$ satisfy $\frac{2}{s_*}+\frac{3}{q} = 1+\frac{1}{q}$. In both cases we get

$$s = s_* \leq q, \quad \frac{2}{s_*}+\frac{3}{q} = 1+\frac{1}{q}, \quad \frac{2}{s_*}+\frac{3}{q_*} = 1, \tag{3.40}$$

since $q \geq 4$. As a second step we may assume after a shift of coordinates in space and time that $x_0 = 0$ and $t_0 = 0$. Next we use a scaling argument and consider

$$u_r(y,\tau) = ru(ry, r^2\tau), \quad p_r(y,\tau) = r^2 p(ry, r^2\tau) \tag{3.41}$$

on $Q_1 = B_1(0) \times (-1, 0)$ instead of (u, p) on Q_r. Note that u_r, p_r solve the Navier–Stokes system with the same viscosity ν and that u_r satisfies (3.39) in the form

$$\|u_r\|_{L^s L^q(Q_1)} \leq \varepsilon_* \min\big(\nu, \nu^{1-1/s}\big). \tag{3.42}$$

Hence, without loss of generality, we assume that u satisfies (3.39) on Q_1 with $r = 1$ and $s = s_*$.

The idea of the proof is to construct with the help of Theorem 2.18 a very weak solution v in $Q' = B_{r'} \times (t', 0)$ for suitable $r' \in (\frac{1}{2}, 1)$ and $t' \in (-1, -\frac{1}{2})$ with data

$$v(t') = u(t'), \quad v|_{\partial B_{r'}} = u|_{\partial B_{r'}}$$

and to identify v with u on Q'; hence

$$v = u \in L^{s_*} L^{q_*}(Q') \quad \text{and} \quad v = u \text{ in } L^{s_*} L^{q_*}\big(B_{1/2} \times (-\frac{1}{2}, 0)\big).$$

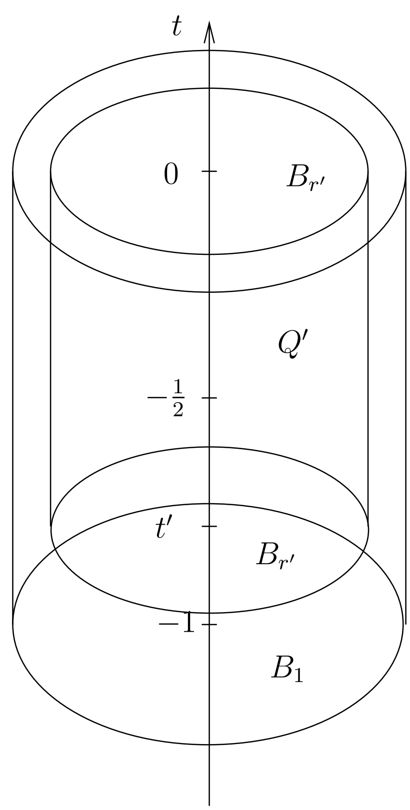

FIGURE 3. The space-time cylinders Q_1 and Q'.

For this purpose we have to find $r' \in (\frac{1}{2}, 1)$ and $t' \in (-1, -\frac{1}{2})$ such that the smallness conditions

$$\int_0^{-t'} \|A_{q_*} e^{-\nu\tau A_{q_*}} A_{q_*}^{-1} P_{q_*} u(t')\|_{q_*}^{s_*} \, d\tau \quad \leq \quad \varepsilon_*^{s_*} \nu^{s_*-1} \tag{3.43}$$

$$\int_{t'}^0 \|u|_{\partial B_{r'}}\|_{W^{-1/q_*,q_*}(\partial B_{r'})}^{s_*} \, d\tau \quad \leq \quad \varepsilon_*^{s_*} \nu^{s_*-1}, \tag{3.44}$$

cf. (2.50), are fulfilled; here A_{q_*} and P_{q_*} denote the Stokes operator and the Helmholtz projection, respectively, on $B_{r'}$.

Concerning (3.43) we find $t' \in (-1, -\frac{1}{2})$ satisfying

$$\|u(t')\|_{L^q(B_1)}^s \leq ⨍_{-1}^{-1/2} \|u\|_{L^q(B_1)}^s \, d\tau \leq 2\,\|u\|_{L^s L^q(Q_1)}^s \leq 2\varepsilon_*^s \nu^s .$$

Then Lemma 1.11 (3), (4) with $\alpha = \frac{1}{2q}$, $\frac{1}{q} + \frac{3}{q_*} = \frac{3}{q}$, and the property $\frac{s_*}{2q} = \frac{1}{q-2} < 1$ imply that

$$\begin{aligned}
&\int_0^{-t'} \|A_{q_*} e^{-\nu\tau A_{q_*}} A_{q_*}^{-1} P_{q_*} u(t')\|_{q_*}^{s_*} \, d\tau \\
&= \int_0^{-t'} \|A_q^{1/2q} e^{-\nu\tau A_q} A_q^{-1/2q} P_q u(t')\|_{q_*}^{s_*} \, d\tau \\
&\leq c \int_0^{-t'} \frac{e^{-\nu\delta_0 s_* \tau}}{(\nu\tau)^{s_*/2q}} \, \|u(t')\|_q^{s_*} \, d\tau \\
&\leq \frac{c}{\nu} \|u(t')\|_q^{s_*} \leq c\,\varepsilon_*^{s_*} \nu^{s-1} .
\end{aligned}$$

Hence (3.43) is satisfied for a sufficiently small constant ε_* in (3.42).

Now consider the problem of finding $r' \in (\frac{1}{2}, 1)$ such that (3.44) is satisfied. By the mean value argument as before, there exists $r' \in (\frac{1}{2}, 1)$ such that

$$\begin{aligned}
\|u\|_{L^{s_*}(-1,0;L^q(\partial B_{r'}))}^{s_*} &= \int_{-1}^0 \|u\|_{L^q(\partial B_{r'})}^{s_*} \, d\tau \\
&\leq ⨍_{1/2}^1 \Big(\int_{-1}^0 \|u\|_{L^q(\partial B_r)}^{s_*} \Big) \, d\tau \, dr \\
&= 2 \int_{-1}^0 \Big(\int_{1/2}^1 \|u\|_{L^q(\partial B_r)}^{s_*} \, dr \Big) \, d\tau.
\end{aligned}$$

Since $s_* \leq q$, see (3.40), we apply Hölder's inequality to the inner integral and get from (3.42) that

$$\begin{aligned}
\|u\|_{L^{s_*}(-1,0;L^q(\partial B_{r'}))}^{s_*} &\leq 2 \int_{-1}^0 \Big(\int_{1/2}^1 \|u\|_{L^q(\partial B_r)}^q dr \Big)^{s_*/q} d\tau \\
&\leq 2 \int_{-1}^0 \|u\|_{L^q(B_1)}^{s_*} d\tau \\
&\leq 2\varepsilon_*^{s_*} \nu^{s_*-1}.
\end{aligned}$$

Finally, using the embedding $L^q(\partial B_{r'}) \subset W^{-1/q_*,q_*}(\partial B_{r'})$ with an embedding constant uniformly bounded in $r' \in \left(\frac{1}{2}, 1\right)$, we get that (3.44) is satisfied for a slightly different constant $\varepsilon_* > 0$.

Now Theorem 2.18 yields a unique very weak solution v in $L^{s_*}L^{q_*}(Q')$ with data $v(t') = u(t')$ and $v = u$ on $\partial B_{r'} \times (t', 0)$. For this argument it is important to note that the smallnes constant ε_* in the application of Theorem 2.18 in the space-time domain Q' may be chosen independently of $r' \in (\frac{1}{2}, 1)$ and $t' \in (-1, -\frac{1}{2})$; for its proof we have to refer to the scaling argument (3.41).

As the final step of the proof it suffices to show that $v = u$ on Q'. First consider the case $s \geq 4$ in which $s_* = 4$, $q_* = 6$, $v \in L^4L^6(Q')$ and $u \in L^4L^4(Q')$. Let γ denote the very weak solution of the Stokes system

$$\begin{aligned} \gamma_t - \nu\Delta\gamma + \nabla p &= 0, \qquad \operatorname{div}\gamma = 0 \text{ in } Q', \\ \gamma(t') &= u(t'), \quad \gamma_{|_{\partial B_{r'}}} = u_{|_{\partial B_{r'}}}. \end{aligned}$$

By Theorem 2.14 $\gamma \in L^4L^6(Q') \subset L^4L^4(Q')$ so that $v - \gamma$ and $u - \gamma$ solve the instationary Stokes system

$$\begin{aligned} &U_t - \nu\Delta U + \nabla p = -\operatorname{div}(vv) \qquad \text{and} \qquad = -\operatorname{div}(uu) \text{ in } Q', \\ &\operatorname{div} U = 0 \text{ in } Q', \quad U(t') = 0, \quad U_{|_{\partial B_{r'}}} = 0, \end{aligned}$$

respectively. Since $vv \in L^2L^2(Q')$ and $uu \in L^2L^2(Q')$, in both cases the very weak solution U is even a *weak* solution satisfying the energy identity. Hence $u - v = u - \gamma - (v - \gamma)$ is a weak solution of the Stokes system

$$\begin{aligned} U_t - \nu\Delta U + \nabla p &= -\operatorname{div}(Uu + vU), \quad \operatorname{div} U = 0 \text{ in } Q', \\ U(t') &= 0, \quad u_{|_{\partial B_{r'}}} = 0, \end{aligned} \tag{3.45}$$

where Uu, $vU \in L^2L^2(Q')$. Let $\|\cdot\|_{[\tau,t]}$, $\tau < t$, denote the norm

$$\|w\|_{[\tau,t]} = \left(\|w\|^2_{L^\infty(\tau,t;L^2(B_{r'}))} + \nu\|\nabla w\|^2_{L^2(\tau,t;L^2(B_{r'}))}\right)^{1/2}.$$

Testing (3.45) in $B_{r'} \times [t', t' + \varepsilon]$, $\varepsilon > 0$, with U we get the estimate

$$\|U\|^2_{[t',t'+\varepsilon]} \leq c\|U\|^2_{[t',t'+\varepsilon]}\,\|v\|_{L^4(t',t'+\varepsilon;L^6(B_{r'}))} \tag{3.46}$$

with a constant $c > 0$ independent of t' and $\varepsilon > 0$ as well as of U, u and v; here we used that $\int_{B_{r'}} Uu \cdot \nabla U\,dx = 0$ and that

$$\left|\int_{B_{r'}} vU \cdot \nabla U\,dx\right| \leq c\|\nabla U\|_2\,\|U\|_3\,\|v\|_6 \leq c\|\nabla U\|_2^{3/2}\,\|U\|_2^{1/2}\,\|v\|_6.$$

Since $v \in L^4L^6(Q')$, we may choose $\varepsilon > 0$ sufficiently small so that (3.46) yields $U \equiv 0$ on $[t', t' + \varepsilon]$. Repeating this argument a finite number of times with the same $\varepsilon > 0$ we conclude that $U \equiv 0$ on $[t', 0]$, i.e., $u = v \in L^4L^6(Q')$. This proves Theorem 3.11 in the case $s \geq 4$. Note that u was not assumed to be a suitable weak solution in this case.

Secondly, let $2 < s = s_* < 4$ and consequently $q > 4$. In this case an approximation procedure is used to apply the localized energy inequality in a similar way as in Serrin's uniqueness criterion concerning the usual energy inequality. Moreover, regularity results for v allow to conclude that $U = u - v$ satisfies the inequality

$$\frac{1}{2}\,\|U(t)\|_2^2 + \nu \int_{t'}^{t} \|\nabla U\|_2^2 \, d\tau \le \int_{t'}^{t} (U \cdot \nabla U, v) \, d\tau \, ;$$

we omit further details of these technical arguments. Since $v \in L^{s_*}L^{q_*}(Q_{r'})$, the absorption principle may be used to get in a finite number of steps on consecutive intervals $t' = t_1 < t_2 < \ldots < t_m = 0$ that $u = v$ in Q', cf. (3.46). $\square$

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Part 2

Qualitative and quantitative aspects of curvature driven flows of planar curves

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Abstract. In this lecture notes we are concerned with evolution of plane curves satisfying a geometric equation $v = \beta(k, x, \nu)$ where v is the normal velocity of an evolving family of planar closed curves. We assume the normal velocity to be a function of the curvature k, tangential angle ν and the position vector x of a plane curve Γ. We follow the direct approach and we analyze the so-called intrinsic heat equation governing the motion of plane curves obeying such a geometric equation. We show how to reduce the geometric problem to a solution of fully nonlinear parabolic equation for important geometric quantities. Using a theory of fully nonlinear parabolic equations we present results on local in time existence of classical solutions. We also present an approach based on level set representation of curves evolved by the curvature. We recall basic ideas from the theory of viscosity solutions for the level set equation. We discuss numerical approximation schemes for computing curvature driven flows and we present various examples of application of theoretical results in practical problems.

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Preface

The lecture notes on Qualitative and quantitative aspects of curvature driven flows of plane curves are based on the series of lectures given by the author in the fall of 2006 during his stay at the Nečas Center for Mathematical Modeling at Charles University in Prague. The principal goal was to present basic facts and known results in this field to PhD students and young researchers of NCMM.

The main purpose of these notes is to present theoretical and practical topics in the field of curvature driven flows of planar curves and interfaces. There are many recent books and lecture notes on this topic. My intention was to find a balance between presentation of subtle mathematical and technical details and ability of the material to give a comprehensive overview of possible applications in this field. This is often a hard task but I tried to find this balance.

I am deeply indebted to Karol Mikula for long and fruitful collaboration on the problems of curvature driven flows of curves. A lot of the material presented in these lecture notes has been jointly published with him. I want to acknowledge a recent collaboration with V. Srikrishnan who brought to my attention important problems arising in tracking of moving boundaries. I also wish to thank Josef Málek from NCMM of Charles University in Prague for giving me a possibility to visit NCMM and present series of lectures and for his permanent encouragement to prepare these lecture notes.

Daniel Ševčovič
Bratislava, July 2007.

CHAPTER 1

Introduction

In this lecture notes we are concerned with evolution of plane curves satisfying a geometric equation

$$v = \beta(k, x, \nu) \tag{1.1}$$

where v is the normal velocity of an evolving family of planar closed curves. We assume the normal velocity to be a function of the curvature k, tangent angle ν and the position vector x of a plane curve Γ.

Geometric equations of the form (1.1) can be often found in variety of applied problems like e.g. the material science, dynamics of phase boundaries in thermomechanics, in modeling of flame front propagation, in combustion, in computations of first arrival times of seismic waves, in computational geometry, robotics, semiconductors industry, etc. They also have a special conceptual importance in image processing and computer vision theories. A typical case in which the normal velocity v may depend on the position vector x can be found in image segmentation [CKS97, KKO$^+$96]. For a comprehensive overview of other important applications of the geometric Eq. (1.1) we refer to recent books by Sethian, Sapiro and Osher and Fedkiw [Set96, Sap01, OF03].

1. Mathematical models leading to curvature driven flows of planar curves

1.1. Interface dynamics. If a solid phase occupies a subset $\Omega_s(t) \subset \Omega$ and a liquid phase - a subset $\Omega_l(t) \subset \Omega$, $\Omega \subset \mathbb{R}^2$, at a time t, then the phase interface is the set $\Gamma^t = \partial\Omega_s(t) \cap \partial\Omega_l(t)$ which is assumed to be a closed smooth embedded curve. The sharp-interface description of the solidification process is then described by the Stefan problem with a surface tension

$$\begin{aligned} \rho c \partial_t U &= \lambda \Delta U \qquad \text{in } \Omega_s(t) \text{ and } \Omega_l(t), \\ [\lambda \partial_n U]_s^l &= -Lv \qquad \text{on } \Gamma^t, && (1.2) \\ \frac{\delta e}{\sigma}(U - U^*) &= -\delta_2(\nu)k + \delta_1(\nu)v \text{ on } \Gamma^t, && (1.3) \end{aligned}$$

subject to initial and boundary conditions for the temperature field U and initial position of the interface Γ (see e.g. [Ben01]). The constants ρ, c, λ represent material characteristics (density, specific heat and thermal conductivity), L is the latent heat per unit volume, U^* is a melting point and v is the normal velocity of an interface. Discontinuity in the heat flux on the interface Γ^t is described by the Stefan condition (1.2). The relationship (1.3) is referred to as the Gibbs – Thomson law on the interface Γ^t, where δe is difference in entropy per unit volume between liquid and solid phases, σ is a constant surface tension, δ_1 is a coefficient of attachment kinetics and dimensionless function δ_2 describes anisotropy of the interface. One can see that the Gibbs–Thomson condition can be viewed as a geometric equation having the form (1.1). In this application the normal velocity $v = \beta(k, x, \nu)$ has a special form

$$\beta = \beta(k, \nu) = \delta(\nu)k + F$$

In the theory of phase interfaces separating solid and liquid phases, the geometric equation (1.1) with $\beta(k, x, \nu) = \delta(\nu)k + F$ corresponds to the so-called Gibbs–Thomson law governing the crystal growth in an undercooled liquid [Gur93, BM98]. In the series of papers [AG89, AG94, AG96]. Angenent and Gurtin studied motion of phase interfaces. They proposed to study the equation of the form

$$\mu(\nu, v)v = h(\nu)k - g$$

where μ is the kinetic coefficient and quantities h, g arise from constitutive description of the phase boundary. The dependence of the normal velocity v on the curvature k is related to surface tension effects on the interface, whereas the dependence on ν (orientation of interface) introduces anisotropic effects into the model. In general, the kinetic coefficient μ may also depend on the velocity v itself giving rise to a nonlinear dependence of the function $v = \beta(k, \nu)$ on k and ν. If the motion of an interface is very slow, then $\beta(k, x, \nu)$ is linear in k (cf. [AG89]) and (1.1) corresponds to the classical mean curvature flow studied extensively from both the mathematical (see, e.g., [GH86, AL86, Ang90a, Gra87]) and numerical point of view (see, e.g., [Dzi94, Dec97, MK96, NPV93, OS88]).

In the series of papers [AG89, AG96], Angenent and Gurtin studied perfect conductors where the problem can be reduced to a single equation on the interface. Following their approach and assuming a constant kinetic coefficient one obtains the equation

$$v = \beta(k, \nu) \equiv \delta(\nu)k + F$$

describing the interface dynamics. It is often referred to as the *anisotropic curve shortening equation* with a constant driving force F (energy difference between bulk phases) and a given anisotropy function δ.

1.2. Image segmentation. A similar equation to (1.1) arises from the theory of image segmentation in which detection of object boundaries in the analyzed image plays an important role. A given black and white image can be represented by its intensity function $I : R^2 \to [0, 255]$. The aim is to detect edges of the image, i.e. closed planar curves on which the gradient ∇I is large (see [KM95]). The idea behind the so-called *active contour models* is to construct an evolving family of plane curves converging to an edge (see [KWT87]). One can construct a family of curves evolved by the normal velocity $v = \beta(k, x, \nu)$ of the form

$$\beta(k, x, \nu) = \delta(x, \nu)k + c(x, \nu)$$

where $c(x, \nu)$ is a driving force and $\delta(x, \nu) > 0$ is a smoothing coefficient. These functions depend on the position vector x as well as orientation angle ν of a curve. Evolution starts from an initial curve which is a suitable approximation of the edge and then it converges to the edge provided that δ, c are suitable chosen functions.

In the context of level set methods, edge detection techniques based on this idea were first discussed by Caselles et al. and Malladi et al. in [CCCD93, MSV95]. Later on, they have been revisited and improved in [CKS97, CKSS97, KKO+96].

1.3. Geodesic curvature driven flow of curves on a surface. Another interesting application of the geometric equation (1.1) arises from the differential geometry. The purpose is to investigate evolution of curves on a given surface driven by the geodesic curvature and prescribed external force. We restrict our attention to the case when the normal velocity $\mathcal{V}$ is a linear function of the geodesic curvature $\mathcal{K}_g$ and external force $\mathcal{F}$, i.e. $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ and the surface $\mathcal{M}$ in $\mathbb{R}^3$ can be represented by a smooth graph. The idea how to analyze a flow of curves on a surface $\mathcal{M}$ consists in vertical projection of surface curves into the plane. This allows for reducing the problem to the analysis of evolution of planar curves instead of surface ones. Although the geometric equation $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ is simple the description of the normal velocity v of the family of projected planar curves is rather involved. Nevertheless, it can be written in the form of equation (1.1). The precise form of the function β can be found in the last section.

2. Methodology

Our methodology how to solve (1.1) is based on the so-called direct approach investigated by Dziuk, Deckelnick, Gage and Hamilton, Grayson, Mikula and Ševčovič and other authors (see e.g. [Dec97, Dzi94, Dzi99, GH86, Gra87, MK96, Mik97, MS99, MS01, MS04a, MS04b] and references therein). The main idea is to use the so-called Lagrangean description of motion and to represent the flow of planar curves by a position vector x which is a solution to the geometric equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T}$$

where $\vec{N}, \vec{T}$ are the unit inward normal and tangent vectors, resp. It turns out that one can construct a closed system of parabolic-ordinary differential equations for relevant geometric quantities: the curvature, tangential angle, local length and position vector. Other well-known techniques, like e.g. level-set method due to Osher and Sethian [Set96, OF03] or phase-field approximations (see e.g. Caginalp, Nochetto et al., Beneš [Cag90, NPV93, Ben01]) treat the geometric equation (1.1) by means of a solution to a higher dimensional parabolic problem. In comparison to these methods, in the direct approach one space dimensional evolutionary problems are solved only. Notice that the direct approach for solving (1.1) can be accompanied by a proper choice of tangential velocity α significantly improving and stabilizing numerical computations as it was documented by many authors (see e.g. [Dec97, HLS94, HKS98, Kim97, MS99, MS01, MS04a, MS04b]).

3. Numerical techniques

Analytical methods for mathematical treatment of (1.1) are strongly related to numerical techniques for computing curve evolutions. In the *direct approach* one seeks for a parameterization of the evolving family of curves. By solving the so-called *intrinsic heat equation* one can directly find a position vector of a curve (see e.g. [Dzi91, Dzi94, Dzi99, MS99, MS01, MS04a]). There are also other direct methods based on solution of a porous medium–like equation for curvature of a curve

[MK96, Mik97], a crystalline curvature approximation [Gir95, GK94, UY00], special finite difference schemes [Kim94, Kim97], and a method based on erosion of polygons in the affine invariant scale case [Moi98]. By contrast to the direct approach, *level set methods* are based on introducing an auxiliary function whose zero level sets represent an evolving family of planar curves undergoing the geometric equation (1.1) (see, e.g., [OS88, Set90, Set96, Set98, HMS98]). The other indirect method is based on the phase-field formulations (see, e.g., [Cag90, NPV93, EPS96, BM98]). The level set approach handles implicitly the curvature-driven motion, passing the problem to higher dimensional space. One can deal with splitting and/or merging of evolving curves in a robust way. However, from the computational point of view, level set methods are much more expensive than methods based on the direct approach.

CHAPTER 2

Preliminaries

The purpose of this section is to review basic facts and results concerning a curvature driven flow of planar curves. We will focus our attention on the so-called Langrangean description of a moving curve in which we follow an evolution of point positions of a curve. This is also referred to as a direct approach in the context of curvature driven flows of planar curves ([AL86, Dzi91, Dzi94, Dec97, MK96, MS99, MS01]).

First we recall some basic facts and elements of differential geometry. Then we derive a system of equations for important geometric quantities like e.g. a curvature, local length and tangential angle. With help of these equations we shall be able to derive equations describing evolution of the total length, enclosed area of an evolving curve and transport of a scalar function quantity.

1. Notations and elements of differential geometry

An embedded regular plane curve (a Jordan curve) Γ is a closed C^1 smooth one dimensional nonselfintersecting curve in the plane $\mathbb{R}^2$. It can be parameterized by a smooth function $x : S^1 \to \mathbb{R}^2$. It means that $\Gamma = \mathrm{Img}(x) := \{x(u), u \in S^1\}$ and $g = |\partial_u x| > 0$. Taking into account the periodic boundary conditions at $u = 0, 1$ we can hereafter identify the unit circle S^1 with the interval $[0, 1]$. The unit arc-length parameterization of a curve $\Gamma = \mathrm{Img}(x)$ is denoted by s and it satisfies $|\partial_s x(s)| = 1$ for any s. Furthermore, the arc-length parameterization is related to the original parameterization u via the equality $\mathrm{d}s = g\,\mathrm{d}u$. Notice that the interval of values of the arc-length parameter depends on the curve Γ. More precisely, $s \in [0, L(\Gamma)]$ where $L(\Gamma)$ is the length of the curve Γ. Since s is the arc-length parameterization the tangent vector $\vec{T}$ of a curve Γ is given by $\vec{T} = \partial_s x = g^{-1}\partial_u x$. We choose orientation of the unit inward normal vector $\vec{N}$ in such a way that $\det(\vec{T}, \vec{N}) = 1$ where $\det(\vec{a}, \vec{b})$ is the determinant of the 2×2 matrix with column vectors $\vec{a}, \vec{b}$. Notice that $1 = |\vec{T}|^2 = (\vec{T}.\vec{T})$. Therefore, $0 = \partial_s(\vec{T}.\vec{T}) = 2(\vec{T}.\partial_s\vec{T})$. Here $a.b$ denotes the standard Euclidean scalar product in $\mathbb{R}^2$. Thus the direction of the normal vector $\vec{N}$ must be proportional to $\partial_s\vec{T}$. It means that there is a real number $k \in \mathbb{R}$ such that $\vec{N} = k\partial_s\vec{T}$. Similarly, as $1 = |\vec{N}|^2 = (\vec{N}.\vec{N})$ we have $0 = \partial_s(\vec{N}.\vec{N}) = 2(\vec{N}.\partial_s\vec{N})$ and so $\partial_s\vec{N}$ is collinear to the vector $\vec{T}$. Since $(\vec{N}.\vec{T}) = 0$ we have $0 = \partial_s(\vec{N}.\vec{T}) = (\partial_s\vec{N}.\vec{T}) + (\vec{N}.\partial_s\vec{T})$. Therefore, $\partial_s\vec{N} = -k\vec{T}$. In summary, for the arc-length derivative of the unit tangent and normal vectors to a curve Γ we have

$$\partial_s\vec{T} = k\vec{N}, \qquad \partial_s\vec{N} = -k\vec{T} \tag{2.1}$$

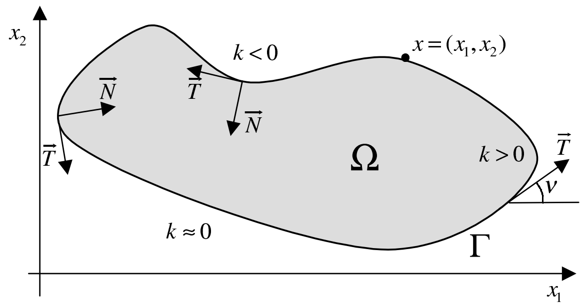

FIGURE 1. Description of a planar curve Γ enclosing a domain Ω, its signed curvature k, unit inward normal $\vec{N}$ and tangent vector $\vec{T}$, position vector x.

where the scalar quantity $k \in \mathbb{R}$ is called a curvature of the curve Γ at a point $x(s)$. Equations (2.1) are referred to as Frenét formulae. The quantity k obeying (2.1) is indeed a curvature in the sense that it is a reciprocal value of the radius of an oscullating circle having C^2 smooth contact with Γ point at a point $x(s)$. Since $\partial_s \vec{T} = \partial_s^2 x$ we obtain a formula for the signed curvature

$$k = \det(\partial_s x, \partial_s^2 x) = g^{-3} \det(\partial_u x, \partial_u^2 x)\,. \tag{2.2}$$

Notice that, according to our notation, the curvature k is positive on the convex side of a curve Γ whereas it is negative on its concave part (see Fig. 1). By ν we denote the tangent angle to Γ, i.e. $\nu = \arg(\vec{T})$ and $\vec{T} = (\cos\nu, \sin\nu)$. Then, by Frenét's formulae, we have

$$k(-\sin\nu, \cos\nu) = k\vec{N} = \partial_s \vec{T} = \partial_s \nu(-\sin\nu, \cos\nu)$$

and therefore

$$\partial_s \nu = k\,.$$

For an embedded planar curve Γ, its tangential angle ν varies from 0 to 2π and so we have $2\pi = \nu(1) - \nu(0) = \int_0^1 \partial_u \nu \, \mathrm{d}u = \int_0^1 kg \, \mathrm{d}u = \int_\Gamma k \, \mathrm{d}s$ and hence the total curvature of an embedded curve satisfies the following equality:

$$\int_\Gamma k \mathrm{d}s = 2\pi\,. \tag{2.3}$$

We remind ourselves that the above equality can be generalized to the case when a closed nonselfintersecting smooth curve Γ belongs to an orientable two dimensional surface $\mathcal{M}$. According to the Gauss-Bonnet formula we have

$$\int_{int(\Gamma)} K \mathrm{d}x + \int_\Gamma k \, \mathrm{d}s = 2\pi\, \chi(\mathcal{M})$$

where K is the Gaussian curvature of am orientable two dimensional surface $\mathcal{M}$ and $\chi(\mathcal{M})$ is the Euler characteristics of the surface $\mathcal{M}$. In a trivial case when $\mathcal{M} = \mathbb{R}^2$ we have $K \equiv 0$ and $\chi(\mathcal{M}) = 1$ and so the equality (2.3) is a consequence of the Gauss-Bonnet formula.

2. Governing equations

In these lecture notes we shall assume that the normal velocity v of an evolving family of plane curves $\Gamma^t, t \geq 0$, is equal to a function β of the curvature k, tangential angle ν and position vector $x \in \Gamma^t$,

$$v = \beta(x, k, \nu)\,.$$

(see (1.1)). Hereafter, we shall suppose that the function $\beta(k, x, \nu)$ is a smooth function which is increasing in the k variable, i.e.

$$\beta'_k(k, x, \nu) > 0\,.$$

An idea behind the direct approach consists of representation of a family of embedded curves Γ^t by the position vector $x \in \mathbb{R}^2$, i.e.

$$\Gamma^t = \mathrm{Img}(x(.,t)) = \{x(u,t),\ u \in [0,1]\}$$

where x is a solution to the geometric equation

$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \tag{2.4}$$

where $\beta = \beta(x, k, \nu)$, $\vec{N} = (-\sin\nu, \cos\nu)$ and $\vec{T} = (\cos\nu, \sin\nu)$ are the unit inward normal and tangent vectors, respectively. For the normal velocity $v = \partial_t x.\vec{N}$ we have $v = \beta(x, k, \nu)$. Notice that the presence of arbirary tangential velocity functional α has no impact on the shape of evolving curves.

The goal of this section is to derive a system of PDEs governing the evolution of the curvature k of $\Gamma^t = \mathrm{Img}(x(.,t)), t \in [0,T)$, and some other geometric quantities where the family of regular plane curves where $x = x(u,t)$ is a solution to the position vector equation (2.4). These equations will be used in order to derive a priori estimates of solutions. Notice that such an equation for the curvature is well known for the case when $\alpha = 0$, and it reads as follows: $\partial_t k = \partial_s^2 \beta + k^2 \beta$ (cf. [GH86, AG89]). Here we present a brief sketch of the derivation of the corresponding equations for the case of a nontrivial tangential velocity α.

Let us denote $\vec{p} = \partial_u x$. Since $u \in [0,1]$ is a fixed domain parameter we commutation relation $\partial_t \partial_u = \partial_u \partial_t$. Then, by using Frenét's formulae, we obtain

$$\begin{aligned}
\partial_t \vec{p} &= |\partial_u x|((\partial_s \beta + \alpha k)\vec{N} + (-\beta k + \partial_s \alpha)\vec{T}), \\
\vec{p}\,.\,\partial_t \vec{p} &= |\partial_u x|\ \vec{T}\,.\,\partial_t \vec{p} = |\partial_u x|^2(-\beta k + \partial_s \alpha), \\
\det(\vec{p}, \partial_t \vec{p}) &= |\partial_u x|\ \det(\vec{T}, \partial_t \vec{p}) = |\partial_u x|^2\ (\partial_s \beta + \alpha k), \\
\det(\partial_t \vec{p}, \partial_u \vec{p}) &= -|\partial_u x| \partial_u |\partial_u x| (\partial_s \beta + \alpha k) + |\partial_u x|^3\ (-\beta k + \partial_s \alpha),
\end{aligned} \tag{2.5}$$

because $\partial_u \vec{p} = \partial_u^2 x = \partial_u(|\partial_u x|\ \vec{T}) = \partial_u |\partial_u x|\ \vec{T} + k|\partial_u x|^2\ \vec{N}$. Since $\partial_u \det(\vec{p}, \partial_t \vec{p}) = \det(\partial_u \vec{p}, \partial_t \vec{p}) + \det(\vec{p}, \partial_u \partial_t \vec{p})$, we have $\det(\vec{p}, \partial_u \partial_t \vec{p}) = \partial_u \det(\vec{p}, \partial_t \vec{p}) + \det(\partial_t \vec{p}, \partial_u \vec{p})$. As $k = \det(\vec{p}, \partial_u \vec{p})\,|\vec{p}|^{-3}$ (see (2.2)), we obtain

$$\begin{aligned}
\partial_t k &= -3|p|^{-5}(\vec{p}\,.\,\partial_t \vec{p}) \det(\vec{p}, \partial_u \vec{p}) + |\vec{p}|^{-3} (\det(\partial_t \vec{p}, \partial_u \vec{p}) + \det(\vec{p}, \partial_u \partial_t \vec{p})) \\
&= -3k|\vec{p}|^{-2}(\vec{p}\,.\,\partial_t \vec{p}) + 2|\vec{p}|^{-3} \det(\partial_t \vec{p}, \partial_u \vec{p}) + |\vec{p}|^{-3} \partial_u \det(\vec{p}, \partial_t \vec{p}).
\end{aligned}$$

Finally, by applying identities (2.5), we end up with the second-order nonlinear parabolic equation for the curvature:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta\,. \tag{2.6}$$

The identities (2.5) can be used in order to derive an evolutionary equation for the local length $|\partial_u x|$. Indeed, $\partial_t |\partial_u x| = (\partial_u x \,.\, \partial_u \partial_t x)/|\partial_u x| = (\vec{p} \,.\, \partial_t \vec{p})/|\partial_u x|$. By (2.5) we have the

$$\partial_t |\partial_u x| = -|\partial_u x|\, k\beta \; + \partial_u \alpha \tag{2.7}$$

where $(u,t) \in Q_T = [0,1] \times [0,T)$. In other words, $\partial_t \mathrm{d}s = (-k\beta + \partial_s \alpha)\mathrm{d}s$. It yields the commutation relation

$$\partial_t \partial_s - \partial_s \partial_t = (k\beta - \partial_s \alpha)\partial_s . \tag{2.8}$$

Next we derive equations for the time derivative of the unit tangent vector $\vec{T}$ and tangent angle ν. Using the above commutation relation and Frenét formulae we obtain

$$\begin{aligned} \partial_t \vec{T} &= \partial_t \partial_s x = \partial_s \partial_t x + (k\beta - \partial_s \alpha)\partial_s x \,, \\ &= \partial_s (\beta \vec{N} + \alpha \vec{T}) + (k\beta - \partial_s \alpha)\vec{T} \,, \\ &= (\partial_s \beta + \alpha k)\vec{N} \,. \end{aligned}$$

Since $\vec{T} = (\cos\nu, \sin\nu)$ and $\vec{N} = (-\sin\nu, \cos\nu)$ we conclude that $\partial_t \nu = \partial_s \beta + \alpha k$. Summarizing, we end up with evolutionary equations for the unit tangent and normal vectors $\vec{T}, \vec{N}$ and the tangent angle ν

$$\begin{aligned} \partial_t \vec{T} &= (\partial_s \beta + \alpha k)\vec{N} \,, \\ \partial_t \vec{N} &= -(\partial_s \beta + \alpha k)\vec{T} \,, \\ \partial_t \nu &= \partial_s \beta + \alpha k \,. \end{aligned} \tag{2.9}$$

Since $\partial_s \nu = k$ and $\partial_s \beta = \beta'_k \partial_s k + \beta'_\nu k + \nabla_x \beta . \vec{T}$ we obtain the following closed system of parabolic-ordinary differential equations:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta \,, \tag{2.10}$$

$$\partial_t \nu = \beta'_k \partial_s^2 \nu + (\alpha + \beta'_\nu)\partial_s \nu + \nabla_x \beta . \vec{T} \,, \tag{2.11}$$

$$\partial_t g = -gk\beta + \partial_u \alpha \,, \tag{2.12}$$

$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \,, \tag{2.13}$$

where $(u,t) \in Q_T = [0,1] \times (0,T)$, $\mathrm{d}s = g\,\mathrm{d}u$ and $\vec{T} = \partial_s x = (\cos\nu, \sin\nu), \vec{N} = \vec{T}^\perp = (-\sin\nu,\, \cos\nu)$. The functional α may depend on the variables k, ν, g, x. A solution (k, ν, g, x) to (2.10) – (2.13) is subject to initial conditions

$$k(.,0) = k_0 \,, \quad \nu(.,0) = \nu_0 \,, \quad g(.,0) = g_0 \,, \quad x(.,0) = x_0(.) \,,$$

and periodic boundary conditions at $u = 0, 1$ except of the tangent angle ν for which we require that the tangent vector $\vec{T}(u,t) = (\cos(\nu(u,t)), \sin(\nu(u,t)))$ is 1-periodic in the u variable, i.e. $\nu(1,t) = \nu(0,t) + 2\pi$. Notice that the initial conditions for k_0, ν_0, g_0 and x_0 (the curvature, tangent angle, local length element and position vector of the initial curve Γ_0) must satisfy the following compatibility constraints:

$$g_0 = |\partial_u x_0| > 0 \,, \quad k_0 = \det(g_0^{-3} \partial_u x_0, \partial_u^2 x_0) \,, \quad \partial_u \nu_0 = g_0 k_0 \,.$$

3. First integrals for geometric quantities

The aim of this section is to derive basic identities for various geometric quantities like e.g. the length of a closed curve and the area enclosed by a Jordan curve in the plane. These identities (first integrals) will be used later in the analysis of the governing system of equations.

3.1. The total length equation. By integrating (2.7) over the interval $[0,1]$ and taking into account that α satisfies periodic boundary conditions, we obtain the total length equation

$$\frac{\mathrm{d}}{\mathrm{d}t}L^t + \int_{\Gamma^t} k\beta \mathrm{d}s = 0, \tag{2.14}$$

where $L^t = L(\Gamma^t)$ is the total length of the curve Γ^t, $L^t = \int_{\Gamma^t} \mathrm{d}s = \int_0^1 |\partial_u x(u,t)|\,\mathrm{d}u$. If $k\beta \geq 0$, then the evolution of planar curves parameterized by a solution of (1.1) represents a curve shortening flow, i.e., $L^{t_2} \leq L^{t_1} \leq L^0$ for any $0 \leq t_1 \leq t_2 \leq T$. The condition $k\beta \geq 0$ is obviously satisfied in the case $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, where $m > 0$ and γ is a nonnegative anisotropy function. In particular, the Euclidean curvature driven flow ($\beta = k$) is curve shortening flow.

3.2. The area equation. Let us denote by $A = A^t$ the area of the domain Ω^t enclosed by a Jordan curve Γ^t. Then by using Green's formula we obtain, for $P = -x_2/2, Q = x_1/2$,

$$A^t = \iint_{\Omega_t} \mathrm{d}x = \iint_{\Omega_t} \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\mathrm{d}x = \oint_{\Gamma^t} P\mathrm{d}x_1 + Q\mathrm{d}x_2 = \frac{1}{2}\oint_{\Gamma^t} -x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2\,.$$

Since $\mathrm{d}x_i = \partial_u x_i \mathrm{d}u, u \in [0,1]$, we have

$$A^t = \frac{1}{2}\int_0^1 \det(x, \partial_u x)\,\mathrm{d}u\,.$$

Clearly, integration of the derivative of a quantity along a closed curve yields zero. Therefore $0 = \int_0^1 \partial_u \det(x,\partial_t x)\mathrm{d}u = \int_0^1 \det(\partial_u x, \partial_t x) + \det(x, \partial_u\partial_t x)\mathrm{d}u$, and so $\int_0^1 \det(x,\partial_u\partial_t x)\mathrm{d}u = \int_0^1 \det(\partial_t x, \partial_u x)\mathrm{d}u$ because $\det(\partial_u x, \partial_t x) = -\det(\partial_t x, \partial_u x)$. As $\partial_t x = \beta\vec{N} + \alpha\vec{T}$, $\partial_u x\mathrm{d}u = \vec{T}\mathrm{d}s$ and $\frac{\mathrm{d}}{\mathrm{d}t}A^t = \frac{1}{2}\int_0^1 2\det(\partial_t x, \partial_u x)\mathrm{d}u$ we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}A^t + \int_{\Gamma^t} \beta\mathrm{d}s = 0. \tag{2.15}$$

Remark. In the case when a curve is evolved according to the curvature, i.e. $\beta = k$, then it follows from (2.3) and (2.15) that $\frac{\mathrm{d}}{\mathrm{d}t}A^t = -2\pi$ and so

$$A^t = A^0 - 2\pi t\,.$$

It means that the curve Γ^t ceases to exists for $t = T_{max} = \frac{A^0}{2\pi}$, i.e. the lifespan of curve evolution with $\beta = k$ is finite.

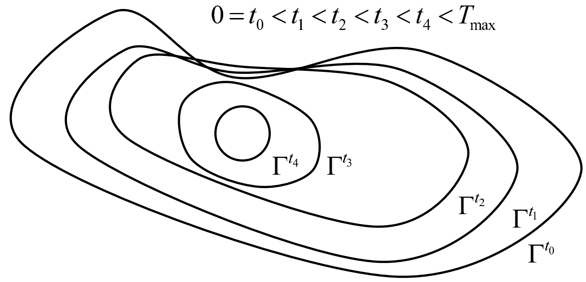

FIGURE 2. A closed curve evolving by the curvature becomes convex in finite time and then it converges to a point.

3.3. Brakke's motion by curvature. The above first integrals can be generalized for computation of the time derivative of the quantity $\int_{\Gamma^t} \phi(x,t)\,\mathrm{d}s$ where $\phi \in C_0^\infty(\mathbb{R}^2, [0,T))$ is a compactly supported test function. It represents a total value of a transported quantity represented by a scalar function ϕ. Since the value of the geometric quantity $\int_{\Gamma^t} \phi(x,t)\,\mathrm{d}s$ is independent of a particular choice of a tangential velocity α we may take $\alpha = 0$ for simplicity. Since $\partial_t x = \beta \vec{N}$ and $\partial_t \mathrm{d}s = \partial_t g \mathrm{d}u = -k\beta g \mathrm{d}u = -k\beta \mathrm{d}s$ we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma^t} \phi(x,t)\,\mathrm{d}s &= \int_{\Gamma^t} \partial_t \phi(x,t) + \nabla_x \phi . \partial_t x - k\beta\phi\,\mathrm{d}s \\ &= \int_{\Gamma^t} \partial_t \phi(x,t) + \beta \nabla_x \phi . \vec{N} - k\beta\phi\,\mathrm{d}s\,. \end{aligned} \tag{2.16}$$

The above integral identity (2.16) can be used in description of a more general flow of rectifiable subsets of $\mathbb{R}^2$ with a distributional notion of a curvature which is refereed to as varifold. Let $\Gamma^t, t \in [0,T)$, be a flow of one dimensional countably rectifiable subsets of the plane $\mathbb{R}^2$. Brakke in [Bra78, Section 3.3] introduced a notion of a mean curvature flow (i.e. $\beta = k$) as a solution to the following integral inequality

$$\overline{\frac{\mathrm{d}}{\mathrm{d}t}} \int_{\Gamma^t} \phi(x,t)\,\mathrm{d}\mathcal{H}^1(x) \le \int_{\Gamma^t} \left(\partial_t \phi(x,t) + k \nabla_x \phi . \vec{N} - k^2 \phi \right) \mathrm{d}\mathcal{H}^1(x) \tag{2.17}$$

for any smooth test function $\phi \in C_0^\infty(\mathbb{R}^2, [0,T))$. Here we have denoted by $\overline{\frac{\mathrm{d}}{\mathrm{d}t}}$ the upper derivative and $\mathcal{H}^1(x)$ the one dimensional Hausdorff measure.

4. Gage-Hamilton and Grayson's theorems

Assume that a smooth, closed, and embedded curve is evolved along its normal vector at a normal velocity proportional to its curvature, i.e. $\beta = k$. This curve evolution is known as the Euclidean curve shortening flow, and is depicted in Fig. 2. Since the curvature is positive on the convex side and it is negative on the concave side one may expect that the evolving curve becomes more convex and less concave as time t increases. Finally, it becomes convex shape and it shrinks to a circular point in finite time. This natural observation has been rigorously proved by M.

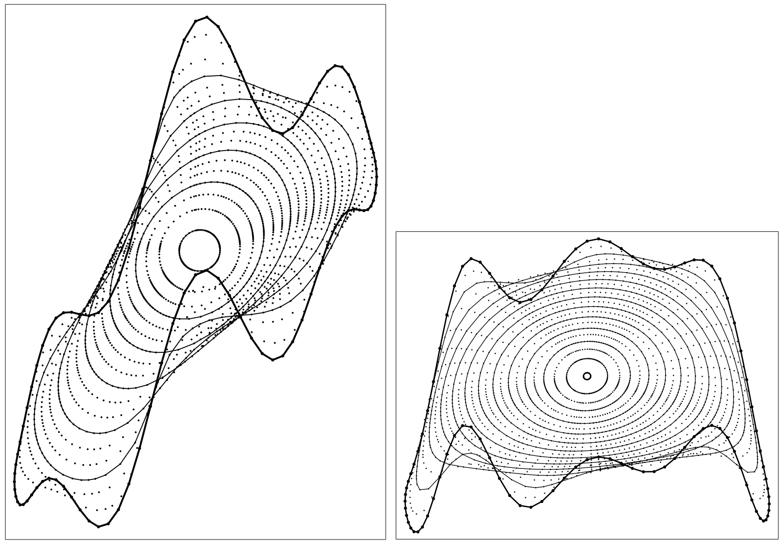

FIGURE 3. Motion by the curvature. Numerically computed evolution of various initial curves.

Grayson in [Gra87]. He used already known result due to Gage and Hamilton. They considered evolution of convex curves in the plane and proved that evolved curves shrink to a circular point in finite time.

THEOREM 2.1 (Gage and Hamilton [GH86]). *Any smooth closed convex curve embedded in* $\mathbb{R}^2$ *evolved by the curvature converges to a point in finite time with asymptotic circular shape.*

What Grayson added to this proof was the statement that any embedded smooth planar curve (not necessarily convex) when evolving according to the curvature becomes convex in finite time, stays embedded and then it shrinks to a circular point in finite time.

THEOREM 2.2 (Grayson [Gra87]). *Any smooth closed curve embedded in* $\mathbb{R}^2$ *evolve by the curvature becomes convex in finite time and then it converges to a point in finite time with asymptotical circular shape.*

Figure 3 shows computational results of curvature driven evolution of two initial planar curve evolved with the normal velocity $\beta = k$.

Although we will not go into the details of proofs of the above theorems it is worthwile to note that the proof of Grayson's theorem consists of several steps. First one needs to prove that an embedded initial curve Γ^0 when evolved according to the curvature stays embedded for $t > 0$, i.e. selfintersections cannot occur for $t > 0$. Then it is necessary to prove that eventual concave parts of a curve decrease they length. To this end, one can construct a partition a curve into its convex and concave part and show that concave parts are vanishing when time increases. The curve eventually becomes convex. Then Grayson applied previous result due to Gage and Hamilton. Their result says that any initial convex curve asymptotically approaches a circle when $t \to T_{max}$ where T_{max} is finite. To interpret their result in the language of parabolic partial differential equations we notice that the solution to (2.10) with $\beta = k$ remains positive provided that the initial value k^0 was nonnegative. This is a direct consequence of the maximum principle for parabolic equations. Indeed, let us denote by $y(t) = \min_{\Gamma^t} k(.,t)$. With regard to the envelope theorem we may assume that there exists $s(t)$ such that $y(t) = k(s(t),t)$. As $\partial_s^2 k \ge 0$ and $\partial_s k = 0$ at $s = s(t)$ we obtain from (2.10) that $y'(t) \ge y^3(t)$. Solving this ordinary differential inequality with positive initial condition $y(0) = \min_{\Gamma^0} k^0 > 0$ we obtain $\min_{\Gamma^t} k(.,t) = y(t) > 0$ for $0 < t < T_{max}$. Thus Γ^t remains convex provided Γ^0 was convex. The proof of the asymptotic circular profile is more complicated. However, it can be very well understood when considering selfsimilarly rescaled dependent and independent variables in equation (2.10). In these new variables, the statement of Gage and Hamilton theorem is equivalent to the proof of asymptotical stability of the constant unit solution.

In the proof of Grayson's theorem one can find another nice application of the parabolic comparison principle. Namely, if one wants to prove embeddednes property of an evolved curve Γ^t it is convenient to inspect the following distance function between arbitrary two points $x(s_1,t), x(s_2,t)$ of a curve Γ^t:

$$f(s_1, s_2, t) = |x(s_1,t) - x(s_2,t)|^2$$

where $s_1, s_2 \in [0, L(\Gamma^t)]$ and $t > 0$. Assume that $x = x(s,t)$ satisfies (2.4). Without loss of generality we may assume $\alpha = 0$ as α does not change the shape of the curve. Hence the embeddednes property is independent of α. Let us compute partial derivatives of f with respect to its variables. With help of Frenét formulae we obtain

$$\begin{aligned}
\partial_t f &= 2((x(s_1,t) - x(s_2,t)).(\partial_t x(s_1,t) - \partial_t x(s_2,t))) \\
&= 2((x(s_1,t) - x(s_2,t)).(k(s_1,t)\vec{N}(s_1,t) - k(s_2,t)\vec{N}(s_2,t))) \\
\partial_{s_1} f &= 2((x(s_1,t) - x(s_2,t)).\vec{T}(s_1,t)) \\
\partial_{s_2} f &= -2((x(s_1,t) - x(s_2,t)).\vec{T}(s_2,t)) \\
\partial_{s_1}^2 f &= 2(\vec{T}(s_1,t).\vec{T}(s_1,t)) + 2k(s_1,t)((x(s_1,t) - x(s_2,t)).\vec{N}(s_1,t)) \\
\partial_{s_2}^2 f &= 2(\vec{T}(s_1,t).\vec{T}(s_1,t)) - 2k(s_2,t)((x(s_1,t) - x(s_2,t).\vec{N}(s_2,t))\,.
\end{aligned}$$

Hence

$$\partial_t f = \Delta f - 4$$

where Δ is the Laplacian operator with respect to variables s_1, s_2. Using a clever application of a suitable barrier function (a circle) and comparison principle for the above parabolic equation Grayson proved that $f(s_1,s_2,t) \ge \delta > 0$ whenever $|s_1 - s_2| \ge \epsilon > 0$ where $\epsilon, \delta > 0$ are sufficiently small. But this is equivalent to the statement that the curve Γ^t is embedded. Notice that the above "trick" works only for the case $\beta = k$ and this is why it is still an open question whether embedded initial curve remains embedded when it is evolved by a general normal velocity $\beta = \beta(k)$.

4.1. Asymptotic profile of shrinking curves for other normal velocities. There are some partial results in this direction. If $\beta = k^{1/3}$ then the corresponding flow of planar curves is called affine space scale flow. It has been studied

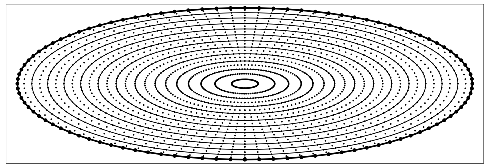

FIGURE 4. An initial ellipse evolved with the normal velocity $\beta = k^{1/3}$.

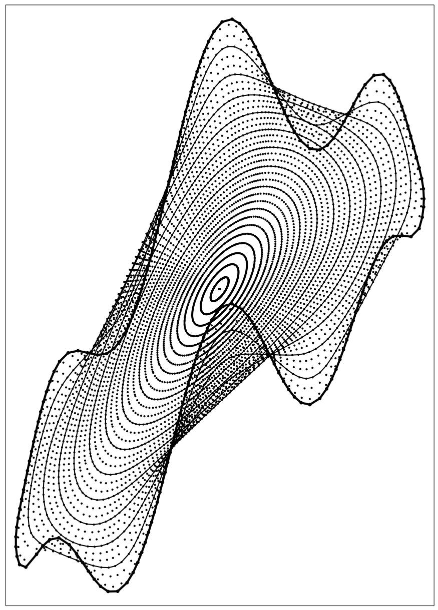

FIGURE 5. An example of evolution of planar curves evolved by the normal velocity $\beta = k^{1/3}$.

and analyzed by Angenent, Shapiro and Tannenbaum in [AST98] and [ST94]. In this case the limiting profile of a shrinking family of curves is an ellipse. Selfsimilar property of shrinking ellipses in the case $\beta = k^{1/3}$ has been also addressed in [MS99]. In Fig. 4 we present a computational result of evolution of shrinking ellipses. Fig. 5 depicts evolution of the same initial curve as in Fig. 3 (left) but now the curve is evolved with $\beta = k^{1/3}$. Finally. Fig. 6 shows computational results of curvature driven evolution of an initial spiral-like curve. Notice that the normal velocity of form $\beta(k) = k^\omega$ has been investigated by Ushijima and Yazaki in [UY00] in the context of crystaline curvature numerical approximation of the flow. It can be shown that $\omega = 1/3, 1/8, 1/15, ..., 1/(n^2-1), ...$, are bifurcation values for which one can prove the existence of branches of selfsimilar solutions of evolving curves shrinking to a point as a rounded polygon with n faces.

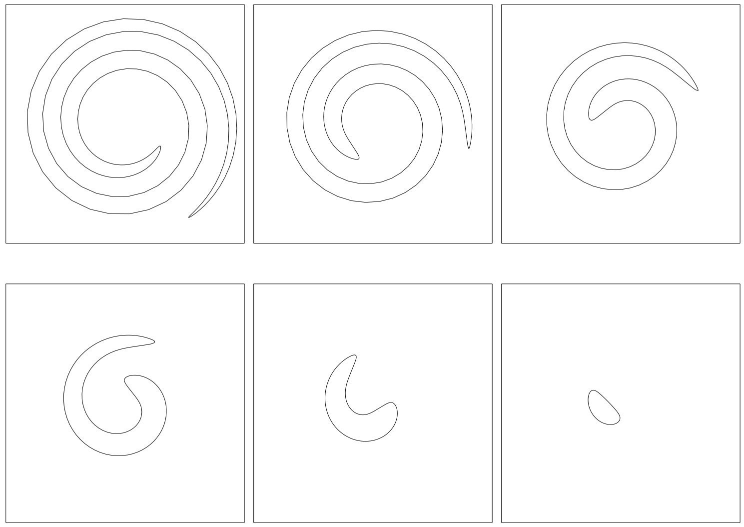

FIGURE 6. The sequence of evolving spirals for $\beta = k^{1/3}$.

CHAPTER 3

Qualitative behavior of solutions

In this chapter we focus our attention on qualitative behavior of curvature driven flows of planar curves. We present techniques how to prove local in time existence of a smooth family of curves evolved with the normal velocity given by a general function $\beta = \beta(k, x, \nu)$ depending on the curvature k, position vector x as well as the tangential angle ν. The main idea is to transform the geometric problem into the language of a time depending solution to an evolutionary partial differential equation like e.g. (2.10)–(2.13). First we present an approach due to Angenent describing evolution of an initial curve by a fully nonlinear parabolic equation for the distance function measuring the normal distance of the initial curve Γ^0 the evolved curve Γ^t for small values of $t > 0$. The second approach presented in this chapter is based on solution to the system of nonlinear parabolic-ordinary differential equations (2.10)–(2.13) also proposed by Angenent and Gurtin [AG89, AG94] and further analyzed and applied by Mikula and Ševčovič in the series of papers [MS01, MS04a, MS04b]. Both approaches are based on the solution to a certain fully nonlinear parabolic equation or system of equations. To provide a local existence and continuation result we have apply the theory of nonlinear analytic semiflows due to Da Prato and Grisvard, Lunardi [DPG75, DPG79, Lun82] and Angenent [Ang90a, Ang90b].

1. Local existence of smooth solutions

The idea of the proof of a local existence of an evolving family of closed embedded curves is to transform the geometric problem into a solution to a fully nonlinear parabolic equation for the distance $\phi(u, t)$ of a point $x(u, t) \in \Gamma^t$ from its initial value position $x^0(u) = x(u, 0) \in \Gamma^0$. This idea is due to Angenent [Ang90b] who derived the fully nonlinear parabolic equation for ϕ and proved local existence of smooth solutions by method of abstract nonlinear evolutionary equations in Banach spaces [Ang90b].

1.1. Local representation of an embedded curve. Let $\Gamma^0 = \mathrm{Img}(x^0)$ be a smooth initial Jordan curve embedded in $\mathbb{R}^2$. Because of its smoothness and embeddednes one can construct a local parameterization of any smooth curve $\Gamma^t = \mathrm{Img}(x(., t))$ lying in the thin tubular neighborhood along Γ^0, i.e. $\mathrm{dist}_H(\Gamma^t, \Gamma^0) < \varepsilon$ where dist_H is the Hausdorff set distance function. This is why there exists a small number $0 < \varepsilon \ll 1$ and a smooth immersion function $\sigma : S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ such that

- $x^0(u) = \sigma(u, 0)$ for any $u \in S^1$

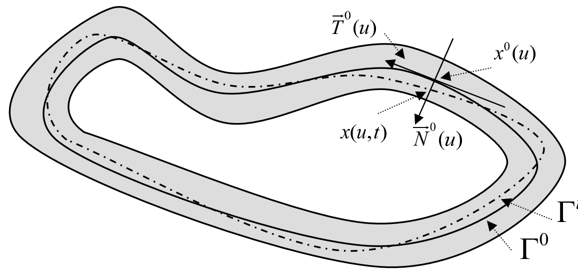

FIGURE 1. Description of a local parameterization of an embedded curve Γ^t in the neighborhood of the initial curve Γ^0.

- for any $u \in S^1$ there exists a unique $\phi = \phi(u,t) \in (-\varepsilon, \varepsilon)$ such that $\sigma(u, \phi(u,t)) = x(u,t)$.
- the implicitly defined function $\phi = \phi(u,t)$ is smooth in its variables provided the function $x = x(u,t)$ is smooth.

It is easy to verify that the function $\sigma(u,\phi) = x^0(u) + \phi\vec{N}^0(u)$ is the immersion having the above properties. Here $\vec{N}^0(u)$ is the unit inward vector to the curve Γ^0 at the point $x^0(u)$ (see Fig. 1).

Now we can evaluate $\partial_t x, \partial_u x, \partial_u^2 x$ and $|\partial_u x|$ as follows:

$$\begin{aligned}
\partial_t x &= \sigma'_\phi \partial_t \phi\,, \\
\partial_u x &= \sigma'_u + \sigma'_\phi \partial_u \phi\,, \\
\partial_u^2 x &= \sigma''_{uu} + 2\sigma''_{u\phi}\partial_u\phi + \sigma''_{\phi\phi}(\partial_u\phi)^2 + \sigma'_\phi \partial_u^2 \phi\,, \\
g = |\partial_u x| &= \left(|\sigma'_u|^2 + 2(\sigma'_u.\sigma'_\phi)\partial_u\phi + |\sigma'_\phi|^2(\partial_u\phi)^2\right)^{\frac{1}{2}}\,.
\end{aligned}$$

Hence we can express the curvature $k = \det(\partial_u x, \partial_u^2 x)/|\partial_u x|^3$ as follows:

$$\begin{aligned}
g^3 k &= \det(\partial_u x, \partial_u^2 x) = \partial_u^2\phi\partial_u\phi \det(\sigma'_\phi, \sigma'_\phi) + \partial_u^2\phi \det(\sigma'_u, \sigma'_\phi) \\
&+ (\partial_u\phi)^2 \left[\det(\sigma'_u, \sigma''_{\phi\phi}) + \partial_u\phi \det(\sigma'_\phi, \sigma''_{\phi\phi})\right] + 2\partial_u\phi \det(\sigma'_u, \sigma''_{u\phi}) \\
&+ 2(\partial_u\phi)^2 \det(\sigma'_\phi, \sigma''_{u\phi}) + \det(\sigma'_u, \sigma''_{uu}) + \partial_u\phi \det(\sigma'_\phi, \sigma''_{uu})\,.
\end{aligned}$$

Clearly, $\det(\sigma'_\phi, \sigma'_\phi) = 0$. Since $\sigma'_\phi = \vec{N}^0$ and $\sigma'_u = \partial_u x^0 + \phi\partial_u\vec{N}^0 = g^0(1-k^0\phi)\vec{T}^0$ we have $\det(\sigma'_u, \sigma'_\phi) = g^0(1-k^0\phi)$ and $(\sigma'_u.\sigma'_\phi) = 0$. Therefore the local length $g = |\partial_u x|$ and the curvature k can be expressed as

$$\begin{aligned}
g &= |\partial_u x| = \left((g^0(1-k^0\phi))^2 + (\partial_u\phi)^2\right)^{\frac{1}{2}}\,, \\
k &= \frac{g^0(1-k^0\phi)}{g^3}\partial_u^2\phi + R(u,\phi,\partial_u\phi)
\end{aligned}$$

where $R(u,\phi,\partial_u\phi)$ is a smooth function.

We proceed with evaluation of the time derivative $\partial_t x$. Since $\partial_u x = \sigma'_u + \sigma'_\phi\partial_u\phi$ we have $\vec{T} = \frac{1}{g}(\sigma'_u + \sigma'_\phi\partial_u\phi)$. The vectors $\vec{N}$ and $\vec{T}$ are perpendicular to each other. Thus

$$\partial_t x.\vec{N} = -\det(\partial_t x, \vec{T}) = \frac{1}{g}\det(\sigma'_u, \sigma'_\phi)\partial_t\phi = \frac{g^0(1-k^0\phi)}{g}\partial_t\phi$$

because $\det(\sigma'_\phi, \sigma'_\phi) = 0$. Hence, a family of embedded curves $\Gamma^t, t \in [0,T)$, evolves according to the normal velocity

$$\beta = \mu k + c$$

if and only if the function $\phi = \phi(u,t)$ is a solution to the nonlinear parabolic equation

$$\partial_t \phi = \frac{\mu}{g^2}\partial_u^2\phi + \frac{g}{g^0(1-k^0\phi)}(\mu R(u,\phi,\partial_u\phi) + c)$$

where

$$g = \left(|g^0|^2(1-k^0\phi)^2 + (\partial_u\phi)^2\right)^{\frac{1}{2}} .$$

In a general case when the normal velocity $\beta = \beta(k, x, \vec{N})$ is a function of curvature k, position vector x and the inward unit normal vector $\vec{N}$, ϕ is a solution to a fully nonlinear parabolic equation of the form:

$$\partial_t\phi = F(\partial_u^2\phi, \partial_u\phi, \phi, u), \quad u \in S^1, t \in (0,T)\,. \tag{3.1}$$

The right-hand side function $F = F(q,p,\phi,u)$ is C^1 is a smooth function of its variables and

$$\frac{\partial F}{\partial q} = \frac{\beta'_k}{g^2} > 0$$

and so equation (3.1) is a nonlinear strictly parabolic equation. Equation (3.1) is subject to an initial condition

$$\phi(u,0) = \phi^0(u) \equiv 0\,, \quad u \in S^1\,. \tag{3.2}$$

1.2. Nonlinear analytic semiflows. In this section we recall basic facts from the theory of nonlinear analytic semiflows which can be used in order to prove local in time existence of a smooth solutions to the fully nonlinear parabolic equation (3.1) subject to the initial condition (3.2). The theory has been developed by S. Angenent in [Ang90b] and A. Lunardi in [Lun82].

Equation (3.1) can be rewritten as an abstract evolutionary equation

$$\partial_t\phi = \mathcal{F}(\phi) \tag{3.3}$$

subject to the initial condition

$$\phi(0) = \phi^0 \in E_1 \tag{3.4}$$

where $\mathcal{F}$ is a C^1 smooth mapping between two Banach spaces E_1, E_0, i.e. $\mathcal{F} \in C^1(E_1, E_0)$. For example, if we take

$$E_0 = h^\varrho(S^1), \qquad E_1 = h^{2+\varrho}(S^1)\,,$$

where $h^{k+\varrho}(S^1), k = 0,1,...,$ is a little Hölder space, i.e. the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{k+\sigma}(S^1)$ (see [Ang90b]), then the mapping F defined as in the right-hand side of (3.1) is indeed a C^1 mapping from E_1 into E_0. Its Frechét derivative $d\mathcal{F}(\phi^0)$ is being given by the linear operator

$$d\mathcal{F}(\phi^0)\phi = a^0\partial_u^2\phi + b^0\partial_u\phi + c^0\phi$$

where

$$a^0 = F'_q(\partial_u^2\phi^0, \partial_u\phi^0, \phi^0, u) = \frac{\beta'_k}{(g^0)^2}, \quad b^0 = F'_p(\partial_u^2\phi^0, \partial_u\phi^0, \phi^0, u),$$

$$c^0 = F'_\phi(\partial_u^2\phi^0, \partial_u\phi^0, \phi^0, u)\,.$$

Suppose that the initial curve $\Gamma^0 = \mathrm{Img}(x^0)$ is sufficiently smooth, $x^0 \in \left(h^{2+\varrho}(S^1)\right)^2$ and regular, i.e. $g^0(u) = |\partial_u x^0(u)| > 0$ for any $u \in S^1$. Then $a^0 \in h^{1+\varrho}(S^1)$. A standard result from the theory of analytic semigroups (c.f. [Hen81]) enables us to conclude that the principal part $A := a^0\partial_u^2$ of the linearization $d\mathcal{F}(\phi^0)$ is a generator of a analytic semigroup $\exp(tA), t \geq 0$, in the Banach space $E_0 = h^\varrho(S^1)$.

1.2.1. *Maximal regularity theory.* In order to proceed with the proof of local in time existence of a classical solution to the abstract nonlinear equation (3.3) we have to recall a notion of a maximal regularity pair of Banach spaces.

Assume that (E_1, E_0) is a pair of Banach spaces with E_1 densely included into E_0. By $L(E_1, E_0)$ we shall denote the Banach space of all linear bounded operators from E_1 into E_0. An operator $A \in L(E_1, E_0)$ can be considered as an unbounded operator in the Banach space E_0 with a dense domain $D(A) = E_1$. By $Hol(E_1, E_0)$ we shall denote a subset of $L(E_1, E_0)$ consisting of all generators A of an analytic semigroup $\exp(tA), t \geq 0$, of linear operators in the Banach space E_0 (c.f. [Hen81]).

The next lemma is a standard perturbation result concerning the class of generators of analytic semigroups.

Lemma 3.1. *[Paz83, Theorem 2.1] The set $Hol(E_1, E_0)$ is an open subset of the Banach space $L(E_1, E_0)$.*

The next result is also related to the perturbation theory for the class of generators of analytic semigroups.

Definition 3.2. We say that the linear bounded operator $B : E_1 \to E_0$ has a relative zero norm if for any $\varepsilon > 0$ there is a constant $k_\varepsilon > 0$ such that

$$\|Bx\|_{E_0} \leq \varepsilon\|x\|_{E_1} + k_\varepsilon\|x\|_{E_0}$$

for any $x \in E_1$.

As an example of such an operator we may consider an operator $B \in L(E_1, E_0)$ satisfying the following inequality of Gagliardo-Nirenberg type:

$$\|Bx\|_{E_0} \leq C\|x\|_{E_1}^\lambda \|x\|_{E_0}^{1-\lambda}$$

for any $x \in E_1$ where $\lambda \in (0, 1)$. Then using Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1\,,$$

with $p = 1/\lambda$ and $q = 1/(1-\lambda)$. it is easy to verify that B has zero relative norm.

Lemma 3.3. *[Paz83, Section 2.1] The set $Hol(E_1, E_0)$ is closed with respect to perturbations by linear operators with zero relative norm, i.e. if $A \in Hol(E_1, E_0)$ and $B \in L(E_1, E_0)$ has zero relative norm then $A + B \in Hol(E_1, E_0)$.*

Neither the theory of C^0 semigroups (c.f. Pazy [Paz83]) nor the theory of analytic semigroups (c.f. Henry [Hen81]) are able to handle fully nonlinear parabolic equations. This is mainly due to the method of integral equation which is suitable for semilinear equations only. The second reason why these methods cannot provide a local existence result is due to the fact that semigroup theories are working with function spaces which are fractional powers of the domain of a generator of an analytic semigroup (see [Hen81]). Therefore we need a more robust theory capable of handling fully nonlinear parabolic equations. This theory is due to Angenent and Lunardi [Ang90a, Lun82] and it is based on abstract results by Da Prato and Grisvard [DPG75, DPG79]. The basic idea is the linearization technique where one can linearize the fully nonlinear equation at the initial condition ϕ^0. Then one sets up a linearized semilinear equation with the right hand side which is of the second order with respect to deviation from the initial condition. In what follows, we shall present key steps of this method. First we need to introduce the maximal regularity class which will enable us to construct an inversion operator to a nonhomogeneous semilinear equation.

Let $E = (E_1, E_0)$ be a pair of Banach spaces for which E_1 is densely included in E_0. Let us define the following function spaces

$$X = C([0,1], E_0), \qquad Y = C([0,1], E_1) \cap C^1([0,1], E_0)\,.$$

We shall identify ∂_t with the bounded differentiation operator from Y to X defined by $(\partial_t\phi)(t) = \phi'(t)$. For a given linear bounded operator $A \in L(E_1, E_0)$ we define the extended operator $\mathcal{A} : Y \to X \times E_1$ defined by $\mathcal{A}\phi = (\partial_t\phi - A\phi, \phi(0))$. Next we define a class $\mathcal{M}_1(E)$ as follows:

$$\mathcal{M}_1(E) = \{A \in Hol(E),\ \mathcal{A} \text{ is an isomorphism between } Y \text{ and } X \times E_1\}\,.$$

It means that the class $\mathcal{M}_1(E)$ consist of all generators of analytic semigroups A such that the initial value problem for the semilinear evolution equation

$$\partial_t\phi - A\phi = f(t), \qquad \phi(0) = \phi^0,$$

has a unique solution $\phi \in Y$ for any right-hand side $f \in X$ and the initial condition $\phi^0 \in E_1$ (c.f. [Ang90a]). For such an operator A we obtain boundedness of the inverse of the operator $\phi \mapsto (\partial_t - A)\phi$ mapping the Banach space $Y^{(0)} = \{\phi \in Y, \phi(0) = 0\}$ onto the Banach space X, i.e.

$$\|(\partial_t - A)^{-1}\|_{L(X,Y^{(0)})} \le C < \infty\,.$$

The class $\mathcal{M}_1(E)$ is refereed to as maximal regularity class for the pair of Banach spaces $E = (E_1, E_0)$.

An analogous perturbation result to Lemma 3.3 has been proved by Angenent.

LEMMA 3.4. *[Ang90a, Lemma 2.5] The set $\mathcal{M}_1(E_1, E_0)$ is closed with respect to perturbations by linear operators with zero relative norm.*

Using properties of the class $\mathcal{M}_1(E)$ we are able to state the main result on the local existence of a smooth solution to the abstract fully nonlinear evolutionary problem (3.3)–(3.4).

THEOREM 3.5. *[Ang90a, Theorem 2.7] Assume that $\mathcal{F}$ is a C^1 mapping from some open subset $\mathcal{O} \subset E_1$ of the Banach space E_1 into the Banach space E_0. If the Frechét derivative $A = d\mathcal{F}(\phi)$ belongs to $\mathcal{M}_1(E)$ for any $\phi \in \mathcal{O}$ and the initial condition ϕ^0 belongs to $\mathcal{O}$ then the abstract fully nonlinear evolutionary problem (3.3)–(3.4) has a unique solution $\phi \in C^1([0,T], E_0) \cap C([0,T], E_1)$ on some small time interval $[0,T], T > 0$.*

Proof. The proof is based on the Banach fixed point theorem. Without loss of generality (by shifting the solution $\phi(t) \mapsto \phi^0 + \phi(t)$) we may assume $\phi^0 = 0$. Taylor's series expansion of $\mathcal{F}$ at $\phi = 0$ yields $\mathcal{F}(\phi) = \mathcal{F}_0 + A\phi + R(\phi)$ where $\mathcal{F}_0 \in E_0$, $A \in \mathcal{M}_1(E)$ and the remainder function R is quadratically small, i.e. $\|R(\phi)\|_{E_0} = O(\|\phi\|^2_{E_1})$ for small $\|\phi\|_{E_1}$. Problem (3.3)–(3.4) is therefore equivalent to the fixed point problem

$$\phi = (\partial_t - A)^{-1}(R(\phi) + \mathcal{F}_0)$$

on the Banach space $Y_T^{(0)} = \{\phi \in C^1([0,T], E_0) \cap C([0,T], E_1), \phi(0) = 0\}$. Using boundedness of the operator $(\partial_t - A)^{-1}$ and taking $T > 0$ sufficiently one can prove that the right hand side of the above equation is a contraction mapping on the space $Y_T^{(0)}$ proving thus the statement of theorem. □

1.2.2. *Application of the abstract result for the fully nonlinear parabolic equation for the distance function.* Now we are in a position to apply the abstract result contained in Theorem 3.5 to the fully nonlinear parabolic equation (3.1) for the distance function ϕ subject to a zero initial condition $\phi^0 = 0$. Notice that one has to carefully choose function spaces to work with. Baillon in [Bai80] showed that, if we exclude the trivial case $E_1 = E_0$, the class $\mathcal{M}_1(E_1, E_0)$ is nonempty only if the Banach space E_0 contains a closed subspace isomorphic to the sequence space (c_0). As a consequence of this criterion we conclude that $\mathcal{M}_1(E_1, E_0)$ is empty for any reflexive Banach space E_0. Therefore the space E_0 cannot be reflexive. On the other hand, one needs to prove that the linearization $A = d\mathcal{F}(\phi) : E_1 \to E_0$ generates an analytic semigroup in E_0. Therefore it is convenient to work with little Hölder spaces satisfying these structural assumptions.

Applying the abstract result from Theorem 3.5 we are able to state the following theorem which is a special case of a more general result by Angenent [Ang90b, Theorem 3.1] to evolution of planar curves.

Theorem 3.6. *[Ang90b, Theorem 3.1] Assume that the normal velocity $\beta = \beta(k, \nu)$ is a $C^{1,1}$ smooth function such that $\beta'_k > 0$ for all $k \in \mathbb{R}$ and $\nu \in [0, 2\pi]$. Let Γ^0 be an embedded smooth curve with Hölder continuous curvature. Then there exists a unique maximal solution $\Gamma^t, t \in [0, T_{max})$, consisting of curves evolving with the normal velocity equal to $\beta(k, \nu)$.*

Remark. Verification of nonemptyness of the set $\mathcal{M}_1(E_1, E_0)$ might be difficult for a particular choice of Banach pair (E_1, E_0). There is however a general construction of the Banach pair (E_1, E_0) such that a given linear operator A belongs to $\mathcal{M}_1(E_1, E_0)$. Let $F = (F_1, F_0)$ be a Banach pair. Assume that $A \in Hol(F_1, F_0)$. We define the Banach space $F_2 = \{\phi \in F_1, A\phi \in F_1\}$ equipped with the graph norm $\|\phi\|_{F_2} = \|\phi\|_{F_1} + \|A\phi\|_{F_1}$. For a fixed $\sigma \in (0, 1)$ we introduce the continuous interpolation spaces $E_0 = F_\sigma = (F_1, F_0)_\sigma$ and $E_1 = F_{1+\sigma} = (F_2, F_1)_\sigma$. Then, by result due to Da Prato and Grisvard [DPG75, DPG79] we have $A \in \mathcal{M}_1(E_1, E_0)$.

1.3. Local existence, uniqueness and continuation of classical solutions. In this section we present another approach for the proof a of a local existence of a classical solution. Now we put our attention to a solution of the system of parabolic-ordinary differential equations (2.10) – (2.13). Let a regular smooth initial curve $\Gamma_0 = \mathrm{Img}(x_0)$ be given. Recall that a family of planar curves

$\Gamma^t = \text{Img}(x(.,t)), \quad t \in [0,T)$, satisfying (1.1) can be represented by a solution $x = x(u,t)$ to the position vector equation (2.4). Notice that $\beta = \beta(x,k,\nu)$ depends on x, k, ν and this is why we have to provide and analyze a closed system of equations for the variables k, ν as well as the local length $g = |\partial_u x|$ and position vector x. In the case of a nontrivial tangential velocity functional α the system of parabolic–ordinary governing equations has the following form:

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta\,, \tag{3.5}$$

$$\partial_t \nu = \beta_k' \partial_s^2 \nu + (\alpha + \beta_\nu')\partial_s \nu + \nabla_x \beta . \vec{T}\,, \tag{3.6}$$

$$\partial_t g = -gk\beta + \partial_u \alpha\,, \tag{3.7}$$

$$\partial_t x = \beta \vec{N} + \alpha \vec{T} \tag{3.8}$$

where $(u,t) \in Q_T = [0,1] \times (0,T)$, $\mathrm{d}s = g\,\mathrm{d}u$, $\vec{T} = \partial_s x = (\cos\nu, \sin\nu)$, $\vec{N} = \vec{T}^\perp = (-\sin\nu,\ \cos\nu)$, $\beta = \beta(x,k,\nu)$. A solution (k,ν,g,x) to (3.5) – (3.8) is subject to initial conditions

$$k(.,0) = k_0\,,\ \nu(.,0) = \nu_0\,,\ g(.,0) = g_0\,,\ x(.,0) = x_0(.)$$

and periodic boundary conditions at $u = 0,1$ except of ν for which we require the boundary condition $\nu(1,t) \equiv \nu(0,t) \bmod(2\pi)$. The initial conditions for k_0, ν_0, g_0 and x_0 have to satisfy natural compatibility constraints: $g_0 = |\partial_u x_0| > 0\,,\ k_0 = \det(g_0^{-3}\partial_u x_0, \partial_u^2 x_0)\,,\ \partial_u \nu_0 = g_0 k_0$ following from the equation $k = \det(\partial_s x, \partial_s^2 x)$ and Frenét's formulae applied to the initial curve $\Gamma_0 = \text{Img}(x_0)$. Notice that the system of governing equations consists of coupled parabolic-ordinary differential equations.

Since α enters the governing equations a solution k,ν,g,x to (3.5) – (3.8) does depend on α. On the other hand, the family of planar curves $\Gamma^t = \text{Img}(x(.,t)), t \in [0,T)$, is independent of a particular choice of the tangential velocity α as it does not change the shape of a curve. The tangential velocity α can be therefore considered as a free parameter to be suitably determined later. For example, in the Euclidean curve shortening equation $\beta = k$ we can write equation (2.4) in the form $\partial_t x = \partial_s^2 x = g^{-1}\partial_u(g^{-1}\partial_u x) + \alpha g^{-1}\partial_u x$ where $g = |\partial_u x|$. Epstein and Gage [EG87] showed how this degenerate parabolic equation (g need not be smooth enough) can be turned into the strictly parabolic equation $\partial_t x = g^{-2}\partial_u^2 x$ by choosing the tangential term α in the form $\alpha = g^{-1}\partial_u(g^{-1})\partial_u x$. This trick is known as "De Turck's trick" named after De Turck (see [DeT83]) who use this approach to prove short time existence for the Ricci flow. Numerical aspects of this "trick" has been discussed by Dziuk and Deckelnick in [Dzi94, Dzi99, Dec97]. In general, we allow the tangential velocity functional α appearing in (3.5) – (3.8) to be dependent on k,ν,g,x in various ways including nonlocal dependence, in particular (see the next section for details).

Let us denote $\Phi = (k,\nu,g,x)$. Let $0 < \varrho < 1$ be fixed. By E_k we denote the following scale of Banach spaces (manifolds)

$$E_k = h^{2k+\varrho} \times h_*^{2k+\varrho} \times h^{1+\varrho} \times (h^{2+\varrho})^2 \tag{3.9}$$

where $k = 0, 1/2, 1$, and $h^{2k+\varrho} = h^{2k+\varrho}(S^1)$ is the "little" Hölder space (see [Ang90a]). By $h_*^{2k+\varrho}(S^1)$ we have denoted the Banach manifold $h_*^{2k+\varrho}(S^1) = \{\nu : \mathbb{R} \to \mathbb{R}\,, \vec{N} = (-\sin\nu, \cos\nu) \in (h^{2k+\varrho}(S^1))^2\}$. [1]

Concerning the tangential velocity α we shall make a general regularity assumption:

$$\alpha \in C^1(\mathcal{O}_{\frac{1}{2}}, h^{2+\varrho}(S^1)) \tag{3.10}$$

for any bounded open subset $\mathcal{O}_{\frac{1}{2}} \subset E_{\frac{1}{2}}$ such that $g > 0$ for any $(k, \nu, g, x) \in \mathcal{O}_{\frac{1}{2}}$.

In the rest of this section we recall a general result on local existence and uniqueness a classical solution of the governing system of equations (3.5) – (3.8). The normal velocity β depending on k, x, ν belongs to a wide class of normal velocities for which local existence of classical solutions has been shown in [MS04a, MS04b]. This result is based on the abstract theory of nonlinear analytic semigroups developed by Angenent in [Ang90a] an it utilizes the so-called maximal regularity theory for abstract parabolic equations.

THEOREM 3.7. *([MS04a, Theorem 3.1] Assume $\Phi_0 = (k_0, \nu_0, g_0, x_0) \in E_1$ where k_0 is the curvature, ν_0 is the tangential vector, $g_0 = |\partial_u x_0| > 0$ is the local length element of an initial regular closed curve $\Gamma_0 = Img(x_0)$ and the Banach space E_k is defined as in (3.9). Assume $\beta = \beta(x, k, \nu)$ is a C^4 smooth and 2π-periodic function in the ν variable such that $\min_{\Gamma_0} \beta'_k(x_0, k_0, \nu_0) > 0$ and α satisfies (3.10). Then there exists a unique solution $\Phi = (k, \nu, g, x) \in C([0,T], E_1) \cap C^1([0,T], E_0)$ of the governing system of equations (3.5) – (3.8) defined on some small time interval $[0,T]$, $T > 0$. Moreover, if Φ is a maximal solution defined on $[0, T_{max})$ then we have either $T_{max} = +\infty$ or $\liminf_{t\to T_{max}^-} \min_{\Gamma^t} \beta'_k(x, k, \nu) = 0$ or $T_{max} < +\infty$ and $\max_{\Gamma^t} |k| \to \infty$ as $t \to T_{max}$.*

PROOF. Since $\partial_s \nu = k$ and $\partial_s \beta = \beta'_k \partial_s k + \beta'_\nu k + \nabla_x \beta . \vec{T}$ the curvature equation (3.5) can be rewritten in the divergent form

$$\partial_t k = \partial_s(\beta'_k \partial_s k) + \partial_s(\beta'_\nu k) + k \nabla_x \beta . \vec{N} + \partial_s(\nabla_x \beta . \vec{T}) + \alpha \partial_s k + k^2 \beta\,.$$

Let us take an open bounded subset $\mathcal{O}_{\frac{1}{2}} \subset E_{\frac{1}{2}}$ such that $\mathcal{O}_1 = \mathcal{O}_{\frac{1}{2}} \cap E_1$ is an open subset of E_1 and $\Phi_0 \in \mathcal{O}_1$, $g > 0$, and $\beta'_k(x, k, \nu) > 0$ for any $(k, \nu, g, x) \in \mathcal{O}_1$. The linearization of f at a point $\bar{\Phi} = (\bar{k}, \bar{\nu}, \bar{g}, \bar{x}) \in \mathcal{O}_1$ has the form $df(\bar{\Phi}) = d_\Phi F(\bar{\Phi}, \bar{\alpha}) + d_\alpha F(\bar{\Phi}, \bar{\alpha})\, d_\Phi \alpha(\bar{\Phi})$ where $\bar{\alpha} = \alpha(\bar{\Phi})$ and

$$d_\Phi F(\bar{\Phi}, \bar{\alpha}) = \partial_u \bar{D} \partial_u + \bar{B} \partial_u + \bar{C}\,, \quad d_\alpha F(\bar{\Phi}, \bar{\alpha}) = \left(\bar{g}^{-1} \partial_u \bar{k}\,,\ \bar{k}\,,\ \partial_u\,,\ \vec{\bar{T}}\right)$$

$\bar{D} = \mathrm{diag}(\bar{D}_{11}, \bar{D}_{22}, 0, 0, 0)$, $\bar{D}_{11} = \bar{D}_{22} = \bar{g}^{-2} \beta'_k(\bar{x}, \bar{k}, \bar{\nu}) \in C^{1+\varrho}(S^1)$ and $\bar{B}, \bar{C}$ are 5×5 matrices with $C^\varrho(S^1)$ smooth coefficients. Moreover, $\bar{B}_{ij} = 0$ for $i = 3, 4, 5$ and $\bar{C}_{3j} \in C^{1+\varrho}$, $\bar{C}_{ij} \in C^{2+\varrho}$ for $i = 4, 5$ and all j. The linear operator A_1 defined by $A_1 \Phi = \partial_u(\bar{D} \partial_u \Phi)$, $D(A_1) = E_1 \subset E_0$ is a generator of an analytic semigroup on E_0 and, moreover, $A_1 \in \mathcal{M}_1(E_0, E_1)$ (see [Ang90a, Ang90b]). Notice that $d_\alpha F(\bar{\Phi}, \bar{\alpha})$ belongs to $\mathcal{L}(C^{2+\varrho}(S^1), E_{\frac{1}{2}})$ and this is why we can write $d_\Phi f(\bar{\Phi})$ as a sum $A_1 + A_2$ where $A_2 \in L(E_{\frac{1}{2}}, E_0)$. By Gagliardo–Nirenberg inequality we have $\|A_2 \Phi\|_{E_0} \le C \|\Phi\|_{E_{\frac{1}{2}}} \le C \|\Phi\|_{E_0}^{1/2} \|\Phi\|_{E_1}^{1/2}$ and so the linear operator A_2 is a relatively

[1] Alternatively, one may consider the normal velocity β depending directly on the unit inward normal vector $\vec{N}$ belonging to the linear vector space $(h^{2k+\varrho}(S^1))^2$, i.e. $\beta = \beta(k, x, \vec{N})$.

bounded linear perturbation of A_1 with zero relative bound (cf. [Ang90a]). With regard to Lemma 3.4 (see also [Ang90a, Lemma 2.5]) the class $\mathcal{M}_1$ is closed with respect to such perturbations. Thus $d_\Phi f(\bar{\Phi}) \in \mathcal{M}_1(E_0, E_1)$. The proof of the short time existence of a solution Φ now follows from Theorem 3.5 (see also [Ang90a, Theorem 2.7]).

Finally, we will show that the maximal curvature becomes unbounded as $t \to T_{max}$ in the case $\liminf_{t\to T_{max}^-} \min_{\Gamma^t} \beta'_k > 0$ and $T_{max} < +\infty$. Suppose to the contrary that $\max_{\Gamma^t} |k| \le M < \infty$ for any $t \in [0, T_{max})$. According to [Ang90b, Theorem 3.1] there exists a unique maximal solution $\Gamma : [0, T'_{max}) \to \Omega(\mathbb{R}^2)$ satisfying the geometric equation (1.1). Recall that $\Omega(\mathbb{R}^2)$ is the space of C^1 regular Jordan curves in the plane (cf. [Ang90b]). Moreover, Γ^t is a C^∞ smooth curve for any $t \in (0, T'_{max})$ and the maximum of the absolute value of the curvature tends to infinity as $t \to T'_{max}$. Thus $T_{max} < T'_{max}$ and therefore the curvature and subsequently ν remain bounded in $C^{2+\varrho'}$ norm on the interval $[0, T_{max}]$ for any $\varrho' \in (\varrho, 1)$. Applying the compactness argument one sees that the limit $\lim_{t\to T_{max}} \Phi(., t)$ exists and remains bounded in the space E_1 and one can continue the solution Φ beyond T_{max}, a contradiction. □

Remark. In a general case where the normal velocity may depend on the position vector x, the maximal time of existence of a solution can be either finite or infinite. Indeed, as an example one can consider the unit ball $B = \{|x| < 1\}$ and function $\delta(x) = (|x| - 1)^\gamma$ for $x \notin B$, $\gamma > 0$. Suppose that $\Gamma_0 = \{|x| = R_0\}$ is a circle with a radius $R_0 > 1$ and the family $\Gamma^t, t \in [0, T)$, evolves according to the normal velocity function $\beta(x, k) = \delta(x)k$. Then, it is an easy calculus to verify that the family Γ^t approaches the boundary $\partial B = \{|x| = 1\}$ in a finite time $T_{max} < \infty$ provided that $0 < \gamma < 1$ whereas $T_{max} = +\infty$ in the case $\gamma = 1$.

CHAPTER 4

Level set methods for curvature driven flows of planar curves

By contrast to the direct approach, *level set methods* are based on introducing an auxiliary shape function whose zero level sets represent a family of planar curves which is evolved according to the geometric equation (1.1) (see e.g. [OS88, Set90, Set96, Set98]). The level set approach handles implicitly the curvature-driven motion, passing the problem to higher dimensional space. One can deal with splitting and/or merging of evolving curves in a robust way. However, from the computational point of view, level set methods are much more computationaly expensive than methods based on the direct approach. The purpose of this chapter is to present basic ideas and results concerning the level set approach in curvature driven flows of planar curves.

Other indirect method is based on the phase-field formulations. In these lecture notes we however do not go into details of these methods and interested reader is referred to extensive literature in this topic (see e.g. [Cag90, EPS96, BM98] and references therein).

1. Level set representation of Jordan curves in the plane

In the level set method the evolving family of planar curves $\Gamma^t, t \geq 0$, is represented by the zero level set of the so-called shape function $\phi : \Omega \times [0, T] \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^2$ is a simply connected domain containing the whole family of evolving curves $\Gamma^t, t \in [0, T]$. We adopt a notation according to which the interior of a curve is described as: $int(\Gamma^t) = \{x \in \mathbb{R}^2, \phi(x,t) < 0\}$ and, consequently,

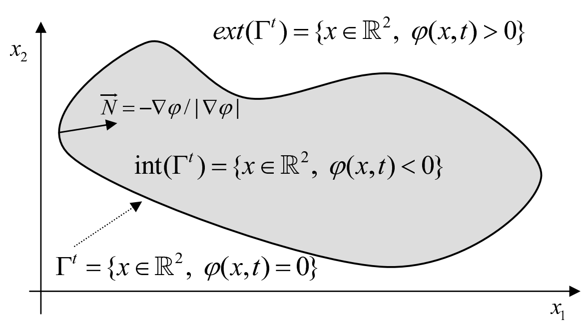

FIGURE 1. Description of the level set representation of a planar embedded curve by a shape function $\phi : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$.

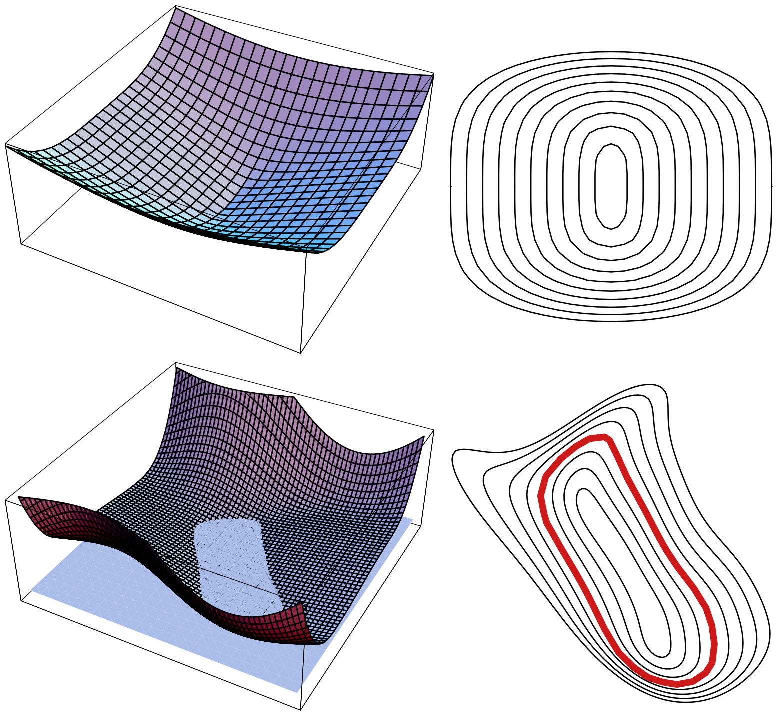

FIGURE 2. Description of the representation of planar embedded curves by level sets of two functions $\phi : \mathbb{R}^2 \to \mathbb{R}$. The level set functions (left) and their level cross-section (right).

$ext(\Gamma^t) = \{x \in \mathbb{R}^2, \phi(x,t) > 0\}$ and $\Gamma^t = \{x \in \mathbb{R}^2, \phi(x,t) = 0\}$ (see Fig. 1). With this convection, the unit inward normal vector $\vec{N}$ can be expressed as

$$\vec{N} = -\nabla\phi/|\nabla\phi|\,.$$

In order to express the signed curvature k of the curve Γ^t we make use of the identity $\phi(x(s,t),t) = 0$. Differentiating this identity with respect to the arc-length parameter s we obtain $0 = \nabla\phi.\partial_s x = \nabla\phi.\vec{T}$. Differentiating the latter identity with respect to s again and using the Frenét formula $\partial_s\vec{T} = k\vec{N}$ we obtain $0 = k(\nabla\phi.\vec{N}) + \vec{T}^{\perp}\nabla^2\phi\vec{T}$. Since $\vec{N} = -\nabla\phi/|\nabla\phi|$ we have

$$k = \frac{1}{|\nabla\phi|}\vec{T}^T\nabla^2\phi\vec{T}\,. \tag{4.1}$$

It is a long but straightforward computation to verify the identity

$$|\nabla\phi|\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = \vec{T}^{\perp}\nabla^2\phi\vec{T}\,.$$

Hence the signed curvature k is given by the formula

$$k = \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\,.$$

In other words, the curvature k is just the minus the divergence of the normal vector $\vec{N} = \nabla\phi/|\nabla\phi|$, i.e. $k = -\mathrm{div}\vec{N}$.

Let us differentiate the equation $\phi(x(s,t),t) = 0$ with respect to time. We obtain $\partial_t\phi + \nabla\phi.\partial_t x = 0$. Since the normal velocity of x is $\beta = \partial_t x.\vec{N}$ and $\vec{N} = -\nabla\phi/|\nabla\phi|$ we obtain

$$\partial_t\phi = |\nabla\phi|\beta\,.$$

Combining the above identities for $\partial_t\phi, \vec{N}$, and k we conclude that the geometric equation (1.1) can be reformulated in terms of the evolution of the shape function $\phi = \phi(x,t)$ satisfying the following fully nonlinear parabolic equation:

$$\partial_t\phi = |\nabla\phi|\beta\left(\mathrm{div}\left(\nabla\phi/|\nabla\phi|\right), x, -\nabla\phi/|\nabla\phi|\right), \qquad x \in \Omega,\ t \in (0,T)\,. \tag{4.2}$$

Here we assume that the normal velocity β may depend on the curvature k, the position vector x and the tangent angle ν expressed through the unit inward normal vector $\vec{N}$, i.e. $\beta = \beta(k, x, \vec{N})$. Since the behavior of the shape function ϕ in a far distance from the set of evolving curves $\Gamma^t, t \in [0,T]$, does not influence their evolution, it is usual in the context of the level set equation to prescribe homogeneous Neumann boundary conditions at the boundary $\partial\Omega$ of the computational domain Ω, i.e.

$$\phi(x,t) = 0 \ \text{ for } x \in \partial\Omega\,. \tag{4.3}$$

The initial condition for the level set shape function ϕ can be constructed as a signed distance function measuring the signed distance of a point $x \in \mathbb{R}^2$ and the initial curve Γ^0, i.e.

$$\phi(x,0) = \mathrm{dist}(x,\Gamma^0) \tag{4.4}$$

where $\mathrm{dist}(x,\Gamma^0)$ is a signed distance function defined as

$$\begin{aligned}
\mathrm{dist}(x,\Gamma^0) &= \inf_{y\in\Gamma^0} |x-y|, \qquad \text{for } \ x \in ext(\Gamma^0)\,,\\
\mathrm{dist}(x,\Gamma^0) &= -\inf_{y\in\Gamma^0} |x-y|, \qquad \text{for } \ x \in int(\Gamma^0)\,,\\
\mathrm{dist}(x,\Gamma^0) &= 0, \qquad \text{for } \ x \in \Gamma^0\,.
\end{aligned}$$

If we assume that the normal velocity of an evolving curve Γ^t is an affine in the k variable, i.e.

$$\beta = \mu k + f$$

where $\mu = \mu(x, \vec{N})$ is a coefficient describing dependence of the velocity speed on the position vector x and the orientation of the curve Γ^t expressed through the unit inward normal vector $\vec{N}$ and $f = f(x, \vec{N})$ is an external forcing term.

$$\partial_t\phi = \mu\,|\nabla\phi|\,\mathrm{div}\,\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + f|\nabla\phi|, \qquad x \in \Omega, t \in (0,T)\,. \tag{4.5}$$

1.1. A-priori bounds for the total variation of the shape function. In this section we derive an important a-priori bound for the total variation of the shape function satisfying the level set equation (4.2). The total variation (or the $W^{1,1}$ Sobolev norm) of the function $\phi(.,t)$ is defined as $\int_\Omega |\nabla\phi(x,t)|\,\mathrm{d}x$ where

$\Omega \subset \mathbb{R}^2$ is a simply connected domain such that $int(\Gamma^t) \subset \Omega$ for any $t \in [0,T]$. Differentiating the total variation of $\phi(.,t)$ with respect to time we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\phi|\,\mathrm{d}x &= \int_\Omega \frac{1}{|\nabla\phi|}(\nabla\phi.\partial_t\nabla\phi)\,\mathrm{d}x = \int_\Omega \frac{\nabla\phi}{|\nabla\phi|}.\nabla\partial_t\phi\,\mathrm{d}x \\ &= -\int_\Omega \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right).\partial_t\phi\,\mathrm{d}x = -\int_\Omega k\beta|\nabla\phi|\,\mathrm{d}x \end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\phi|\,\mathrm{d}x + \int_\Omega k\beta|\nabla\phi|\,\mathrm{d}x = 0 \tag{4.6}$$

where k is expressed as in (4.1) and $\beta = \beta\left(\mathrm{div}\left(\nabla\phi/|\nabla\phi|\right), x, -\nabla\phi/|\nabla\phi|\right)$. With help of the co-area integration theorem, the identity (4.6) can be viewed as a level set analogy to the total length equation (2.14).

In the case of the Euclidean curvature driven flow when curves are evolved in the normal direction by the curvature (i.e. $\beta = k$) we have $\int_\Omega k\beta|\nabla\phi|\,\mathrm{d}x = \int_\Omega k^2|\nabla\phi|\,\mathrm{d}x > 0$ and this is why

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\phi|\,\mathrm{d}x < 0 \quad \text{for any } t \in (0,T)\,,$$

implying thus the estimate

$$\phi \in L^\infty((0,T), W^{1,1}(\Omega))\,. \tag{4.7}$$

The same property can be easily proved by using Gronwall's lemma for a more general form of the normal velocity when $\beta = \mu k + f$ where $\mu = \mu(x,\vec{N}) > 0, f = f(x,\vec{N})$ are globally bounded functions. We presented this estimate because the same estimates can be proved for the gradient flow in the theory of minimal surfaces. Notice that the estimate (4.7) is weaker than the L^2–energy estimate $\phi \in L^\infty((0,T), W^{1,2}(\Omega))$ which can be easily shown for nondegenerate parabolic equation of the form $\partial_t\phi = \Delta\phi$, $\mathrm{d}\phi/\mathrm{d}n = 0$ on $\partial\Omega$, by multiplying the equation with the test function ϕ and integrating over the domain Ω.

2. Viscosity solutions to the level set equation

In this section we briefly describe a concept of viscosity solutions to the level set equation (4.2). We follow the book by Cao (c.f. [Cao03]). For the sake of simplicity of notation we shall consider the normal velocity β of the form $\beta = \beta(k)$. Hence equation (4.2) has a simplified form

$$\partial_t\phi = |\nabla\phi|\beta\left(\mathrm{div}\left(\nabla\phi/|\nabla\phi|\right)\right)\,. \tag{4.8}$$

The concept of viscosity solutions has been introduced by Crandall and Lions in [CL83]. It has been generalized to second order PDEs by Jensen [Jen88] (see also [IS95, FS93]). The proof of the existence and uniqueness of a viscosity solution to (4.8) is a consequence of the maximum principle for viscosity solutions (uniqueness part). Existence part can be proven by the method of sub and supersolutions known as the so-called Perron's method.

Following [Cao03] we first explain the basic idea behind the definition of a viscosity solution. We begin with a simple linear parabolic equation

$$\partial_t\phi = \Delta\phi\,. \tag{4.9}$$

Let ψ be any C^2 smooth function such that $\phi-\psi<0$ except of some point $(\bar{x},\bar{t})$ in which $\phi(\bar{x},\bar{t})=\psi(\bar{x},\bar{t})$, i.e. $(\bar{x},\bar{t})$ is a strict local maximum of the function $\phi-\psi$. Clearly, $\nabla\phi(\bar{x},\bar{t})-\nabla\psi(\bar{x},\bar{t})=0, \partial_t\phi(\bar{x},\bar{t})-\partial_t\psi(\bar{x},\bar{t})=0$, and $\Delta(\phi(\bar{x},\bar{t})-\psi(\bar{x},\bar{t}))\le 0$. Hence

$$\partial_t\psi \le \Delta\psi \quad \text{at} \quad (\bar{x},\bar{t})\,. \tag{4.10}$$

We say that ϕ is a subsolution to (4.9) if the inequality (4.10) hold whenever $\phi-\psi$ has a strict maximum at $(\bar{x},\bar{t})$. Analogously, we say that ϕ is a supersolution to (4.9) if the reverse inequality $\partial_t\psi\ge\Delta\psi$ holds at a point $(\bar{x},\bar{t})$ in which the function $\phi-\psi$ attains a strict minimum. It is important to realize, that such a definition of a sub and supersolution does not explicitly require smoothness of the function ϕ. It has been introduced by Crandall and Lions in [CL83]. Moreover, the above concept of sub and supersolutions can be extended to the case when the second order differential operator contains discontinuities. For the Euclidean motion by mean curvature (i.e. $\beta(k)=k$) the existence and uniqueness of a viscosity solution to (4.8) has been established by Evans and Spruck [ES91] and by Chen, Giga and Goto [CGG91] for the case $\beta(k)$ is sublinear at $\pm\infty$. Finally, Barles, Souganidis and Ishii introduced a concept of a viscosity solution for (4.8) in the case of arbitrary continuous and nondecreasing function $\beta(k)$ and they also proved the existence and uniqueness of a viscosity solution in [IS95, BS91]. Moreover, Souganidis extended a notion of a viscosity solution for the case when the elliptic operator is undefined in a set of critical points of ϕ.

Following Souganidis et al. (c.f. [IS95, BS91]), the class $\mathcal{A}(\beta)$ of admissible test functions consists of those C^2 compactly supported functions $\psi:\mathbb{R}^2\times[0,\infty)\to\mathbb{R}$ having the property: if $(\bar{x},\bar{t})$ is a critical point of ψ, i.e. $\nabla\psi(\bar{x},\bar{t})=0$ then there exists a neighborhood $B_\delta(\bar{x},\bar{t})$ with a radius $\delta>0$, a function $f\in\mathcal{F}(\beta)$, and $\omega\in C((0,\infty))$ satisfying $\lim_{r\to0}\omega(r)/r=0$ such that

$$|\psi(y,s)-\psi(\bar{x},\bar{t})-\partial_t\psi(\bar{x},\bar{t})(s-\bar{t})|\le f(|y-\bar{x}|)+\omega(|s-\bar{t}|), \quad \text{for any } (y,s)\in B_\delta(\bar{x},\bar{t})\,.$$

The class $\mathcal{F}(\beta)$ consists of those C^2 functions f such that $f(0)=f'(0)=f''(0)=0, f''(r)>0$ for $r>0$ and $\lim_{r\to0}f'(|r|)\beta(1/r)=0$.

The idea behind a relatively complicated definition of the set of admissible function is simple. It consists in the requirement that test functions must be enough flat to absorb singularities of the function β at their critical points. With this concept of the set of admissible test functions we are in a position to introduce a notion of a viscosity sub and super solution to the level set equation (4.8).

Definition 4.1. [Cao03, Definition 4.3.2] We say that a bounded function $\phi:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is a viscosity subsolution to (4.8) if for all admissible functions $\psi\in\mathcal{A}(\beta)$, if $\phi^*-\psi$ admits a strict maximum at a point $(\bar{x},\bar{t})$ then

$$\begin{aligned}\partial_t\psi(\bar{x},\bar{t}) &\le |\nabla\psi(\bar{x},\bar{t})|\beta\left(\operatorname{div}\left(\nabla\psi(\bar{x},\bar{t})/|\nabla\psi(\bar{x},\bar{t})|\right)\right), \qquad \text{if } \nabla\psi(\bar{x},\bar{t})\ne0,\\ \partial_t\psi(\bar{x},\bar{t}) &\le 0, \qquad \text{if } \nabla\psi(\bar{x},\bar{t})=0\,.\end{aligned}$$

We say that a bounded function $\phi:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is a viscosity supersolution to (4.8) if for all admissible functions $\psi\in\mathcal{A}(\beta)$, if $\phi_*-\psi$ admits a strict minimum at a point $(\bar{x},\bar{t})$ then

$$\begin{aligned}\partial_t\psi(\bar{x},\bar{t}) &\ge |\nabla\psi(\bar{x},\bar{t})|\beta\left(\operatorname{div}\left(\nabla\psi(\bar{x},\bar{t})/|\nabla\psi(\bar{x},\bar{t})|\right)\right), \qquad \text{if } \nabla\psi(\bar{x},\bar{t})\ne0,\\ \partial_t\psi(\bar{x},\bar{t}) &\ge 0, \qquad \text{if } \nabla\psi(\bar{x},\bar{t})=0\,.\end{aligned}$$

We say that ϕ is a viscosity solution if it both viscosity sub and supersolution.

Here we have denoted by ϕ^* and ϕ_* the upper and lower semicontinuous envelope of the function ϕ, i.e. $\phi^*(x,t) = \limsup_{(y,s)\to(x,t)} \phi(y,s)$ and $\phi_*(x,t) = \liminf_{(y,s)\to(x,t)} \phi(y,s)$.

THEOREM 4.2. *[IS95],[Cao03, Theorem 4.3.2] Let $\phi^0 \in BUC(\mathbb{R}^2)$. Assume the function $\beta : \mathbb{R} \to \mathbb{R}$ is nondecreasing and continuous. Then there exists a unique viscosity solution $\phi = \phi(x,t)$ to*

$$\begin{aligned} \partial_t \phi &= |\nabla \phi| \beta \left(div (\nabla \phi / |\nabla \phi|) \right), \qquad x \in \mathbb{R}^2, t \in (0,T) \\ \phi(x,0) &= \phi^0(x), \quad x \in \mathbb{R}^2 \end{aligned}$$

PROOF. The proof of this theorem is rather complicated and relies on several results from the theory of viscosity solutions. The hardest part is the proof of the uniqueness of a viscosity solution. It is based on the comparison (maximum) principle (see e.g. [Cao03, Theorem 4.3.1]) for viscosity sub and supersolutions to (4.8). It uses a clever result in this field which is referred to as the Theorem on Sums proved by Ishii (see [Cao03, Lemma 4.3.1] for details). The proof of existence is again due to Ishii and is based on the Perron method of sub and supersolutions. First one has to prove that, for a set S of uniformly bounded viscosity subsolutions to (4.8), their supremum

$$\bar{\psi}(x,t) = \sup\{\psi(x,t),\ \psi \in S\}$$

is also a viscosity subsolution. If there are bounded viscosity sub and supersolutions $\underline{\psi}, \bar{\psi}$ to (4.8) such that $\underline{\psi} \le \bar{\psi}$ then it can be shown that

$$\phi(x,t) = \sup\{\psi(x,t), \psi \text{ is a viscosity subsolution}, \underline{\psi} \le \psi \le \bar{\psi}\}$$

is a viscosity solution to (4.8) (c.f. [Cao03, Propositions 4.3.3, 4.3.4]). Finally, one has to construct suitable viscosity sub and supersolutions $\underline{\psi}, \bar{\psi}$ satisfying $\underline{\psi} \le \phi^0 \le \bar{\psi}$ for an initial condition ϕ^0 belonging to the space BUC of all bounded uniformly continuous functions in $\mathbb{R}^2$. The statement of the Theorem then follows. □

3. Numerical methods

Although these lecture notes are not particularly concerned with numerical methods for level set methods we present results obtained by a comprehensive Matlab toolbox ToolboxLS-1.1 which can be used for numerical approximation of level set methods in two or three spatial dimensions. It has been developed by Ian Mitchell and its latest version can be freely downloaded from his web page `www.cs.ubc.ca/~mitchell`.

3.1. Examples from Mitchell's Level set Matlab toolbox. In Fig. 3 we present an output of Mitchell's ToolboxLS-1.0 for two different level set function evolution (left) for some time $t > 0$. On the right side we can see corresponding zero level sets.

The Matlab toolbox can be used for tracking evolution of two dimensional embedded surfaces in $\mathbb{R}^3$. In Fig. 4 we present evolution of a two dimensional dumb-bell like surface which is evolved by the mean curvature. Since the mean curvature for a two dimensional surface is a sum of two principal cross-sectional curvatures one can conclude that the mean curvature at the bottle-neck of the

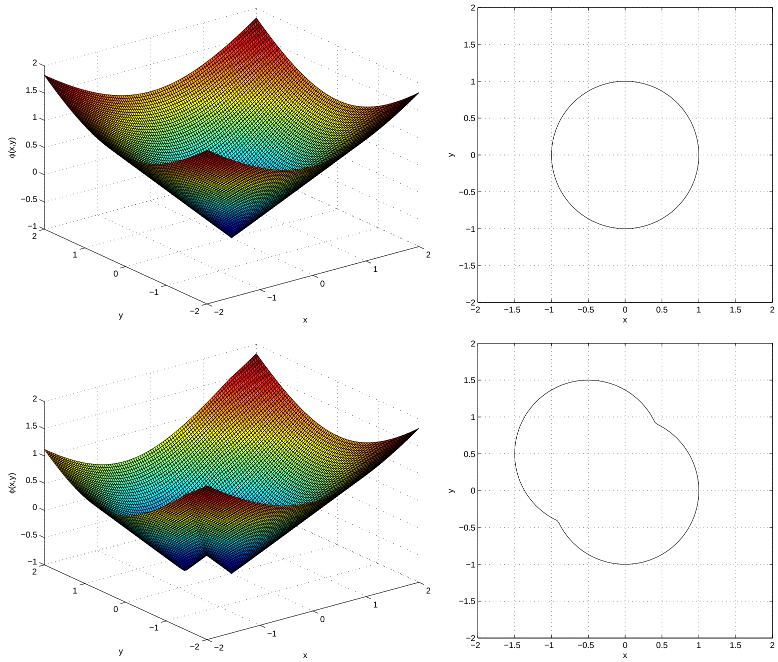

FIGURE 3. Two examples of level set functions $\phi(., t)$ (left) and their zero level set (right) plotted at some positive time $t > 0$.

surface is positive because of the dominating principal curvature of the section plane perpendicular to the axis of a rotational symmetry of the dumb-bell. Thus the flow of a surface tends to shrink the bottle-neck. Notice that this is purely three dimensional feature and can not be observed in two dimensions. Furthermore, we can see from Fig. 4 that dumb-bell's bottle-neck shrinks to a pinching point in a finite time. After that time evolution continues in two separate sphere–like surfaces which shrink to two points in finite time. This observation enables us to conclude that a three dimensional generalization of Grayson's theorem (see Section 2) is false.

Another intuitive explanation for the failure of the Grayson theorem in three dimensions comes from the description of the mean curvature flow of two dimensional embedded surfaces in $\mathbb{R}^3$. According to Huisken [Hui90] the mean curvature H of the surface is a solution to the following system of nonlinear parabolic equations

$$\begin{aligned} \partial_t H &= \Delta_{\mathcal{M}} H + |A|^2 H\,, \\ \partial_t |A|^2 &= \Delta_{\mathcal{M}} |A|^2 - 2|\nabla_{\mathcal{M}} A|^2 + 2|A|^4 \end{aligned}$$

where $|A|^2$ is the second trace (Frobenius norm) of the second fundamental form of the embedded manifold $\mathcal{M}$. Here $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator with respect to the surface $\mathcal{M}$. The above system of equations is a two dimensional generalization of the simple one dimensional parabolic equation $\partial_t k = \partial_s^2 k + k^3$ describing the Euclidean flow of planar curves evolved by the curvature. Now, one can interpret Grayson's theorem for embedded curves in terms of nonincrease

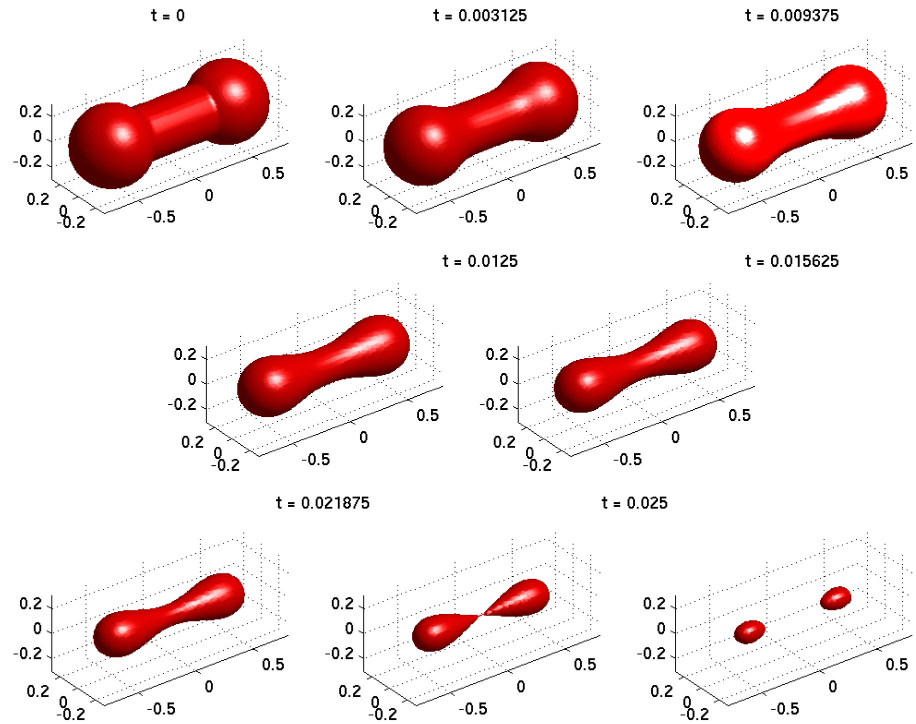

FIGURE 4. Time evolution of a dumb-bell initial surface driven by the mean curvature.

of nodal points of the curvature k. This result is known in the case of a scalar reaction diffusion equation and is refererred to as Sturm's theorem or Nonincrease of lap number theorem due to Matano. However, in the case of a system of two dimensional equations for the mean curvature H and the second trace $|A|^2$ one cannot expect similar result which is known to be an intrinsic property of scalar parabolic equations and cannot be extended for systems of parabolic equations.

CHAPTER 5

Numerical methods for the direct approach

In this part we suggest a fully discrete numerical scheme for the direct approach for solving the geometric equation (1.1). It is based on numerical approximation of a solution to the system of governing equations (2.10)–(2.13). The numerical scheme is semi-implicit in time, i.e. all nonlinearities are treated from the previous time step and linear terms are discretized at the current time level. Then we solve tridiagonal systems in every time step in a fast and simple way. We emphasize the role of tangential redistribution. The direct approach for solving (1.1) can be accompanied by a suitable choice of a tangential velocity α significantly improving and stabilizing numerical computations as it was documented by many authors (see e.g. [Dec97, HLS94, HKS98, MS99, MS01, MS04a, MS04b]). We show that stability constraint for our semi-implicit scheme with tangential redistribution is related to an integral average of $k\beta$ along the curve and not to pointwise values of $k\beta$. The pointwise influence of this term would lead to severe time step restriction in a neighborhood of corners while our approach benefits from an overall smoothness of the curve. Thus the method allows the choosing of larger time steps without loss of stability.

We remind ourselves that other popular techniques, like e.g. level-set method due to Osher and Sethian [Set96, OF03] or phase-field approximations (see e.g. Caginalp, Elliott et al. or Beneš [Cag90, EPS96, Ben01, BM98]) treat the geometric equation (1.1) by means of a solution to a higher dimensional parabolic problem. In comparison to these methods, in the direct approach one space dimensional evolutionary problems are solved only.

1. A role of the choice of a suitable tangential velocity

The main purpose of this section is to discuss various possible choices of a tangential velocity functional α appearing in the system of governing equations (2.10)–(2.13). In this system α can be viewed still as a free parameter which has to be determined in an appropriate way. Recall that k, ν, g, x do depend on α but the family $\Gamma_t = \mathrm{Img}(x(., t)), t \in [0, T)$, itself is independent of a particular choice of α.

To motivate further discussion, we recall some of computational examples in which the usual choice $\alpha = 0$ fails and may lead to serious numerical instabilities like e.g. formation of so-called swallow tails. In Figures 1 and 2 we computed the mean curvature flow of two initial curves (bold faced curves). We chose $\alpha = 0$ in the experiment shown in Fig. 1. It should be obvious that numerically computed grid points merge in some parts of the curve Γ_t preventing thus numerical approximation of $\Gamma_t, t \in [0, T)$, to be continued beyond some time T which is still far away from the maximal time of existence T_{max}. These examples also showed that a suitable

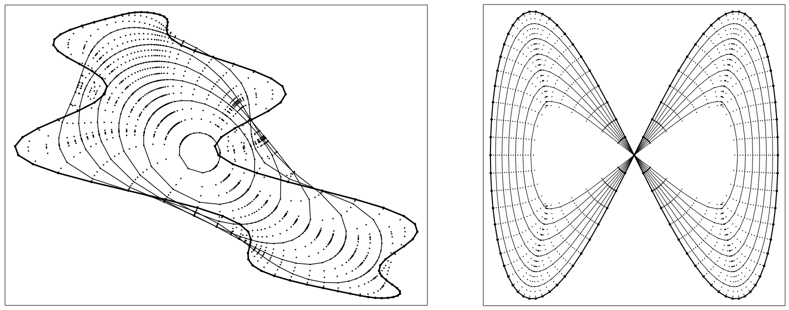

FIGURE 1. Merging of numerically computed grid points in the case of the vanishing tangential velocity functional $\alpha = 0$.

grid points redistribution governed by a nontrivial tangential velocity functional α is needed in order to compute the solution on its maximal time of existence.

The idea behind construction of a suitable tangential velocity functional α is rather simple and consists in the analysis of the quantity θ defined as follows:

$$\theta = \ln(g/L)$$

where $g = |\partial_u x|$ is a local length and L is a total length of a curve $\Gamma = \mathrm{Img}(x)$. The quantity θ can be viewed as the logarithm of the relative local length g/L. Taking into account equations (2.12) and (2.14) we have

$$\partial_t \theta + k\beta - \langle k\beta \rangle_\Gamma = \partial_s \alpha \,. \tag{5.1}$$

By an appropriate choice of $\partial_s \alpha$ in the right hand side of (5.1) we can therefore control behavior of θ. Equation (5.1) can be also viewed as a kind of a constitutive relation determining redistribution of grid points along a curve.

1.1. Non-locally dependent tangential velocity functional. We first analyze the case when $\partial_s \alpha$ (and so does α) depends on other geometric quantities k, β and g in a nonlocal way. The simplest possible choice of $\partial_s \alpha$ is:

$$\partial_s \alpha = k\beta - \langle k\beta \rangle_\Gamma \tag{5.2}$$

yielding $\partial_t \theta = 0$ in (5.1). Consequently,

$$\frac{g(u,t)}{L_t} = \frac{g(u,0)}{L_0} \quad \text{for any } u \in S^1,\ t \in [0, T_{max})\,.$$

Notice that α can be uniquely computed from (5.2) under the additional renormalization constraint: $\alpha(0,t) = 0$. In the sequel, tangential redistribution driven by a solution α to (5.2) will be refereed to as *a parameterization preserving relative local length*. It has been first discovered and utilized by Hou et al. in [HLS94, HKS98] and independently by Mikula and Ševčovič in [MS99, MS01, MS04a, MS04b].

A general choice of α is based on the following setup:

$$\partial_s \alpha = k\beta - \langle k\beta \rangle_\Gamma + \left(e^{-\theta} - 1\right) \omega(t) \tag{5.3}$$

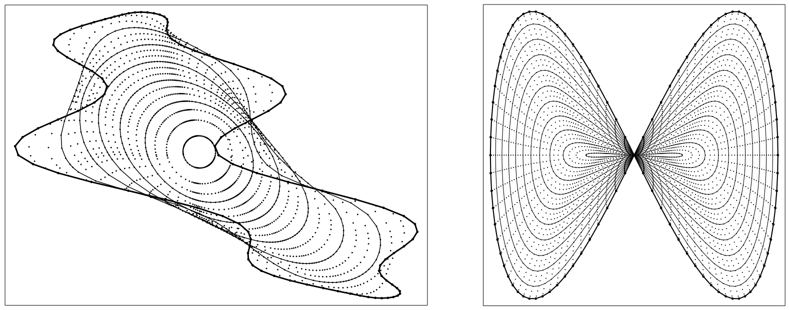

FIGURE 2. Impact of suitably chosen tangential velocity functional α on enhancement of spatial grids redistribution.

where $\omega \in L^1_{loc}([0, T_{max}))$. If we additionally suppose

$$\int_0^{T_{max}} \omega(\tau)\, d\tau = +\infty \tag{5.4}$$

then, after insertion of (5.3) into (5.1) and solving the ODE $\partial_t \theta = \left(e^{-\theta} - 1\right)\omega(t)$, we obtain $\theta(u,t) \to 0$ as $t \to T_{max}$ and hence

$$\frac{g(u,t)}{L_t} \to 1 \quad \text{as } t \to T_{max} \quad \text{uniformly w.r. to } u \in S^1.$$

In this case redistribution of grid points along a curve becomes uniform as t approaches the maximal time of existence T_{max}. We will refer to the parameterization based on (5.3) to as *an asymptotically uniform parameterization*. The impact of a tangential velocity functional defined as in (5.2) on enhancement of redistribution of grid points can be observed from two examples shown in Fig. 2 computed by Mikula and Ševčovič in [MS01].

Asymptotically uniform redistribution of grid points is of a particular interest in the case when the family $\{\Gamma_t, t \in [0,T)\}$ shrinks to a point as $t \to T_{max}$, i.e. $\lim_{t \to T_{max}} L_t = 0$. Then one can choose $\omega(t) = \kappa_2 \langle k\beta \rangle_{\Gamma_t}$ where $\kappa_2 > 0$ is a positive constant. By (2.14), $\int_0^t \omega(\tau)\, d\tau = -\kappa_2 \int_0^t \ln L_\tau d\tau = \kappa_2 (\ln L_0 - \ln L_t) \to +\infty$ as $t \to T_{max}$. On the other hand, if the length L_t is away from zero and $T_{max} = +\infty$ one can choose $\omega(t) = \kappa_1$, where $\kappa_1 > 0$ is a positive constant in order to meet the assumption (5.4).

Summarizing, in both types of grid points redistributions discussed above, a suitable choice of the tangential velocity functional α is given by a solution to

$$\partial_s \alpha = k\beta - \langle k\beta \rangle_\Gamma + (L/g - 1)\,\omega\,, \ \alpha(0) = 0\,, \tag{5.5}$$

where $\omega = \kappa_1 + \kappa_2 \langle k\beta \rangle_\Gamma$ and $\kappa_1, \kappa_2 \geq 0$ are given constants.

If we insert tangential velocity functional α computed from (5.5) into (2.10)–(2.13) and make use of the identity $\alpha \partial_s k = \partial_s(\alpha k) - k \partial_s \alpha$ then the system of governing equations can be rewritten as follows:

$$\partial_t k = \partial_s^2 \beta + \partial_s(\alpha k) + k\langle k\beta\rangle_\Gamma + (1 - L/g)\, k\omega\,, \tag{5.6}$$

$$\partial_t \nu = \beta_k' \partial_s^2 \nu + (\alpha + \beta_\nu')\partial_s \nu + \nabla_x \beta . \vec{T}\,, \tag{5.7}$$

$$\partial_t g = -g\langle k\beta\rangle_\Gamma + (L - g)\omega\,, \tag{5.8}$$

$$\partial_t x = \beta\vec{N} + \alpha\vec{T}\,. \tag{5.9}$$

It is worth to note that the strong reaction term $k^2\beta$ in (2.10) has been replaced by the averaged term $k\langle k\beta\rangle_\Gamma$ in (5.6). A similar phenomenon can be observed in (5.8). This is very important feature as it allows for construction of an efficient and stable numerical scheme.

1.2. Locally dependent tangential velocity functional. Another possibility for grid points redistribution along evolved curves is based on a tangential velocity functional defined locally. If we take $\alpha = \partial_s\theta$, i.e. $\partial_s\alpha = \partial_s^2\theta$ then the constitutive equation (5.1) reads as follows: $\partial_t\theta + k\beta - \langle k\beta\rangle_\Gamma = \partial_s^2\theta$. Since this equation has a parabolic nature one can expect that variations in θ are decreaing during evolution and θ tends to a constant value along the curve Γ due to the diffusion process. The advantage of the particular choice

$$\alpha = \partial_s\theta = \partial_s \ln(g/L) = \partial_s \ln g \tag{5.10}$$

has been already observed by Deckelnick in [Dec97]. He analyzed the mean curvature flow of planar curves (i.e. $\beta = k$) by means of a solution to the intrinsic heat equation

$$\partial_t x = \frac{\partial_u^2 x}{|\partial_u x|^2}\,, \quad u \in S^1, t \in (0, T),$$

describing evolution of the position vector x of a curve $\Gamma_t = \mathrm{Img}(x(., t))$. By using Frenét's formulae we obtain $\partial_t x = k\vec{N} + \alpha\vec{T}$ where $\alpha = \partial_s \ln g = \partial_s \ln(g/L) = \partial_s\theta$.

Inserting the tangential velocity functional $\alpha = \partial_s\theta = \partial_s(\ln g)$ into (2.10)–(2.13) we obtain the following system of governing equations:

$$\partial_t k = \partial_s^2\beta + \alpha\partial_s k + k^2\beta, \tag{5.11}$$

$$\partial_t \nu = \beta_k' \partial_s^2 \nu + (\alpha + \beta_\nu')\partial_s \nu + \nabla_x \beta . \vec{T}, \tag{5.12}$$

$$\partial_t g = -gk\beta + g\partial_s^2(\ln g), \tag{5.13}$$

$$\partial_t x = \beta\vec{N} + \alpha\vec{T}\,. \tag{5.14}$$

Notice that equation (5.13) is a nonlinear parabolic equation whereas (5.8) is a nonlocal ODE for the local length g.

2. Flowing finite volume approximation scheme

The aim of this part is to review numerical methods for solving the system of equations (2.10)–(2.13). We begin with a simpler case in which we assume the normal velocity to be an affine function of the curvature with coefficients depending on the tangent angle only. Next we consider a slightly generalized form of the normal velocity in which coefficients may also depend on the position vector x.

2.0.1. *Normal velocity depending on the tangent angle.* First, we consider a simpler case in which the normal velocity β has the following form:

$$\beta = \beta(k, \nu) = \gamma(\nu)k + F \tag{5.15}$$

with a given anisotropy function $\gamma(\nu) > 0$ and a constant driving force F. The system of governing equations is accompanied by the tangential velocity α given by

$$\partial_s \alpha = k\beta - \frac{1}{L}\int_\Gamma k\beta \mathrm{d}s - \omega(1 - \frac{L}{g}) \tag{5.16}$$

where L is the total length of the curve Γ and ω is a relaxation function discussed in Section 1.1. Since there is no explicit dependence of flow on spatial position x the governing equations are simpler and the evolving curve Γ_t is given (uniquely up to a translation) by reconstruction

$$x(u, .) = \int_0^u g\vec{T}\mathrm{d}u = \int_0^s \vec{T}\mathrm{d}s. \tag{5.17}$$

Before performing temporal and spatial discretization we insert (5.16) into (2.10) and (2.12) to obtain

$$\partial_t k = \partial_s^2 \beta + \partial_s(\alpha k) + k\langle k\beta\rangle + k\omega(1 - \frac{L}{g}), \tag{5.18}$$

$$\partial_t \nu = \beta'_k \partial_s^2 \nu + (\alpha + \beta'_\nu)\partial_s \nu\,, \tag{5.19}$$

$$\partial_t g = -g\langle k\beta\rangle - \omega(g - L). \tag{5.20}$$

From the numerical discretization point of view, critical terms in Eqs. (2.10) – (2.12) are represented by the reaction term $k^2\beta$ in (2.10) and the decay term $k\beta$ in (2.12). In Eqs. (5.18) – (5.20) these critical terms were replaced by the averaged value of $k\beta$ along the curve, thus computation of a local element length in the neighborhood of point with a high curvature is more stable.

In our computational method a solution of the evolution Eq. (1.1) is represented by discrete plane points $x_i^j, i = 0, ..., n,\ j = 0, ..., m$, where index i represents space discretization and index j a discrete time stepping. Since we only consider closed initial curves the periodicity condition $x_0^0 = x_n^0$ is required at the beginning. If we take a uniform division of the time interval $[0, T]$ with a time step $\tau = T/m$ and a uniform division of the fixed parameterization interval $[0, 1]$ with a step $h = 1/n$, a point x_i^j corresponds to $x(ih, j\tau)$. Difference equations will be given for discrete quantities $k_i^j,\ \nu_i^j,\ r_i^j,\ i = 1, ..., n,\ j = 1, ..., m$ representing piecewise constant approximations of the curvature, tangent angle and element length for the segment $\left[x_{i-1}^j, x_i^j\right]$ and for α_i^j representing tangential velocity of the flowing node x_i^{j-1}. Then, at the j-th discrete time level, $j = 1, ..., m$, approximation of a curve is given by a discrete version of the reconstruction formula (5.17)

$$x_i^j = x_0^j + \sum_{l=1}^{i} r_l^j(\cos(\nu_l^j), \sin(\nu_l^j)), \quad i = 1, ..., n. \tag{5.21}$$

In order to construct a discretization scheme for solving (5.18) – (5.20) we consider time dependent functions $k_i(t)$, $\nu_i(t)$, $r_i(t)$, $x_i(t)$, $\alpha_i(t)$; k_i^j, ν_i^j, r_i^j, x_i^j, α_i^j, described above, represent their values at time levels $t = j\tau$. Let us denote $B = \frac{1}{L}\int_\Gamma k\beta ds$.

We integrate Eqs. (5.16) and (5.18) – (5.20) at any time t over the so-called *flowing control volume* $[x_{i-1}, x_i]$. Using the Newton-Leibniz formula and constant approximation of the quantities inside flowing control volumes, at any time t we get

$$\alpha_i - \alpha_{i-1} = r_i k_i \beta(k_i, \nu_i) - r_i B - \omega \left(r_i - \frac{L}{n} \right).$$

By taking discrete time stepping, for *values of the tangential velocity* α_i^j we obtain

$$\alpha_i^j = \alpha_{i-1}^j + r_i^{j-1} k_i^{j-1} \beta(k_i^{j-1}, \nu_i^{j-1}) - r_i^{j-1} B^{j-1} - \omega(r_i^{j-1} - M^{j-1}), \qquad (5.22)$$

$i = 1, ..., n$, with $\alpha_0^j = 0$ (x_0^j is moving only in the normal direction) where

$$M^{j-1} = \frac{1}{n} L^{j-1},\ L^{j-1} = \sum_{l=1}^{n} r_l^{j-1},\ B^{j-1} = \frac{1}{L^{j-1}} \sum_{l=1}^{n} r_l^{j-1} k_l^{j-1} \beta(k_l^{j-1}, \nu_l^{j-1})$$

and $\omega = \kappa_1 + \kappa_2 B^{j-1}$, with input redistribution parameters κ_1, κ_2. Using similar approach as above, Eq. (5.20) gives us

$$\frac{dr_i}{dt} + r_i B + r_i \omega = \omega \frac{L}{n}.$$

By taking a backward time difference we obtain an update for local lengths

$$r_i^j = \frac{r_i^{j-1} + \tau \omega M^{j-1}}{1 + \tau(B^{j-1} + \omega)}, \quad i = 1, ..., n, \quad r_0^j = r_n^j, \quad r_{n+1}^j = r_1^j. \qquad (5.23)$$

Subsequently, new local lengths are used for approximation of intrinsic derivatives in (5.18) – (5.19). Integrating the curvature Eq. (5.18) in flowing control volume $[x_{i-1}, x_i]$ we have

$$r_i \frac{dk_i}{dt} = [\partial_s \beta(k, \nu)]_{x_{i-1}}^{x_i} + [\alpha k]_{x_{i-1}}^{x_i} + k_i (r_i (B + \omega) - \omega \frac{L}{n}).$$

Now, by replacing the time derivative by time difference, approximating k in nodal points by the average value of neighboring segments, and using semi-implicit approach we obtain a *tridiagonal system* with periodic boundary conditions imposed for new discrete values of the curvature

$$a_i^j k_{i-1}^j + b_i^j k_i^j + c_i^j k_{i+1}^j = d_i^j, \quad i = 1, ..., n, \quad k_0^j = k_n^j, \quad k_{n+1}^j = k_1^j, \qquad (5.24)$$

where

$$a_i^j = \frac{\alpha_{i-1}^j}{2} - \frac{\gamma(\nu_{i-1}^{j-1})}{q_{i-1}^j}, \quad c_i^j = -\frac{\alpha_i^j}{2} - \frac{\gamma(\nu_{i+1}^{j-1})}{q_i^j}, \quad d_i^j = \frac{r_i^j}{\tau} k_i^{j-1},$$

$$b_i^j = r_i^j \left(\frac{1}{\tau} - (B^{j-1} + \omega) \right) + \omega M^{j-1} - \frac{\alpha_i^j}{2} + \frac{\alpha_{i-1}^j}{2} + \frac{\gamma(\nu_i^{j-1})}{q_{i-1}^j} + \frac{\gamma(\nu_i^{j-1})}{q_i^j}$$

where $q_i^j = \frac{r_i^j + r_{i+1}^j}{2}$, $i = 1, ..., n$. Finally, by integrating the tangent angle Eq. (5.19) we get

$$r_i \frac{d\nu_i}{dt} = \gamma(\nu_i) [\partial_s \nu]_{x_{i-1}}^{x_i} + [\alpha \nu]_{x_{i-1}}^{x_i} - \nu_i (\alpha_i - \alpha_{i-1}) + \gamma'(\nu_i) k_i [\nu]_{x_{i-1}}^{x_i}.$$

Again, values of the tangent angle ν in nodal points are approximated by the average of neighboring segments values, the time derivative is replaced by the time difference and using a semi-implicit approach we obtain *tridiagonal system* with periodic boundary conditions *for new values of the tangent angle*

$$A_i^j \nu_{i-1}^j + B_i^j \nu_i^j + C_i^j \nu_{i+1}^j = D_i^j, \quad i = 1, ..., n, \quad \nu_0^j = \nu_n^j, \quad \nu_{n+1}^j = \nu_1^j, \tag{5.25}$$

where

$$A_i^j = \frac{\alpha_{i-1}^j}{2} + \frac{\gamma'(\nu_i^{j-1})k_i^j}{2} - \frac{\gamma(\nu_i^{j-1})}{q_{i-1}^j}, \quad B_i^j = \frac{r_i^j}{\tau} - (A_i^j + C_i^j),$$

$$C_i^j = -\frac{\alpha_i^j}{2} - \frac{\gamma'(\nu_i^{j-1})k_i^j}{2} - \frac{\gamma(\nu_i^{j-1})}{q_i^j}, \quad D_i^j = \frac{r_i^j}{\tau}\nu_i^{j-1}.$$

The initial quantities for the algorithm are computed as follows:

$$R_i = (R_{i_1}, R_{i_2}) = x_i^0 - x_{i-1}^0, \quad i = 1, ..., n, \quad R_0 = R_n, \quad R_{n+1} = R_1,$$

$$r_i^0 = |R_i|, \quad i = 0, ..., n+1, \tag{5.26}$$

$$k_i^0 = \frac{1}{2r_i^0}\mathrm{sgn}\left(\det(R_{i-1}, R_{i+1})\right) \arccos\left(\frac{R_{i+1}.R_{i-1}}{r_{i+1}^0 r_{i-1}^0}\right), \tag{5.27}$$

$$i = 1, ..., n, \ k_0^0 = k_n^0, \ k_{n+1}^0 = k_1^0,$$

$$\nu_0^0 = \arccos(R_{i_1}/r_i^0) \text{ if } R_{i_2} \geq 0, \quad \nu_0^0 = 2\pi - \arccos(R_{i_1}/r_i) \text{ if } R_{i_2} < 0,$$

$$\nu_i^0 = \nu_{i-1}^0 + r_i^0 k_i^0, \quad i = 1, ..., n+1. \tag{5.28}$$

Remark (*Solvability and stability of the scheme.*) Let us first examine discrete values of the tangent angle ν computed from (5.25). One can rewrite it into the form

$$\nu_i^j + \frac{\tau}{r_i^j} C_i^j (\nu_{i+1}^j - \nu_i^j) + \frac{\tau}{r_i^j} A_i^j (\nu_{i-1}^j - \nu_i^j) = \nu_i^{j-1}. \tag{5.29}$$

Let $\max_k \nu_k^j$ be attained at the i-th node. We can always take a fine enough resolution of the curve, i.e. take small $q_i^j \ll 1$, $i = 1, .., n$, such that both A_i^j and C_i^j are nonpositive and thus the second and third terms on the left hand side of (5.29) are nonnegative. Then $\max_k \nu_k^j = \nu_i^j \leq \nu_i^{j-1} \leq \max_k \nu_k^{j-1}$. By a similar argument we can derive an inequality for the minimum. In this way we have shown the L^∞-stability criterion, namely

$$\min_k \nu_k^0 \leq \min_k \nu_k^j \leq \max_k \nu_k^j \leq \max_k \nu_k^0, \quad j = 1, .., m.$$

Notice that in the continuous case the above comparison inequality is a consequence of the parabolic maximum principle for equation (5.7) in which the term $\nabla_x \beta.\vec{T}$ is vanishing as β does not explicitly depend on the position vector x.

Having guaranteed non-positivity of A_i^j and C_i^j we can conclude positivity and diagonal dominance of the diagonal term B_i^j. In particular, it implies that the tridiagonal matrix of the system (5.25) is an M-matrix and hence a solution to (5.25) always exists and is unique.

In the same way, by taking q_i^j small enough, we can prove nonpositivity of the off-diagonal terms a_i^j and c_i^j in the system (5.24) for discrete curvature values. Then the diagonal term b_i^j is positive and dominant provided that $\tau(B^{j-1} + \omega) < 1$. Again we have shown that the corresponding matrix is an M-matrix and therefore there exists a unique solution to the system (5.24).

Another natural stability requirement of the scheme is related to the positivity of local lengths r_i^j during computations. It follows from (5.23) that the positivity of r_i^j is equivalent to the condition $\tau(B^{j-1}+\omega) > -1$. Taking into account both inequalities for the time step we end up with the following stability restriction on the time step τ:

$$\tau \leq \frac{1}{|B^{j-1}+\omega|} \tag{5.30}$$

related to B^{j-1} (a discrete average value of $k\beta$ over a curve).

2.0.2. *Normal velocity depending on the tangent angle and the position vector.* Next we consider a more general motion of the curves with explicit dependence of the flow on position x and suggest numerical scheme for such a situation. We consider (1.1) with a linear dependence of β on the curvature, i.e.

$$\beta(k,x,\nu) = \delta(x,\nu)k + c(x,\nu)$$

where $\delta(x,\nu) > 0$. By using Frenét's formulae one can rewrite the position vector Eq. (2.13) as an intrinsic convection-diffusion equation for the vector x and we get the system

$$\partial_t k = \partial_s^2\beta + \partial_s(\alpha k) + k\frac{1}{L}\int_\Gamma k\beta ds + k\omega(1-\frac{L}{g})\,, \tag{5.31}$$

$$\partial_t \nu = \beta_k'\partial_s^2\nu + (\alpha+\beta_\nu')\partial_s\nu + \nabla_x\beta.\vec{T}\,, \tag{5.32}$$

$$\partial_t g = -g\frac{1}{L}\int_\Gamma k\beta ds - \omega(g-L)\,, \tag{5.33}$$

$$\partial_t x = \delta(x,\nu)\partial_s^2 x + \alpha\partial_s x + \vec{c}(x,\nu)\,, \tag{5.34}$$

where $\vec{c}(x,\nu) = c(x,\nu)\vec{N} = (-c(x,\nu)\sin\nu, c(x,\nu)\cos\nu)$. In comparison to the scheme given above, two new tridiagonal systems have to be solved at each time level in order to update the curve position vector x. The curve position itself and all geometric quantities entering the model are resolved from their own intrinsic Eqs. (5.31) – (5.34). In order to construct a discretization scheme, Eqs. (5.31) – (5.33) together with (5.16) are integrated over a flowing control volume $[x_{i-1}, x_i]$. We also construct a time dependent dual volumes $\left[\tilde{x}_{i-1}^j, \tilde{x}_i^j\right]$, $i = 1,..,n, j = 1,..,m$, where $\tilde{x}_i^j = \frac{x_{i-1}^j + x_i^j}{2}$ over which the last Eq. (5.34) will be integrated. Then, for *values of the tangential velocity* we obtain

$$\alpha_i^j = \alpha_{i-1}^j + r_i^{j-1}k_i^{j-1}\beta(\tilde{x}_i^{j-1}, k_i^{j-1}, \nu_i^{j-1}) - r_i^{j-1}B^{j-1} - \omega(r_i^{j-1} - M^{j-1}),$$
$$i = 1,...,n, \quad \alpha_0^j = 0, \tag{5.35}$$

with M^{j-1}, L^{j-1}, ω given as above and

$$B^{j-1} = \frac{1}{L^{j-1}}\sum_{l=1}^n r_l^{j-1}k_l^{j-1}\beta(\tilde{x}_l^{j-1}, k_l^{j-1}, \nu_l^{j-1}).$$

Local lengths are updated by the formula:

$$r_i^j = \frac{r_i^{j-1} + \tau\omega M^{j-1}}{1+\tau(B^{j-1}+\omega)}, \quad i = 1,...,n, \quad r_0^j = r_n^j, \quad r_{n+1}^j = r_1^j. \tag{5.36}$$

The *tridiagonal system for discrete values of the curvature* reads as follows:

$$a_i^j k_{i-1}^j + b_i^j k_i^j + c_i^j k_{i+1}^j = d_i^j, \quad i = 1, ..., n, \quad k_0^j = k_n^j, \quad k_{n+1}^j = k_1^j, \tag{5.37}$$

where

$$a_i^j = \frac{\alpha_{i-1}^j}{2} - \frac{\delta(\tilde{x}_{i-1}^{j-1}, \nu_{i-1}^{j-1})}{q_{i-1}^j}, \quad c_i^j = -\frac{\alpha_i^j}{2} - \frac{\delta(\tilde{x}_{i+1}^{j-1}, \nu_{i+1}^{j-1})}{q_i^j},$$

$$b_i^j = r_i^j \left(\frac{1}{\tau} - (B^{j-1} + \omega)\right) + \omega M^{j-1} - \frac{\alpha_i^j}{2} + \frac{\alpha_{i-1}^j}{2} + \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_{i-1}^j} + \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_i^j},$$

$$d_i^j = \frac{r_i^j}{\tau} k_i^{j-1} + \frac{c(\tilde{x}_{i+1}^{j-1}, \nu_{i+1}^{j-1}) - c(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_i^j} - \frac{c(\tilde{x}_i^{j-1}, \nu_i^{j-1}) - c(\tilde{x}_{i-1}^{j-1}, \nu_{i-1}^{j-1})}{q_{i-1}^j}.$$

The *tridiagonal system for new values of the tangent angle* is given by

$$A_i^j \nu_{i-1}^j + B_i^j \nu_i^j + C_i^j \nu_{i+1}^j = D_i^j, \quad i = 1, ..., n, \quad \nu_0^j = \nu_n^j, \quad \nu_{n+1}^j = \nu_1^j, \tag{5.38}$$

where

$$A_i^j = \frac{\alpha_{i-1}^j + \beta'_\nu(\tilde{x}_i^{j-1}, k_i^j, \nu_i^{j-1})}{2} - \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_{i-1}^j},$$

$$C_i^j = -\frac{\alpha_i^j + \beta'_\nu(\tilde{x}_i^{j-1}, k_i^j, \nu_i^{j-1})}{2} - \frac{\delta(\tilde{x}_i^{j-1}, \nu_i^{j-1})}{q_i^j},$$

$$B_i^j = \frac{r_i^j}{\tau} - (A_i^j + C_i^j),$$

$$D_i^j = \frac{r_i^j}{\tau} \nu_i^{j-1} + r_i^j \nabla_x \beta(\tilde{x}_i^{j-1}, \nu_i^{j-1}, k_i^j).(\cos(\nu_i^{j-1}), \sin(\nu_i^{j-1})).$$

Finally, we end up with *two tridiagonal systems for updating the position vector*

$$\mathcal{A}_i^j x_{i-1}^j + \mathcal{B}_i^j x_i^j + \mathcal{C}_i^j x_{i+1}^j = \mathcal{D}_i^j, \quad i = 1, ..., n, \quad x_0^j = x_n^j, \quad x_{n+1}^j = x_1^j, \tag{5.39}$$

where

$$\mathcal{A}_i^j = -\frac{\delta(\tilde{x}_i^{j-1}, \frac{1}{2}(\nu_i^j + \nu_{i+1}^j))}{r_i^j} + \frac{\alpha_i^j}{2}, \quad \mathcal{C}_i^j = -\frac{\delta(\tilde{x}_i^{j-1}, \frac{1}{2}(\nu_i^j + \nu_{i+1}^j))}{r_{i+1}^j} - \frac{\alpha_i^j}{2},$$

$$\mathcal{B}_i^j = \frac{q_i^j}{\tau} - (\mathcal{A}_i^j + \mathcal{C}_i^j), \quad \mathcal{D}_i^j = \frac{q_i^j}{\tau} x_i^{j-1} + q_i^j \vec{c}(x_i^{j-1}, \frac{1}{2}(\nu_i^j + \nu_{i+1}^j)).$$

The *initial quantities* for the algorithm are given by (5.26) – (5.28).

CHAPTER 6

Applications of curvature driven flows

1. Computation of curvature driven evolution of planar curves with external force

In following figures we present numerical solutions computed by the scheme; initial curves are plotted with a thick line and the numerical solution is given by further solid lines with points representing the motion of some grid points during the curve evolution. In Figure 1 we compare computations with and without tangential redistribution for a large driving force F. As an initial curve we chose $x_1(u) = \cos(2\pi u)$, $x_2(u) = 2\sin(2\pi u) - 1.99\sin^3(2\pi u)$, $u \in [0,1]$. Without redistribution, the computations are collapsing soon because of the degeneracy in local element lengths in parts of a curve with high curvature leading to a merging of the corresponding grid points. Using the redistribution the evolution can be successfully handled. We used $\tau = 0.00001$, 400 discrete grid points and we plotted every $150th$ time step. In Figure 2 we have considered an initial curve $x_1(u) = (1 - C\cos^2(2\pi u))\cos(2\pi u)$, $x_2(u) = (1 - C\cos^2(2\pi u))\sin(2\pi u)$, $u \in [0,1]$ with $C = 0.7$. We took $\tau = 0.00001$ and 800 (Figure 2 left) and 1600 (Figure 2 right) grid points for representation of a curve. In Figure 2 left we plot each $500th$ time step, and in Figure 2 right each $100th$ step. It is natural that we have to use small time steps in case of strong driving force. However, the time step is not restricted by the point-wise values of the almost singular curvature in the corners which would lead to an un-realistic time step restriction. According to (5.30), the time step is restricted by the average value of $k\beta$ computed over the curve which is much more weaker restriction because of the regularity of the curve outside the corners. In Figure 3 we present experiments with three-fold anisotropy starting with unit circle. We used $\tau = 0.001$, 300 grid points and we plotted every $50th$ time step (left) and every $750th$ time step (right). In all experiments we chose redistribution parameters $\kappa_1 = \kappa_2 = 10$.

2. Flows of curves on a surface driven by the geodesic curvature

The purpose of this section is to analytically and numerically investigate a flow of closed curves on a given graph surface driven by the geodesic curvature. We show how such a flow can be reduced to a flow of vertically projected planar curves governed by a solution of a fully nonlinear system of parabolic differential equations. We present various computational examples of evolution of surface curves driven by the geodesic curvature are presented in this part. The normal velocity $\mathcal{V}$ of the evolving family of surface curves $\mathcal{G}^t, t \geq 0$, is proportional to the geodesic curvature

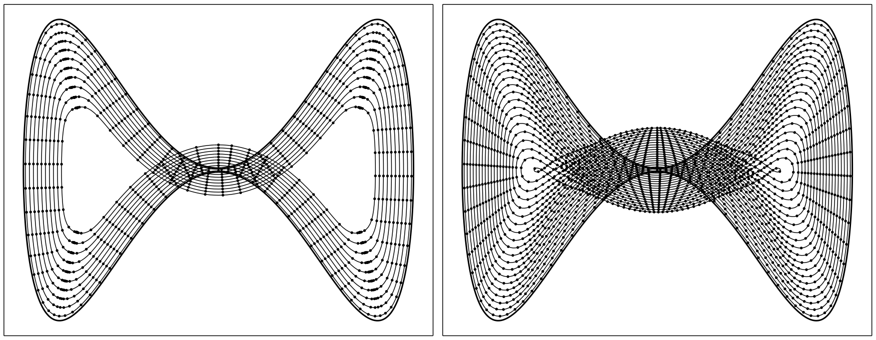

FIGURE 1. Isotropic curvature driven motion, $\beta(k, \nu) = \varepsilon k + F$, with $\varepsilon = 1$, $F = 10$, without (left) and with (right) uniform tangential redistribution of grid points.

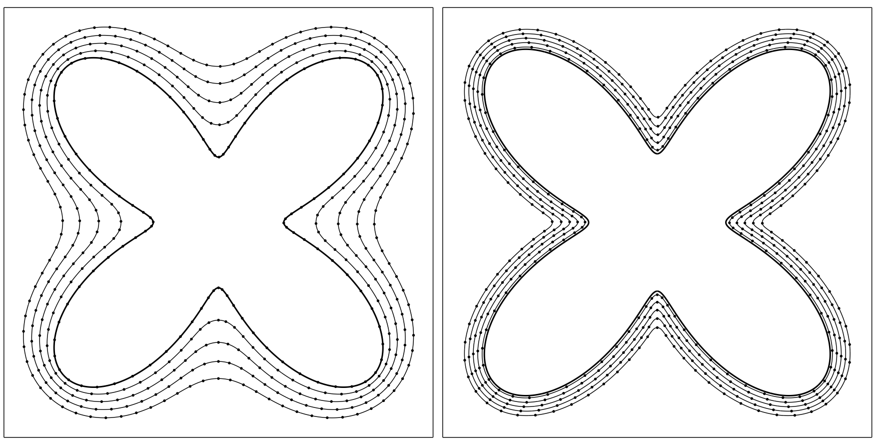

FIGURE 2. Isotropic curvature driven motion of an initial nonconvex curve including uniform tangential redistribution of grid points; $\beta(k, \nu) = \varepsilon k + F$, with $\varepsilon = 1$, $F = -10$ (left) and $\varepsilon = 0.1$, $F = -10$ (right). Resolution of sharp corners in the case of a highly dominant forcing term using the algorithm with redistribution is possible.

$\mathcal{K}_g$ of $\mathcal{G}^t$, i.e.

$$\mathcal{V} = \delta \mathcal{K}_g \tag{6.1}$$

where $\delta = \delta(X, \vec{\mathcal{N}}) > 0$ is a smooth positive coefficient describing anisotropy depending on the position X and the orientation of the unit inward normal vector $\vec{\mathcal{N}}$ to the curve on a surface.

The idea how to analyze and compute numerically such a flow is based on the so-called direct approach method applied to a flow of vertically projected family of planar curves. Vertical projection of surface curves on a simple surface $\mathcal{M}$ into the plane $\mathbb{R}^2$. It allows for reducing the problem to the analysis of evolution of planar

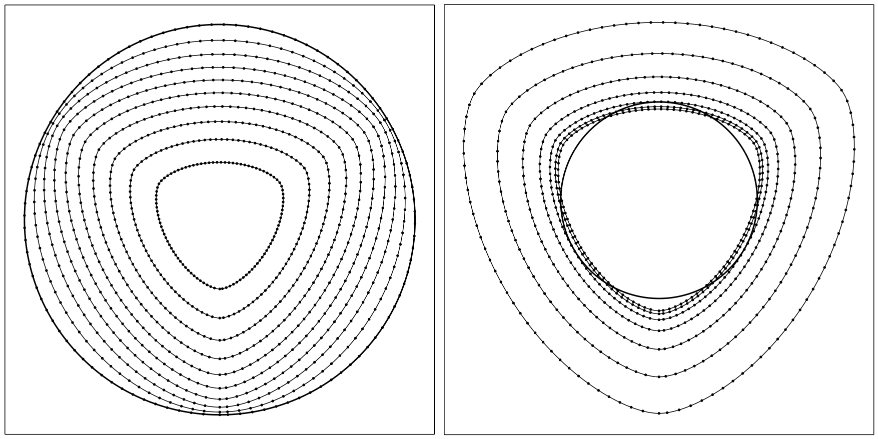

FIGURE 3. Anisotropic curvature driven motion of the initial unit circle including uniform tangential redistribution of grid points; $\beta(k,\nu) = \gamma(\nu)k + F$, with $\gamma(\nu) = 1 - \frac{7}{9}\cos(3\nu)$, $F = 0$ (left) and $\gamma(\nu) = 1 - \frac{7}{9}\cos(3\nu)$, $F = -1$ (right).

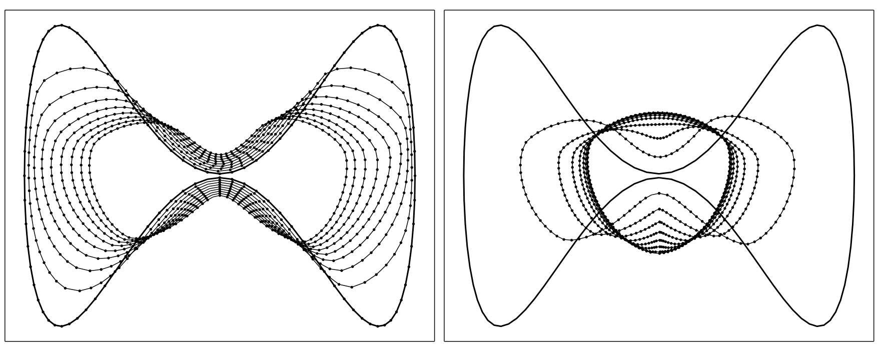

FIGURE 4. Curve evolution governed by $v = (1 - \frac{8}{9}\cos(3\nu))(x_1^2 + x_2^2)\ k + (-x_1, -x_2).(-\sin\nu, \cos\nu) - 0.5$.

curves $\Gamma^t : S^1 \to \mathbb{R}^2$, $t \geq 0$ driven by the normal velocity v given as a nonlinear function of the position vector x, tangent angle ν and as an affine function of the curvature k of Γ^t, i.e.

$$v = \beta(x, \nu, k) \tag{6.2}$$

where $\beta(x,\nu,k) = a(x,\nu)k + c(x,\nu)$ and $a(x,\nu) > 0, c(x,\nu)$ are bounded smooth coefficients.

2.1. Planar projection of the flow on a graph surface. Throughout this section we will always assume that a surface $\mathcal{M} = \{(x,z)) \in \mathbb{R}^3,\ z = \phi(x),\ x \in \Omega\}$ is a smooth graph of a function $\phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ defined in some domain $\Omega \subset \mathbb{R}^2$. Hereafter, the symbol (x,z) stands for a vector $(x_1, x_2, z) \in \mathbb{R}^3$ where

$x = (x_1, x_2) \in \mathbb{R}^2$. In such a case any smooth closed curve $\mathcal{G}$ on the surface $\mathcal{M}$ can be then represented by its vertical projection to the plane, i.e. $\mathcal{G} = \{(x, z) \in \mathbb{R}^3,\ x \in \Gamma,\ z = \phi(x)\}$ where Γ is a closed planar curve in $\mathbb{R}^2$. Recall, that for a curve $\mathcal{G} = \{(x, \phi(x)) \in \mathbb{R}^3,\ x \in \Gamma\}$ on a surface $\mathcal{M} = \{(x_1, x_2, \phi(x_1, x_2)) \in \mathbb{R}^3,\ (x_1, x_2) \in \Omega\}$ the geodesic curvature $\mathcal{K}_g$ is given by

$$\mathcal{K}_g = -\sqrt{EG - F^2}\Big(x_1'' x_2' - x_1' x_2'' - \Gamma^2_{11} x_1'^3 + \Gamma^1_{22} x_2'^3$$

$$-(2\Gamma^2_{12} - \Gamma^1_{11}) x_1'^2 x_2' + (2\Gamma^1_{12} - \Gamma^2_{22}) x_1' x_2'^2 \Big)$$

where E, G, F are coefficients of the first fundamental form and Γ^k_{ij} are Christoffel symbols of the second kind. Here $(.)'$ denotes the derivative with respect to the unit speed parameterization of a curve on a surface. In terms of geometric quantities related to a vertically projected planar curve we obtain, after some calculations, that

$$\mathcal{K}_g = \frac{1}{\left(1 + (\nabla\phi.\vec{T})^2\right)^{\frac{3}{2}}} \left(\left(1 + |\nabla\phi|^2\right)^{\frac{1}{2}} k + \frac{\vec{T}^T \nabla^2 \phi \vec{T}}{(1 + |\nabla\phi|^2)^{\frac{1}{2}}} \nabla\phi.\vec{N} \right) \tag{6.3}$$

(see [MS04b]). Moreover, the unit inward normal vector $\vec{\mathcal{N}} \perp T_x(\mathcal{M})$ to a surface curve $\mathcal{G} \subset \mathcal{M}$ relative to $\mathcal{M}$ can be expressed as

$$\vec{\mathcal{N}} = \frac{\left((1 + (\nabla\phi.\vec{T})^2)\vec{N} - (\nabla\phi.\vec{T})(\nabla\phi.\vec{N})\vec{T},\ \nabla\phi.\vec{N}\right)}{\left((1 + |\nabla\phi|^2)(1 + (\nabla\phi.\vec{T})^2)\right)^{\frac{1}{2}}}$$

(see also [MS04b]). Hence for the normal velocity $\mathcal{V}$ of $\mathcal{G}_t = \{(x, \phi(x)),\ x \in \Gamma^t\}$ we have

$$\mathcal{V} = \partial_t (x, \phi(x)).\vec{\mathcal{N}} = (\vec{N}, \nabla\phi.\vec{N}).\beta\vec{\mathcal{N}} = \left(\frac{1 + |\nabla\phi|^2}{1 + (\nabla\phi.\vec{T})^2} \right)^{\frac{1}{2}} \beta$$

where β is the normal velocity of the vertically projected planar curve Γ^t having the unit inward normal $\vec{N}$ and tangent vector $\vec{T}$. Following the so-called direct approach (see [Dec97, Dzi94, Dzi99, HLS94, Mik97, MS99, MS01, MS04a, MS04b, MS06]) the evolution of planar curves $\Gamma^t, t \geq 0$, can be described by a solution $x = x(., t) \in \mathbb{R}^2$ to the position vector equation $\partial_t x = \beta\vec{N} + \alpha\vec{T}$ where β and α are normal and tangential velocities of Γ^t, resp. Assuming the family of surface curves $\mathcal{G}_t$ satisfies (6.1) it has been shown in [MS04b] that the geometric equation $v = \beta(x, k, \nu)$ for the normal velocity v of the vertically projected planar curve Γ^t can be written in the following form:

$$v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla\phi(x).\vec{N} \tag{6.4}$$

where $a = a(x, \nu) > 0$ and $b = b(x, \nu)$ are smooth functions given by

$$a(x, \nu) = \frac{\delta}{1 + (\nabla\phi.\vec{T})^2}\,, \quad b(x, \nu) = -a(x, \nu) \frac{\vec{T}^T \nabla^2 \phi \vec{T}}{1 + |\nabla\phi|^2}\,, \tag{6.5}$$

where $\delta(X, \vec{\mathcal{N}}) > 0, X = (x, \phi(x)), \phi = \phi(x)$, k is the curvature of Γ^t, and $\vec{N} = (-\sin\nu, \cos\nu)$ and $\vec{T} = (\cos\nu, \sin\nu)$ are the unit inward normal and tangent vectors to a curve Γ^t.

We can also consider a more general flow of curves on a given surface driven by the normal velocity

$$\mathcal{V} = \mathcal{K}_g + \mathcal{F} \tag{6.6}$$

where $\mathcal{F}$ is the normal component of a given external force $\vec{G}$, i.e. $\mathcal{F} = \vec{G}.\vec{\mathcal{N}}$. The external vector field $\vec{G}$ is assumed to be perpendicular to the plane $\mathbb{R}^2$ and it may depend on the vertical coordinate $z = \phi(x)$ only, i.e.

$$\vec{G}(x) = -(0, 0, \gamma)$$

where $\gamma = \gamma(z) = \gamma(\phi(x))$ is a given scalar "gravity" functional.

Assuming the family of surface curves $\mathcal{G}^t$ satisfies (6.6) it has been shown in [MS04b] that the geometric equation $v = \beta(x, k, \nu)$ for the normal velocity v of the vertically projected planar curve Γ^t can be written in the following form:

$$v = \beta(x, k, \nu) \equiv a(x, \nu)\, k - b(x, \nu)\, \nabla\phi(x).\vec{N}$$

where $a = a(x, \nu) > 0$ and $b = b(x, \nu)$ are smooth functions given by

$$a(x, \nu) = \frac{1}{1 + (\nabla\phi.\vec{T})^2}, \quad b(x, \nu) = a(x, \nu)\left(\gamma(\phi) - \frac{\vec{T}^T \nabla^2\phi\, \vec{T}}{1 + (\nabla\phi.\vec{T})^2}\right). \tag{6.7}$$

In order to compute evolution of surface curves driven by the geodesic curvature and external force we can use numerical approximation scheme developed in Chapter 5 for the flow of vertically projected planar curves driven by the normal velocity given as in (6.4).

The next couple of examples illustrate a geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps. In Fig. 5 we show an example of an evolving family of surface curves shrinking to a point in finite time. In this example the behavior of evolution of surface curve is similar to that of planar curves for which Grayson's theorem holds. On the other hand, in Fig. 6 we present the case when the surface has two sufficiently high humps preventing evolved curve to pass through them. As it can be seen from Fig. 6 the evolving family of surface curves approaches a closed geodesic curve $\bar{\mathcal{G}}$ as $t \to \infty$.

The initial curve with large variations in the curvature is evolved according to the normal velocity $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$ where the external force $\mathcal{F} = \vec{G}.\vec{\mathcal{N}}$ is the normal projection of $\vec{G} = -(0, 0, \gamma)$ (see Fig. 7). In the numerical experiment we considered a strong external force coefficient $\gamma = 30$. The evolving family of surface curves approaches a stationary curve $\bar{\Gamma}$ lying in the bottom of the sharp narrow valley.

In the examples shown in Fig. 8 we present numerical results of simulations of a surface flow driven by the geodesic curvature and gravitational like external force, $\mathcal{V} = \mathcal{K}_g + \mathcal{F}$, on a wave-let surface given by the graph of the function $\phi(x) = f(|x|)$ where $f(r) = \sin(r)/r$ and $\gamma = 2$. In the first example shown in Fig. 8 (left-up) we started from the initial surface curve having large variations in the geodesic curvature. The evolving family converges to the stable stationary curve $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ with the second smallest stable radius. Vertical projection of the evolving family to the plane driven by the normal velocity $v = \beta(x, k, \nu)$ is shown

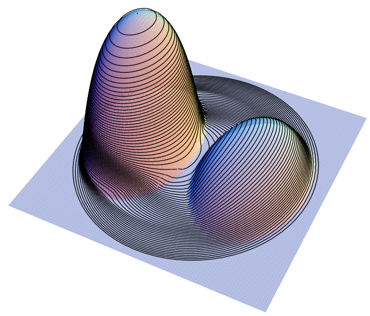

FIGURE 5. A geodesic flow $\mathcal{V} = \mathcal{K}_g$ on a surface with two humps having different heights.

in Fig. 8 (right-up). In Fig. 8 (left-bottom) we study a surface flow on the same surface the same external force. The initial curve is however smaller compared to that of the previous example. In this case the evolving family converges to the stable stationary curve with the smallest stable radius.

3. Applications in the theory of image segmentation

3.1. Edge detection in static images. A similar equation to (1.1) arises from the theory of image segmentation in which detection of object boundaries in the analyzed image plays an important role. A given black and white image can be represented by its intensity function $I : R^2 \to [0, 255]$. The aim is to detect edges of the image, i.e. closed planar curves on which the gradient ∇I is large (see [KM95]). The method of the so-called active contour models is to construct an evolving family of plane curves converging to an edge (see [KWT87]).

One can construct a family of curves evolved by the normal velocity $v = \beta(k, x, \nu)$ of the form

$$\beta(k, x, \nu) = \delta(x, \nu)k + c(x, \nu)$$

where $c(x, \nu)$ is a driving force and $\delta(x, \nu) > 0$ is a smoothing coefficient. These functions depend on the position vector x as well as orientation angle ν of a curve. Evolution starts from an initial curve whcih is a suitable approximation of the edge and then it converges to the edge. If $c > 0$ then the driving force shrinks the curve whereas the impact of c is reversed in the case $c < 0$. Let us consider an auxiliary function $\phi(x) = h(|\nabla I(x)|)$ where h is a smooth edge detector function like e.g. $h(s) = 1/(1 + s^2)$. The gradient $-\nabla\phi(x)$ has the important geometric property: it points towards regions where the norm of the gradient ∇I is large (see Fig. 9 right). Let us therefore take $c(x, \nu) = -b(\phi(x))\nabla\phi(x).\vec{N}$ and $\delta(x, \nu) = a(\phi(x))$ where $a, b > 0$ are given smooth functions. Now, if an initial curve belongs to a neighborhood of an edge of the image and it is evolved according to the geometric equation

$$v = \beta(x, k, \nu) \equiv a(\phi(x))k - b(\phi(x))\nabla\phi.\vec{N}$$

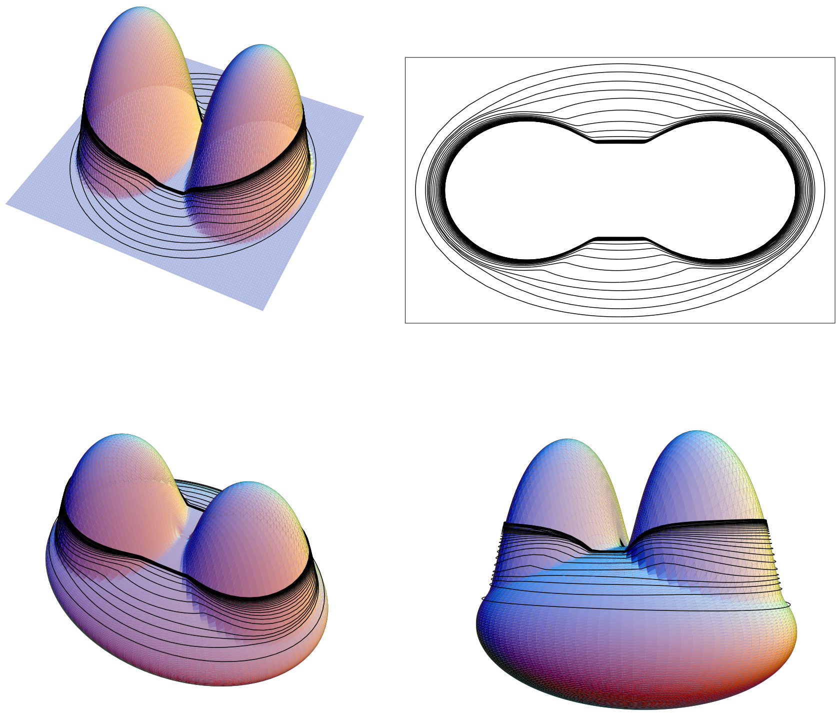

FIGURE 6. A geodesic flow on a surface with two sufficiently high humps (left-up) and its vertical projection to the plane (right-up). The evolving family of surface curves approaches a closed geodesic as $t \to \infty$. The same phenomenon of evolution on a compact manifold without boundary (below).

then it is driven towards this edge. In the context of level set methods, edge detection techniques based on this idea were first discussed by Caselles et al. and Malladi et al. in [CCCD93, MSV95] (see also [CKS97, CKSS97, KKO$^+$96]).

We apply our computational method to the image segmentation problem. First numerical experiment is shown in Fig. 10. We look for an edge in a 2D slice of a real 3D echocardiography which was prefiltered by the method of [SMS99]. The testing data set (the image function I) is a courtesy of Prof. Claudio Lamberti, DEIS, University of Bologna. We have inserted an initial ellipse into the slice close to an expected edge (Fig. 10 left). Then it was evolved according to the normal velocity described above using the time stepping $\tau = 0.0001$ and nonlocal redistribution strategy from Chaper 5. with parameters $\kappa_1 = 20$, $\kappa_2 = 1$ until the limiting curve

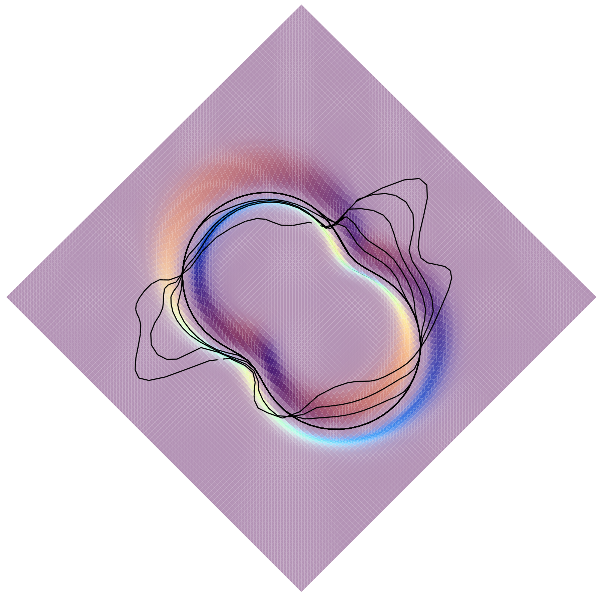

FIGURE 7. A geodesic flow on a flat surface with a sharp narrow valley.

has been formed (400 time steps). The final curve representing the edge in the slice can be seen in Fig. 10 right.

Next we present results for the image segmentation problem computed by means of a geodesic flow with external force discussed in Section 6.3. We consider an artificial dumb-bell image.from Fig. 9. If we take $\phi(x) = 1/(1 + |\nabla I(x)|^2)$ then the surface $\mathcal{M}$ defined as a graph of ϕ has a sharp narrow valley corresponding to points of the image in which the gradient $|\nabla I(x)|$ is very large representing thus an edge in the image. In contrast to the previous example shown in Fig. 10 we will make use of the flow of curves on a surface $\mathcal{M}$ driven by the geodesic curvature and strong "gravitational-like" external force $\mathcal{F}$. According to section 6.3 such a surface flow can be represented by a family of vertically projected plane curves driven by the normal velocity

$$v = a(x, \nu)k - b(x, \nu)\nabla\phi(x).\vec{N}$$

where coefficients a, b are defined as in (6.5) with strong external force coefficient $\gamma = 100$. Results of computation are presented in Fig. 11.

3.2. Tracking moving boundaries. In this section we describe a model for tracking boundaries in a sequence of moving images. Similarly as in the previous section the model is based on curvature driven flow with an external force depending on the position vector x.

Parametric active contours have been used extensively in computer vision for different tasks like segmentation and tracking. However, all parametric contours are known to suffer from the problem of frequent bunching and spacing out of curve

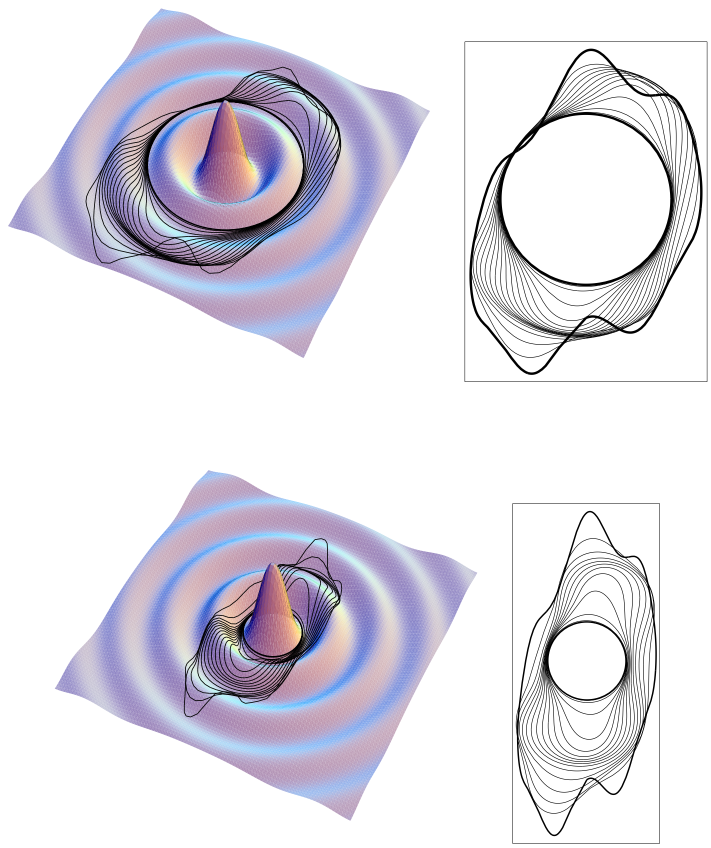

FIGURE 8. A surface flow on a wavelet like surface (left) and its vertical projection to the plane (right). Surface curves converge to the stable stationary circular curve $\bar{\Gamma} = \{x, |x| = \bar{r}\}$ with the smallest stable radius $\bar{r}$ (bottom) and the second smallest radius (up).

points locally during the curve evolution. In this part, we discuss a mathematical basis for selecting such a suitable tangential component for stabilization. We demonstrate the usefulness of the proposed choice of a tangential velocity method with a number of experiments. The results in this section can be found in a recent papers by Srikrishnan et al. [SCDR07, SCDRS07].

The force at each point on the curve can be resolved into two components: along the local tangent and normal denoted by α and β, respectively. This is written as:

$$\frac{\partial x}{\partial t} = \beta \vec{N} + \alpha \vec{T}\,. \tag{6.8}$$

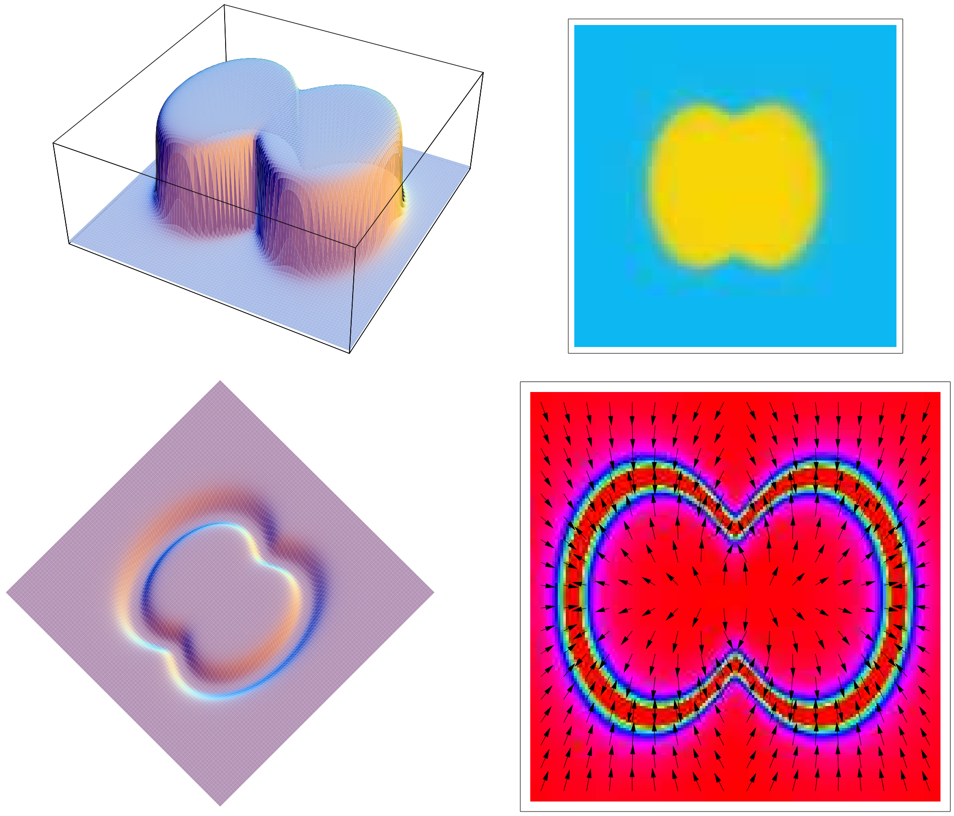

FIGURE 9. An image intensity function $I(x)$ (left-up) corresponding to a "dumb-bell" image (right-up). The the function ϕ (bottom-left) and corresponding vector field $-\nabla\phi(x)$ (bottom-right).

In this application, the normal velocity β has the form: $\beta = \mu\kappa + f(x)$ where f is a bounded function depending on the position of a curve point x. For the purpose of tracking we use the function $f(x) = \log\left(\frac{Prob_B(I(x))}{Prob_T(I(x))}\right)$ and we smoothly cut-off this function if either $Prob_B(I(x))$ or $Prob_B(I(x))$ are less than a prescribed tolerance. Here $Prob_B(I(x))$ stands for the probability that the point x belongs to a background of the image represented by the image intensity function I whereas $Prob_T(I(x))$ represents the probability that the point x belongs to a target in the image to be tracked. Both probabilities can be calculated from the image histogram (see [SCDR07, SCDRS07] for details).

In this field of application of a curvature driven flow of planar curves representing tracked boundaries in moving images it is very important to propose a suitable tangential redistribution of numerically computed grid points. Let us demonstrate the importance of tangential velocity by the following motivational example. In Fig. 12, we show two frames from a tracking sequence of a hand. Without any

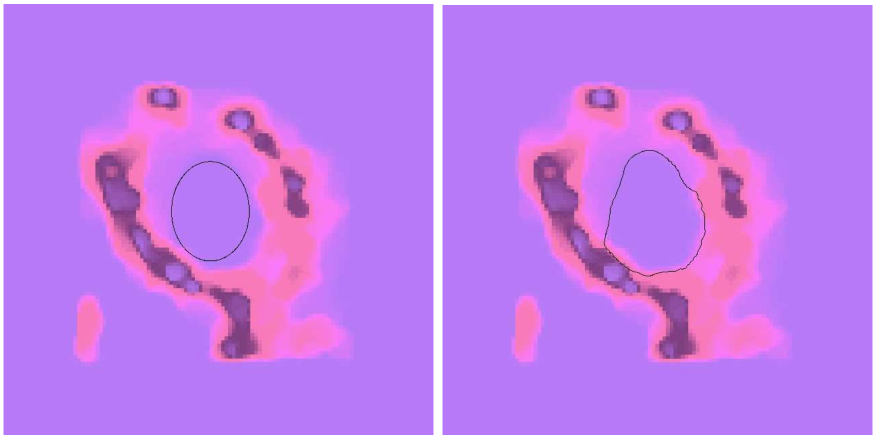

FIGURE 10. An initial ellipse is inserted into the 2D slice of a prefiltered 3D echocardiography (left), the slice together with the limiting curve representing the edge (right).

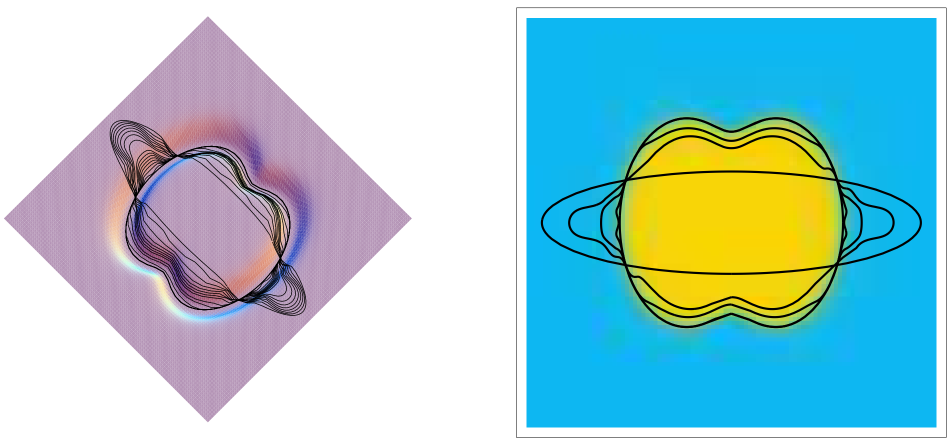

FIGURE 11. A geodesic flow on a flat surface with a sharp narrow valley (left) and its vertical projection to the plane with density plot of the image intensity function $I(x)$ (right).

tangential velocity (i.e. $\alpha = 0$) one can observe formation of small loops in the right picture which is a very next frame to the initial left one. These loops blow up and the curve becomes unstable within the next few frames.

In [SCDRS07] we proposed a suitable tangential velocity functional α capable of preventing evolved family of curves (image contours) from formation such undesirable loops like in Fig. 12 (right). Using a tangential velocity satisfying

$$\frac{\partial \alpha}{\partial u} = K - g + g\kappa\beta.$$

FIGURE 12. Illustration of curve degeneration. Left: The initial curve in red. Right: Bunching of points (in red) starts due to target motion leading to a loop formation.

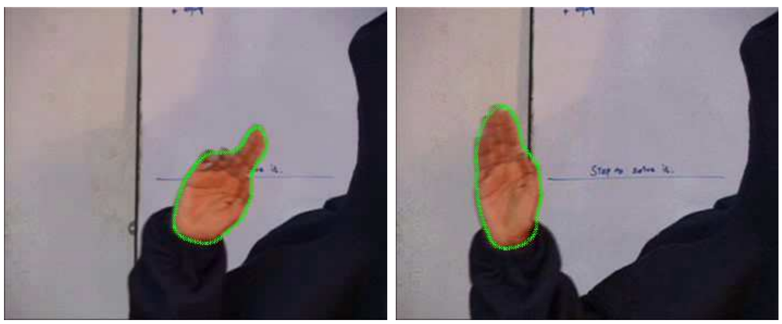

FIGURE 13. Tracking results for the same sequence as in Fig. 12 using a nontrivial tangential redistribution.

where $K = L(\Gamma) - \int_\Gamma \kappa\beta \, \mathrm{d}s$ we are able to significantly improve the results of tracking boundaries in moving images. If we compare tracking results in Fig. 13 and those from Fig. 12 we can conclude that the presence of a nontrivial suitably chosen tangential velocity α significantly improved tracking results.

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Part 3

The Navier–Stokes equations with particle methods

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Abstract. The non-stationary nonlinear Navier–Stokes equations describe the motion of a viscous incompressible fluid flow for $0 < t \leqslant T$ in some bounded three-dimensional domain. Up to now it is not known whether these equations are well-posed or not. Therefore we use a particle method to develop a system of approximate equations. We show that this system can be solved uniquely and globally in time and that its solution has a high degree of spatial regularity. Moreover we prove that the system of approximate solutions has an accumulation point satisfying the Navier–Stokes equations in a weak sense.

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CHAPTER 1

The Navier–Stokes equations with particle methods

1. Introduction

Let $T > 0$ be given and $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with a smooth compact boundary $\partial\Omega$. In Ω we consider a non-stationary viscous incompressible fluid flow and assume that it can be described by the Navier–Stokes equations

$$\begin{aligned} \partial_t v - \nu\Delta v + \nabla p + v \cdot \nabla v &= F \\ \nabla \cdot v &= 0 \\ v_{|\partial\Omega} &= 0 \\ v_{|t=0} &= v_0 \end{aligned} \qquad (N_0)$$

These equations represent a system of nonlinear partial differential equations concerning four unknown functions: the velocity vector $v = (v_1(t,x), v_2(t,x), v_3(t,x))$ and the (scalar) kinematic pressure function $p = p(t,x)$ of the fluid at the time $t \in (0,T)$ in the point $x \in \Omega$. The constant $\nu > 0$ (kinematic viscosity), the external force density F, and the initial velocity v_0 are given data. In (N_0) $\partial_t v$ means the partial derivative with respect to the time t, Δ is the Laplace operator in $\mathbb{R}^3$, and $\nabla = (\partial_1, \partial_2, \partial_3)$ the gradient, where $\partial_j = \frac{\partial}{\partial x_j}$ denotes the partial derivative with respect to x_j $(j = 1, 2, 3)$. From the physical point of view, the nonlinear convective term $v \cdot \nabla v$ is a result of the total derivative of the velocity field. Here the operator $v \cdot \nabla$ has to be applied to each component v_j of v. In the fourth equation $\nabla \cdot v = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$ defines the divergence of v, which vanishes due to the incompressibility of the fluid. Finally, the no-slip boundary condition $v_{|\partial\Omega} = 0$ expresses that the fluid adheres to the boundary $\partial\Omega$.

Let us assume that smooth data are given without any smallness assumptions. Then the problem to construct a solution v, ∇p of (N_0), which is uniquely determined and exists globally in time, has not been solved in the 3-d case considered here (see for example [6], [7], [8]). Consequently, there is no globally stable approximation scheme for (N_0) up to now.

In the present paper we use particle methods to approximate the Navier–Stokes equations by globally and uniquely solvable systems. To do so, let us consider, in particular, the nonlinear convective term $v \cdot \nabla v$, which is responsable for the non-global existence of the solution. From the physical point of view, this term results from the total (material) derivative of the velocity field v, and therefore the use of total differences in connection with particle methods seems to be reasonable. This leads to an approximation of the nonlinear term by some kind of central total difference quotient, which does not destroy the conservation of energy. The corresponding particle method and the properties of the trajectories are studied in Section 2 and Section 3. Using an additional time delay, the resulting system can be linearized. This requires a certain initial procedure to start, which is carried out in Section 4. Constructing sufficiently regular solutions even at initial time $t = 0$, a non-local compatibility condition arises, not checkable for given data. This condition can be satisfied, however, by a construction of suitable initial velocities from a prescribed initial acceleration vanishing on the boundary $\partial\Omega$. In the following Sections 5, 6, and 7 the approximate system is investigated with energy methods: A Galerkin ansatz based on the eigenfunctions of the Stokes operator $-P\Delta$ leads to a unique, for $0 \le t \le T$ globally existing, strongly H_4 - continuous regular solution. In Section 8 we prove that the Navier Stokes equations (N_0) can be re-obtained from this system in a certain sense, if the finite differences tend to zero: In this case there always exists a subsequence of the solution sequence with limit function v such that v is a weak solution of (N_0). Finally, in Section 9 local convergence properties of the whole sequence to the locally in time existing strong solution on the Navier Stokes system (N_0) are proved.

At this stage let us outline the notation: We use $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and denote by $\mathbb{R}$ the real numbers.

For $x, y \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, let

$$x \cdot y := \sum_{i=1}^{3} x_i y_i$$

be the scalar product of x, y and $|x| := \sqrt{x \cdot x}$ the Euclidean norm of x.

Throughout the paper, Ω denotes a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ at least of class C^2. We set $\overline{\Omega} := \Omega \cup \partial\Omega$ and $\Omega^c := \mathbb{R}^3 \backslash \overline{\Omega}$. For $T > 0$ let $\Omega_T := (0, T) \times \Omega$. By $N(x)$ we mean the exterior unit normal with respect to Ω in $x \in \partial\Omega$.

We use the same symbols for scalar-valued and vector-valued functions. The partial derivative of some functions v with respect to the i-th coordinate is denoted by $\partial_i v$, for a multi-index $\alpha \in \mathbb{N}_0^3$ let $\partial^\alpha v := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} v$ and

$$|\alpha| := \sum_{i=1}^{3} \alpha_i$$

the length of α.

Setting $\nabla := (\partial_1, \partial_2, \partial_3)$ we denote by

$$\mathrm{div} v := \nabla \cdot v := \sum_{i=1}^{3} \partial_i v_i$$

the divergence of the function $v = (v_1, v_2, v_3)$, and ∇v is the 3×3-matrix defined by $(\partial_i v_j)_{ji}$.

For a domain $A \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}_0$, let $C^m(A)$ be the space of functions being m-times continuously differentiable in A, and let $C^m(\overline{A})$ denote the subspace of functions, which—together with all their derivatives up to and including order

m—can be extended continuously onto ∂A. We set

$$C^\infty(A) := \bigcap_{m\in\mathbb{N}} C^m(A),$$

and define $C_0^\infty(A)$ to be the subspace of $C^\infty(A)$ containing functions with a compact support in A. The subspace $C_{0,\sigma}^\infty(A)$ contains vector functions in $C_0^\infty(A)$, which are divergence free, in addition.

If $v : (t,x) \to v(t,x)$ is a function defined in Ω_T we denote by $v(t) := v(t,\cdot)$ the function defined by $x \to (v(t))(x) := v(t,x)$ in Ω, $t \in (0,T)$. For $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$, and some Banach space B let $C([T_2,T_2],B)$ be the space of B-valued function being uniformly continuous in $[T_1,T_2]$.

By $L_p(\Omega)(1 \le p < \infty)$ we denote the usual (Lebesgue) Banach space, equipped with the norm

$$||v||_{0,p} := \left(\int_\Omega |v(x)|^p dx \right)^{\frac{1}{p}}.$$

The space $L_\infty(\Omega)$ with the norm

$$||v||_\infty := \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|$$

is the Banach space of all functions being essentially bounded in Ω. Setting $p = 2$, the space $L_2(\Omega)$ is a Hilbert space with the scalar product

$$(u,v) := \int_\Omega u(x)\cdot v(x)dx$$

and the norm

$$||v|| := ||v||_{0,2}.$$

For $m \in \mathbb{N}_0$ and $1 \le p < \infty$ let $H_{m,p}(\Omega)$ denote the Sobolev space of all functions $v \in L_p(\Omega)$ having distributional derivatives up to and including the order m in $L_p(\Omega)$, and let

$$||v||_{m,p} := \left(\sum_{|\alpha|=0}^m ||\partial^\alpha v||_{0,p}^p \right)^{\frac{1}{p}}$$

denote the corresponding norm. The spaces $H_{m,p}(\Omega)$ are Banach spaces, for $p = 2$ Hilbert spaces, and we define in this case $H_m(\Omega) := H_{m,2}(\Omega)$ and

$$||v||_m := ||v||_{m,2}.$$

The closure of the space $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||_1$ is defined by $\overset{\circ}{H}_1(\Omega)$, and the closure of the space $C_{0,\sigma}^\infty(\Omega)$ with respect to the norms $||\cdot||$ and $||\cdot||_1$ by $\mathcal{H}_0(\Omega)$ and $\mathcal{H}_1(\Omega)$, respectively.

The operator $P : L_2(\Omega) \to \mathcal{H}_0(\Omega)$ denotes the orthogonal projection such that

$$L_2(\Omega) = \mathcal{H}_0(\Omega) \oplus \{v \in L_2(\Omega) | v = \nabla p \quad \text{for some} \quad p \in H_1(\Omega)\}. \tag{1.1}$$

In $\overset{\circ}{H}_1(\Omega)$ and $\mathcal{H}_1(\Omega)$ we also use

$$(\nabla u, \nabla v) := \sum_{i,j=1}^3 (\partial_j u_i, \partial_j v_i), \qquad ||\nabla v|| := (\nabla v, \nabla v)^{\frac{1}{2}}$$

as a scalar product and a norm ([13]), respectively, as well as $(P\Delta u, P\Delta v)$ and $||P\Delta v||$ in $H_2(\Omega) \cap \mathcal{H}_1(\Omega)$ (see [4]).

In the notation of the function spaces, the symbol Ω is sometimes omitted: $\mathcal{H}_1 := \mathcal{H}_1(\Omega), \ldots$

Throughout the paper, for the estimates we often use the continuity of the imbedding

$$H_{j,p}(\Omega) \longrightarrow C^m(\overline{\Omega}), \tag{1.2}$$

valid for $j, m \in \mathbb{N}$ with $j > m + \frac{3}{p}$, and the compactness of the imbedding

$$H_{m,r}(\Omega) \longrightarrow H_{j,p}(\Omega), \tag{1.3}$$

valid for $1 \leq p, r < \infty$ and $j, m \in \mathbb{N}_0$ with $0 \leq j < m$, $\frac{3}{p} + m > \frac{3}{r} + j$ (see [1]).

Finally, by $C_\Omega, C_M, \ldots$ we denote positive constants depending on the terms indicated as subscripts. The values of theses constants may differ in different calculations.

Without loss of generality, throughout the paper we assume conservative external forces, i.e. in the system (N_0) we set $F = 0$.

2. An initial value problem

Let $J \subseteq \mathbb{R}$ denote a compact time interval and $v : J \times \overline{\Omega} \to \mathbb{R}^3$ a continuous velocity field. Moreover, for all $t \in J$ and $x \in \partial\Omega$ let $v(t, x) = 0$, and assume $v(t) := v(t, \cdot) \in C^l(\overline{\Omega})$, $1 \leq l \leq 4$.

Consider for fixed $(s, x_s) \in J \times \overline{\Omega}$ in the time interval J the initial value problem

$$\begin{array}{rcl} \dot{x}(t) & = & v(t, x(t)) \\ x(x) & = & x_s \end{array} \tag{1.4}$$

Here the function $t \to x(t) = X(t, s, x_s)$ denotes a solution of (1.4). It represents the trajectory of a particle of the fluid, which at initial time $t = s \in J$ is located in $x_s \in \overline{\Omega}$.

Due to $v(t, x) = 0$ for $(t, x) \in J \times \partial\Omega$ we find that for all $(s, x_s) \in J \times \overline{\Omega}$ the solution $x(t)$ of (1.4) exists globally in J and is uniquely determined there (see [16]).

For all $k = 0, 1, \ldots, l$ and all multi-indices α with $|\alpha| = 0, 1, \ldots, l$, respectively, the partial derivatives $\partial_s^k X, \partial_t \partial_s^k X, \partial^\alpha X$ and $\partial_t \partial^\alpha X$ exist and are continuous functions in $J \times J \times \overline{\Omega}$.

Due to the uniqueness of the solution, for the mappings

$$X_{t,s} : \begin{cases} \overline{\Omega} \to \overline{\Omega} \\ x \to X_{t,s}(x) := X(t, s, x) \end{cases} \tag{1.5}$$

the composition rule

$$X_{t,s} \circ X_{s,r} = X_{t,r}$$

holds true for all $t, s, r \in J$. In particular, we find that for all $t, s \in J$ the mapping $X_{t,s}$ is a diffeomorphism in $\overline{\Omega}$, and we have

$$X_{t,s}^{-1} = X_{s,t}. \tag{1.6}$$

Using $v(t, x) = 0$ for $(t, x) \in J \times \partial\Omega$ we obtain, moreover, $X_{t,x}(\Omega) = \Omega$ for all $t, s \in J$.

Now consider Liouville's differential equation

$$\partial_t \det \nabla X_{t,s}(x) = -\mathrm{div}_X v(t, X_{t,s}(X)) \det \nabla X_{t,s}(x)$$

concerning the functional determinant $\det \nabla X_{t,s}(x)$.

If we assume $\mathrm{div} v(t,x) = 0$ for all $(t,x) \in J \times \Omega$, then it follows

$$\det \nabla X_{t,s}(x) = \det \nabla X_{s,s}(x) = \det \nabla x = 1$$

for all $t, s \in J$ and $x \in \Omega$.

Throughout this paper we call this important property of the mappings $X_{t,s}$ the conservation of measure.

It implies, in particular, that for divergence-free vector functions $v(t,\cdot)$ and all $t, s, r \in J$ we have

$$||v(t, X_{s,r})||_{o,p} = ||v(t)||_{o,p} \tag{1.7}$$

for all $1 \leq p \leq \infty$.

Lemma 1.1. *Let $J \subseteq \mathbb{R}$ denote a compact time interval, and let $v \in C(J, H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ for $m \in \{3, 4\}$ be given. Then we have*

$$a_1 := \max_{t\in J} ||v(t)||_\infty < \infty, \quad a_2 := \max_{t\in J} ||\nabla v(t)||_\infty < \infty,$$

and for $m = 4$, in addition,

$$a_3 := \max_{t\in J} ||\nabla^2 v(t)||_\infty < \infty.$$

For every $(s, x_s) \in J \times \overline{\Omega}$ there is a uniquely determined solution $t \to x(t) = X(t, s, x_s)$ of the initial value problem (1.4), which exists in the whole J. For the mappings $X_{t,s}$ defined by (1.5) for all $t, s \in J$ we have the estimates

$$||X_{t,s}||_\infty \leq |t-s|a_1 + c_\Omega, \quad ||\nabla X_{t,s}||_\infty \leq e^{|t-s|a_2},$$

and for $m = 4$, in addition,

$$||\nabla^2 X_{t,s}||_\infty \leq \frac{a_3}{a_2} e^{|t-s|a_2}(e^{|t-s|a_2} - 1).$$

Here the constant c_Ω depends only on Ω.

Proof. The existence of the norms a_i follows from well-known imbedding theorems, as well as the existence and uniqueness of the solution of (1.4): For $l = m-2$ all the above required properties of v are fulfilled, where the no-slip boundary condition and the vanishing divergence (solenoidality) follow from $v(t) \in \mathcal{H}_1(\Omega)$ for all $t \in J$.

Because for every half norm $||\cdot||_h$ and every absolutely continuous function $f : t \to f(t)$ the inequality $\frac{d}{dt}||f(t)||_h \leq ||f'(t)||_h$ holds true (see [15]), we obtain using the conservation of measure of the mappings $X_{t,s}$ for all $t, s \in J$ the following estimates:

$$\begin{aligned}
\frac{d}{dt}||X_{t,s}||_\infty &\le ||\partial_t X_{t,s}||_\infty = ||v(t,X_{t,s})||_\infty = ||v(t)||_\infty,\\
\frac{d}{dt}||\nabla X_{t,s}||_\infty &\le ||\partial_t \nabla X_{t,s}||_\infty = ||\nabla_X v(t,X_{t,s})\nabla X_{t,s}||_\infty\\
&\le ||\nabla v(t)||_\infty ||\nabla X_{t,s}||_\infty,\\
\frac{d}{dt}||\nabla^2 X_{t,s}||_\infty &\le ||\partial_t \nabla^2 X_{t,s}||_\infty = ||\nabla_X^2 v(t,X_{t,s})(\nabla X_{t,s})^2 + \nabla_X v(t,X_{t,s})\nabla^2 X_{t,s}||_\infty\\
&\le ||\nabla^2 v(t)||_\infty ||\nabla X_{t,s}||_\infty^2 + ||\nabla v(t)||_\infty ||\nabla^2 X_{t,s}||_\infty.
\end{aligned}$$

These are three differential inequalities concerning the L_∞-norms of the derivatives of the mappings $X_{t,s}$.

Due to $X_{s,s}(x_s) = x_s, \nabla X_{s,s}(x_s) = I$ and $\nabla^2 X_{s,s}(x_s) = 0$, where I denotes the identity matrix and 0 the zero tensor for all $x_s \in \overline{\Omega}$, the corresponding initial values are also well known, and the estimates follow in both cases $s \le t$ and $t \le s$ from Gronwall's Lemma (see [15]). ☐

3. Approximation of the convective term

Up to now it is not known whether the Navier–Stokes initial boundary value problem (N_0) in three dimensions is well-posed or not: We only know the existence and uniqueness of a strong solution locally in time. So in the following we want to derive a suitable smoothing procedure to end up with a modified Navier–Stokes-like system of equations, which can even be solved globally in time.

This system of equations $(\tilde{N}_\varepsilon)$ depends on a certain regularizing parameter $\varepsilon > 0$ in the nonlinear term. In the following we shall develop this regularized system step by step.

To do so, let us first recall the physical deduction of the Navier–Stokes equations: The nonlinear convective term $v(t,x) \cdot \nabla v(t,x)$, which is responsable for the non-global solvability of these equations, results from the total derivative of the velocity field $v(t,x)$. Thus a so-called total or Lagrangian difference quotient could be used for an approximation of the nonlinear convective term:

Definition 1.2. Let $J \subseteq \mathbb{R}$ denote a compact time interval and let $v \in C(J, H_3(\Omega) \cap \mathcal{H}_1(\Omega))$ be given. Let $\varepsilon > 0$ and let $t, s, s+\varepsilon \in J, x \in \overline{\Omega}$. Then the quotients

$$\frac{1}{\varepsilon}\{v(t, X_{s+\varepsilon,s}(x)) - v(t,x)\}$$

and

$$\frac{1}{\varepsilon}\{v(t,x) - v(t, X_{s,s+\varepsilon}(x))\}$$

are well defined and denoted by an upwards and a backwards total (Lagrangian) difference quotient, respectively. Summing up both quotients and dividing by two we obtain

$$\frac{1}{2\varepsilon}\{v(t, X_{s+\varepsilon,s}(x)) - v(t, X_{s,s+\varepsilon}(x))\}$$

and call it a central total (Lagrangian) difference quotient.

REMARK 1.3. *Using a mean value theorem, as* $\varepsilon \to 0$ *all the above mentioned difference quotients converge to* $v(s,x) \cdot \nabla v(t,x)$. *For example, for the upwards quotient we find*

$$\begin{aligned}\frac{1}{\varepsilon}\{v(t, X_{s+\varepsilon,s}(x)) - v(t,x)\} &= \frac{1}{\varepsilon}\{v(t, X_{s+\varepsilon,s}(x)) - v(t, X_{s,s}(x))\}\\ &= \frac{1}{\varepsilon}\int_s^{s+\varepsilon} \partial_\tau X_{\tau,s}(x) \cdot \nabla v(t, X_{\tau,s}(x))d\tau\\ &= \frac{1}{\varepsilon}\int_s^{s+\varepsilon} v(\tau, X_{\tau,s}(x)) \cdot \nabla v(t, X_{\tau,s}(x))d\tau,\end{aligned}$$

where the term on the right hand side tends to $v(s, X_{s,s}(x)) \cdot \nabla v(t, X_{s,s}(x)) = v(s,x) \cdot \nabla v(t,x)$ *as* $\varepsilon \to 0$.

It is well known (see [13]) that for vector functions $u \in \mathcal{H}_1(\Omega), w \in \overset{\circ}{H}_1(\Omega)$ the orthogonality relation $(u \cdot \nabla w, w) = 0$ holds true. This important relation is used by Hopf (see [5]) to prove the existence of weak solutions of the Navier–Stokes equations (N_0) global in time. Using an approximation of the convective term $v(t,x) \cdot \nabla v(t,x)$ by a central total difference quotient, we can prove the following analogue of this orthogonality relation:

LEMMA 1.4. *Under the assumptions of Definition* 1.2*, for the central total difference quotient the following orthogonality relation holds true:*

$$\left(\frac{1}{2\varepsilon}[v(t, X_{s+\varepsilon,s}(\cdot)) - v(t, X_{s,s+\varepsilon}(\cdot))], v(t,\cdot)\right) = 0.$$

PROOF. For all $s_1, s_2 \in J$ the mappings X_{s_1,s_2} are measure conserving, and by (1.6), we obtain $X_{s_1,s_2} \circ X_{s_2,s_1}(x) = x$ for all $x \in \overline{\Omega}$. Using the symmetry of the scalar product this implies the orthogonality:

$$\begin{aligned}&(v(t, X_{s+\varepsilon,s}(\cdot)) - v(t, X_{s,s+\varepsilon}(\cdot)), v(t,\cdot))\\ &\quad = (v(t, X_{s+\varepsilon,s}(\cdot)), v(t,\cdot)) - (v(t, X_{s,s+\varepsilon} \circ X_{s+\varepsilon,s}(\cdot)), v(t, X_{s+\varepsilon,s}(\cdot))) = 0.\end{aligned}$$

□

REMARK 1.5. *From Lemma 1.4 it follows that for sufficiently regular solutions of an approximate system regularized by central total differences the energy equation is satisfied. As seen from the proof above, this important equation does not hold true if only a one-sided total difference quotient is used to approximate the convective term.*

To avoid fixed point considerations—it is clear that in general both the velocity field v as well as the trajectories X are not known—in the following we use in addition a time delay: The convective term $v(t,x) \cdot \nabla v(t,x)$ is replaced by a central total difference quotient of the form

$$\frac{1}{2\varepsilon}\{v(t, X_{s+\varepsilon,s}(x)) - v(t, X_{s,s+\varepsilon}(x))\}$$

with times $s, s+\varepsilon < t$. In these time points - using a step by step construction - the velocity field is known, already.

Let us now formulate the regularized problem $(\tilde{N}_\varepsilon)$ as follows:

Let $T \in \mathbb{R}(T > 0)$ and $N \in \mathbb{N}(N \geq 2)$ be given. Setting $\varepsilon := \frac{T}{N} > 0$ we define by $t_k = k\varepsilon$ $(k = 0, \pm 1, \dots, \pm N)$ an equidistant time grid on the compact time interval $[-T, +T]$.

Construct a velocity field $v = (v_1, v_2, v_3)$ and some pressure function p as solution of the regularized Navier–Stokes initial boundary value problem

$$\begin{aligned} \partial_t v - \nu \Delta v + \nabla p &= -Z_\varepsilon v \\ \nabla \cdot v &= 0 \qquad (t,x) \in (0,T] \times \Omega \\ v_{|\partial\Omega} &= 0 \\ \partial_t v_{|t=0} &= f. \end{aligned} \tag{$\tilde{N}_\varepsilon$}$$

Here for $(t,x) \in [t_k, t_{k+1}] \times \overline{\Omega}$ and $k = 0, 1, \dots, N-1$ we define

$$\begin{aligned} Z_\varepsilon v(t,x) &:= Z_\varepsilon^k v(t,x) \\ &:= \frac{t - t_k}{\varepsilon} \cdot \frac{1}{2\varepsilon} \cdot \{v(t, X_{t_k, t_{k-1}}(x)) - v(t, X_{t_{k-1}, t_k}(x))\} \\ &\quad + \frac{t_{k+1} - t}{\varepsilon} \cdot \frac{1}{2\varepsilon} \cdot \{v(t, X_{t_{k-1}, t_{k-2}}(x)) - v(t, X_{t_{k-2}, t_{k-1}}(x))\}, \end{aligned} \tag{1.8}$$

where the mappings X_{t_i, t_j} have to be constructed from the solution $t \to x(t) = X(t, s, x_s)$ of a corresponding initial value problem (1.4) with a velocity field already known.

Remark 1.6. *To compute the solution $v(t)$ of $(\tilde{N}_\varepsilon)$ in the first subinterval $[t_0, t_1]$ we have to construct the mappings $X_{t_0, t_{-1}}$ and $X_{t_{-1}, t_{-2}}$ together with the inverse mappings $X_{t_0, t_{-1}}^{-1} = X_{t_{-1}, t_0}$ and $X_{t_{-1}, t_{-2}}^{-1} = X_{t_{-2}, t_{-1}}$, respectively. This construction will be carried out in the next section.*

Remark 1.7. *The global construction step by step requires certain regularity properties of the solution $v(t)$ of $(\tilde{N}_\varepsilon)$ on the subintervals $J_k := [t_k, t_{k+1}]$ for $k = 0, 1, \dots, N-1)$. These regularity properties are necessary to imply the unique solvability of the initial value problem (1.4) in J_k and thus the existence of the mappings $X_{t_{k+1}, t_k}, X_{t_k, t_{k-1}}$, which are needed for the construction of the solution on subsequent time intervals.*

This high degree of regularity—$v \in C(J_k, H_3(\Omega) \cap \mathcal{H}_1(\Omega))$ is sufficient, but $v \in C(J_k, H_2(\Omega) \cap \mathcal{H}_1(\Omega))$ is not sufficient—leads to compatibility conditions arising on the boundary of the time-space cylinder (see [4] and [14]), as usual for parabolic problems. In our case, due to the stepwise construction, these conditions appear in the points (t_k, x) with $k = 0, 1, \dots, N-1$ and $x \in \partial\Omega$.

The compatibility conditions for $t_k > 0$ are fulfilled because of the continuity of the functions $Z_\varepsilon v(\cdot, x)$ in $[0, T]$ for $x \in \overline{\Omega}$, and the condition at time $t_0 = 0$ holds by a special initial construction using an idea of Solonnikov: Instead of the initial velocity $v(0)$ the initial acceleration $\partial_t v(0)$ has to be prescribed in a suitable way. This construction is carried out in the next section.

4. Construction of the initial data

Let $T > 0, 2 \leq N \in \mathbb{N}, \varepsilon = \frac{T}{N} > 0$ and $t_k = k\varepsilon$ for $k = 0, \pm 1, \ldots, \pm N$ as fixed above in problem $(\tilde{N}_\varepsilon)$.

Let $\overline{v} \in C([-T, 0], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ for $m \in \{3, 4\}$ be given. Then by Lemma 1.1 it follows that the mapping

$$X_{t_{-1},t_{-2}} : \begin{cases} \overline{\Omega} \to \overline{\Omega} \\ x \to X_{t_{-1},t_{-2}}(x) = X(-\varepsilon, -2\varepsilon, x) \end{cases} \tag{1.9}$$

defined by (1.5) exists, as well as its inverse

$$X_{t_{-2},t_{-1}} : \begin{cases} \overline{\Omega} \to \overline{\Omega} \\ x \to X_{t_{-2},t_{-1}}(x) = X(-2\varepsilon, -\varepsilon, x). \end{cases} \tag{1.10}$$

Denoting by $P : L_2(\Omega) \to \mathcal{H}_0(\Omega)$ the orthogonal projection (see [13]) we consider in Ω the stationary regularized Navier–Stokes boundary value problem

$$\nu P \Delta v_0 - \frac{1}{2\varepsilon} P\{v_0 \circ X_{t_{-1},T_{-2}} - v_0 \circ X_{t_{-2},t_{-1}}\} = Pf. \tag{1.11}$$

LEMMA 1.8. *Let $\overline{v} \in C([-T, 0], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ for $m \in \{3, 4\}$, and let the mappings $X_{t_{-1},t_{-2}}, X_{t_{-2},t_{-1}}$ be constructed as above. Let $f \in H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega)$. Then there exists a uniquely determined solution $v_0 \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$ of problem (1.11).*

PROOF. Let us set $X := X_{t_{-1},t_{-2}}$ for abbreviation. Then from the linearity of the problem, for the difference $w_0 := v_0^1 - v_0^2$ of two solutions v_0^1 and v_0^2 it follows the identity

$$\nu P \Delta w_0 - \frac{1}{2\varepsilon} P\{w_0 \circ X - w_0 \circ X^{-1}\} = 0.$$

Here we find $(w_0 \circ X - w_0 \circ X^{-1}, w_0) = 0$ due to the measure conserving property of the mappings X, and the uniqueness follows from the inequality of Poincaré:

$$\nu ||w_0||^2 \leq \nu c_\Omega ||\nabla w_0||^2 = 0.$$

The existence of a solution $v_0 \in \mathcal{H}_1(\Omega)$ can be obtained from the theory of the stationary Navier–Stokes equations (see [13]), and for the regularity statement $v_0 \in H_m(\Omega)$ we can use the estimate of Cattabriga ([2]), which means that we only have to show $||v_0 \circ X||_{m-2} < \infty$, $||v_0 \circ X^{-1}||_{m-2} < \infty$. This indeed follows from $v_0 \in \mathcal{H}_1(\Omega)$ and the regularity properties of the mappings X and X^{-1} following Lemma 1.1. □

Now using the function $\overline{v}$ prescribed above together with the solution v_0 of the system (1.11) we can define for example by linear interpolation some function $\tilde{v} \in C([-T, 0], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ for $m \in \{3, 4\}$:

$$\tilde{v}(t) := \begin{cases} \overline{v}(t) & \text{for} \quad t \in [-T, -\varepsilon] \\ \frac{1}{\varepsilon}\{(t+\varepsilon)v_0 - t \cdot \overline{v}(-\varepsilon)\} & \text{for} \quad t \in [-\varepsilon, 0]. \end{cases} \tag{1.12}$$

Now we use the function $\tilde{v}$ and Lemma 1.1 to construct the mapping

$$X_{t_0,t_{-1}} : \begin{cases} \overline{\Omega} \to \overline{\Omega} \\ x \to X_{t_0,t_{-1}}(x) = X(0, -\varepsilon, x) \end{cases} \tag{1.13}$$

together with its inverse

$$X_{t_{-1},t_0} : \begin{cases} \overline{\Omega} \to \overline{\Omega} \\ x \to X_{t_{-1},t_0}(x) = X(-\varepsilon, 0, x). \end{cases} \tag{1.14}$$

All the mappings (1.9), (1.10), (1.13) and (1.14) constructed in this way are used in the next section where we start the investigation of problem $(\tilde{N}_\varepsilon)$ on the first subinterval $[t_0, t_1]$. Due to Lemma 1.8 we can replace the initial condition $\partial_t v_{|t=0} = f$ from problem $(\tilde{N}_\varepsilon)$ (note we assume $f \in H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega)$ with $m \in \{3, 4\}$) by the initial condition $v_{|t=0} = v_0$ (in this case we have $v_0 \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$).

Moreover, the above construction implies that the initial acceleration $\partial_t v(0) = f$ is contained in $\mathcal{H}_1(\Omega)$, if the differential equations still hold for $t = 0$. This ensures - as we shall see later on - that the important compatibility condition at time $t = 0$ is satisfied.

5. Strongly H_2-continuous solutions

We consider the problem $(\tilde{N}_\varepsilon)$ restricted to $t \in [t_0, t_1] = [0, \varepsilon]$ in the following form:

Find a velocity field $v = (v_1, v_2, v_3)$ and some pressure function p as a solution of the regularized equations

$$\begin{aligned} \partial_t v - \nu \Delta v + \nabla p &= -Z_\varepsilon^0 v \\ \nabla \cdot v &= 0 \qquad\qquad (t,x) \in (0,\varepsilon] \times \Omega \\ v_{|\partial\Omega} &= 0 \\ v_{|t=0} &= v_0. \end{aligned} \tag{N_ε^0}$$

Here for $(t, x) \in [0, \varepsilon] \times \overline{\Omega}$ we define

$$Z_\varepsilon^0 v(t,x) := \frac{t}{2\varepsilon^2}\{v(t, X(x)) - v(t, X^{-1}(x))\} + \frac{\varepsilon - t}{2\varepsilon^2}\{v(t, Y(x)) - v(t, Y^{-1}(x))\} \tag{1.15}$$

with some given measure conserving homomorphisms $X : \overline{\Omega} \to \overline{\Omega}$ and $Y : \overline{\Omega} \to \overline{\Omega}$. The function $v_0 \in H_2(\Omega) \cap \mathcal{H}_1(\Omega)$ is a given initial velocity distribution.

In this section we show the existence of a solution $t \to v(t) := v(t, \cdot)$ to the linear system (N_ε^0) being strongly H_2-continuous in $[0, \varepsilon]$. This solution is uniquely determined and satisfies the energy equation (Theorem 1.15). Similar to [3] we prove the existence with help of a Galerkin ansatz based on the eigenfunctions of the Stokes operator $-P\Delta$. This is done in the following way: First we derive suitable a-priori estimates for the Galerkin approximations and then we use compactness arguments to proceed to the limit and prove the existence of a solution.

(a) Galerkin ansatz

The Stokes operator $-P\Delta : H_2(\Omega) \cap \mathcal{H}_1(\Omega) \to \mathcal{H}_0(\Omega)$ defines in $\mathcal{H}_1(\Omega)$ a symmetric positive definite operator with compact inverse $(-P\Delta)^{-1} : \mathcal{H}_0(\Omega) \to \mathcal{H}_0(\Omega)$ (see [12]). Hence there is a sequence $(\lambda_i)_i$ of positive eigenvalues satisfying $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty$, and the corresponding sequence $(e_i)_i$ of eigenfunctions represents a complete orthonormal system in $\mathcal{H}_0(\Omega)$. Due to the regularity of the boundary ($\partial\Omega \in C^4$) we have $e_i \in H_4(\Omega)$ for all $i \in \mathbb{N}$ (see [13]).

Now for $i = 1, 2, \ldots, n (n \in \mathbb{N})$ and $t > 0$ we consider the following initial value problem for ordinary differential equations:

$$\begin{aligned} c'_{\text{in}}(t) &= -\nu\lambda_i c_{\text{in}}(t) \\ &\quad - \frac{1}{2\varepsilon^2}\sum_{j=1}^{n} c_{\text{jn}}(t)(t\{e_j \circ X - e_j \circ X^{-1}\} + (\varepsilon - t)\{e_j \circ Y - e_j \circ Y^{-1}\}, e_i), \\ c_{\text{in}}(0) &= (v_0, e_i). \end{aligned}$$

This is a linear system for the functions c_{in}, and it is globally and uniquely solvable with $c_{\text{in}} \in C^\infty([0, \varepsilon])$ for all $i = 1, \ldots, n$.

The function

$$v^n : \begin{cases} [0,\varepsilon] \times \overline{\Omega} \to \mathbb{R}^3 \\ (t,x) \to v^n(t,x) := \sum\limits_{i=1}^{n} c_{\text{in}}(t) \cdot e_i(x) \end{cases} \tag{1.16}$$

is denoted as Galerkin approximation of order n for the solution v of the problem (N_ε^0).

Due to the construction, for the functions $v^n(t) := v^n(t, \cdot)$ for all $t \in (0, \varepsilon]$ and all $i = 1, 2, \ldots, n$ the following equations hold true:

$$(\partial_t v^n(t), e_i) - \nu(P\Delta v^n(t), e_i) = -(Z_\varepsilon^0 v^n(t), e_i), \tag{1.17}$$

$$v^n(0) = \sum_{j=1}^{n} (v_0, e_j)e_j. \tag{1.18}$$

(b) Estimates of the Galerkin approximations

All estimates of the Galerkin approximations $v^n(t)$ can be obtained from the following two lemmata and are valid—depending on the regularity of the data—uniformly for $t \in [0, \varepsilon]$ or only for $t \in (0, \varepsilon]$.

Lemma 1.9. *Let $k, n \in \mathbb{N}$. Then for the Galerkin approximation $v^n(t)$ defined for $t \in [0, \varepsilon]$ by* (1.16) *the following identities hold true:*

$$\frac{d}{dt}||v^n(t)||^2 + 2\nu||\nabla v^n(t)||^2 = 0,$$

$$\begin{aligned} \frac{d}{dt}||\partial_t^k v^n(t)||^2 + 2\nu||\nabla\partial_t^k v^n(t)||^2 &= -2(\partial_t^k Z_\varepsilon^0 v^n(t), \partial_t^k v^n(t)) \\ &= -\frac{k}{\varepsilon^2}\left(\partial_t^{k-1} v^n(t,X) - \partial_t^{k-1} v^n(t,X^{-1}), \partial_t^k v^n(t)\right) \\ &\qquad - \frac{k}{\varepsilon^2}\left(-\partial_t^{k-1} v^n(t,Y) + \partial_t^{k-1} v^n(t,Y^{-1}), \partial_t^k v^n(t)\right) \end{aligned} \tag{1.19}$$

$$\frac{d}{dt}||\nabla v^n(t)||^2 + 2\nu||P\Delta v^n(t)||^2 = -2(Z_\varepsilon^0 v^n(t), -P\Delta v^n(t)),$$

$$\begin{aligned}\frac{d}{dt}||\nabla\partial_t^k v^n(t)||^2 + 2\nu||P\Delta\partial_t^k v^n(t)||^2 &= -2(\partial_t^k Z_\varepsilon^0 v^n(t), -P\Delta\partial_t^k v^n(t))\\ &= -\frac{k}{\varepsilon^2}\left(\partial_t^{k-1}v^n(t,X) - \partial_t^{k-1}v^n(t,X^{-1}), -P\Delta\partial_t^k v^n(t)\right)\\ &\quad - \frac{k}{\varepsilon^2}\left(-\partial_t^{k-1}v^n(t,Y) + \partial_t^{k-1}v^n(t,Y^{-1}), -P\Delta\partial_t^k v^n(t)\right) \quad (1.20)\end{aligned}$$

Proof. Let $k \geq 0$. Then the first sequence of identities in the second line of (1.19) and (1.20), respectively, follows from differentiating the equation (1.17) k times with respect to t, multiplying the result scalar by $\partial_t^k c_{\text{in}}(t)$ and $\lambda_i\partial_t^k c_{\text{in}}(t)$, respectively, and afterwards summing up for $i = 1, \dots, n$. Concerning the first line of (1.19) we use in addition the orthogonality $(Z_\varepsilon^0 v^n(t), v^n(t)) = 0$, which follows from the measure conserving property of the mappings X, Y.

The second sequence of identities is obtained from

$$\begin{aligned}\partial_t^k Z_\varepsilon^0 v^n(t) &= \partial_t^{k-1}\Bigg(Z_\varepsilon^0\partial_t v^n(t)\\ &\quad + \frac{1}{2\varepsilon^2}\{v^n(t,X) - v^n(t,X^{-1}) - v^n(t,Y) + v^n(t,Y^{-1})\}\Bigg) = \ldots = Z_\varepsilon^0\partial_t^k v^n(t)\\ &+ \frac{k}{2\varepsilon^2}\{\partial_t^{k-1}v^n(t,X) - \partial_t^{k-1}v^n(t,X^{-1}) - \partial_t^{k-1}v^n(t,Y) + \partial_t^{k-1}v^n(t,Y^{-1})\}. \quad (1.21)\end{aligned}$$

□

Estimating the right hand sides in (1.19) and (1.20) using the inequalities of Hölder, Poincaré and Young (see [7]), the measure conserving property of the mappings X and Y, and, finally, the absorption of terms arising on both sides of the inequalities, the following corollary can be proved:

Corollary 1.10. *Let $k, n \in \mathbb{N}$. Then the Galerkin approximations $v^n(t)$ defined for $t \in [0,\varepsilon]$ by* (1.16) *satisfy the following differential inequalities:*

$$\frac{d}{dt}||\partial_t^k v^n(t)||^2 + \nu||\nabla\partial_t^k v^n(t)||^2 \leq c_{\Omega,\nu,\varepsilon,k}||\partial_t^{k-1}v^n(t)||^2, \quad (1.22)$$

$$\frac{d}{dt}||\nabla v^n(t)||^2 + \nu||P\Delta v^n(t)||^2 \leq c_{\nu,\varepsilon}||v^n(t)||^2, \quad (1.23)$$

$$\frac{d}{dt}||\nabla\partial_t^k v^n(t)||^2 + \nu||P\Delta\partial_t^k v^n(t)||^2 \leq c_{\nu,\varepsilon,k}(||\partial_t^{k-1}v^n(t)||^2 + ||\partial_t^k v^n(t)||^2). \quad (1.23')$$

Here and in the following, the constants $c_{\Omega,\ldots}$ depend on the items indicated in the subscripts. Finally, from the Galerkin equations (1.17) we quote immediately:

Lemma 1.11. *Let $k, n \in \mathbb{N}$. Then for the Galerkin approximation $v^n(t)$ defined by* (1.16) *for $t \in [0,\varepsilon]$ the following inequalities hold true:*

$$||\partial_t v^n(t)||^2 \leq c_{\nu,\varepsilon}(||P\Delta v^n(t)||^2 + ||v^n(t)||^2), \quad (1.24)$$

$$||\partial_t^{k+1}v^n(t)||^2 \leq c_{\nu,\varepsilon,k}(||P\Delta\partial_t^k v^n(t)||^2 + ||\partial_t^k v^n(t)|| + ||\partial_t^{k-1}v^n(t)||^2), \quad (1.24')$$

$$||P\Delta v^n(t)||^2 \leq c_{\nu,\varepsilon}(||\partial_t v^n(t)||^2 + ||v^n(t)||^2), \quad (1.25)$$

$$||P\Delta\partial_t^k v^n(t)||^2 \leq c_{\nu,\varepsilon,k}(||\partial_t^{k+1}v^n(t)||^2 + ||\partial_t^k v^n(t)||^2 + ||\partial_t^{k-1}v^n(t)||^2). \quad (1.25')$$

Now we are ready to prove the above mentioned a-priori estimates for the Galerkin approximations $v^n(t)$ for $t \in [0, \varepsilon]$:

LEMMA 1.12. *Let $n \in \mathbb{N}$. Then the Galerkin approximation $v^n(t)$ defined by* (1.16) *satisfies for all $t \in [0, \varepsilon]$ the following a-priori estimates:*

$$||v^n(t)||^2 + 2\nu \int_0^t ||\nabla v^n(\tau)||^2 d\tau = ||v^n(0)||^2 \leq ||v_0||^2, \tag{1.26}$$

$$\begin{aligned} ||\nabla v^n(t)||^2 + \nu \int_0^t ||P\Delta v^n(\tau)||^2 d\tau &\leq ||\nabla v^n(0)||^2 + c_{\nu,\varepsilon}||v_0||^2 t \\ &\leq ||\nabla v_0||^2 + c_{\nu,\varepsilon}||v_0||^2 t, \end{aligned} \tag{1.27}$$

$$||\partial_t v^n(t)||^2 + \nu \int_0^t ||\nabla \partial_\tau v^n(\tau)||^2 d\tau \leq ||\partial_t v^n(0)||^2 + c_{\Omega,\nu,\varepsilon}||v_0||^2 t, \tag{1.28}$$

$$||\partial_t v^n(0)||^2 \leq c_{\Omega,\nu,\varepsilon}||P\Delta v_0||^2, \tag{1.29}$$

$$||P\Delta v^n(t)||^2 \leq c_{\Omega,\nu,\varepsilon}||P\Delta v_0||^2. \tag{1.30}$$

Here all appearing constants are independent of n.

PROOF. The estimates (1.26), (1.27), (1.28) follow by integration from (1.19), (1.23) and (1.22). The estimates (1.29) and (1.30) can be obtained from (1.24) and (1.25), using the estimate of Cattabriga in the form

$$||w||^2 \leq c_\Omega ||P\Delta w||^2, \tag{1.31}$$

valid for functions $w \in H_2(\Omega) \cap \mathcal{H}_1(\Omega)$ (compare [2] and [4]). □

In Lemma 1.12 all norm estimates of the Galerkin approximations $v^n(t)$, which are valid uniformly for all $t \in [0, \varepsilon]$, are listed. Due to the regularity of the initial value $v_0 \in H_2(\Omega) \cap \mathcal{H}_1(\Omega)$, higher order estimates uniformly in time cannot be expected.

Nevertheless, higher order regularity statements about the solution of (N_ε^0) can be proved, if norm estimates for higher order derivatives of the functions $v^n(t)$ independent of n are available. This is only possible for $t > 0$ or uniformly for $t \in [\alpha, \varepsilon]$ with $\alpha > 0$:

LEMMA 1.13. *Let $\alpha \in \mathbb{R}$ with $0 < \alpha < \varepsilon$, and let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$. Then for the Galerkin approximation $v^n(t)$ defined by* (1.16) *the following estimates hold true for all $t \in [\alpha, \varepsilon]$:*

$$||\partial_t^k v^n(t)||^2 + 2\nu \int_\alpha^t ||\nabla \partial_\tau^k v^n(\tau)||^2 d\tau \leq K_1, \tag{1.32}$$

$$||\nabla \partial_t^k v^n(t)||^2 + \nu \int_\alpha^t ||P\Delta \partial_\tau^k v^n(\tau)||^2 d\tau \leq K_2, \tag{1.33}$$

$$||P\Delta \partial_t^k v^n(t)||^2 \leq K_3. \tag{1.34}$$

Here the constants K_1, K_2, K_3 depend only on α, ε, ν, k, Ω and on the H_2-norm of the initial value v_0, but not on n.

The proof follows as in [3] by combining mean value theorems with methods of bootstrapping to increase the regularity.

At the end of this subsection we shall prove a continuity statement which is needed for later use. Taking into account (1.26) and (1.27), we find that for the function $v^n(t)$ the inequalities

$$\limsup_{t\searrow 0} ||v^n(t)|| \leq ||v^n(0)||$$

and

$$\limsup_{t\searrow 0} ||\nabla v^n(t)|| \leq ||\nabla v^n(0)||.$$

hold true, respectively. A corresponding statement for the norm $||P\Delta v^n(t)||$ can be obtained from the next lemma:

LEMMA 1.14. *Let $n \in \mathbb{N}$. Then for the Galerkin approximation $v^n(t)$ defined by* (1.16) *for all $t \in [0, \varepsilon]$ the following inequality holds:*

$$||P\Delta v^n(t)||^2 + \frac{2}{\nu}\int_0^t ||\nabla\partial_\tau v^n(\tau)||^2 d\tau \leq ||P\Delta v^n(0)||^2 + Kt$$
$$+ c_{\varepsilon,\nu}\left(v^n(\tau, Y(\cdot)) - v^n(\tau, Y^{-1}(\cdot)), P\Delta v^n(\tau,\cdot)\right)\Big|_{\tau=0}^{\tau=t} \quad (1.35)$$

Here the constant K depends only on Ω, ε, ν and the H_2-norm of the initial value v_0.

PROOF. From (1.17) we obtain by multiplication with $\lambda_i c'_{\text{in}}(t)$, by summing up for $i = 1, \ldots, n$ and by integration from 0 to t the identity

$$\int_0^t ||\nabla\partial_\tau v^n(\tau)||^2 d\tau + \frac{\nu}{2}(||P\Delta v^n(t)||^2 - ||P\Delta v^n(0)||^2)$$
$$= -\int_0^t (Z_\varepsilon^0 v^n(\tau), -P\Delta\partial_\tau v^n(\tau))d\tau.$$

Using partial integration and (1.21) it follows for the right hand side

$$(Z_\varepsilon^0 v^n(\tau), P\Delta v^n(\tau))\Big|_{\tau=0}^{\tau=t} + \int_0^t (Z_\varepsilon^0 \partial_\tau v^n(\tau), -P\Delta v^n(\tau))d\tau$$
$$+\frac{1}{2\varepsilon^2}\int_0^t (v^n(\tau, X) - v^n(\tau, X^{-1}) - v^n(\tau, Y) + v^n(\tau, Y^{-1}), -P\Delta v^n(\tau))d\tau =: \sum_{i=1}^3 s_i.$$

Finally, the terms s_1, s_2, s_3 can be estimated using (1.28) and (1.30):

$$s_1 \leq Kt + c_\varepsilon \left(v^n(\tau, Y) - v^n(\tau, Y^{-1}), P\Delta v^n(\tau)\right)\Big|_{\tau=0}^{\tau=t},$$
$$s_2 \leq Kt,$$
$$s_3 \leq Kt.$$

□

(c) Existence of the solution

Based on the estimates of the Galerkin approximations v^n independent of $n \in \mathbb{N}$ we can prove a first main result:

THEOREM 1.15. *Let $v_0 \in H_2(\Omega) \cap \mathcal{H}_1(\Omega)$ and let X, Y be measure conserving homomorphisms in $\overline{\Omega}$. Then there is a uniquely determined function $v \in C([0,\varepsilon], H_2(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t v \in C([0,\varepsilon], \mathcal{H}_0(\Omega))$ and a uniquely determined function $\nabla p \in C([0,\varepsilon],\ L_2(\Omega))$ as a solution of the equations (N_ε^0). The function $v(t)$ satisfies for all $t \in [0,\varepsilon]$ the energy equation*

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(\tau)||^2 d\tau = ||v_0||^2 \tag{1.36}$$

and the estimates

$$||\nabla v(t)||^2 + \nu \int_0^t ||P\Delta v(\tau)||^2 d\tau \leq ||\nabla v_0||^2 + c_{\nu,\varepsilon}||v_0||^2 t, \tag{1.37}$$

$$||\partial_t v(t)||^2 + \nu \int_0^t ||\nabla \partial_\tau v(\tau)||^2 d\tau \leq ||\partial_t v(0)||^2 + c_{\Omega,\nu,\varepsilon}||v_0||^2 t. \tag{1.38}$$

Moreover, for all $t \in [\alpha, \varepsilon]$, $0 < \alpha < \varepsilon$, the estimates (1.32), (1.33) and (1.34) also hold true, where here the constants on the right hand sides, in particular, depend on α.

PROOF. Let $v^n, n \in \mathbb{N}$ the Galerkin approximation defined by (1.16). Then we obtain from the estimates of Lemma 1.13 using the theorem of Arzela and Ascoli that for every $k \in \mathbb{N}_0$ the sequence $(\partial_t^k v^n)_n$ is relatively compact in $C([\alpha,\varepsilon], \mathcal{H}_1(\Omega))$, $0 < \alpha < \varepsilon$.

Using Lemma 1.12, in the case $k = 0$ even $\alpha = 0$ is allowed. By subsequently $(k = 0, 1, \ldots)$ extracting subsequences we finally obtain a subsequence, denoted by $(v^{\tilde{n}})_{\tilde{n}}$, and a function $\tilde{v} \in C([0,\varepsilon], \mathcal{H}_1(\Omega))$ with $\partial_t^k \tilde{v} \in C([\alpha,\varepsilon], \mathcal{H}_1(\Omega))$ for all $k \in \mathbb{N}$, satisfying

$$\sup_{[0,\varepsilon]} ||\nabla v^{\tilde{n}}(t) - \nabla \tilde{v}(t)|| \xrightarrow{\tilde{n}\to\infty} 0$$

and

$$\sup_{[\alpha,\varepsilon]} ||\nabla \partial_t^k v^{\tilde{n}}(t) - \nabla \partial_t^k \tilde{v}(t)|| \xrightarrow{\tilde{n}\to\infty} 0$$

for every α with $0 < \alpha < \varepsilon$ and every $k \in \mathbb{N}$.

Using the properties of weakly convergent subsequences in Sobolev spaces we find that every bound being independent of n of the functions $v^n(t)$ from Lemma

1.12 and Lemma 1.13 also holds true for the limit function $\tilde{v}(t)$, as well as the estimate (1.35). This means that $\tilde{v}$ is a solution of the equations (N_ε^0) such that the equation

$$\partial_\tau \tilde{v} - \nu P\Delta\tilde{v} = -PZ_\varepsilon^0\tilde{v} \tag{1.39}$$

is satisfied as identity in $L_2(0,t,\mathcal{H}_0(\Omega))$ for all $0 < t < \varepsilon$ with

$$\lim_{t\searrow 0} ||\nabla\tilde{v}(t) - \nabla v_0|| = 0. \tag{1.40}$$

Due to the measure conserving property of the mappings X, Y we find $(Z_\varepsilon^0\tilde{v}(t), \tilde{v}(t)) = 0$ for all $t \in [0,\varepsilon]$, and the energy equation follows from (1.39).

Moreover, the existence of a uniquely determined pressure gradient $\nabla\tilde{p}(t)$ is obtained from the projection theorem (compare [13]).

All the properties derived for the function $\tilde{v}$ are also valid for any other accumulation point, which is obtained by extracting some other different subsequences. Since, due to the linearity of the equations (N_ε^0), such solutions are uniquely determined there is only one accumulation point $v = \tilde{v}$, and the whole sequence of Galerkin approximations v^n converges to v in the corresponding spaces.

Thus for the proof of the theorem it remains to show that the regularity $v \in C([0,\varepsilon], H_2(\Omega))$ holds true. Using the estimate of Cattabriga ([2]), for any function $w \in H_2(\Omega) \cap \mathcal{H}_1(\Omega) (\partial\Omega \in C^2)$ we have

$$||w||_2 \leq c_\Omega ||P\Delta w||. \tag{1.41}$$

From (1.33) we obtain by the Cauchy-Schwarz inequality the strong H_2-continuity of the function $t \to v(t)$ for every $t > 0$:

$$||P\Delta(t+h) - P\Delta v(t)||^2 \leq h \int_t^{t+h} ||P\Delta\partial_\tau v(\tau)||^2 d\tau \leq hK_2 \quad (h > 0).$$

To prove the strong L_2-continuity of the function $t \to P\Delta v(t)$ at time $t = 0$ we firstly show the weak L_2-continuity of this function at $t = 0$ as in ([3]).

From here we quote (see [11])

$$||P\Delta v_0|| \leq \liminf_{t\searrow 0} ||P\Delta v(t)||. \tag{1.42}$$

On the other hand, from (1.35) we obtain

$$\limsup_{t\searrow 0} ||P\Delta v(t)|| \leq ||P\Delta v_0||, \tag{1.43}$$

due to $||P\Delta v^n(0)|| \leq ||P\Delta v_0||$ for $n \in \mathbb{N}$ and the strong L_2-continuity of the functions $t \to v(t, Y)$ and $t \to v(t, Y^{-1})$.

From the continuity of the function $t \to ||P\Delta v(t)||$ in $t = 0$ due to (1.42) and (1.43) finally it follows $v \in C([0,\varepsilon],\ H_2(\Omega))$, and the theorem is proved. □

6. Solutions compatible at initial time $t = 0$

In Section 4 we proved the existence and uniqueness of a solution $v \in C([0,\varepsilon], H_2(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t v \in C([0,\varepsilon], \mathcal{H}_0(\Omega))$ of the problem (N_ε^0).

In this case we assumed the regularity $v_0 \in H_2(\Omega) \cap \mathcal{H}_1(\Omega)$ for the initial value and, in addition, the mappings X, Y to be measure conserving homomorphisms in $\overline{\Omega}$.

Next we want to show that the solution v together with its time derivatives $\partial_t^k v (k \in \mathbb{N})$ of general order is contained in $C([0,\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ for $m \in \{3,4\}$, if, in addition, the mappings X, Y are C^{m-2}-diffeomorphisms in $\overline{\Omega}$.

This statement follows from the estimate of Cattabriga ([2]), which can be applied in Ω for $t > 0$ and $k \in \mathbb{N}_0$ to the system

$$P\Delta\partial_t^k v(t) = \frac{1}{\nu} P\{\partial_t^{k+1} v(t) + \partial_t^k Z_\varepsilon^0 v(t)\} =: R_k(t). \tag{1.44}$$

After that we shall investigate a necessary and sufficient condition for $v \in C([0,\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ with $m \in \{3,4\}$. In particular, we shall see that the assumption $v_0 \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$ without further requirements is not sufficient for the strong H_m-continuity of the solution $t \to v(t)$ of (N_ε^0) uniformly in time (compare [14]).

(a) H_4-regularity for $t > 0$

The next lemma shows regularity properties of the solution $v(t)$ from Theorem 1.15 for $t > 0$. Here no additional assumptions on the initial value v_0 are necessary.

LEMMA 1.16. *Let the assumptions of Theorem* 1.15 *be satisfied, and suppose, in addition, that the mappings in* (N_ε^0) *satisfy* $X, X^{-1}, Y, Y^{-1} \in C^{m-2}(\overline{\Omega})$ *with* $m \in \{3,4\}$. *Then for the solution* v *of* (N_ε^0) *constructed in Theorem* 1.15 *we find* $\partial_t^k v \in C((0,\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ *for all* $k \in \mathbb{N}_0$.

PROOF. Let $t \in (0,\varepsilon]$, $k \in \mathbb{N}_0$, and $m \in \{3,4\}$ be given. Using (1.44) and the estimate of Cattabriga we have to verify $R_k(t) \in H_{m-2}(\Omega)$. For the term $\partial_t^{k+1} v(t)$ this follows from (1.34) using (1.41).

To prove $\partial_t^k Z_\varepsilon^0 v(t) \in H_{m-2}(\Omega)$ we use the representation (1.21) of $\partial_t^k Z_\varepsilon^0 v(t)$ to derive suitable bounds for the terms $||\partial_t^k Z_\varepsilon^0 v(t)||^2$, $||\nabla(\partial_t^k Z_\varepsilon^0 v(t))||^2$, and $||\nabla^2(\partial_t^k Z_\varepsilon^0 v(t))||^2$ here.

Due to the measure conserving property of X, Y and (1.32) it follows

$$||\partial_t^k Z_\varepsilon^0 v(t)||^2 \leq c_{k,\varepsilon}\{||\partial_t^k v(t)||^2 + ||\partial_t^{k-1} v(t)||^2\} \leq \tilde{K}_1, \tag{1.45}$$

where in the case $k = 0$ the norm $||\partial_t^{k-1} \cdot ||$ can be neglected.

To estimate the term $||\nabla(\partial_t^k Z_\varepsilon^0 v(t))||^2$, let a denote some constant satisfying $||\nabla g||_\infty \leq a$ for every mapping $g \in \mathcal{M} = \{X, X^{-1}, Y, Y^{-1}\}$. Because of

$$||\nabla(\partial_t^k v(t,g))|| \leq ||\nabla\partial_t^k v(t)|| \, ||\nabla g||_\infty,$$

from (1.33) we find

$$||\nabla(\partial_t^k Z_\varepsilon^0 v(t))||^2 \leq c_{k,\varepsilon} a^2 \{||\nabla\partial_t^k v(t)||^2 + ||\nabla\partial_t^{k-1} v(t)||^2\} \leq a^2 \tilde{K}_2. \tag{1.46}$$

Here as above, for $k = 0$ the norm $||\nabla\partial_t^{k-1} \cdot ||$ can be neglected.

Now let $m = 4$ and denote by b some constant satisfying $||\nabla^2 g||_\infty \leq b$ for $g \in \mathcal{M}$, in addition. From

$$\begin{aligned} ||\nabla^2(\partial_t^k v(t))|| &= ||\nabla_g^2(\partial_t^k v(t,g)) \cdot (\nabla_g)^2 + \nabla_g \partial_t^k v(t,g) \cdot \nabla^2 g|| \\ &\leq C_\Omega a^2 ||P\Delta\partial_t^k v(t)|| + b||\nabla\partial_t^k v(t)|| \end{aligned}$$

using (1.41), we find by (1.33) and (1.34) the estimate

$$||\nabla^2(\partial_t^k Z_\varepsilon^0 v(t))||^2 \le C_{k,\varepsilon,\Omega}\left\{a^4\{||P\Delta\partial_t^k v(t)||^2 + ||P\Delta\partial_t^{k-1}v(t)||^2\}\right.$$
$$\left. + b^2\{||\nabla\partial_t^k v(t)||^2 + ||\nabla\partial_t^{k-1}v(t)||^2\}\right\} \le a^4\tilde{K}_3 + b^2\tilde{K}_2, \quad (1.47)$$

where for $k = 0$ the norms $|| \cdot \partial_t^{k-1} \cdot ||$ can be neglected.

Collecting the estimates (1.45), (1.46), and (1.47) we obtain $\partial_t^k v(t) \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$ for $m \in \{3, 4\}$, $k \in \mathbb{N}_0$, and $t \in (0, \varepsilon]$, and the asserted continuity for $t > 0$ follows from

$$||\partial^\alpha\partial_t^k(v(t+h) - v(t))||^2 \le \left|h\int_t^{t+h} ||\partial^\alpha\partial_\tau^k v(\tau)||^2 d\tau\right| \quad (h \in \mathbb{R})$$

for any spatial derivative of order $|\alpha| \le m$. □

(b) The compatibility condition at time $t = 0$

The regularity property $v \in C([0, T^*], H_3(\Omega) \cap \mathcal{H}_1(\Omega))$ of a strong solution v of the Navier–Stokes initial value problem (N_0)—here $[0, T^*]$ denotes the maximum existence interval of the strong solution—leads to an over-determined Neumann problem for the pressure $p_0 = p(0)$ at time $t = 0$ (initial pressure). Due to its nonlocal character it is in general not checkable for given data (see [4] and [9]).

For the construction of solutions of the Navier–Stokes system (N_0) with such a high degree of regularity it is necessary that some compatibility conditions are satisfied by the velocity field on the parabolic boundary (see [14]). Similar conditions for the corresponding linear problem can be find in [14] and [7]. The compatibility condition for the problem (N_ε^0) is formulated in the next theorem. It can be satisfied—as shown in the next section—due to a special construction of the initial values (compare Remark 1.7).

THEOREM 1.17. *Let the assumptions of Lemma* 1.16 *be satisfied and assume, in addition,* $v_0 \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$ *for* $m \in \{3, 4\}$. *Let* v *denote the solution of Problem* (N_ε^0) *from Theorem* 1.15. *Then it holds* $v \in C([0, \varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ *if and only if*

$$\nu P\Delta v_{0_{|\partial\Omega}} - \frac{1}{2\varepsilon}P\{v_0 \circ Y - v_0 \circ Y^{-1}\}_{|\partial\Omega} = 0. \quad (1.48)$$

Moreover, from (1.48) *it follows* $\partial_t v \in C([0, \varepsilon], H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega))$, *as well as, in addition,* $\partial_t^2 v \in C([0, \varepsilon], \mathcal{H}_0(\Omega))$ *for* $m = 4$.

REMARK 1.18. *Using the projection* P *in the equations* (N_ε^0), *we obtain at time* $t = 0$

$$\nu P\Delta v_0 - \frac{1}{2\varepsilon}P\{v_0 \circ Y - v_0 \circ Y^{-1}\} = \partial_t v(0). \quad (1.49)$$

Hence for $\partial_t v(0) \in \mathcal{H}_1(\Omega)$ *the condition* (1.48) *is satisfied* (*compare* (1.11) *and Lemma* 1.8).

REMARK 1.19. *The proof of the above theorem uses some statements about the solution* $w = (w_1, w_2, w_3)$ *of the non-stationary Stokes initial boundary value problem*

$$\begin{array}{rcll} \partial_t w - \nu P\Delta w &=& g & (t, x) \in (0, T] \times \Omega \\ w_{|t=0} &=& w_0. & \end{array} \quad (1.50)$$

Here we recall:

(a) For $w_0 \in \mathcal{H}_1(\Omega)$ and $g \in L_2(0,T,\mathcal{H}_0(\Omega))$ there exists a uniquely determined solution w of Problem (1.50), and it holds $w \in C([0,T])$, $\mathcal{H}_1(\Omega)$ with $\partial_t w \in L_2(0,T,\mathcal{H}_0(\Omega))$ (see [7]).

(b) If, in addition, $w_0 \in H_2(\Omega)$ and $\partial_t g \in L_2(0,T,\mathcal{H}_0(\Omega))$, then, in addition, we even find $w \in C([0,T], H_2(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t w \in C([0,T],\mathcal{H}_0(\Omega))$ (see [7]).

Proof of Theorem 1.17. Let $v \in C([0,\varepsilon], H_3(\Omega) \cap \mathcal{H}_1(\Omega))$ be a solution of the problem (N^0_ε). Then it follows $v(0) \in \mathcal{H}_1(\Omega)$ and $\partial_t v(0) \in H_1(\Omega)$. Since $\mathcal{H}_1(\Omega)$ is a closed subspace of $H_1(\Omega)$ we find $\partial_t v(0) \in \mathcal{H}_1(\Omega)$, and (1.48) follows from Remark 1.18.

Let us assume now that the equation (1.48) holds. Differentiate the equation (N^0_ε) with respect to t and consider the resulting equations as a problem of the type (1.50) for $w = \partial_t v$ with $T = \varepsilon, g = -P\partial_t Z^0_\varepsilon v$ and w_0 given by (1.49).

For this problem the assumptions in Remark 1.19 (a) are satisfied: Due to $\partial_t v \in L_2(0,\varepsilon,\mathcal{H}_0(\Omega))$ using (1.39) it follows $g \in L_2(0,\varepsilon,\mathcal{H}_0(\Omega))$, and since $v_0 \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$ due to the assumption we find $w_0 = \partial_t v(0) \in H_{m-2}(\Omega) \cap \mathcal{H}_0(\Omega)$, hence using (1.48) we have $w_0 \in \mathcal{H}_1(\Omega)$. Therefore we quote $w = \partial_t v \in C([0,\varepsilon],\mathcal{H}_1(\Omega))$ with $\partial_t w = \partial_t^2 v \in L_2(0,\varepsilon,\mathcal{H}_0(\Omega))$. This implies $\partial_t g \in L_2(0,\varepsilon,\mathcal{H}_0(\Omega))$, and in the case $m = 4$ also $w = \partial_t v \in ([0,\varepsilon], H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t w = \partial_t^2 v \in C([0,\varepsilon],\mathcal{H}_0(\Omega))$, using Remark 1.19 (b).

Due to

$$\nu P\Delta v = \partial_t v + PZ^0_\varepsilon v \in C([0,\varepsilon], H_{m-2}(\Omega) \cap \mathcal{H}_0(\Omega))$$

we finally obtain $v \in C([0,\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ with help of Cattabriga's estimate. □

7. Global solutions

In the first theorem of this section we prove the existence of a solution v to the problem $(\tilde{N}_\varepsilon)$ formulated in §2 with $v \in C([0,T], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ for $m \in \{3,4\}$.

This high degree of regularity can be proved since all compatibility conditions can be satisfied: At time $t_0 = 0$ we use a special construction of the initial values as shown in Section 3 (we prescribe $\partial_t v(0) = f \in \mathcal{H}_1(\Omega)$) as initial acceleration), and at time $t_k > 0 (k = 1,\dots,N-1)$ we use the continuity of the mappings $Z_\varepsilon v(\cdot,x)$ for $x \in \overline{\Omega}$ (the function $\partial_t v(t_k)$ as final value of the k-th partial problem coincides with the initial value of the next partial problem).

Moreover we show that the solution is uniquely determined if the initial values are constructed as indicated in Section 3, and that the solution satisfies the energy equation.

Theorem 1.20. *Let $f \in H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega)$ for $m \in \{3,4\}$. With f and some function $\overline{v} \in C([-T,0], H_m(\Omega) \cap \mathcal{H}_2(\Omega))$ let the initial values for the problem $(\tilde{N}_\varepsilon)$ be constructed uniquely as indicated in Section 3. Then there exists a uniquely determined function $v \in C([0,T], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t v \in C([0,T], H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega))$ and a uniquely determined function $\nabla p \in C([0,T], H_{m-2}(\Omega))$ as solution of the problem $(\tilde{N}_\varepsilon)$.*

Proof. Using v and $\overline{v}$, first we construct the initial mappings $Y := X_{t_{-1},t_{-2}}$, $Y^{-1} := X_{t_{-2},t_{-1}}$, v_0, $X := X_{t_0,t_{-1}}$ and $X^{-1} := X_{t_{-1},t_0}$ as done in Section 3.

Then, with these functions as given data, we consider the problem $(\tilde{N}_\varepsilon)$ restricted to the interval $[0,\varepsilon]$ as a problem of the type (N_ε^0).

Due to Section 1 and Lemma 1.8 all the assumptions of Theorem 1.17 are fulfilled, and, following Remark 1.18, due to $f \in \mathcal{H}_1(\Omega)$ also the compatibility condition (1.48) is satisfied. It follows that there is a uniquely determined function v^0 as solution of (N_ε^0) (and thus of $(\tilde{N}_\varepsilon)$, restricted to $[0,\varepsilon]$) with the following regularity properties:

$v^0 \in C([0,\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t v \in C([0,\varepsilon], H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega))$ and for $m = 4$ also $\partial_t^2 v \in C([0,\varepsilon], \mathcal{H}_0(\Omega))$; moreover we have $v^0(0) = v_0$ and $\partial_t v^0(0) = f$.

This solution procedure for the problem $(\tilde{N}_\varepsilon)$ can be repeated: First we construct the mappings X_{t_1,t_0} and X_{t_0,t_1} from the function v^0, since they are needed in the next subinterval $[\varepsilon, 2\varepsilon]$ (according to Lemma 1.1, this is possible due to $v^0 \in C([0,\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega)))$, and then we consider the problem $(\tilde{N}_\varepsilon)$ restricted to the subinterval $[\varepsilon, 2\varepsilon]$ as problem

$$\begin{array}{rcll} \partial_t v - \nu\Delta v + \nabla p &=& -Z_\varepsilon^1 v & \\ \nabla \cdot v &=& 0 & (t,x) \in (\varepsilon, 2\varepsilon] \times \Omega \\ v_{|\partial\Omega} &=& 0 & \\ v_{|t=\varepsilon} &=& v^0(\varepsilon), & \end{array} \qquad (N_\varepsilon^1)$$

where $Z_\varepsilon^1 v$ is defined by 1.6.

We denote the solution of this problem by v^1. This solution has the same properties as the solution v^0 of the problem (N_ε^0): Due to $v^1(\varepsilon) = v^0(\varepsilon)$, $Z_\varepsilon^1 v^1(\varepsilon) = Z_\varepsilon^0 v^0(\varepsilon)$ and the unique solvability of Problem (1.11) we find $\partial_t v^1(\varepsilon) = \partial_t v^0(\varepsilon) \in H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega)$. Hence the compatibility condition corresponding to (1.48) for the problem (N_ε^1) at time $t_1 = \varepsilon$ is satisfied, too. It follows that the solution v^1 of this problem is uniquely determined, and we have $v^1 \in C([\varepsilon, 2\varepsilon], H_m(\Omega) \cap \mathcal{H}_1(\Omega))$ with $\partial_t v^1 \in C([\varepsilon, 2\varepsilon], H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega))$, and for $m = 4$ also $\partial_t^2 v^1 \in C([\varepsilon, 2\varepsilon], \mathcal{H}_0(\Omega))$. Moreover, we find $v^1(\varepsilon) = v^0(\varepsilon)$ and $\partial_t v^1(\varepsilon) = \partial_t v^0(\varepsilon)$, but not in general $\partial_t^2 v^1(\varepsilon) = \partial_t^2 v^0(\varepsilon)$ $(m = 4)$.

By repeating this solution procedure we finally obtain functions v^k defined on subsequent subintervals $[t_k, t_{k-1}]$, which we can put together to a uniquely determined solution v of the problem $(\tilde{N}_\varepsilon)$ in such a way that all properties asserted in Theorem 1.20 are satisfied. □

To investigate convergence in the next section we need the regularized problem (N_ε), which is obtained from $(\tilde{N}_\varepsilon)$ by changing the initial condition: Let $T > 0, N \in \mathbb{N}(N \geq 2)$, $\varepsilon := \frac{T}{N}$ and $t_k = k \cdot \varepsilon$ for $k = 0, \pm 1, \ldots, \pm N$ as in Problem $(\tilde{N}_\varepsilon)$.

Construct a velocity field $v = (v_1, v_2, v_3)$ and some pressure function p as a solution of the problem

$$\begin{array}{rcll} \partial_t v - \nu\Delta v + \nabla p &=& -Z_\varepsilon v & \\ \nabla \cdot v &=& 0 & (t,x) \in (0, T] \times \Omega \\ v_{|\partial\Omega} &=& 0 & \\ v_{|t=0} &=& v_0(\varepsilon), & \end{array} \qquad (N_\varepsilon)$$

where $Z_\varepsilon v$ is defined as in 1.6.

Theorem 1.20 leads to a statement about the solvability of the problem (N_ε):

COROLLARY 1.21. *Let the assumptions of Theorem* 1.20 *be satisfied, and let the initial data for the problem* (N_ε) *coincide with the initial data of problem* $(\tilde{N}_\varepsilon)$ *from Theorem* 1.20. *Then, given* $f \in H_{m-2}(\Omega) \cap \mathcal{H}_1(\Omega)$ *for* $m \in \{3, 4\}$ *as initial value in Problem* $(\tilde{N}_\varepsilon)$, *there is a uniquely determined function* $v_0 \in H_m(\Omega) \cap \mathcal{H}_1(\Omega)$ *as the solution of Problem* 1.11. *Using this function* v_0 *as initial value the problem* (N_ε) *is uniquely solvable.*

Its solution v *coincides with the solution of the problem* $(\tilde{N}_\varepsilon)$ *from Theorem 1.20 and satisfies in* $t \in [0, T]$ *the energy equation*

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(\tau)||^2 d\tau = ||v_0||^2. \tag{1.51}$$

PROOF. We only have to show (1.51). This follows from (1.36), since for $t \in [t_k, t_{k+1}]$ and $k = 0, 1, \dots, N-1$ we find

$$||v(t)||^2 - ||v(t_k)||^2 + 2\nu \int_{t_k}^t ||\nabla v(\tau)||^2 d\tau + \sum_{j=1}^k \{||v(t_j)||^2 - ||v(t_{j-1})||^2 + 2\nu \int_{t_{j-1}}^{t_j} |\nabla v(\tau)||^2 d\tau\} = 0.$$

□

The properties

$$v \in C([0, T], H_4(\Omega) \cap \mathcal{H}_1(\Omega)),$$
$$\partial_t v \in C([0, T], H_2(\Omega) \cap \mathcal{H}_1(\Omega))$$

for the solution v of the problem $(\tilde{N}_\varepsilon)$ and (N_ε), respectively, represent the highest degree of regularity—formulated in standard Sobolev spaces of integer order—which is possible without using additional (coupled) compatibility conditions at time $t = 0$ (see [14]). These properties imply, in particular, that the solution v is continuously differentiable one time with respect to t and two times with respect to x in the open cylinder $\Omega_T := (0, T) \times \Omega$, and the corresponding derivatives can be continuously extended to the parabolic boundary of Ω_T. In this sense the function v is a classical solution of the problem $(\tilde{N}_\varepsilon)$, and we finish the regularity investigation of this problem at this stage.

8. Global convergence to a weak solution of (N_0)

In this section we suppose that the initial data for the problems $(\tilde{N}_\varepsilon)$ and (N_ε) are constructed as in Section 3, and that the initial value v_0 given in the problem (N_ε) has been determined as the solution of the problem (1.11) from the initial value f of the problem $(\tilde{N}_\varepsilon)$.

Under these assumptions we showed in the last Section 6, that the problems $(\tilde{N}_\varepsilon)$ and (N_ε) are globally uniquely solvable and that their solutions coincide.

In the following we investigate the behavior of the solution of (N_ε), if for fixed $T > 0$ the step size $\varepsilon = \frac{T}{N}$ for $N \to \infty$ tends to zero.

To do so, we set $\varepsilon_N := \frac{T}{N}$ for $2 \le N \in \mathbb{N}$ and denote the solution of the problem (N_{ε_N}) from Corollary 1.21 by v^N. The corresponding initial value, depending on ε_N as well, is denoted by v_0^N. Moreover, let $\tilde{v}^N$ be the function defined by (1.12), necessary for the construction of the initial values. Here we start with some given function $\overline{v}^N$ for $N = 2$ as in Section 3, and then we choose $\overline{v}^N := \tilde{v}^{N-1}$ for all $N \ge 3$:

The reason for this choice of the functions $\overline{v}^N$ will be explained in the next section. All statements of this section also remain true if we simply choose $\overline{v}^N := \overline{v}^2 =: \overline{v}$.

The definition of a weak solution of the Navier–Stokes problem (N_0) in the sense of Hopf (compare [5], [13], [10], [7]) is given now:

Definition 1.22. Let $v_0 \in \mathcal{H}_0(\Omega)$. A function $v \in L_2(0, T, \mathcal{H}_1(\Omega)) \cap L_\infty(0, T, \mathcal{H}_0(\Omega))$ is a weak solution of the problem (N_0) with initial value v_0, if

$$v : [0, T] \longrightarrow \mathcal{H}_0(\Omega) \quad \text{is weakly continuous,} \tag{1.52}$$

$$\lim_{t \to 0} ||v(t) - v_0|| = 0, \tag{1.53}$$

and for all test functions $\phi \in C_{0,\sigma}^\infty(\Omega_T)$ we have

$$\int_0^T \{(v(t), \partial_t \phi(t)) - \nu(\nabla v(t), \nabla \phi(t))\} dt = -\int_0^T (v(t) \cdot \nabla \phi(t), v(t)) dt. \tag{1.54}$$

Remark 1.23. *It can be shown that every function $v \in L_2(0, T, \mathcal{H}_1(\Omega))$ satisfying* (1.53) *and* (1.54) *already represents a weak solution of the problem* (N_0) *with initial value $v_0 \in \mathcal{H}_0(\Omega)$, according to Definition* 1.22 *(see [13]).*

The following theorem states the main convergence result and shows that only the central total difference quotient leads to an energy conserving regularizing approximation for the convective term of the Navier–Stokes system (compare Remark 1.3).

Theorem 1.24. *Let $T \in \mathbb{R}(T > 0)$ and $N \in \mathbb{N}(N \ge 2)$. Setting $\varepsilon_N := \frac{T}{N}$ let v^N denote the uniquely determined solution of the problem (N_{ε_N}) with initial value v_0^N from Corollary* 1.21*. Then there is a convergent subsequence $(v^{N_k})_k$ of the sequence $(v^N)_N$ of the solutions with limit function v and a convergent subsequence $(v_0^{N_k})_k$ of the sequence $(v_0^N)_N$ of the corresponding initial values with limit function v_0 such that v is a weak solution of the Navier–Stokes problem (N_0) with initial value $v_0 \in \mathcal{H}_0(\Omega)$ and satisfies for $t \in [0, T]$ the energy inequality*

$$||v(t)||^2 + 2\nu \int_0^t ||\nabla v(\tau)||^2 d\tau \le ||v_0||^2. \tag{1.55}$$

To prove Theorem 1.24 we need some estimates of the regularized solutions v^N and their data independent of N. These estimates will be established in the next two lemmata.

LEMMA 1.25. *Let the assumptions of Theorem* 1.24 *be satisfied. Let* f *denote the initial value of the problem* $(\tilde{N}_{\varepsilon_N})$, *and let* $\tilde{v}^N$ *denote the function defined by* (1.12) *and constructed according to Section 3 from* $\overline{v}$. *Then the following estimates hold true independent of* $N \in \mathbb{N}(N \geq 2)$:

$$||v_0^N||^2 \leq c_{\Omega,\nu}||f||^2, \tag{1.56}$$

$$||\nabla v_0^N||^2 \leq \tilde{c}_{\Omega,\nu}||f||^2, \tag{1.57}$$

as well as for all $t \in [0,T]$

$$||v^N(t)||^2 + 2\nu \int_0^t ||\nabla v^N(\tau)||^2 d\tau = ||v_0^n||^2 \leq c_{\Omega,\nu}||f||^2, \tag{1.58}$$

and for all $t \in [-T,0]$

$$||\tilde{v}^N(t)|| \quad \leq \quad \max\left\{ \sup_{s\in[-T,0]} ||\overline{v}(s)||, c_{\Omega,\nu}||f|| \right\}, \tag{1.59}$$

$$||\nabla\tilde{v}^N(t)|| \quad \leq \quad \max\left\{ \sup_{s\in[-T,0]} ||\nabla\overline{v}(s)||, \tilde{c}_{\Omega,\nu}||f|| \right\} \tag{1.60}$$

PROOF. The estimates (1.56) and (1.57) follow using the inequality of Poincaré due to the measure conserving property of the mappings $X_{.,.}$ from (1.11). The estimate (1.58) is obtained from (1.51) and (1.59), and (1.60) follows from (1.12).

LEMMA 1.26. *Let* v^N *denote the solution of the problem* (N_{ε_N}) *from Theorem* 1.24, *and let* $\mathcal{V} := \{a_i | i \in \mathbb{N}\}$ *be a complete orthonormal system in* $\mathcal{H}_0(\Omega)$. *Then for every* $i \in \mathbb{N}$ *we have the estimate*

$$|(Z_{\varepsilon_N} v^N(t), a_i)| \leq K_i, \tag{1.61}$$

where the constant K_i *does not depend on* $N \in \mathbb{N}(N \geq 2)$ *and not on* $t \in [0,T]$.

PROOF. Let $i, N \in \mathbb{N}(N \geq 2)$ and $t \in [0,T]$. For simplification we set $\varepsilon := \varepsilon_N$ and obtain from the measure conserving property of the mappings $X_{.,.}$ analogously to partial integration the identity

$$(Z_\varepsilon v^N(t), a_i) = -(Z_\varepsilon a_i, v^N(t)). \tag{1.62}$$

Here $Z_\varepsilon a_i$ for $t \in [t_k, t_{k+1}]$ and $k = 0, 1, \ldots, N-1$ is defined by

$$Z_\varepsilon a_i = \frac{t-t_k}{2\varepsilon^2}\{a_i \circ X_{t_k,t_{k-1}} - a_i \circ X_{t_{k-1},t_k}\} + \frac{t_{k+1}-t}{2\varepsilon^2}\{a_i \circ X_{t_{k-1},t_{k-2}} - a_i \circ X_{t_{k-2},t_{k-1}}\}. \tag{1.63}$$

By a well-known density argument we can choose $\mathcal{V} \subset C^\infty_{0,\sigma}(\Omega)$. Since the functions $X_{.,.}$ defined by (1.5) have been constructed from the solution of an initial value problem of type (1.4), we obtain, setting for abbreviation

$$X_k := X_{t_k,t_{k-1}} \qquad (k = 1, 0, \ldots, N-1), \tag{1.64}$$

for every $x \in \overline{\Omega}$ the following representation:

$$a_i \circ X_k(x) - a_i \circ X_k^{-1}(x) = a_i \circ X_k(x) - a_i + a_i - a_i \circ X_k^{-1}(x)$$

$$= \int_{t_{k-1}}^{t_k} \{\partial_\tau X(\tau, t_{k-1}, x) \cdot \nabla_X a_i(X(\tau, t_{k-1}, x)) + \partial_\tau X(\tau, t_k, x) \cdot \nabla a_i(X(\tau, t_k, x))\} d\tau$$

$$= \int_{t_{k-1}}^{t_k} \{[v^N(\tau) \cdot \nabla a_i] \circ X(\tau, t_{k-1}, x) + [v^N(\tau) \cdot \nabla a_i] \circ X(\tau, t_k, x)\} d\tau.$$

Here for $k \in \{-1, 0\}$, due to the construction of the initial values according to Section 3, the function v^N has to be replaced by $\tilde{v}^N$. From

$$|a_i \circ X_k(x) - a_i \circ X_k^{-1}(x)| \leq 2\varepsilon \sup_{s_1, s_2 \in [0,T]} |[v^N(s_1) \cdot \nabla a_i] \circ X(s_1, s_2, x)|$$

for $k \in \{1, 2, \ldots, N-1\}$ and from

$$|a_i \circ X_k(x) - a_i \circ X_k^{-1}(x)| \leq 2\varepsilon \sup_{s_1, s_2 \in [-T,0]} |[\tilde{v}^N(s_1) \cdot \nabla a_i] \circ X(s_1, s_2, x)|$$

for $k \in \{-1, 0\}$ it follows by (1.58), (1.59), and (1.63) that the estimate

$$||Z_\varepsilon a_i|| \leq \tilde{K}_i, \tag{1.65}$$

holds true. Here the constant does not depend on ε and thus not on N. The assertion now follows using (1.58) from (1.62), and the lemma is proved. □

Due to (1.61) we obtain by projecting the problem (N_{ε_N}) onto the subspace of $\mathcal{H}_0(\Omega)$ spanned by $a_i \in \mathcal{V}$ the estimate

$$\left| \frac{d}{dt}(v^N(t), a_i) \right| = |(\partial_t v^N(t), a_i)| \leq \overline{K}_i, \tag{1.66}$$

where the constant $\overline{K}_i$, $i \in \mathbb{N}$, is independent of $N \in \mathbb{N}(N \geq 2)$ and $t \in [0, T]$. It depends only on Ω, ν, and the data $f, \overline{v}$, and the basis function $a_i \in \mathcal{V}$, where we may assume $\mathcal{V} \subset C_{0,\sigma}^\infty(\Omega)$ using a density argument, as already mentioned above.

As in [5] we obtain from (1.58) and (1.66)

Lemma 1.27. *Let the assumptions of Theorem* 1.24 *be satisfied. Then there exists a weakly continuous function* $v : [0, T] \to \mathcal{H}_0(\Omega)$ *with*

$$v \in L_2(0, T, \mathcal{H}_1(\Omega)) \cap L_\infty(0, T, \mathcal{H}_0(\Omega))$$

and a subsequence $(v^{N_k})_k$ *of the sequence of the solutions of the problems* (N_{ε_N}) *with the following properties: For* $t \in [0, T]$ *the sequence* $(v^{N_k}(t))_k$ *converges weakly in* $\mathcal{H}(\Omega)$ *to the limit* $v(t)$*, and the sequence* $(v^{N_k})_k$ *converges weakly to* v *in* $L_2(0, T, \mathcal{H}_1(\Omega))$ *and strongly in* $L_2(0, T, \mathcal{H}_0(\Omega))$.

The following theorem shows that the properties of the subsequence $(v^{N_k})_k$ from Lemma 1.27 are already sufficient to proceed to the limit in the nonlinear convective term of the Navier–Stokes equations:

THEOREM 1.28. *Let $\mathcal{V} := \{a_i | i \in \mathbb{N}\}$ denote a complete orthonormal system in $\mathcal{H}_0(\Omega)$. Then for the convergent subsequence $(v^{N_k})_k$ with limit function v according to Lemma 1.27 for all $i \in \mathbb{N}$ the following identity holds true:*

$$\lim_{k\to\infty} \int_0^T (Z_{\varepsilon_{N_k}} a_i, v^{N_k}(t))dt = \int_0^T (v(t) \cdot \nabla a_i, v(t))dt. \tag{1.67}$$

PROOF. For simplification we set $\varepsilon := \varepsilon_{N_k}$, $N := N_k$, $a := a_i$, $v^N(s) := \tilde{v}^N(s)$ for $s \in [-T, 0]$ and write (1.67) in the form

$$\lim_{N\to\infty} \int_0^T \{(Z_\varepsilon a, v^N(t)) - (v(t) \cdot \nabla a, v(t))\}dt = 0. \tag{1.68}$$

We prove (1.68) using a decomposition of the integrand $I^N(t)$:

$$I^N(t) = (Z_\varepsilon a, v^N(t) - v(t)) + (Z_\varepsilon a - v(t) \cdot \nabla a, v(t)) =: \tilde{S}_1^N(t) + \tilde{S}_2^N(t).$$

Due to (1.63) and the notation (1.64) we find

$$\begin{aligned}
\tilde{S}_2^n(t) = \frac{t-t_k}{2\varepsilon} &\left(\frac{1}{\varepsilon}\{a \circ X_k - a\} - v(t) \cdot \nabla a, v(t)\right) \\
&+ \frac{t-t_k}{2\varepsilon} \left(\frac{1}{\varepsilon}\{a - a \circ X_k^{-1}\} - v(t) \cdot \nabla a, v(t)\right) \\
&+ \frac{t_{k+1}-t}{2\varepsilon} \left(\frac{1}{\varepsilon}\{a \circ X_{k-1} - a\} - v(t) \cdot \nabla a, v(t)\right) \\
&+ \frac{t_{k+1}-t}{2\varepsilon} \left(\frac{1}{\varepsilon}\{a - a \circ X_{k-1}^{-1}\} - v(t) \cdot \nabla a, v(t)\right) =: \sum_{j=1}^4 M_j^N(t).
\end{aligned}$$

The term $M_1^N(t)$ is decomposed again ($M_2^N(t)$ to $M_4^N(t)$ analogously):

$$\begin{aligned}
M_1^N(t) &= \frac{t-t_k}{2\varepsilon} \left(\frac{1}{\varepsilon} \int_{t_{k-1}}^{t_k} \partial_\tau X(\tau, t_{k-1}, \cdot) \cdot \nabla a(X(\tau, t_{k-1}, \cdot))d\tau - v(t) \cdot \nabla a, v(t)\right) \\
&= \frac{t-t_k}{2\varepsilon^2} \int_{t_{k-1}}^{t_k} ([v^N(\tau) \cdot \nabla a] \circ X(\tau, t_{k-1}, \cdot) - v(t) \cdot \nabla a, v(t))d\tau \\
&= \frac{t-t_k}{2\varepsilon^2} \int_{t_{k-1}}^{t_k} ([v^N(\tau) - v^N(t)] \cdot \nabla a, v(t) \circ X(t_{k-1}, \tau, \cdot))d\tau \\
&+ \frac{t-t_k}{2\varepsilon^2} \int_{t_{k-1}}^{t_k} ([v^N(t) - v(\tau)] \cdot \nabla a, v(t) \circ X(t_{k-1}, \tau, \cdot))d\tau \\
&- \frac{t-t_k}{2\varepsilon^2} \int_{t_{k-1}}^{t_k} (v(t) \cdot \nabla a, v(t) - v(t) \circ X(t_{k-1}, \tau, \cdot))d\tau =: \sum_{j=1}^3 s_j^N(t).
\end{aligned}$$

The integrals of the summands $\tilde{S}_1^N(t)$ and $s_1^N(t)$, $s_2^N(t)$, $s_3^N(t)$ will be estimated now. In the following, all appearing constants $K_1, K_2, \ldots$ do not depend on N.

$\underline{\tilde{S}_1^N(t)}$: Using (1.65) we find that there is a bound for $||Z_\varepsilon a||$ not depending on $t \in [0,T]$ and not depending on ε. This implies

$$\left|\int_0^T \tilde{S}_1^N(t)dt\right| \le \int_0^T ||Z_\varepsilon a|| ||v^N(t) - v(t)||dt \le K_1 \left(\int_0^T ||v^N(t) - v(t)||^2 dt\right)^{\frac{1}{2}},$$

and using the strong convergence in $L_2(0,T,\mathcal{H}_0(\Omega))$ we obtain

$$\lim_{N\to\infty} \left|\int_0^T \tilde{S}_1^N(t)dt\right| = 0.$$

$\underline{s_2^N(t)}$: Due to $v \in L_2(0,T,\mathcal{H}_1(\Omega))$ according to Lemma 1.27 it follows from the estimate

$$\left|\int_0^T s_2^N(t)dt\right| \le \frac{1}{2}\int_0^T ||v^N(t) - v(t)|| ||\nabla a||_\infty ||v(t)||dt$$

$$\le K_2 \left(\int_0^T ||v^N(t) - v(t)||^2 dt\right)^{\frac{1}{2}}$$

the same behavior as above for $\tilde{S}_1^N(t)$:

$$\lim_{N\to\infty} \left|\int_0^T s_2^N(t)dt\right| = 0.$$

$\underline{s_1^N(t)}$: For every $t \in [0,T]$ the function $v(t)$ is the weak limit of $v^N(t)$ in $\mathcal{H}_0(\Omega)$ (see Lemma 1.27). Hence also the weak limit $v(t)$ satisfies the estimate (1.58), and we find

$$\left|\int_0^T s_1^N(t)dt\right| = \left|\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} s_1^N(t)dt\right|$$

$$\le \frac{1}{2\varepsilon}\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} ||v^N(\tau) - v^N(t)||\, ||\nabla a||_\infty ||v(t)||d\tau dt$$

$$\le \frac{K_3}{\varepsilon}\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{t_{k-1}}^{t_k} ||v^N(\tau) - v^N(t)||d\tau dt.$$

Here we have $\tau \leq 0$ if $k = 0$. This summand is estimated separately: Due to (1.58) and (1.59) we find

$$\frac{K_3}{\varepsilon}\int_0^{\varepsilon}\int_{-\varepsilon}^{0}||v^N(\tau) - v^N(t)||d\tau dt \leq \frac{K_3}{\varepsilon}\int_0^{\varepsilon}\int_{-\varepsilon}^{0}\{||v^N(\tau)|| + ||v^N(t)||\}d\tau dt \leq \frac{K_4}{N}.$$

The rest of the above sum can be treated with the inequality of Friedrichs (see [11]): Let $\delta > 0$ be fixed. Then there is some number $M_\delta \in \mathbb{N}$ with

$$||v^N(\tau) - v^N(t)|| \leq \sum_{j=1}^{M_\delta}|(v^N(\tau) - v^N(t), a_j)| + \delta\{||\nabla v^n(\tau)|| + ||\nabla v^n(t)||\}$$
$$=: g_1^N(\tau, t) + \delta g_2^N(\tau, t).$$

In the first term g_1^N we have $\tau, t \geq 0$. Using (1.66) it follows

$$|(v^N(\tau) - v^N(t), a_j)| \leq \overline{K}_j|\tau - t| \leq 2\overline{K}_j\varepsilon,$$

and we obtain for the integral over g_1^N the estimate

$$\frac{K_5}{\varepsilon}\left(\sum_{j=1}^{M_\delta}\overline{K}_j\right)N\varepsilon^3 = K_5\left(\sum_{j=1}^{M_\delta}\overline{K}_j\right)\frac{T^2}{N}.$$

Thus it remains to show that the integral over g_2^N remains bounded as $N \to \infty$.

Due to

$$\frac{K_3}{\varepsilon}\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}\int_{t_{k-1}}^{t_k}\{||\nabla v^N(\tau)|| + ||\nabla v^N(t)||\}d\tau dt$$
$$\leq K_3\sum_{k=1}^{N-1}\left\{\int_{t_k}^{t_{k+1}}||\nabla v^N(t)||dt + \int_{t_{k-1}}^{t_k}||\nabla v^N(\tau)||d\tau\right\}$$
$$\leq 2K_3\int_0^T||\nabla v^N(t)||dt \leq K_6\left(\int_0^T||\nabla v^N(t)||^2dt\right)^{\frac{1}{2}} \leq K_7$$

according to (1.58), also the limit procedure

$$\lim_{n\to\infty}\left|\int_0^T s_1^N(t)dt\right| = 0$$

holds true.

$\underline{s_3^N(t)}$: Let $x \in \overline{\Omega}$ and $\tau \in [t_{k-1}, t_k]$ for $k \in \{0, 1, \ldots, N-1\}$. From $x = X(\tau, \tau, x)$ we find

$$v(t, x) - v(t, X(t_{k-1}, \tau, x)) = \int_{t_{k-1}}^{\tau}[v^N(\sigma)\cdot\nabla v(t)] \circ X(\sigma, \tau, x)d\sigma.$$

From the Hölder inequality and the Sobolev imbedding theorem it follows

$$\begin{aligned}
|s_3^N(t)| &\leq \frac{t-t_k}{2\varepsilon^2}\left|\int\limits_{t_{k-1}}^{t_k}\int\limits_{t_{k-1}}^{\tau}(v(t)\cdot\nabla a,[v^N(\sigma)\cdot\nabla v(t)]\circ X(\sigma,\tau,\cdot))d\sigma d\tau\right| \\
&\leq \frac{1}{2\varepsilon}\int\limits_{t_{k-1}}^{t_k}\int\limits_{t_{k-1}}^{\tau}||v(t)||_{0,6}||\nabla a||_\infty||v^N(\sigma)||_{0,3}||\nabla v(t)||d\sigma d\tau \\
&\leq K_7||\nabla v(t)||^2\cdot\int\limits_{t_{k-1}}^{t_k}||\nabla v^N(\sigma)||d\sigma \\
&\leq K_z\sqrt{\varepsilon}||\nabla v(t)||^2\left(\int\limits_{-T}^{+T}||\nabla v^N(\sigma)||^2d\sigma\right)^{\frac{1}{2}}\leq K_8\sqrt{\varepsilon}||\nabla v(t)||^2.
\end{aligned}$$

Here the last estimate is due to (1.58) and (1.60), and using (1.66) also the last limit transition is proved:

$$\lim_{N\to\infty}\left|\int\limits_0^T s_3^N(t)dt\right|\leq\lim_{N\to\infty}\frac{K_g}{\sqrt{N}}\cdot\int\limits_0^T||\nabla v(t)||^2dt=0.$$

After these preparations we are now ready to prove that the function v from Lemma 1.27 satisfies the assertions of Theorem 1.24:

Proof of Theorem 1.24: Using (1.57) and the compactness of the imbedding

$$\mathcal{H}_1(\Omega)\hookrightarrow\mathcal{H}_0(\Omega)$$

we find without any restriction of generality that the subsequence $(v_0^{N_k})_k$ of the initial values of the solutions v^{N_k} from Lemma 1.27 converges to some function $v_0\in\mathcal{H}_0(\Omega)$ strongly in $\mathcal{H}_0(\Omega)$. From here we quote with the weak convergence according to Lemma 1.27 that the limit function v satisfies the energy inequality: For all $t\in[0,T]$ we have

$$\begin{aligned}
||v(t)||^2+2\nu\int\limits_0^t||\nabla v(\tau)||^2d\tau &\leq\liminf_{k\to\infty}||v^{N_k}(t)||^2+2\nu\liminf_{k\to\infty}\int\limits_0^t||\nabla v^{N_k}(\tau)||^2d\tau \\
&\leq\lim_{k\to\infty}||v_0^{N_k}-v_0+v_0||^2=||v_0||^2.
\end{aligned}$$

Together with the weak continuity of the function $t\to v(t)$ in $\mathcal{H}_0(\Omega)$ at time $t=0$ it follows the strong continuity of this function at time $t=0$ as in (1.42), (1.43), hence (1.53) holds true. It remains to prove (1.54), where we can restrict our considerations to test functions of the form $\phi_i=\varphi\cdot a_i$ with $\varphi\in C_0^\infty((0,T))$ and $a_i\in\mathcal{V}\subset C_{0,\sigma}^\infty(\Omega)$, using again a density argument. To do so, we multiply the first equation of the problem $(N_{\varepsilon_{N_k}})$ by φ_i, integrate over Ω_T and obtain with help of partial integration

$$\int_0^T \{(v^{N_k(t),a_i})\varphi'(t) - \nu(\nabla v^{N_k}(t), \nabla a_i)\varphi(t)\}dt = -\int_0^T (v^{N_k}(t)\cdot\nabla a_i, v^{N_k}(t))\varphi(t)dt.$$

Finally, using the weak convergence of the subsequence in $L_2(0,T,\mathcal{H}_0(\Omega))$ as well as in $L_2(0,T,\,\mathcal{H}_1(\Omega))$ to the limit function v, respectively, and Theorem 1.28 we obtain (1.54) as $k\to\infty$, and the theorem is proved. □

9. Local strong convergence

Let $T>0, N\in\mathbb{N}(N\geq 2), \varepsilon_N := \frac{T}{N}$, and let v^N denote the solution of the problem (N_{ε_N}) with initial value $v_0^N\in H_m(\Omega)\cap\mathcal{H}_1(\Omega)$ for $m\in\{3,4\}$. According to Corollary 1.21, the function v^N coincides with the solution of the problem $(\tilde{N}_{\varepsilon_N})$ with initial value $f\in H_{m-2}(\Omega)\cap\mathcal{H}_1(\Omega)$.

In the last section we have shown that the sequences $(v^N)_N$ and $(v_0^N)_N$ have accumulation points v and v_0, respectively, such that v is a weak solution of the problem (N_0) with initial value $v_0\in\mathcal{H}_0(\Omega)$.

The next theorem describes local properties of such a pair of accumulation points.

THEOREM 1.29. *Let $T>0, N\in\mathbb{N}(N\geq 2)$ and $\varepsilon_N := \frac{T}{N}$. Let the assumptions of Theorem* 1.24 *be satisfied, and let $f\in H_{m-2}(\Omega)\cap\mathcal{H}_1(\Omega)$ for $m\in\{3,4\}$ denote the initial value of the problem $(\tilde{N}_{\varepsilon_N})$.*
Then there exists some number $T^*=T^*(\Omega,\nu,f)$ with $0<T^*\leq T$ such that the following statements hold true: Every problem (N_0) from Theorem 1.24 is uniquely solvable in $[0,T^*]$, and for its solution v we find

$$v\in C([0,T^*],H_m(\Omega)\cap\mathcal{H}_1(\Omega)), \partial_t v\in C([0,T^*],H_{m-2}(\Omega)\cap\mathcal{H}_1(\Omega)),$$

and for $m=4$ even $\partial_t^2 v\in C([0,T^*],\mathcal{H}_0(\Omega))$. Moreover we have

$$\partial_t v(0) = f. \tag{1.69}$$

To prove this theorem we consider first in Ω the stationary nonlinear Navier–Stokes boundary value problem

$$\nu P\Delta u_0 - P(u_0\cdot\nabla u_0) = f \tag{1.70}$$

and show in the next lemma that every function v_0 from Theorem 1.24 is a weak solution of this problem. To do so let us recall:

DEFINITION 1.30. A function $u_0\in\mathcal{H}_1(\Omega)$ is a weak solution of the Navier–Stokes problem (1.70), if for all test functions $\varphi\in C_{0,\sigma}^\infty(\Omega)$ the identity

$$-\nu(\nabla u_0,\nabla\varphi) + (u_0\cdot\nabla\varphi, u_0) = (f,\varphi) \tag{1.71}$$

is valid.

LEMMA 1.31. *Let $f\in H_{m-2}(\Omega)\cap\mathcal{H}_1(\Omega)$. Then every function v_0 from Theorem* 1.24 *is a weak solution of the problem* (1.70) *according to the definition* 1.30*, and we find $v_0\in H_m(\Omega)\cap\mathcal{H}_1(\Omega)$.*

PROOF. For simplification let us assume the convergence of the whole sequence $(v_0^N)_N$ from Therorem 1.24 to the limit function v_0. From (1.57) we know that this convergence is weak with respect to $\mathcal{H}_1(\Omega)$ and strong with respect to $\mathcal{H}_0(\Omega)$, and that $v_0 \in \mathcal{H}_1(\Omega)$ is true. Now let $\mathcal{V} := \{a_i | i \in \mathbb{N}\}$ be a complete orthonormal system in $\mathcal{H}_1(\Omega)$. Then we can choose $\mathcal{V} \subset C_{0,\sigma}^\infty(\Omega)$ and restrict us in (1.71) for φ on basis functions of the type $a = a_i$. Due to the weak convergence in $\mathcal{H}_1(\Omega)$ the first part of the lemma is proved, if, using (1.64), we can show

$$\lim_{N\to\infty} \left\{ \frac{1}{2\varepsilon}(a \circ X_{-1} - a \circ X_{-1}^{-1}, v_0^N) - (v_0 \cdot \nabla a, v_0) \right\} = 0. \tag{1.72}$$

According to Section 7 here the measure conserving mapping $X_{-1} := X_{-\varepsilon,-2\varepsilon}$ is constructed from the velocity field $\overline{v}^N = \tilde{v}^{N-1}$, and we obtain as in the proof of Lemma 1.26 for every $x \in \overline{\Omega}$ the identity

$$\begin{aligned} &a \circ X_{-1}(x) - a \circ X_{-1}^{-1}(x) \\ &= \int_{-2\varepsilon}^{-\varepsilon} \{[\tilde{v}^{N-1}(\tau) \cdot \nabla a] \circ X(\tau, -2\varepsilon, x) + [\tilde{v}^{n-1}(\tau) \cdot \nabla a] \circ X(\tau, -\varepsilon, x)\} d\tau. \end{aligned} \tag{1.73}$$

The assertion (1.72) then follows by a suitable decomposition as in Theorem 1.28.

For the asserted regularity statement, using the estimate of Cattabriga for the equation

$$P\Delta v_0 = \frac{1}{\nu}\{f + P(v_0 \cdot \nabla v_0)\},$$

we have to prove $v_0 \cdot \nabla v_0 \in H_{m-2}(\Omega)$. This can be done with help of the Sobolev imbedding theorem and a suitable *bootstrapping procedure* as in ([14]): Due to $v_0 \in \mathcal{H}_1(\Omega)$ we have $v_0 \in L_6(\Omega)$, and using

$$||v_0 \cdot \nabla v_0||_{0,\frac{3}{2}} \leq ||\nabla v_0|| \cdot ||v_0||_{0,6}$$

it follows $v_0 \cdot \nabla v_0 \in L_{\frac{3}{2}}(\Omega)$, hence $v_0 \in H_{2,\frac{3}{2}}(\Omega)$ and thus $v_0 \in L_\infty(\Omega)$. Since $||v_0 \cdot \nabla v_0|| \leq ||v_0||_\infty \cdot ||\nabla v_0||$ it follows $v_0 \cdot \nabla v_0 \in L_2(\Omega)$, hence $v_0 \in H_2(\Omega)$.

From

$$||\nabla(v_0 \cdot \nabla v_0)|| = ||\nabla v_0||_{0,4}^2 + ||v_0||_\infty ||\nabla^2 v_0|| \leq c_\Omega ||v_0||_{2,2}^2$$

we quote $v_0 \cdot \nabla v_0 \in H_1(\Omega)$, hence $v_0 \in H_3(\Omega)$ and thus $\nabla v_0 \in L_\infty(\Omega)$. Using

$$||\nabla^2(v_0 \cdot \nabla v_0)|| \leq 3||\nabla v_0||_\infty ||\nabla^2 v_0|| + ||v_0|| ||v_0||_{3,2} \leq c_\Omega ||v_0||_{3,2}^2$$

we finally obtain $v_0 \cdot \nabla v_0 \in H_{m-2}(\Omega)$. □

From (1.73) it follows that the first statement of Lemma 1.31 and thus the assertion (1.69) from Theorem 1.29 does not necessarily hold true, if the same function $\overline{v}$, which is needed for the construction of the initial data of the problem (N_{ε_N}), is used for all $N \in \mathbb{N}$.

Proof of Theorem 1.29: For every weakly solvable problem from Theorem 1.24, due to the regularity of its initial value v_0, there exists a number $\tilde{T} = \tilde{T}(\Omega, \nu, v_0)$ with $0 < \tilde{T} \leq T$ such that this problem can be solved uniquely in $[0, \tilde{T}]$, and that its strong solution, due to the validity of the compatibility condition at time $t = 0$, has all the properties asserted in 1.29 (compare also [14]). Since the norms of the initial data can be estimated by the norm of f there is a unique existence interval $[0, T^*]$ of positive length. □

Thus the solution of problem (N_0) from Theorem 1.29 is a solution of the following Navier–Stokes initial boundary value problem$(\tilde{N}_0)$in $[0, T^*]$:

Let $T > 0$. Construct a velocity field $v = (v_1, v_2, v_3)$ and some pressure function p as a solution of the system

$$\begin{array}{rcll} \partial_t v - \nu\Delta v + \nabla p &=& -v \cdot \nabla v & \\ \nabla \cdot v &=& 0 & (t, x) \in (0, T] \times \Omega \\ v_{|\partial\Omega} &=& 0 & \\ \partial_t v_{|t=0} &=& f. & \end{array} \qquad (\tilde{N}_0)$$

Since the stationary problem (1.70) can be solved uniquely for sufficiently large ν or sufficiently small f (see [13]) we find that in this case the problem$(\tilde{N}_0)$is locally, i.e. in $[0, T^*]$, uniquely solvable and the whole sequence $(v_0^N)_N$ of the initial values of the problems (N_{ε_N}) from Theorem 1.24 converges to the uniquely determined initial value v_0.

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Part 4

Qualitative theory of semilinear parabolic equations and systems

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Abstract. The chapter presents some principal results and methods in qualitative theory of semilinear parabolic problems. Notions of well-posedness, local and global existence, large time behavior and blow-up are thoroughly discussed.

The text is intended as a first introduction to qualitative theory of semilinear parabolic equations.

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CHAPTER 1

Qualitative theory of semilinear parabolic equations and systems

1. Introduction

The main aim of these notes is to present some typical results and methods in the qualitative theory of semilinear parabolic problems. We are mainly interested in local and global existence, blow-up and large time behavior of classical solutions. In order to explain the main ideas we consider simple model examples only. Most of our assertions can be easily generalized; for example, we could consider more general second-order elliptic operators instead of the Laplace operator, time-dependent nonlinearities etc.

The simplest equations or systems that we study are of the form

$$u_t - \Delta u = f(u), \qquad x \in \Omega,\ t > 0, \tag{1.1}$$

complemented by suitable boundary and initial conditions. Here Ω is a domain in $\mathbb{R}^n$, f is a C^1 function and u is either a scalar or a vector function. Such problems often appear in natural sciences. The operator Δ usually represents diffusion (of heat, chemical substance or biological habitat, for example) and $f(u)$ reaction so that (1.1) is often called reaction-diffusion equation or system. We will also consider problems with $f = f(u, \nabla u)$, including the system of incompressible Navier-Stokes equations. However, since the global existence of classical solutions of the Navier-Stokes system is still an open problem in the physically most interesting case $n = 3$, we will restrict ourselves to the proof of well-posedness of this system in the space of divergence-free functions in $(L^n(\Omega))^n$ and some related properties.

The qualitative theory of the ODE $u' = f(u)$ (including the theory of invariant manifolds, periodic solutions, stability issues etc) is well known even for undergraduate students. The parabolic counterpart of this theory for problems of the form (1.1) (and more general ones) was first systematically developed in [4]. One of the main tools in that book is the theory of analytic semigroups. We also use this tool in order to prove the well-posedness and some simple stability criteria. On the other hand, we are much more interested in typical PDE questions like existence for singular initial data or formation of singularities and these questions require a somewhat different approach even in the use of the semigroup theory (for example, in order to get optimal well-posedness results we use interpolation and extrapolation spaces instead of fractional power spaces). Concerning global existence we first consider typical scalar problems whose solutions exhibit an L^∞ or gradient blow-up and then we study the role of diffusion in global existence and blow-up for systems. In particular we present examples of systems of ODEs, in which the addition of diffusion prevents or induces blow-up. The understanding of the role of diffusion on the long-time behavior seems to be particularly important in chemical kinetics, where many models are considered without diffusion just because of the complexity of the problem (the number of equations is too large).

All our examples (except for the Navier-Stokes system and a problem with nonlinear boundary conditions) and many more can be found in [6]. That monograph also contains complete proofs of the (nonlinear) assertions whose proofs are just sketched or omitted in this text. The proofs of the statements concerning the linear semigroup theory in interpolation and extrapolation spaces can be found in [1] and [5], for the proofs of results on the Navier-Stokes equations we refer to [3] and the references therein. Very general results on equations and systems with nonlinear boundary conditions can be found in papers by H. Amann.

Finally let us mention that throughout this text Ω denotes a (bounded or unbounded) domain in $\mathbb{R}^n$ which is either the whole of $\mathbb{R}^n$ or its boundary is uniformly smooth (say $C^{2,\alpha}$).

If X, Y are Banach spaces then we write $X \hookrightarrow Y$ if X is continuously embedded in Y. If both $X \hookrightarrow Y$ and $Y \hookrightarrow X$ (that is X and Y coincide and carry equivalent norms) then we write $X \doteq Y$. By $\mathcal{L}(X, Y)$ we denote the space of bounded linear operators $A : X \to Y$, $\mathcal{L}(X) = \mathcal{L}(X, X)$. We also use standard notation for frequently used function spaces, for example $\mathcal{D}(\Omega)$ denotes the space of all (C^∞-)smooth functions with compact support in Ω, $BUC(\Omega)$ denotes the space of bounded and uniformly continuous functions endowed with the sup-norm (hence $BUC(\Omega) = C(\overline{\Omega})$ if Ω is bounded), etc.

2. The role of Δ in modeling

As mentioned in the previous section, the typical equation that we study contains the Laplace operator. Since this text is primarily meant to be used by PhD students of the Nečas center for mathematical modeling it will be worthwhile to present typical models where this operator occurs. We will mention two examples which were presented by J. Nečas at the beginning of his undergraduate course on PDEs in the academic year 1980-1981.

The first example is a model of heat conduction. Consider a body which occupies an n-dimensional domain Ω and let $u(x, t)$ denote the temperature at the point x and time t. Consider a smooth subdomain O such that $\overline{O} \subset \Omega$. The heat produced in O in the time interval (t_1, t_2) can be written as

$$\int_{t_1}^{t_2} \int_O f(x, t)\, dx\, dt,$$

where f is the source function. Using Green's theorem, the heat which flows from O into $\Omega \setminus O$ can be computed as

$$-\int_{t_1}^{t_2} \int_{\partial\Omega} k \frac{\partial u}{\partial \nu}\, dS\, dt = -\int_{t_1}^{t_2} \int_O \sum_i \frac{\partial}{\partial x_i}\Big(k \frac{\partial u}{\partial x_i}\Big)\, dx\, dt,$$

where ν denotes the outward unit normal on the boundary ∂O and $k = k(x, u, \nabla u, \dots)$ represents the coefficient of internal heat conduction. The difference of these two quantities is responsible for the increase (or decrease) of the temperature in O and can be written as

$$\int_O \big(u(x,t_2)-u(x,t_1)\big)\rho(x)c(x)\,dx = \int_{t_1}^{t_2}\int_O \frac{\partial u}{\partial t}\rho(x)c(x)\,dx\,dt,$$

where ρ is density and c specific heat capacity. If we divide the balance equation by (t_2-t_1) and by the measure of O and pass to the limit as $t_2 \to t_1$ and O shrinks (in a suitable way) to a point, we arrive at the equation

$$\rho c\frac{\partial u}{\partial t} - \sum_i \frac{\partial}{\partial x_i}\Big(k\frac{\partial u}{\partial x_i}\Big) = f.$$

In particular, if c, ρ and k are constants equal to 1 then we obtain the linear heat equation $u_t - \Delta u = f$. Consider this linear problem in the case $\Omega = \mathbb{R}^n$, $f = 0$ and assume that the initial datum $u(x,0) = u_0(x)$ is a nonnegative smooth function with compact support, $u_0 \not\equiv 0$. Then the (only bounded) solution of this problem is given by the explicit formula $u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy$, where $G(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ is the heat kernel. In particular, $u(x,t) > 0$ for any $x \in \mathbb{R}^n$ and $t > 0$ which means that the heat is propagating with an infinite speed. Since this violates the relativity theory, many people try to derive other models of heat conduction. For example, taking $k = k(u) = u^m$ we obtain the porous medium equation with finite speed of propagation. On the other hand, comparison of experimental results and numerical computations for many models based on the linear heat equation show that this equation is in fact a very good approximation of the real (Newtonian) world.

The second example describes a deformation of an elastic membrane. Assume that the membrane initially occupies a two-dimensional domain Ω in the horizontal plane $\{x_3 = 0\}$ and is exposed to a vertical external force f. Assume that the membrane is deformed only in the vertical direction and let $u = u(x_1, x_2)$ denote the height of this deformation. Denoting $x = (x_1, x_2)$, the potential energy of the membrane can be computed as

$$\Phi(u) = \int_\Omega dW - \int_\Omega fu\,dx + \mathrm{BI},$$

where dW is the Lagrangian of the potential of internal forces, the second integral represents the work of external forces and "BI" stands for some boundary integrals which depend on boundary conditions (and which will not be important in the following considerations). By the linear Hook law, we have $dW = TP$, where T is a positive constant and $P := \sqrt{1+|\nabla u|^2}$ represents the increase of the surface of the membrane. The minimizer of Φ satisfies the necessary condition $\Phi'(u)\varphi = 0$ for all "suitable" test functions φ. Taking φ a smooth function with compact support in Ω we obtain

$$\int_\Omega \Big(\frac{T}{P}\nabla u\cdot\nabla\varphi - f\varphi\Big)\,dx = 0$$

and applying Green's theorem (and assuming that u is smooth enough) we easily arrive at the Euler equation

$$-T\sum_i \frac{\partial}{\partial x_i}\Big(\frac{1}{P}\frac{\partial u}{\partial x_i}\Big) = f \qquad \text{in } \Omega.$$

The left hand side of this equation is (a multiple of) the mean curvature of u and it is known that this equation need not be solvable. In addition, a careful analysis shows that the equation violates both forces and momentum balance equations. This is due to our erroneous assumption that the deformation occurs in the vertical direction only. A physically correct model has to consider a vector valued deformation u. On the other hand, if we assume that the deformation is small then we can approximate P by the constant 1, which leads to the linear equation $-T\Delta u = f$. This model represents a good approximation of the corresponding physical phenomenon for small deformations.

3. Basic tools

There exist many important tools and methods which are used in the study of semilinear problems of the form (1.1). In this part we describe in more detail a few of them: comparison principle, Lyapunov function, scaling, analytic semigroups and interpolation spaces. Let us only mention some of other useful methods: the *moving planes technique* (based on the symmetry of the Laplace operator), *multiplication of the equation* by a suitable test function (for example, a power of u or the first eigenfunction of the Dirichlet Laplacian), linear *L^p and Schauder estimates* and various *bootstrap arguments* which are usually used in order to prove regularity or a priori estimates of the solutions.

Comparison and maximum principles (if available) represent a very powerful tool in the study of semilinear parabolic problems. Let us first consider solutions u_1, u_2 of the linear Cauchy problems

$$\partial_t u_i - \Delta u_i = f_i \quad \text{in } \mathbb{R}^n \times (0,\infty), \qquad u_i(\cdot,0) = u_{0,i}, \qquad i = 1,2.$$

If $u_{0,i}, f_i$ are bounded and measurable, for example, then the unique solutions satisfying $u_i(\cdot,t) \in L^\infty(\mathbb{R}^n)$ for $t > 0$ are given by the formulas

$$u_i(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_{0,i}(y)\,dy + \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)f_i(y,s)\,dy\,ds,$$

where G is the heat kernel. In particular, if $u_{0,1} \geq u_{0,2}$ and $f_1 \geq f_2$ then $u_1 \geq u_2$ due to the positivity of G. In addition, $u_1(x,t) > u_2(x,t)$ for any $x \in \mathbb{R}^n$ and $t > 0$ whenever $u_{0,1} \not\equiv u_{0,2}$. Analogous comparison principles are also true for nonlinear equations (and so called cooperative systems). Unfortunately, it is probably not possible to formulate a general comparison principle which would be applicable in each case. Therefore we refer to [6] for various useful formulations of comparison principles (and the proofs or references to the proofs of such statements) and we just mention (a simplified version of) one of them.

Proposition 1.1. *Let $Q := \Omega \times (0,T)$, $u_1, u_2 \in C^{2,1}(Q) \cap C(\overline{Q})$ be scalar functions, $|u_i|, |\nabla u_i| \leq C$, $f \in C^1$, $u_1 \leq u_2$ on the parabolic boundary $\Sigma := \big(\Omega \times \{0\}\big) \cup \big(\partial\Omega \times (0,T)\big)$ and*

$$\partial_t u_1 - \Delta u_1 - f(u_1, \nabla u_1) \leq \partial_t u_2 - \Delta u_2 - f(u_2, \nabla u_2) \quad \text{in } Q.$$

Then $u_1 \leq u_2$ in Q.

Let us also mention a strongly related tool, the *zero number*, which can be used for scalar problems if $n = 1$ (or radial solutions of scalar problems if $n > 1$). The main property of the zero number roughly says that if u is a nontrivial solution of a linear parabolic equation of the form $u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u$, $x \in (0,1)$, $t > 0$, (and satisfies the homogeneous Dirichlet boundary conditions, for example) then the number of zeroes of the function $u(\cdot,t)$ is finite, nonincreasing in time and it drops at each time t when $u(\cdot,t)$ has a multiple zero. We again refer to [6] for a precise formulation of this statement and applications to semilinear parabolic problems.

Similarly as in the case of ODEs, the *Lyapunov function* (when available) plays a very important role in the study of the dynamical properties of a given parabolic problem. For example, consider a problem which generates a (local) semiflow in a Banach space X and assume that global bounded trajectories of this semiflow are relatively compact. If there exists a strict Lyapunov function then the ω-limit set of any global bounded trajectory is a nonempty compact connected set consisting of equilibria (see [6]). All these assumptions are satisfied, for example, if we consider the semiflow generated by the scalar equation (1.1) (complemented by the homogeneous Dirichlet boundary conditions) in the Sobolev space $W_0^{1,q}(\Omega)$, $q \in (n,\infty)$, $q \geq 2$, provided Ω is bounded. In fact, to see that this problem possesses a strict Lyapunov function, set

$$E(v) := \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 - F(v)\Big)\,dx, \qquad \text{where } \ F(v) := \int_0^v f(s)\,ds.$$

Then

$$\begin{aligned} \frac{d}{dt}E(u(\cdot,t)) &= \int_\Omega (\nabla u \cdot \nabla u_t - f(u)u_t)\,dx \\ &= \int_\Omega (-\Delta u - f(u))u_t\,dx = -\int_\Omega u_t^2\,dx \leq 0. \end{aligned} \tag{1.2}$$

(See [6] for justification of this formal computation.) We will see that the existence of a Lyapunov function (sometimes called *energy*) can be also used in various blow-up and a priori estimates.

Scaling represents a powerful tool, in particular in the study of singularities and asymptotic behavior of global solutions. First notice that if u is a solution of the linear heat equation $u_t - \Delta u = 0$ then so is the rescaled function $v(x,t) = \lambda^\alpha u(\lambda x, \lambda^2 t)$, where $\lambda, \alpha > 0$. The same is true for the nonlinear heat equation $u_t - \Delta u = |u|^{p-1}u$ if we fix $\alpha = 2/(p-1)$ or for the homogeneous system of incompressible Navier-Stokes equations if $\alpha = 1$. Scaling is often used in the proofs of a priori estimates of positive solutions. Assuming on the contrary that there exists a sequence of points (x_k, t_k) such that the sequence $u(x_k,t_k)$ (or $\nabla u(x_k,t_k)$) is unbounded, one rescales the solutions around the points (x_k,t_k) and passes to the limit with the rescaled solutions in order to construct a nontrivial solution of the limiting problem. The contradiction is achieved if the limiting problem does not possess nontrivial solutions (which is ususally guaranteed by a *Liouville type theorem*).

A solution is called *self-similar* if it is invariant under (some) parabolic scaling of its arguments, for example $u(x,t) = u(\lambda x, \lambda^2 t)$. Such solution can be obviously expressed in the form $u(x,t) = U(x/\sqrt{t})$. If in addition $u(\cdot,t)$ is a radial function then U solves an ODE so that it is possible to use ODE methods for the study of radial self-similar solutions. Note that many important phenomena occur within this important subclass of solutions so that the study of such solutions is often the first step in the study of the PDE. Let us also mention that the use of self-similar solutions combined with the comparison principle (or zero number arguments) is one of the most efficient tools in the study of solutions of nonlinear heat equations with supercritical nonlinearities (since the standard functional analytic methods fail).

The last tool which we describe in more detail is the *semigroup theory* combined with *interpolation spaces*. This tool (unlike the others mentioned in detail above) really requires the semilinear structure of the problem (that is, the operators with the highest order derivatives have to be linear). On the other hand, our restriction to autonomous (time independent) operators is not important (see [1] for the corresponding generalization to the non-autonomous case). We will see in the next section that in addition to interpolation spaces we will also have to work in *extrapolation spaces*.

If we consider the solution $u(t) = u(\cdot, t)$ at time t as an element of a Banach space X then the equation (1.1) complemented by the boundary and initial conditions can be written as an abstract Cauchy problem in X:

$$u_t + Au = F(u), \quad t > 0, \qquad u(0) = u_0. \tag{1.3}$$

Here $F(u)(x) = f(u(x))$ (or $F(u)(x) = f(u(x), \nabla u(x))$ if f depends on the gradient of u) is the Nemytskii operator corresponding to the local function f and A is an unbounded operator in X defined by $Au = -\Delta u$ and by its domain of definition $D(A)$ which has to reflect the prescribed boundary conditions. Assume that $-A$ generates a semigroup $\{e^{-tA}; t \geq 0\}$ of bounded linear operators in X. Similarly as in the case of ODEs we would like to solve (1.3) by using the variation-of-constants formula

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(u(s))\,ds. \tag{1.4}$$

More precisely, fix u_0 and set

$$\Phi(u)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(u(s))\,ds. \tag{1.5}$$

We look for a space Y of functions $u : [0, T] \to X$ such that Φ will possess a unique fixed point in Y, hence a unique solution of (1.4) in Y.

If we consider F as a map from X to X then its domain of definition $D(F)$ will be a proper subset of X, in general (consider the case where f depends on ∇u). Since in the fixed point argument we need $\Phi(u)(t) \in D(F)$, the operators e^{-tA} must have some additional properties which will guarantee $e^{-tA}(X) \subset D(F)$, in particular. Therefore we restrict ourselves to sectorial operators: these operators generate so called analytic semigroups.

Definition 1.2. Let A be a closed linear operator in the Banach space X. The operator $-A$ is said to be *sectorial* if there are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$\sigma(-A) \subset \Sigma := \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| \geq \theta\} \cup \{\omega\}$$

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|} \qquad \forall \lambda \in \mathbb{C} \setminus \Sigma.$$

If $-A$ is sectorial then we set $e^{0A}u := u$ for all $u \in X$ and

$$e^{-tA} := \frac{1}{2\pi i}\int_{\omega+\Gamma_{r,\eta}} e^{t\lambda}(\lambda + A)^{-1}\,d\lambda, \qquad t > 0,$$

where $r > 0$, $\eta \in (\pi/2, \theta)$ and $\Gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} : |\arg\lambda| = \eta,\ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \eta,\ |\lambda| = r\}$$

oriented counterclockwise. We also set

$$\omega(-A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(-A)\}.$$

The operators e^{-tA} are bounded linear operators in X since the above Bochner integral converges in $\mathcal{L}(X)$. In the following proposition we collect the most important properties of e^{-tA}. The proofs of these properties can be found in [5]. By X_1 we denote the set $D(A)$ endowed with the graph norm $|u|_1 := \|u\|_X + \|Au\|_X$. Since A is closed, the space X_1 is a Banach space.

PROPOSITION 1.3. *Let $-A$ be sectorial in X and e^{-tA} be defined as above. Then*

(i) $e^{-(t+s)A} = e^{-tA}e^{-sA}$ *for all* $t, s \geq 0$,
(ii) *the mapping* $(0,\infty) \to \mathcal{L}(X) : t \mapsto e^{-tA}$ *is analytic,*
(iii) $lim_{t\to 0+} e^{-tA}u = u$ *if and only if* $u \in \overline{D(A)}$,
(iv) *if* $t > 0$ *then* $e^{-tA}(X) \subset D(A)$ *and* $(e^{-tA})' = -Ae^{-tA}$,
(v) $Ae^{-tA}u = e^{-tA}Au$ *for all* $u \in D(A)$,
(vi) *if* $\omega > \omega(-A)$ *then there exists* $M > 0$ *such that*

$$\left.\begin{aligned} \|e^{-tA}\|_{\mathcal{L}(X)} &\leq Me^{\omega t}, \qquad t \geq 0,\\ \|e^{-tA}\|_{\mathcal{L}(X,X_1)} &\leq \frac{M}{t}e^{\omega t}, \qquad t > 0. \end{aligned}\right\} \tag{1.6}$$

Property (iv) guarantees that the function $u(t) := e^{-tA}u_0$ is a solution of the linear equation $u_t + Au = 0$ for $t > 0$. In addition we see that e^{-tA} maps X into $D(A)$. Hence, if $D(A) \subset D(F)$ then there is a hope for $\Phi(u)(t) \in D(F)$. However, if we try to prove $\Phi(u)(t) \in D(A)$ then we have to estimate the integrand $e^{-(t-s)A}F(u(s))$ (appearing in the definition of Φ) in the norm of X_1 and, using the second part of (1.6), we get a function $1/(t-s)$ which has a non-integrable singularity at $s = t$. Therefore we have to use an intermediate space X_θ between X and X_1 such that the integral in the definition of Φ converges in X_θ and $X_\theta \subset D(F)$.

Set $X_0 = X$ and $|u|_0 := \|u\|_X$ and assume that $(X_\theta, |\cdot|_\theta)$ is a Banach space such that

$$\left.\begin{aligned} &X_1 \hookrightarrow X_\theta \hookrightarrow X_0,\\ &|u|_\theta \leq C|u|_0^{1-\theta}|u|_1^\theta \quad \text{for all } u \in X_1,\\ &\|e^{-tA}\|_{\mathcal{L}(X_\theta)} \leq \tilde{M}e^{\omega t} \quad \text{for all } t \geq 0, \end{aligned}\right\} \tag{1.7}$$

where $C, \tilde{M} \geq 1$. Notice that (1.6) and the second assumption in (1.7) imply the estimate

$$\|e^{-tA}\|_{\mathcal{L}(X_0,X_\theta)} \leq \frac{CM}{t^\theta}e^{\omega t}, \quad t > 0. \tag{1.8}$$

Consequently, enlarging $\tilde{M}$ if necessary we have

$$\|e^{-tA}\|_{\mathcal{L}(X_0)} + \|e^{-tA}\|_{\mathcal{L}(X_\theta)} + t^\theta\|e^{-tA}\|_{\mathcal{L}(X,X_\theta)} \leq \tilde{M}, \qquad t \in [0,1]. \tag{1.9}$$

REMARK 1.4. *Let* $\theta \in (0,1)$ *and* $1 \le p \le \infty$. *Let* $(\cdot,\cdot)_{\theta,p}$ *and* $[\cdot,\cdot]_\theta$ *denote the real and the complex interpolation functors, respectively (see Appendix or [2]). Then all the spaces* $X_\theta = (X_0, X_1)_{\theta,p}$, $1 \le p \le \infty$, *and* $X_\theta = [X_0, X_1]_\theta$ *satisfy* (1.7). *In addition,* (1.7) *is also true if* X_θ *is the fractional power space* X^θ, *that is the domain of the fractional power* $(\omega + A)^\theta$ *endowed with the norm* $|u|_\theta := |(\omega + A)^\theta|_0$. *Here* $\omega > \omega(-A)$ *and* $(\omega + A)^\theta$ *is the inverse of the operator*

$$(\omega + A)^{-\theta} := \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-\omega t} e^{-tA}\, dt.$$

Note that $X^\theta \doteq [X_0, X_1]_\theta$ *if* $\omega + A$ *has bounded imaginary powers (see [1]). Finally, let us mention that if* X_θ *is any of the interpolation spaces or the fractional power space mentioned above (except for the real interpolation space* $(X_0, X_1)_{\theta,\infty}$*) then the density of* $D(A)$ *in* $X = X_0$ *guarantees the density of* $D(A)$ *in* X_θ. □

EXAMPLE 1.5. *Set* $X = X_0 = L^q(\Omega)$, *where* $1 < q < \infty$, *and let* $Au = -\Delta u$ *with* $D(A) = \{u \in W^{2,q}(\Omega) : u = 0 \text{ on } \partial\Omega\}$. *Then* $-A$ *is sectorial and* $D(A)$ *is dense in* X. *In addition,* X *is reflexive. Given* $\theta \in (0,1)$, *set* $X_\theta := (X_0, X_1)_\theta$, *where*

$$(\cdot,\cdot)_\theta := \begin{cases} [\cdot,\cdot]_\theta & \text{if } \theta = 1/2, \\ (\cdot,\cdot)_{\theta,q} & \text{otherwise.} \end{cases} \tag{1.10}$$

Then

$$X_\theta \doteq \begin{cases} \{u \in W^{2\theta,q}(\Omega) : u = 0 \text{ on } \partial\Omega\} & \text{if } 2\theta > 1/q, \\ W^{2\theta,q}(\Omega) & \text{if } 1/q > 2\theta \ge 0. \end{cases}$$

Similarly, if we take $D(A) = \{u \in W^{2,q}(\Omega) : \partial u/\partial\nu = 0 \text{ on } \partial\Omega\}$ *then*

$$X_\theta \doteq \begin{cases} \{u \in W^{2\theta,q}(\Omega) : \partial u/\partial\nu = 0 \text{ on } \partial\Omega\} & \text{if } 2\theta > 1 + 1/q, \\ W^{2\theta,q}(\Omega) & \text{if } 1 + 1/q > 2\theta \ge 0. \end{cases}$$

□

EXAMPLE 1.6. *Let* $X = X_0$ *be any of the spaces* $L^\infty(\Omega)$, $BC(\overline{\Omega})$, $BUC(\Omega)$ *and* $BUC_*(\Omega) := \{u \in BUC(\Omega) : \lim_{|x|\to\infty} u(x) = 0\}$. *Let* $Au = -\Delta u$ *and*

$$D(A) = \{u \in \bigcap_{q \ge 1} W^{2,q}_{loc}(\overline{\Omega}) : u, \Delta u \in X,\ u = 0 \text{ on } \partial\Omega\}.$$

Then $-A$ *is sectorial and* $\overline{D(A)}^X = \{u \in BUC(\Omega) : u = 0 \text{ on } \partial\Omega\}$ *in the first three cases,* $\overline{D(A)}^X = \{u \in BUC_*(\Omega) : u = 0 \text{ on } \partial\Omega\}$ *if* $X = BUC_*(\Omega)$. *In particular,* $D(A)$ *is not dense in* X, *except for the case* $X \in \{BUC(\mathbb{R}^n), BUC_*(\mathbb{R}^n)\}$.

Let X_θ *be any of the interpolation spaces or the fractional power space mentioned in Remark 1.4 and let* $\varepsilon > 0$. *Then*

$$X_\theta \hookrightarrow BC^{2\theta-\varepsilon}(\overline{\Omega}).$$

If, for example, $X = BC(\overline{\Omega})$ *and* $\theta \in [1/2, 1)$ *then even* $(X_0, X_1)_{\theta,1} \hookrightarrow BC^{2\theta}(\overline{\Omega})$ *and the space* $X_{1/2} := \{u \in BC^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ *also satisfies* (1.7) *with* $\theta = 1/2$.

Similar assertions are true also for Neumann boundary conditions. In that case, one has

$$D(A) = \{u \in \bigcap_{q\geq 1} W^{2,q}_{loc}(\overline{\Omega}) : u, \Delta u \in X,\ \partial u/\partial\nu = 0 \text{ on } \partial\Omega\}$$

and $\overline{D(A)}^X = BUC(\Omega)$ *or* $\overline{D(A)}^X = BUC_*(\Omega)$. □

EXAMPLE 1.7. *Consider the system of incompressible Navier-Stokes equations in a bounded domain* Ω *complemented by the non-slip boundary conditions:*

$$\left.\begin{aligned} u_t - \Delta u &= -(u\cdot\nabla)u + f - \nabla p, && x\in\Omega,\ t>0,\\ div\, u &= 0, && x\in\Omega,\ t>0,\\ u &= 0, && x\in\partial\Omega,\ t>0, \end{aligned}\right\} \tag{1.11}$$

where $u : \Omega \to \mathbb{R}^n$ *is velocity (of a fluid),* f *external force and* p *pressure. Fix* $q \in (1,\infty)$ *and define* X *as the closure of the set* $\{u \in \mathcal{D}(\Omega)^n : div\, u = 0\}$ *in* $(L^q(\Omega))^n$. *Set also* $G = \{\nabla p : p \in W^{1,q}(\Omega)\}$. *Then* $(L^q(\Omega))^n$ *can be written as a direct sum of* X *and* G *(Helmholtz decomposition) and we denote by* $P : (L^q(\Omega))^n \to X$ *the corresponding (Helmholtz) projection. Let* $\tilde{A}$ *denote the operator* $-\Delta$ *in* $(L^q(\Omega))^n$ *with domain* $D(\tilde{A}) = \{u \in (W^{2,q}(\Omega))^n : u = 0 \text{ on } \partial\Omega\}$. *Set* $A = P\tilde{A}$ *with domain* $X \cap D(\tilde{A})$ *(Stokes operator). Then* $-A$ *is sectorial in* X *and* $\omega(-A) < 0$. *If* $\theta \in (0,1)$ *and* $X_\theta := [X_0, X_1]_\theta$ *then* $X_\theta \hookrightarrow (H^{2\theta,q}(\Omega))^n$, *where* $H^{2\theta,q}$ *denotes the Bessel potential space. We have also* $X_\theta \doteq X^\theta := (D(A^\theta), |A^\theta\cdot|_0)$, *cf. Remark 1.4. If* $q \geq 2$ *then* $H^{2\theta,q}(\Omega) \hookrightarrow W^{2\theta,q}(\Omega)$.

Notice that system (1.11) can be written as $u_t + Au = F(t,u)$, $t > 0$, *where* $F(t,u) := P\big(-(u\cdot\nabla)u + f(\cdot,t)\big)$. □

4. Well-posedness

Throughout this section we assume that

$$-A \text{ is sectorial in } X = X_0, \quad \omega > \omega(-A), \tag{1.12}$$

and we denote by X_1 the domain of definition of A endowed with the graph norm. We also assume that

$$\text{the Banach spaces } (X_\theta, |\cdot|_\theta),\ \theta\in(0,1), \text{ satisfy (1.7).} \tag{1.13}$$

LEMMA 1.8. *Assume* $f \in L^\infty((0,T), X_0)$ *and set*

$$v(t) = \int_0^t e^{-(t-s)A} f(s)\,ds, \qquad t\in[0,T].$$

Then $v \in C^{1-\alpha}([0,T], X_\alpha)$ *for any* $\alpha \in (0,1)$.

PROOF. Let $\alpha \in (0,1)$. Since $s \mapsto \|e^{-(t-s)A}\|_{\mathcal{L}(X,X_\alpha)}$ belongs to $L^1(0,t)$ for every $t \in (0,T]$ due to (1.8), we have $v(t) \in X_\alpha$.

Let $0 \leq s \leq t \leq T$ and $\|f\|_\infty := \|f\|_{L^\infty((0,T),X_0)}$. Then

$$\begin{aligned} v(t) - v(s) &= \int_0^s \Big(e^{-(t-\tau)A} - e^{-(s-\tau)A}\Big) f(\tau)\,d\tau + \int_s^t e^{-(t-\tau)A} f(\tau)\,d\tau\\ &= \int_0^s \int_{s-\tau}^{t-\tau} (-A)e^{-\sigma A} f(\tau)\,d\sigma\,d\tau + \int_s^t e^{-(t-\tau)A} f(\tau)\,d\tau. \end{aligned}$$

Since

$$\begin{aligned}|Ae^{-\sigma A}f(\tau)|_\alpha &= |e^{-(\sigma/2)A}Ae^{-(\sigma/2)A}f(\tau)|_\alpha \\ &\le \|e^{-(\sigma/2)A}\|_{\mathcal{L}(X_0,X_\alpha)}\|Ae^{-(\sigma/2)A}\|_{\mathcal{L}(X_0,X_0)}|f(\tau)|_0 \\ &\le C\sigma^{-\alpha-1}\|f\|_\infty,\end{aligned}$$

we obtain

$$\begin{aligned}|v(t)-v(s)|_\alpha &\le C(T)\|f\|_\infty\Big(\int_0^s\int_{s-\tau}^{t-\tau}\sigma^{-1-\alpha}\,d\sigma\,d\tau + \int_s^t (t-\tau)^{-\alpha}\,d\tau\Big) \\ &\le \tilde{C}(T)\|f\|_\infty(t-s)^{1-\alpha}.\end{aligned}$$

□

THEOREM 1.9. *Let $\beta\in[0,1)$, $u_0\in X_\beta$ and $F: X_\beta\to X_0$ be locally Lipschitz continuous, uniformly on bounded sets in X_β. Then there exists $T=T(|u_0|_\beta)>0$ such that the variation-of-constants formula (1.4) has a unique solution $u\in L^\infty((0,T),X_\beta)$. In addition $u\in C^{1-\alpha}_{loc}((0,T],X_\alpha)$ for any $\alpha\in(0,1)$. If u_0 belongs to the closure of $D(A)$ in X_β then $u\in C([0,T],X_\beta)$.*

PROOF. Set $Y := L^\infty((0,T),X_\beta)$ and $B_R := \{u\in Y : \|u\|_Y\le R\}$. We will show that the mapping Φ defined in (1.5) possesses a unique fixed point u in B_R provided $R=R(|u_0|_\beta)>0$ is large enough and $T=T(R)>0$ is small enough, $T\le 1$. In addition, $u\in C^{1-\alpha}_{loc}((0,T],X_\alpha)$ for $\alpha\in(0,1)$. Since R can be arbitrarily large, this will prove the uniqueness in the whole space Y.

Fix $R\ge 2\tilde{M}|u_0|_\beta$, where $\tilde{M}$ is the constant from (1.9). Since $F: X_\beta\to X_0$ is uniformly Lipschitz continuous on bounded subsets of X_β, there exists a constant $C_R=C_R(F)$ such that $|F(u)|_0\le C_R$ and $|F(u)-F(v)|_0\le C_R|u-v|_\beta$ for all $u,v\in X_\beta$ satisfying $|u|_\beta,|v|_\beta\le R$.

First let us show that Φ maps B_R into B_R if T is small enough. Let $u\in B_R$ and $t\in(0,T]$. Then

$$\begin{aligned}|\Phi(u)(t)|_\beta &\le \|e^{-tA}\|_{\mathcal{L}(X_\beta)}|u_0|_\beta + \int_0^t \|e^{-(t-s)A}\|_{\mathcal{L}(X_0,X_\beta)}|F(u(s))|_0\,ds \\ &\le \tilde{M}|u_0|_\beta + \int_0^t \tilde{M}(t-s)^{-\beta}C_R\,ds \\ &\le R/2 + \tilde{M}C_R T^{1-\beta}/(1-\beta) \le R\end{aligned}$$

provided $T=T(R)$ is small enough.

Next let us show that $\Phi: B_R\to B_R$ is a contraction for T small. Let $u,v\in B_R$ and $t\in(0,T]$. Then

$$\begin{aligned}|\Phi(u)(t)-\Phi(v)(t)|_\beta &\le \int_0^t \|e^{-(t-s)A}\|_{\mathcal{L}(X_0,X_\beta)}|F(u(s))-F(v(s))|_0\,ds \\ &\le \tilde{M}C_R\int_0^t (t-s)^{-\beta}\|u-v\|_Y\,ds \\ &\le \tilde{M}C_R T^{1-\beta}\|u-v\|_Y/(1-\beta) \le \|u-v\|_Y/2\end{aligned}$$

provided T is small enough.

Consequently, Φ possesses a unique fixed point in B_R. The local Hölder continuity of $u : (0,T] \to X_\alpha$ for $\alpha \in (0,1)$ follows from Lemma 1.8 and Proposition 1.3.

Finally, let u_0 belong to the closure of $D(A)$ in X_β. Fix $\varepsilon > 0$ and choose $v_0 \in D(A)$ such that $|u_0 - v_0|_\beta < \varepsilon$. If $t \le 1$ then (1.9) guarantees

$$|e^{-tA}u_0 - e^{-tA}v_0|_\beta \le \tilde{M}|u_0 - v_0|_\beta < \tilde{M}\varepsilon.$$

Next

$$|e^{-tA}v_0 - v_0|_1 = |e^{-tA}v_0 - v_0|_0 + |e^{-tA}Av_0 - Av_0|_0 \le (\tilde{M}+1)|v_0|_1$$

and $e^{-tA}v_0 \to v_0$ in X_0 as $t \to 0+$, hence

$$|e^{-tA}v_0 - v_0|_\beta \le C|e^{-tA}v_0 - v_0|_0^{1-\beta}|e^{-tA}v_0 - v_0|_1^\beta \to 0$$

as $t \to 0+$. The estimates above imply now $|e^{-tA}u_0 - u_0|_\beta \to 0$ as $t \to 0+$, which concludes the proof. □

Remarks 1.10. *(i) Let u be the solution from Theorem 1.9. Since $u \in C^{1-\alpha}_{loc}((0,T], X_\alpha)$ for any $\alpha > 1$ and $F : X_\beta \to X_0$ is Lipschitz continuous, we see that the RHS $f(s) := F(u(s))$ belongs to $C^\rho_{loc}((0,T], X_0)$ for any $\rho < 1$ satisfying $\rho \le 1 - \beta$. Now similar linear estimates as in Lemma 1.8 (see [5, Theorem 4.3.4]) guarantee $u \in C^\rho_{loc}((0,T], X_1) \cap C^{1+\rho}_{loc}((0,T], X_0)$, hence u is a classical solution for $t > 0$.*

(ii) The same results as in Theorem 1.9 can obviously be obtained if $D_\beta \subset X_\beta$ is open in X_β, $u_0 \in D_\beta$ and $F = F(t,u) : [0,T_0] \times D_\beta \to X_0$ is locally Lipschitz continuous, uniformly on the sets of the form $[0,T_0] \times B_\beta$, where $B_\beta \subset D_\beta$ is bounded in X_β and bounded away from ∂D_β. However, if $D_\beta \ne X_\beta$ then in addition to the hypotheses of Theorem 1.9 we have to assume $u_0 \in \overline{D(A)}^{X_\beta}$ (which guarantees $u \in C([0,T], X_\beta)$) or, at least, that the set $\{e^{-tA}u_0 : t \in [0,T_0]\}$ is a subset of D_β and is bounded away from ∂D_β for some $T_0 > 0$. Due to Remark 1.4 this additional assumption is satisfied for all $u_0 \in D_\beta$ if $D(A)$ is dense in X and we consider intermediate spaces X_θ as in that Remark.

(iii) Consider the situation in Theorem 1.9 with $u_0 \in X_0$ (instead of $u_0 \in X_\beta$) and assume that the nonlinearity F satisfies

$$|F(u) - F(v)|_0 \le C|u-v|_\beta(1 + |u|_\beta^{p-1} + |v|_\beta^{p-1}), \qquad u, v \in X_\beta, \tag{1.14}$$

where $p > 1$, $\beta p < 1$. Then similarly as in the proof of Theorem 1.9 one can prove the existence of $T = T(|u_0|_0) > 0$ such that (1.4) possesses a unique solution u in the space

$$L^\infty_\beta((0,T), X_\beta) := \{u \in L^\infty_{loc}((0,T], X_\beta) : \|u\|_\beta < \infty\},$$

where $\|u\|_\beta := ess\ \sup_{(0,T)} t^\beta |u(t)|_\beta$. In addition, $u \in C((0,T], X_\beta)$. Finally, $u \in C([0,T], X_0)$ provided $u_0 \in \overline{D(A)}^X$. □

Example 1.11. *Consider the problem*

$$\left.\begin{aligned} u_t - \Delta u &= f(u), && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.15}$$

where $f \in C^1$ *and* $u_0 \in L^\infty(\Omega)$. *Set* $F(u)(x) = f(u(x))$, $X_0 = L^\infty(\Omega)$ *and let* A *and* $D(A)$ *be defined as in Example 1.6. Then* $F : X_0 \to X_0$ *satisfies the assumptions of Theorem 1.9 with* $\beta = 0$, *hence we obtain a unique local solution* $u \in L^\infty((0,T), L^\infty(\Omega))$ *of the corresponding variation-of-constants formula* (1.4). *In addition* u *is a classical solution of* (1.15) *for* $t > 0$ *and* $u(t) - e^{-tA}u_0 \in C^{1-\alpha}([0,T], X_\alpha)$ *for any* $\alpha \in (0,1)$, *provided* X_α *satisfies* (1.7) *with* $\theta = \alpha$.

If the nonlinearity f *depends also on* ∇u, $f = f(u, \nabla u)$, *then we set* $F(u)(x) = f(u(x), \nabla u(x))$, $X_0 = BC(\overline{\Omega})$, *we define* A *and* $D(A)$ *again as in Example 1.6 and fix* $X_{1/2} = \{u \in BC^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ *(cf. Example 1.6). Then* $F : X_{1/2} \to X_0$ *is locally Lipschitz continuous (uniformly on bounded sets), hence Theorem 1.9 with* $\beta = 1/2$ *guarantees the local solvability for* $u_0 \in X_{1/2}$. *If in addition we assume suitable growth condition on the derivatives of* f *then we may also consider* $u_0 \in X_0$ *due to Remark 1.10(iii).* □

EXAMPLE 1.12. *Consider the Navier-Stokes system* (1.11) *from Example 1.7 in a bounded domain* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *and let* X_0, X_1, X_θ, A *and* F *be as in that example. Fix* $q > n$, $\beta = 1/2$ *and assume that* $f \in (L^q(\Omega))^n$ *is independent of* t. *Since* $X_{1/2} = [X_0, X_1]_{1/2} \hookrightarrow (W^{1,q}(\Omega))^n \subset (C(\overline{\Omega}))^n$, *it is easy to see that* $F : X_{1/2} \to X_0$ *is locally Lipschitz continuous (uniformly on bounded sets). Consequently, Theorem 1.9 guarantees the local solvability of this system for initial data* $u_0 \in X_{1/2}$. *If* f *depends also on* t *then one can use Remark 1.10(ii). Notice that we cannot use Remark 1.10(iii) to get the solution for* $u_0 \in X_0$ *since the growth assumption* (1.14) *is true only with* $p = 2$, *hence* $\beta p \not< 1$. □

Using Theorem 1.9 one can easily construct a maximal solution $u \in C((0, T_{\max}), X_\beta)$ with the following property:

$$\text{either} \quad T_{\max} = \infty \quad \text{or} \quad \lim_{t \to T_{\max}} |u(t)|_\beta = \infty. \tag{1.16}$$

In fact, assume on the contrary that $T_{\max} < \infty$ and $\liminf_{t \to T_{\max}} |u(t)|_\beta < \infty$ and choose $t_k \to T_{\max}$ such that $|u(t_k)|_\beta < C$. Then Theorem 1.9 with initial data $u_0 := u(t_k)$ shows that u can be prolongated on the intervals $[t_k, t_k + T]$, where T does not depend on k, which contradicts the maximality of $T_{\max}$. Similarly, in the situation of Remark 1.10(iii) we have either $T_{\max} = \infty$ or $\lim_{t \to T_{\max}} |u(t)|_0 = \infty$. These facts yield a simple sufficient criterion for global existence: if $|u(t)|_\beta$ (or $|u(t)|_0$ in the case of Remark 1.10(iii)) remain bounded for $t < T_{\max}$ then $T_{\max} = \infty$. Since global existence is an important issue and a priori estimates of solutions are ususally available only in low regularity spaces (typically some Lebesgue spaces) we wish to be able to choose the value of β as small as possible (or even consider $u_0 \in X_0$). The remark at the end of Example 1.12 shows that this is not always possible in the current interpolation setting. On the other hand, we will see that we can obtain much better results if we consider our problems in extrapolation spaces. This more general setting also allows us to deal with singular initial data (for example measures) and consider problems with nonlinear Neumann boundary conditions.

Extrapolation requires some duality arguments and therefore, in addition to (1.12), in the rest of this section we will assume that

$$X \text{ is reflexive and } D(A) \text{ is dense in } X. \tag{1.17}$$

In addition, for any $\theta \in (0,1)$ we consider an interpolation functor $(\cdot,\cdot)_\theta$ which can be any of the interpolation functors $(\cdot,\cdot)_{\theta,p}$, $1 < p < \infty$, or $[\cdot,\cdot]_\theta$, and we set $X_\theta := (X_0, X_1)_\theta$. Notice that all these abstract assumptions are satisfied in the case of Examples 1.5 and 1.7. Note also that our choice of X_θ implies (1.13) due to Remark 1.4.

Let A_θ denote the X_θ-realization of A in X_θ, that is $A_\theta u = Au$ and $D(A_\theta) = \{u \in D(A) : Au \in X_\theta\}$. Let X_{-1} be the completion of the normed linear space $(X_0, |\cdot|_{-1})$, where $|u|_{-1} := |(\omega + A)^{-1}u|_0$. Given $\theta \in (0,1)$, set $X_{-1+\theta} = (X_{-1}, X_0)_\theta$ and let $A_{-1+\theta}$ be the closure of A in $X_{-1+\theta}$ (it can be proved that A is closable in $X_{-1+\theta}$).

Proposition 1.13. *Let $-1 \le \beta \le \alpha \le 1$. Then the following assertions are true.*

(i) *The space X_α is densely embedded in X_β. If A has compact resolvent and $\alpha > \beta$ then the embedding $X_\alpha \hookrightarrow X_\beta$ is compact.*
(ii) *A_α is the X_α-realization of A_β and $\sigma(A_\alpha) = \sigma(A_\beta)$.*
(iii) *$-A_\alpha$ generates a C^0 analytic semigroup e^{-tA_α} in X_α. In addition, $e^{-tA_\alpha} = e^{-tA_\beta}|_{X_\alpha}$, and there exists $C = C(\omega, A) > 0$ such that*

$$\|e^{-tA_\beta}\|_{\mathcal{L}(X_\beta, X_\alpha)} \le C t^{\beta-\alpha} e^{\omega t} \qquad \text{for all } t > 0. \tag{1.18}$$

If $\alpha \in [-1,1]$ and no confusion seems likely then we will shortly write A and e^{-tA} instead of A_α and e^{-tA_α}, respectively. Similarly as in the case $\alpha \ge 0$, we will denote by $|\cdot|_\alpha$ the norm in X_α also for $\alpha < 0$.

Example 1.14. *Let X_0, X_1 and $(\cdot,\cdot)_\theta$ be as in Example 1.5 and notice that $X_\theta = X_\theta(q)$, $\theta \in [0,1]$. Using duality arguments one can prove $X_\theta(q) \doteq [X_{-\theta}(q')]'$ for $\theta \in [-1,0)$, where $q' = q/(q-1)$.*

Analogous statement is true in the case of Example 1.7. □

The next theorem is an analogue of Theorem 1.9 and Remark 1.10(iii) in the extrapolation setting.

Theorem 1.15. *Let $1 \ge \alpha > \beta \ge 0$ and let $F : X_\beta \to X_{\alpha-1}$ be locally Lipschitz continuous, uniformly on bounded sets.*

(i) *If $u_0 \in X_\beta$ then there exists $T = T(|u_0|_\beta) > 0$ such that the variation-of-constants formula (1.4) has a unique solution $u \in C([0,T], X_\beta)$. If $\gamma \in (\beta, \alpha)$ then $u \in C^{\alpha-\gamma}_{loc}((0,T], X_\gamma)$.*
(ii) *Let $p > 1$, $\delta \in (\beta - 1/p, \beta)$, $u_0 \in X_\delta$ and*

$$|F(u) - F(v)|_{\alpha-1} \le C|u - v|_\beta (1 + |u|_\beta^{p-1} + |v|_\beta^{p-1}). \tag{1.19}$$

  (iia) *If $\alpha > (\beta - \delta)p + \delta$ then there exists $T = T(|u_0|_\delta) > 0$ such that (1.4) has a unique solution $u \in L^\infty_{\beta-\delta}((0,T), X_\beta)$. The solution satisfies $u \in Z := C((0,T], X_\beta) \cap C([0,T], X_\delta)$.*
  (iib) *If $\alpha = (\beta - \delta)p + \delta$ then there exists $T = T(u_0) > 0$ such that (1.4) has a unique solution $u \in L^\infty_{\beta-\delta}((0,T), X_\beta) \cap Z$.*

In both cases the solution is unique in Z.

PROOF. The proofs of (i) and (iia) are straightforward analogues of the proofs of Theorem 1.9 and Remark 1.10(iii). In case (iib) one has to use the fact that $t^{\beta-\delta}|e^{-tA}u_0|_\beta \to 0$ as $t \to 0$ and to use the Banach fixed point argument in a *small* ball of the corresponding space. This solution is unique in the whole space Z, but the proof requires some additional arguments, see [6].

Note also that the time $T(u_0)$ in (iib) can chosen uniform for initial data u_0 lying in a compact subset of X_δ. □

EXAMPLE 1.16. *Consider the problem*

$$\left.\begin{aligned} u_t - \Delta u &= |u|^{p-1}u, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.20}$$

where $p > 1$. Choose $q \geq n(p-1)/2$, $q > 1$, and let X_0, X_1, A and $(\cdot,\cdot)_\theta$ be chosen as in Example 1.5, in particular $X_0 = L^q(\Omega)$.

First assume that $q > n(p-1)/2$. Set $\beta = n/2qp'$. Then $0 < \beta < 1/p$ and $X_\beta \hookrightarrow W^{2\beta,q}(\Omega) \hookrightarrow L^{pq}(\Omega)$ which shows the Lipschitz continuity of $F : X_\beta \to X_0$. More precisely, estimate (1.19) *is true with $\alpha = 1$. Consequently, we can use Theorem 1.15(iia) with $\delta = 0$ and $\alpha = 1$ (or Remark 1.10(iii)) in order to prove the well-posedness of* (1.20) *in $L^q(\Omega)$.*

Next consider the case $q = n(p-1)/2$. Set

$$\delta = 0, \qquad \beta = \frac{1}{2}\Big(\frac{n}{q} - \frac{n}{pz}\Big), \qquad \alpha = \frac{1}{2}\Big(2 + \frac{n}{q} - \frac{n}{z}\Big),$$

where $z \in (\max\{1, q/p\}, q)$. Then standard imbedding theorems and duality show $X_\beta \hookrightarrow L^{pz}(\Omega)$ and $L^z(\Omega) \hookrightarrow X_{\alpha-1}$ (since $X_{\alpha-1}(q) \doteq [X_{1-\alpha}(q')]'$ and $X_{1-\alpha}(q') \hookrightarrow W^{2-2\alpha,q'}(\Omega) \hookrightarrow L^{z'}(\Omega)$), hence $F : X_\beta \to X_{\alpha-1}$ satisfies (1.19). *Now Theorem 1.15(iib) guarantees local existence for $u_0 \in L^q(\Omega)$.*

Note that the case $q = n(p-1)/2$ is known to be critical. If $1 \leq q < n(p-1)/2$ then both existence and uniqueness may fail for suitable u_0. It is also known that the time T in the critical case cannot be chosen uniform for bounded initial data, see [6]. □

EXAMPLE 1.17. *Consider the Navier-Stokes system* (1.11) *from Example 1.7 with $f = 0$ (for simplicity). Let X_0, X_1, A, F be as in that example, $(\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta$. and assume $q \geq n \geq 2$, $q < \infty$. Notice that $X_0 = X_0(q)$ and $A = A(q)$ etc. Recall also that $X_\theta \hookrightarrow (W^{2\theta,q}(\Omega))^n$ if $\theta \geq 0$. We want to use Theorem 1.15(iib) with $\delta = 0$, $\alpha = 1/2$, $\beta = 1/4$ and $p = 2$ to get a local solution for initial data $u_0 \in X_0$. We just have to prove that $F : X_{1/4} \to X_{-1/2}$ satisfies estimate* (1.19). *Since $F(u) = -P[(u\cdot\nabla)u]$ can be considered as a bilinear map, it is sufficient to show the estimate*

$$|P(u\cdot\nabla)v|_{-1/2} \leq C|u|_{1/4}|v|_{1/4}. \tag{1.21}$$

Note that $|P(u\cdot\nabla)v|_{-1/2} \leq C|A^{-1/2}P(u\cdot\nabla)v|_0$, since $X_{-1/2} = X_{-1/2}(q) \doteq [X_{1/2}(q')]' \doteq [D(A^{1/2}(q'))]'$ and $A(q') = A(q)'$. Due to density arguments, it is sufficient to prove (1.21) *for smooth, divergence-free u, v with compact support* *in* Ω. *Since* $\operatorname{div} u = 0$, *we have* $(u \cdot \nabla)v = \sum_j \partial(u_j v)/\partial x_j$ *and a duality argument shows that* $A^{-1/2}P(\partial/\partial x_j)$ *extends to an operator in* $\mathcal{L}((L^q(\Omega))^n, X_0)$ *(since* $(\partial/\partial x_j)IA^{-1/2}(q') : X_0(q') \to (L^{q'}(\Omega))^n$ *is continuous, where* $I = P'$ *denotes the identity considered as a map in* $\mathcal{L}(X_0(q'), (L^{q'}(\Omega))^n)$*). Consequently,*

$$|P(u\cdot\nabla)v|_{-1/2} = \Big|\sum_j A^{-1/2}P(\partial/\partial x_j)(u_j v)\Big|_0 \le C\sum_j \|u_j v\|_{(L^q(\Omega))^n}$$

and the conclusion follows from the Cauchy inequality and the embedding $X_{1/4} \hookrightarrow (L^{2q}(\Omega))^n$ *(which is due to* $q \ge n$*). Note that the solution is classical for* $t > 0$*. Notice also that we can choose* $\beta < 1/4$ *(hence we can use Theorem 1.15(iia)) if* $q > n$.

Fix $q = n = 2$. *Multiplying the equation* $u_t + Au = F(u)$ *by* u *easily yields a uniform bound for* $u(t)$ *in* $(L^2(\Omega))^n)$ *(hence in* X_0*) and a bound for* (Au, u) *in* $L^1(0,T)$ *(hence a bound for* u *in* $L^2((0,T), X_{1/2})$ *since* A *is positive, self-adjoint and* $X_{1/2} \doteq \big(D(A^{1/2}), |A^{1/2}\cdot|_0\big)$*). Notice that the existence time* T *in Theorem 1.15(iib) is not uniform on bounded sets in* X_0 *so that the estimate in* X_0 *cannot be directly used in order to prove the global existence. However, a careful inspection of the proof of Theorem 1.15(iib) in [6] shows that the condition on* T *can be written in the form* $t^{\beta-\delta}|e^{-tA}u_0|_\beta \le \varepsilon$ *for* $t \in (0,T]$*, where* $\varepsilon > 0$ *does not depend on* u_0*. Recall that in our setting* $\beta = 1/4$ *and* $\delta = 0$*. Assume on the contrary that* $T_{\max} < \infty$*. Then due to the bound in* $L^2((0,T_{\max}), X_{1/2})$ *one can find a sequence* $t_k \to T_{\max}$ *such that*

$$|u(t_k)|_{1/2} = o((T_{\max} - t_k)^{-1/2}) \text{ as } k \to \infty.$$

Since $u(t_k)$ *is bounded in* X_0*, by interpolation we obtain*

$$|u(t_k)|_{1/4} = o((T_{\max} - t_k)^{-1/4}).$$

Consequently, denoting $J_k = (0, 2(T_{\max} - t_k)]$ *we can estimate*

$$\begin{aligned}\sup_{t\in J_k} t^\beta |e^{-tA}u(t_k)|_\beta &\le \sup_{t\in J_k} Ct^\beta|u(t_k)|_\beta \\ &\le \tilde{C}(T_{\max} - t_k)^{1/4}\, o((T_{\max} - t_k)^{-1/4}) = o(1).\end{aligned}$$

Hence, fixing k *large and considering* $u(t_k)$ *as initial data, we can choose* $T = 2(T_{\max} - t_k)$ *in Theorem 1.15(iib) which yields a contradiction with the maximality of* $T_{\max}$. □

EXAMPLE 1.18. *Consider the problem* (1.20) *with* $p < 1 + 2/n$ *and* $u_0 \in \mathcal{M}(\Omega)$*, where* $\mathcal{M}(\Omega)$ *denotes the space of bounded Radon measures in* Ω*. Assume also that* Ω *is bounded. Choose* $q \in (1,p)$*, and let* X_0, X_1, A *and* $(\cdot,\cdot)_\theta$ *be as in Examples 1.5, 1.16. Choose* δ *such that*

$$n - \frac{n}{q} < -2\delta < \frac{n+2}{p} - \frac{n}{q},$$

set $\alpha := 1 + \delta$ *and fix* $\beta \in \big((n/q - n/p)/2, \delta + 1/p\big)$*. Then* $\mathcal{M}(\Omega) \hookrightarrow X_\delta$ *and it is straightforward to check that all assumptions of Theorem 1.15(iia) are satisfied, hence we obtain the well-posedness of* (1.20) *in the space of measures. Obviously, the same assertion is true if we replace the nonlinearity* $|u|^{p-1}u$ *by* $-|u|^{p-1}u$*, for example.*

It is known that the condition $p < 1 + 2/n$ *is optimal for such assertions.* □

EXAMPLE 1.19. *Consider the problem*

$$\left.\begin{aligned} u_t - \Delta u &= f(u), && x \in \Omega,\ t > 0, \\ \partial u/\partial \nu &= g(u), && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.22}$$

where $f, g \in C^1$ *and* Ω *is bounded. If we consider this problem in* $X_0 = L^q(\Omega)$, $1 < q < \infty$, *then the domain of the corresponding operator* A *in* X_0 *(hence the operator itself) will depend on* $u(t)$ *due to the nonhomogeneous boundary condition and the semigroup approach cannot be used. This difficulty can be overcome if we use a suitable extrapolation setting.*

Multiplying the equation $u_t - \Delta u = f(u)$ *by a smooth function* v *satisfying* $\partial v/\partial \nu = 0$ *on* $\partial\Omega$, *using Green's theorem and the prescribed boundary condition we obtain*

$$\int_\Omega \big(u_t v - u\Delta v - f(u)v\big)\,dx - \int_{\partial\Omega} g(u)v\,dS = 0 \tag{1.23}$$

Fix $q > n$. *Let* $X_0 = X_0(q) := L^q(\Omega)$, $X_1 = X_1(q) := \{u \in W^{2,q}(\Omega) : \partial u/\partial\nu = 0\}$ *and let* $X_\theta = X_\theta(q)$, $\theta \in (0,1)$, *be defined as in Example 1.5,* $X_\theta(q) = [X_{-\theta}(q')]'$ *for* $\theta < 0$. *In particular,* $X_\theta \doteq W^{2\theta,q}(\Omega)$ *if* $0 \le 2\theta < 1 + 1/q$. *Consider* $A(q) : X_1(q) \to X_0(q) : u \mapsto -\Delta u$ *as an unbounded operator in* $X_0(q)$ *and let* $A_{-1/2}(q)$ *be the closure of* $A(q)$ *in* $X_{-1/2}(q)$ *(cf. the construction of the extrapolation scale of spaces and operators). Then the domain of* $A_{-1/2}(q)$ *equals* $X_{1/2}(q)$ *and*

$$\langle A_{-1/2}(q)u, w\rangle_{-1/2} = \langle u, A_{-1/2}(q')w\rangle_{1/2}, \qquad u \in X_{1/2}(q),\ w \in X_{1/2}(q'),$$

where $\langle\cdot,\cdot\rangle_{-1/2}$ *denotes the duality pairing between* $X_{-1/2}(q)$ *and* $X_{1/2}(q')$ *and* $\langle\cdot,\cdot\rangle_{1/2}$ *denotes the duality pairing between* $X_{1/2}(q)$ *and* $X_{-1/2}(q')$, *respectively, see [1]. Assume that* $u(t) \in X_{1/2}(q) = W^{1,q}(\Omega)$ *(hence* $f(u(t)) \in C(\overline{\Omega})$ *and* $g(u(t)) \in C(\partial\Omega)$ *due to* $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$*). Then*

$$\int_\Omega -u\Delta v\,dx = \langle u, A_{-1/2}(q')v\rangle_{1/2} = \langle A_{-1/2}(q)u, v\rangle_{-1/2}$$

so that, assuming $u_t \in X_{-1/2}(q)$, (1.23) *can be written in the form*

$$\langle u_t + Au - f(u), v\rangle_{-1/2} - \int_{\partial\Omega} g(u)v\,dS = 0, \tag{1.24}$$

where $A := A_{-1/2}(q)$. *By density, this identity remains true for all* $v \in X_{1/2}(q') = W^{1,q'}(\Omega)$ *provided we write the boundary integral in the form* $\int_{\partial\Omega} g(u)\,\gamma v\,dS$, *where* $\gamma : W^{1,q'}(\Omega) \to W^{1-1/q',q'}(\partial\Omega)$ *is the trace operator. By duality, this integral can be written as* $\langle \gamma' g(u), v\rangle_{-1/2}$, *hence* (1.24) *implies*

$$\langle u_t + Au - F(u), v\rangle_{-1/2} = 0 \qquad \text{for all } v \in X_{1/2}(q'),$$

where $F(u) := f(u) + \gamma' g(u)$. *(More precisely, we should write* $F(u) = \mathcal{F}(u) + \gamma'\mathcal{G}(u|_{\partial\Omega})$, *where* $\mathcal{F}$ *and* $\mathcal{G}$ *are the Nemytskii mappings in* Ω *and on* $\partial\Omega$ *corresponding to the local functions* f *and* g, *respectively.) Consequently, we look for solutions of the abstract equation*

$$u_t + Au = F(u), \quad t > 0, \qquad \text{in } X_{-1/2}(q).$$

Choosing $2\beta = 1$ *and* $2\alpha \in (1, 1 + 1/q)$ *we easily see that* $F : X_\beta(q) \to X_{\alpha-1}(q)$ *is locally Lipschitz continuous, uniformly on bounded sets, hence we can use Theorem 1.15(i) in order to find a unique local solution* $u \in C([0,T], W^{1,q}(\Omega))$ *of the corresponding variation-of-constants formula for any* $u_0 \in W^{1,q}(\Omega)$. *It can be proved that this solution is a classical solution of* (1.22) *for* $t > 0$, *hence the problem is well posed in* $W^{1,q}(\Omega)$. *Of course, we have a lot of freedom in the choice of spaces and we can also use Theorem 1.15(ii) in order to prove local existence for low regularity data (under suitable growth assumptions on* f' *and* g'*).* □

5. Stability and global existence for small data

Theorems 1.9 and 1.15 in the previous section provide sufficient conditions for the local well-posedness of the abstract Cauchy problem

$$u_t + Au = F(u), \quad t > 0, \qquad u(0) = u_0.$$

In this section we show that if (in addition to the hypotheses of the above theorems) we assume $\omega(-A) < 0$ and some smallness condition on $F(u)$ for u small then $u = 0$ is a (locally) exponentially asymptotically stable solution. In particular, we obtain global existence for small initial data.

The abstract results apply to most of the examples mentioned above including the Navier-Stokes system or problem (1.20) with Ω bounded. As we shall see in subsequent sections, the smallness assumption is often also necessary for global existence, for example in the case of problem (1.20).

Similarly as in the previous section we first formulate and prove the result in the interpolation setting and then we formulate its "extrapolation analogue" (and we refer to [6] for a detailed proof).

Hence assume (1.12) and (1.13). Then we have the following theorem (cf. Theorem 1.9).

THEOREM 1.20. *Let* $\beta \in [0,1)$, $u_0 \in X_\beta$ *and* $F : X_\beta \to X_0$ *be locally Lipschitz continuous, uniformly on bounded sets in* X_β. *Assume, in addition, that* $\omega(-A) < 0$ *and*

$$|F(u)|_0 = o(|u|_\beta) \qquad \text{as } |u|_\beta \to 0.$$

Then the zero solution of (1.4) *is (locally) exponentially asymptotically stable. More precisely, given* $\tilde\omega \in (\omega(-A), 0)$ *there exist* $\eta > 0$ *and* $C > 0$ *such that the solution* u *with initial data* u_0 *satisfying* $|u_0|_\beta < \eta$ *exists globally and*

$$|u(t)|_\beta \le Ce^{\tilde\omega t}|u_0|_\beta \qquad \text{for all } t \ge 0. \tag{1.25}$$

PROOF. Let $\tilde\omega \in (\omega(-A), 0)$. Choose $\omega \in (\omega(-A), \tilde\omega)$. Then (1.7) and (1.8) guarantee

$$\left.\begin{aligned} \|e^{-tA}\|_{\mathcal{L}(X_0,X_\beta)} &\le C_\omega t^{-\beta} e^{\omega t} \\ \|e^{-tA}\|_{\mathcal{L}(X_\beta)} &\le C_\omega e^{\omega t} \end{aligned}\right\} \qquad \text{for all } t > 0, \tag{1.26}$$

where $C_\omega \ge 1$. Set

$$C^* = C_\omega \int_0^\infty \tau^{-\beta} e^{(\omega-\tilde\omega)\tau}\,d\tau$$

and choose $\varepsilon > 0$ such that

$$|F(u)|_0 \le \frac{1}{2C^*}|u|_\beta \qquad \text{whenever } |u|_\beta \le \varepsilon. \tag{1.27}$$

Choose $\eta = \varepsilon/2C_\omega$ and let $|u_0|_\beta < \eta$. We may assume $u_0 \neq 0$. Set

$$T = \sup\{t \in (0, T_{\max}(u_0)) : |u(s)|_\beta \leq 2C_\omega e^{\tilde{\omega}s}|u_0|_\beta \text{ for all } s \in [0,t]\}$$

and notice that $T > 0$ and $|u(s)|_\beta \leq \varepsilon$ for all $s \in [0,T)$. If $T = \infty$ then (1.25) is true. Hence, assume $T < \infty$. Then $T < T_{\max}(u_0)$ due to the uniform bound of $|u(s)|_\beta$ for $s \in [0,T)$, hence

$$|u(T)|_\beta = 2C_\omega e^{\tilde{\omega}T}|u_0|_\beta. \tag{1.28}$$

On the other hand, using (1.26), (1.27), the inequality in the definition of T and the definition of C^* we obtain

$$\begin{aligned} |u(T)|_\beta &\leq C_\omega e^{\omega T}|u_0|_\beta + C_\omega \int_0^T (T-s)^{-\beta} e^{\omega(T-s)}|F(u(s))|_0\, ds \\ &\leq C_\omega e^{\omega T}|u_0|_\beta + \frac{C_\omega^2}{C^*} e^{\tilde{\omega}T}|u_0|_\beta \int_0^T (T-s)^{-\beta} e^{(\omega-\tilde{\omega})(T-s)}\, ds \\ &< C_\omega e^{\omega T}|u_0|_\beta + C_\omega e^{\tilde{\omega}T}|u_0|_\beta \leq 2C_\omega e^{\tilde{\omega}T}|u_0|_\beta, \end{aligned}$$

which contradicts (1.28) and concludes the proof. □

REMARK 1.21. *Under some additional assumptions on A and F one can choose $\tilde{\omega} = \omega(-A)$ in the above theorem, see [6].* □

Next we formulate the analogue of Theorem 1.20 in the extrapolation setting (cf. Theorem 1.15). Hence, in addition to (1.12) we assume (1.17) and $X_\theta = (X_0, X_1)_\theta$, $\theta \in (0,1)$, where $(\cdot,\cdot)_\theta$ is either the interpolation functor $(\cdot,\cdot)_{\theta,q}$ for some $q \in (1,\infty)$ or the complex interpolation functor $[\cdot,\cdot]_\theta$. The spaces X_θ, $\theta \in [-1,0)$ are defined as in the previous section.

THEOREM 1.22. *Let $1 \geq \alpha > \beta \geq 0$ and let $F : X_\beta \to X_{\alpha-1}$ be locally Lipschitz continuous, uniformly on bounded sets. Assume also that $\omega(-A) < 0$ and*

$$|F(u)|_{\alpha-1} = o(|u|_\beta) \qquad \text{as } |u|_\beta \to 0.$$

Then we have the following.

(i) *The zero solution of (1.4) is (locally) exponentially asymptotically stable in X_β. More precisely, given $\tilde{\omega} \in (\omega(-A), 0)$ there exist $\eta > 0$ and $C > 0$ such that the solution u with initial data u_0 satisfying $|u_0|_\beta < \eta$ exists globally and*

$$|u(t)|_\beta \leq Ce^{\tilde{\omega}t}|u_0|_\beta \qquad \text{for all } t \geq 0.$$

(ii) *Let $p > 1$, $\delta \in (\beta - 1/p, \beta)$, $u_0 \in X_\delta$, $\alpha \geq (\beta-\delta)p + \delta$ and*

$$|F(u) - F(v)|_{\alpha-1} \leq C|u-v|_\beta(1 + |u|_\beta^{p-1} + |v|_\beta^{p-1}).$$

If $\alpha = (\beta-\delta)p + \delta$ assume also

$$|F(u)|_{\alpha-1} \leq C|u|_\beta^p.$$

Then, given $\tilde{\omega} \in (\omega(-A), 0)$, there exist $\eta > 0$ and $C > 0$ such that the solution u with initial data u_0 satisfying $|u_0|_\delta < \eta$ exists globally and

$$|u(t)|_\beta \leq Ct^{\delta-\beta}e^{\tilde{\omega}t}|u_0|_\delta \qquad \text{for all } t > 0.$$

Let us mention again that if $\Omega \subset \mathbb{R}^n$ is bounded then we can use Theorem 1.22(ii) both in the case of the Navier-Stokes system (1.11) (with $f = 0$, $n \geq 2$, $\delta = 0$, $\alpha = 1/2$, $\beta = 1/4$ and $p = 2$, cf. Example 1.17) and problem (1.20) (with $\delta = 0$, $X_0 = L^q(\Omega)$, $q \geq n(p-1)/2$, $q > 1$, cf. Example 1.16) since $\omega(-A) < 0$ in both cases.

Global existence in the case of the Navier-Stokes system and small initial data can be shown also for small non-zero f, see [3]. Finally, if we consider the case $q = n = 2$ in Example 1.17 and $u_0 \in X_0$ then the estimate $\int_0^\infty (Au, u)\,dt < \infty$ implies $\liminf_{t\to\infty} |u(t)|_0 = 0$, hence Theorem 1.22(ii) guarantees an exponential decay of the solution $u(t)$ (without any smallness assumption on u_0). Consequently, the zero is globally asymptotically stable.

6. Blow-up in L^∞ and gradient blow-up

Consider the problem

$$\left.\begin{aligned} u_t - \Delta u &= |u|^{p-1}u, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.29}$$

where $p > 1$. It is easy to see that the solution of the ODE $y' = |y|^{p-1}y$, $y(0) = y_0$, blows up in finite time whenever $y_0 \neq 0$. We will see that the diffusion operator Δ together with homogeneous Dirichlet boundary conditions will prevent blow-up of solutions of (1.29) provided the initial data are small enough. On the other hand, if the initial data are "large" then the solution of (1.29) does blow up in finite time. In order to specify what does "large" mean, we have to introduce some notation. Let λ_1 be the first eigenvalue of the negative Dirichlet Laplacian in Ω and let φ_1 be the corresponding eigenfunction normalized by the condition $\int_\Omega \varphi_1(x)\,dx = 1$. Then

$$-\Delta\varphi_1 = \lambda_1\varphi_1 \quad \text{in } \Omega, \qquad \varphi_1 > 0 \quad \text{in } \Omega, \qquad \varphi_1 = 0 \quad \text{on } \partial\Omega.$$

We also denote by E the "energy function" associated to (1.29),

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx. \tag{1.30}$$

Notice that $\lim_{\alpha\to\infty} E(\alpha\varphi) = -\infty$ for any $\varphi \in H^1(\Omega)$, $\varphi \not\equiv 0$, hence $E(\alpha\varphi) < 0$ if α is large enough. Finally, we denote by $\|\cdot\|_q$ the norm in $L^q(\Omega)$.

Theorem 1.23. *Let Ω be bounded and $u_0 \in L^q(\Omega)$, where $q \in (1,\infty]$, $q > n(p-1)/2$. Then the following is true.*

(i) There exists $\eta > 0$ such that $T_{\max}(u_0) = \infty$ and $\|u(t;u_0)\|_q \to 0$ as $t \to \infty$ whenever $\|u_0\|_q < \eta$.

(ii) Assume that either

$$u_0 \geq 0, \qquad \Big(\int_\Omega u_0\varphi_1\,dx\Big)^{p-1} > \lambda_1, \tag{1.31}$$

or

$$u_0 \in H_0^1(\Omega), \qquad E(u_0) < 0. \tag{1.32}$$

Then $T_{\max}(u_0) < \infty$ and $\|u(t;u_0)\|_q \to \infty$ as $t \to T_{\max}(u_0)$.

PROOF. Part (i) is a consequence of Example 1.16 and Theorem 1.22.

Assume (1.31) and set $y(t) := \int_\Omega u(t)\varphi_1\,dx$, where $u(t) = u(t;u_0)$. Then (1.31) guarantees

$$(1-\varepsilon)y(0)^{p-1} > \lambda_1 \qquad \text{for some } \varepsilon > 0. \tag{1.33}$$

Multiplying the differential equation in (1.29) by φ_1, integrating over Ω and writing $\int f$ instead of $\int_\Omega f(x,t)\,dx$ we obtain

$$\begin{aligned} y' &= \int u_t\varphi_1 = \int (\Delta u + u^p)\varphi_1 = \int u^p\varphi_1 + \int u\Delta\varphi_1 \\ &\geq \Big(\int u\varphi_1\Big)^p - \lambda_1\int u\varphi_1 = y^p - \lambda_1 y = \varepsilon y^p + y\big((1-\varepsilon)y^{p-1} - \lambda_1\big), \end{aligned}$$

where we used integration by parts (Green's theorem) and Jensen's inequality. This estimate and (1.33) guarantee $y' \geq \varepsilon y^p$, hence $T_{\max}(u_0) < \infty$. The assertion $\|u(t)\|_q \to \infty$ as $t \to T_{\max}(u_0)$ follows from (1.16).

Next assume (1.32) and set $y(t) := \int_\Omega u(t)^2\,dx$. Then multiplying the differential equation in (1.29) by u and integrating over Ω yields

$$\begin{aligned} \frac{1}{2}y'(t) &= \int uu_t = \int \big(-|\nabla u|^2 + |u|^{p+1}\big) \\ &= -2E(u(t)) + c_p\int |u|^{p+1} \geq -2E(u_0) + \tilde{C}c_p y^{(p+1)/2}, \end{aligned} \tag{1.34}$$

where $c_p := (p-1)/(p+1)$, $\tilde{C} > 0$ and we used integration by parts, monotonicity of $E(u(t))$ (see (1.2)) and Hölder's inequality. Since $E(u_0) < 0$, estimate (1.34) guarantees $T_{\max}(u_0) < \infty$. □

REMARKS 1.24. *(i) Let $T_{\max}(u_0) < \infty$ and let y be one of the functions defined in the proof of Theorem 1.23(ii). Notice that that proof does not imply* $\lim_{t\to T_{\max}(u_0)} y(t) = \infty$.

If $q := 2 > n(p-1)/2$ then the L^2-norm of $u(t;u_0)$ does blow up as $t \to T_{\max}(u_0)$ due to Example 1.16 and (1.16). *On the other hand if $2 < n(p-1)/2$ then the L^2-norm of $u(t;u_0)$ may stay bounded as $t \to T_{\max}(u_0)$.*

In fact, let $\Omega = B_R := \{x : |x| < R\}$ and let $u_0 \in C^1$ be radially symmetric, radially decreasing, satisfy the boundary conditions and $T_{\max}(u_0) < \infty$. Then, given $\alpha > 2/(p-1)$, there exists $C_\alpha > 0$ such that

$$u(x,t) \leq C_\alpha|x|^{-\alpha} \qquad \textit{for all } x \in \Omega,\ t < T_{\max}(u_0); \tag{1.35}$$

in particular $\limsup_{t\to T_{\max}(u_0)} \|u(t)\|_q < \infty$ *for all $q < n(p-1)/2$. Estimate* (1.35) *follows from the maximum principle applied to the function $J := r^{n-1}u_r + \varepsilon r^{n+\delta}u^\gamma$, where $r = |x|$, $\varepsilon,\delta > 0$ are small enough and $\gamma \in (1,p)$: Function J satisfies a parabolic inequality in $Q := \Omega \times (\eta, T_{\max}(u_0))$, $\eta > 0$, and $J \leq 0$ on the corresponding parabolic boundary, hence $J \leq 0$ in Q. Integrating inequality $J \leq 0$ with respect to r one obtaines estimate* (1.35). *Notice that estimate* (1.35) *also guarantees that u blows up only at $x = 0$ (so-called* single point blow-up*).*

(ii) If we replace the homogeneous Dirichlet boundary conditions in (1.29) *with the homogeneous Neumann boundary conditions $\partial_\nu u = 0$ then all positive solutions of* (1.29) *blow up in finite time. In fact, if $u_0 \geq 0$, $u \not\equiv 0$, and $t_0 > 0$ is small then $u(t_0) > \varepsilon$ in $\overline{\Omega}$ for some $\varepsilon > 0$ due to the strong parabolic maximum principle. Now*

the comparison principle shows $u(\cdot, t+t_0) \geq y(t)$, $t \geq 0$, *where* y *is the solution of the ODE* $y' = y^p$, $y(0) = \varepsilon$. *On the other hand, there exist global nontrivial (sign-changing) solutions of* (1.29) *with Neumann boundary conditions: These solutions belong to the stable manifold of the trivial solution which has codimension 1.*

(iii) Consider problem (1.29) *with* $\Omega = \mathbb{R}^n$ *(so that no boundary conditions are prescribed). Notice that choosing* $X_0 = L^q(\mathbb{R}^n)$, $q > n(p-1)/2$, $q > 1$, *and* $Au = -\Delta u$ *with* $D(A) = W^{2,q}(\mathbb{R}^n)$, *we have* $\omega(-A) = 0$ *so that Theorem 1.22 cannot be used. In this case, the existence of global positive solutions depends on the exponent* p. *More precisely, the following is true: If* $p \leq 1 + 2/n$ *then there are no positive global solutions. If* $p > 1 + 2/n$ *then "small" solutions are global while "large" solutions blow up in finite time. The exponent* $p_F := 1 + 2/n$ *is called Fujita's exponent since its role in blow-up was discovered by H. Fujita in 1966. Let us sketch the proof of the blow-up statements.*

In order to prove an analogue of Theorem 1.23(ii), set $\varphi(x) := \pi^{-n/2} e^{-|x|^2}$. *This function is not an eigenfunction of* A *(the spectrum of* Δ *in* $L^q(\mathbb{R}^n)$ *is purely continuous) but* $\Delta\varphi \geq -2n\varphi$ *so that one can repeat the considerations in the first part of the proof of Theorem 1.23(ii) in order to prove blow-up whenever* $\int u_0\varphi > 2n$. *If* $u_0 \in H^1(\mathbb{R}^n)$ *and* $E(u_0) < 0$ *then one can again prove* $T_{\max}(u_0) < \infty$ *but the proof is more complicated than that in Theorem 1.23. One defines* $M(t) := \int_0^t \int_{\mathbb{R}^n} u^2 \, dx \, d\tau$ *and shows that* $(M^{-\varepsilon})'' < 0$ *for some* $\varepsilon > 0$ *and* t *large enough. Since the function* $M^{-\varepsilon}$ *is positive, decreasing and concave for* t *large, it cannot exist globally. This concavity argument was first used by H.A. Levine in 1973.*

Finally, assume that $p \leq 1 + 2/n$ *and fix* $\xi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ *with* $\xi \equiv 1$ *in* $B_{1/2} \times (-1/2, 1/2)$, $\xi \equiv 0$ *outside* $B_1 \times (-1, 1)$, $0 \leq \xi \leq 1$. *Setting* $\varphi_R(x,t) = \xi^{2p/(p-1)}(x/R, t/R^2)$ *we have*

$$|\Delta\varphi_R| + |\partial_t\varphi_R| \leq CR^{-2}\varphi_R^{1/p}. \tag{1.36}$$

Now assume that u *is a nonnegative classical solution of our problem. Multiplying the equation by* φ_R, *integrating over* $\mathbb{R}^n \times (0, \infty)$ *and using estimate* (1.36) *together with Hölder's inequality we obtain*

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} u^p \varphi_R &\leq -\int_0^\infty \int_{\mathbb{R}^n} u(\Delta\varphi_R + \partial_t\varphi_R) \\ &\leq CR^{-\alpha}\Big(\int\int_{R/2<|x|+\sqrt{t}<2R} u^p\varphi_R\Big)^{1/p}, \end{aligned} \tag{1.37}$$

where $\alpha = (n+2)/p - n$. *If* $p < 1 + 2/n$ *then* $\alpha > 0$ *and passing to the limit as* $R \to \infty$ *we obtain* $u \equiv 0$. *If* $p = 1 + 2/n$ *then* $\alpha = 0$ *and* (1.37) *implies* $\int_0^\infty \int_{\mathbb{R}^n} u^p < \infty$ *so that the right-hand side in* (1.37) *converges to zero as* $R \to \infty$ *and we have* $u \equiv 0$ *again.* □

Next we would like to construct a solution u such that

$$\limsup_{t\to T} \|u(t)\|_\infty < \infty \quad \text{and} \quad \lim_{t\to T} \|\nabla u(t)\|_\infty = \infty$$

(so-called *gradient blow-up*). We know from Example 1.11 and (1.16) that if a solution of the equation $u_t - \Delta u = f(u)$ (complemented with suitable boundary conditions) ceases to exist in finite time then it becomes unbounded in $L^\infty(\Omega)$.

Therefore, if we want to find an example of gradient blow-up then we are naturally led to problems with nonlinearities depending on the gradient of the solution. However, Example 1.17 shows that even some systems with gradient-dependent nonlinearities are well posed in $L^\infty(\Omega)$ so that gradient blow-up cannot occur either. In fact, Bernstein-type estimates guarantee that gradient blow-up can never occur for scalar equations if the growth of the nonlinearity in the gradient variable is at most quadratic.

Let us consider one of the simplest possible model problems with gradient-dependent nonlinearities:

$$\left.\begin{aligned} u_t - \Delta u &= |\nabla u|^p, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.38}$$

where $p > 1$. We know from Example 1.11 that this problem is well posed in $X := \{u \in BC^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ and we will assume $u_0 \in X$, $u_0 \geq 0$. Then the parabolic maximum principle guarantees $u \geq 0$ and $\|u(t)\|_\infty \leq \|u_0\|_\infty$, hence the solution can never blow up in $L^\infty(\Omega)$. An easy combination of Example 1.11 and Theorem 1.15(iia) shows that (1.38) is well posed in $L^\infty(\Omega)$ if $p < 2$, hence all positive solutions are global (and bounded) in this case (if Ω is bounded then it is also easy to see that all these solutions decay to zero exponentially fast). The Bernstein-type estimates mentioned above guarantee that the same is true if $p = 2$ (in the particular situation of (1.38) with $p = 2$, the global existence also follows by using the Hopf–Cole transformation $v := e^u - 1$ since v satisfies $v_t - \Delta v = 0$ in Ω, $v = 0$ on $\partial\Omega$). On the other hand, the following theorem shows that gradient blow-up can occur for some solutions of (1.38) if $p > 2$.

Theorem 1.25. *Consider problem* (1.38) *with $u_0 \in X$, $u_0 \geq 0$, $p > 2$, Ω bounded. If $\int_\Omega u_0\varphi_1\,dx$ is sufficiently large then $T_{\max}(u_0) < \infty$.*

Proof. Set $y(t) = \int_\Omega u(t)\varphi_1\,dx$. Then similarly as in the proof of Theorem 1.23 we obtain

$$y' = \int u_t\varphi_1 = \int |\nabla u|^p\varphi_1 + \int u(\Delta\varphi_1) \geq cy^p - \lambda_1 y,$$

where the estimate $\int |\nabla u|^p\varphi_1 \geq cy^p$ folows from

$$\begin{aligned} y = \int u\varphi_1 \leq C_1\int u \leq C_2\int |\nabla u| &= C_2\int \big(|\nabla u|\varphi_1^{1/p}\big)\varphi_1^{-1/p} \\ \leq C_3\Big(\int |\nabla u|^p\varphi_1\Big)^{1/p}\Big(\int \varphi_1^{-1/(p-1)}\Big) &= C_4\Big(\int |\nabla u|^p\varphi_1\Big)^{1/p}. \end{aligned}$$

Here we used the Poincaré and Hölder inequalities and the fact that $p > 2$ and $\varphi_1(x) \geq c\,\mathrm{dist}(x,\partial\Omega)$. □

Remark 1.26. *Instead of assuming that $\int_\Omega u_0\varphi_1$ is large in Theorem 1.25, it would be sufficient to assume that $\|u_0\|_q$ is large for some $q \in [1,\infty)$. In fact, assuming without loss of generality $q \geq 2(p-1)/(p-2)$ and denoting $y(t) = \int_\Omega u^q(t)\,dx$, the Poincaré and Hölder inequalities can be used in order to prove the blow-up inequality $y' \geq c_1 y^{(q+p-1)/q} - c_2$.* □

Let u be a solution of (1.38) which blows up in finite time and $v := u_{x_i}$. Applying the maximum principle to the equation

$$v_t - \Delta v = p \sum_j |u_{x_j}|^{p-2} u_{x_j} v_{x_j}$$

we see that v attains its maximum on the parabolic boundary, hence the gradient blow-up of u also occurs on the boundary. It can even be proved that it occurs only on the boundary: ∇u remains bounded far away from $\partial\Omega$. The following example shows that gradient blow-up can also occur inside the domain (*interior gradient blow-up*).

EXAMPLE 1.27. *Consider the problem*

$$\left.\begin{aligned} &u_t - u_{xx} = u|u_x|^p, && x \in (-1,1),\ t > 0,\\ &u(\pm 1, t) = \pm A, && t > 0,\\ &u(x,0) = u_0(x), && x \in (-1,1), \end{aligned}\right\} \tag{1.39}$$

where $p > 2$, $A > 0$, $u_0 \in C^2([-1,1])$ is odd, $u_0' \geq 0$ and $u_0'' \leq 0$ on $[0,1]$, and u_0 satisfies the following compatibility conditions: $u_0(1) = A$, $u_0''(1) + A u_0'(1)^p = 0$. We will prove that if A is large enough then $T := T_{\max}(u_0)$ is finite, $\lim_{t\to T} u_x(0,t) = \infty$ and $0 \leq u_x(x,t) \leq A/|x|$ for $0 < |x| < 1$, $0 < t < T$.

The maximum principle guarantees $|u| \leq A$ and we also have that $u(\cdot,t)$ is odd for any $t < T$. Denote $v := u_x$, $w = u_{xx}$. Applying the maximum principle to the equation

$$v_t - v_{xx} = (u|v|^p)_x = |v|^p v + pu|v|^{p-2} v v_x$$

we obtain $v \geq 0$ (since $v(1,t) = v(-1,t) \geq 0$). Similarly, since $w(1,t) = -uv^p(1,t) \leq 0$, $w(0,t) = 0$ and

$$w_t - w_{xx} = (p+1) v^p w + p(uv^{p-1} w)_x,$$

we have $w = u_{xx} \leq 0$ for $x \in [0,1]$ and $t \geq 0$. This inequality implies

$$A \geq u(x,t) - u(0,t) \geq x u_x(x,t), \qquad x > 0,\ t > 0,$$

thus $u_x(x,t) \leq A/x$ for $x > 0$ and $t > 0$. Set $y(t) := \int_0^1 u(t)\varphi\,dx$, where $\varphi(x) := \sin(\pi x)$. Then

$$\begin{aligned} y' &= \int_0^1 u_t \varphi = \int_0^1 (u_{xx} + u|u_x|^p)\varphi \\ &= -u\varphi'|_0^1 + \int_0^1 u\varphi'' + \int_0^1 u|u_x|^p \varphi > -\pi^2 A + cA^{p+1} > 1, \end{aligned}$$

where we used the inequalities $-u\varphi'|_0^1 > 0$, $y(t) \leq A$ and

$$A^{1+1/p} = \int_0^1 (u^{1+1/p})_x = (1+1/p)\int_0^1 \big(u^{1/p} u_x \varphi^{1/p}\big)\varphi^{-1/p} \leq C\Big(\int_0^1 u u_x^p \varphi\Big)^{1/p}.$$

In the last estimate we used Hölder's inequality and $\int_0^1 \varphi^{-1/(p-1)} < \infty$ due to $p > 2$. Since $y' > 1$ and $y \leq A$, we see that $T < \infty$. □

The assertion in the previous example is a very special case of a result due to S. Angenent and M. Fila. Their original argument guaranteeing interior gradient blow-up was based on the construction of suitable singular sub- and supersolutions v_- and v_+. In our case, these singular solutions can be found in the form of traveling waves

$$v_-(x,t) = \Psi(x + (t-1)), \qquad v_+(x,t) = -\Psi(-x + (t-1)),$$

where $\Psi : [0,\infty) \to [0,\infty)$ is an increasing function, $\Psi(0) = 0$, $\Psi'(0) = \infty$, $\Psi(1) < A$. Notice that v_- and v_+ meet at $x = 0$ and $t = 1$ (and their spatial derivatives at $x = 0$ are infinite) so that the solution u lying between v_- and v_+ has to blow up at time $T \leq 1$. Additional arguments show that this blow-up can occur only where $f(u, u_x) := u|u_x|^p$ changes sign (which is the point $x = 0$ in our case).

7. The role of diffusion in blow-up

Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz continuous, $N \geq 1$. In this part we will compare the system of ODEs

$$U_t = f(U),\ t > 0, \qquad U(0) = U_0, \tag{1.40}$$

with the system of PDEs

$$u_t - D\Delta u = f(u),\ x \in \Omega,\ t > 0, \qquad u(x,0) = u_0(x), \tag{1.41}$$

complemented with homogeneous Dirichlet (or Neumann) boundary conditions, where $\Omega \subset \mathbb{R}^n$, $u_0 \in L^\infty(\Omega)$ and $D = \operatorname{diag}(d_1, d_2, \dots, d_N)$ with $d_i > 0$, $i = 1, 2, \dots, N$. Theorem 1.23 shows that adding diffusion and Dirichlet boundary conditions can change the stability of the zero solution: If $N = 1$, $f(u) = |u|^{p-1}u$, $p > 1$, and Ω is bounded then zero is an unstable equilibrium of (1.40) but a stable equilibrium of (1.41) with Dirichlet boundary conditions. On the other hand, it was shown by A.M. Turing in 1952 that adding diffusion (with different diffusion coefficients d_i) and Neumann boundary conditions to a suitable system of ODEs of the form (1.40) can destabilize a stable equilibrium. We will be interested in the question whether global existence of (all) solutions of (1.40) guarantees global existence for (1.41) and vice versa.

Notice that the only way how the solution of (1.41) can cease to exist is the L^∞-blow-up. Hence if all solutions of (1.40) are global and $N = 1$ then the maximum principle implies global existence for all solutions of (1.41). On the other hand, if some solutions of (1.40) blow up in finite time then the same is true for (1.41) if we complement it by homogeneous Neumann boundary conditions or if $\Omega = \mathbb{R}^n$ (since any solution of (1.40) is a spatially homogeneous solution of (1.41) in that case). Consequently, if we want to construct examples with blow-up for (1.40) and global existence (of all solutions) for (1.41) then we have to impose Dirichlet boundary conditions. Similarly, if we wish to find an example with global existence of all solutions of (1.40) but blow-up for (1.41) then we have to work with $N > 1$.

It turns out that even for $N = 1$ one can construct a smooth positive function f such that all solutions of (1.40) blow up in finite time but all solutions of (1.41) (with homogeneous Dirichlet boundary conditions and Ω bounded) are global and bounded. However such a function f is somewhat artificial. In what follows we will consider the case $N = 2$ and a system of the form

$$\left.\begin{aligned} u_t - d_1\Delta u &= f(u-v) - \alpha u, \\ v_t - d_2\Delta v &= f(u-v) - v, \end{aligned}\right\} \quad x \in \Omega,\ t > 0, \tag{1.42}$$

complemented with either Dirichlet boundary conditions

$$u = v = 0 \quad x \in \partial\Omega,\ t > 0, \tag{1.43}$$

or Neumann boundary conditions

$$\partial_\nu u = \partial_\nu v = 0 \quad x \in \partial\Omega,\ t > 0. \tag{1.44}$$

Here $f(w) = |w|^{p-1}w$, $p > 1$, $\alpha \geq 0$, Ω is bounded and we consider solutions with initial data $u_0, v_0 \in L^\infty(\Omega)$. Of course, we will also study the corresponding system of ODEs

$$\begin{aligned} U_t &= f(U-V) - \alpha U, \\ V_t &= f(U-V) - V, \end{aligned} \tag{1.45}$$

with initial data $U_0, V_0 \in \mathbb{R}$. The following theorem is due to M. Fila, H. Ninomiya and J.L. Vázquez in case (i) and N. Mizoguchi, H. Ninomiya and E. Yanagida in case (ii).

THEOREM 1.28. *(i) Let $d_1, d_2 > 0$, $d_1 - d_2 > 1/\lambda_1$, $\alpha = 0$, $p(n-2) < n+2$. Then some solutions of* (1.45) *blow up in finite time while all solutions of* (1.42),(1.43) *are global and converge to* $(0,0)$ *as $t \to \infty$.*

(ii) Let $d_2 > d_1 \geq 0$, $\alpha = 1$. Then some solutions of (1.42),(1.44) *blow up in finite time while all solutions of* (1.45) *are global and converge to* $(0,0)$ *as $t \to \infty$.*

PROOF. (i) Let us first prove that some solutions of (1.45) blow up in finite time. Set

$$K := \{(U,V) : (p-1)f(U-V) > pV > U > 0\}.$$

It is easy to check that K is positively invariant for (1.45) (since $\big((p-1)f(U-V) - pV\big)_t > 0$, $(pV-U)_t > 0$ and $U_t > 0$ whenever $(U,V) \in K$). Fix initial data $(U_0, V_0) \in K$, $q \in (1,p)$ and denote $W = U - V$. Then using $(U(t), V(t)) \in K$ we obtain

$$(W^q + V)_t = qW^{q-1}V + W^p - V > qV^{1+(q-1)/p} + \frac{1}{p}W^p > c(W^q+V)^{1+\varepsilon}$$

for some $c, \varepsilon > 0$, where we used $W_t = V$, $U > V > 0$ and $(p-1)W^p > pV$. Consequently, (U,V) has to blow up in finite time.

Next choose $(u_0, v_0) \in (L^\infty(\Omega) \times L^\infty(\Omega))$ and let us show that the corresponding solution of (1.42),(1.43) exists globally and converges to the trivial equilibrium. First let us show the global existence. Since the problem is well posed in $L^q(\Omega) \times L^q(\Omega)$ for any $q > n(p-1)/2$ and $p+1 > n(p-1)/2$, it is sufficient to find a bound in $L^{p+1}(\Omega) \times L^{p+1}(\Omega)$. Due to the smoothing properties of the semiflow generated by (1.42),(1.43), we can assume $u_0, v_0 \in H^1(\Omega)$. Denote $w = u - v$. Then (w,v) solves the system

$$\left.\begin{aligned} w_t - d_1\Delta w &= (d_1 - d_2)\Delta v + v, \\ v_t - d_2\Delta v + v &= |w|^{p-1}w, \end{aligned}\right\} \quad x \in \Omega,\ t > 0, \tag{1.46}$$

and satisfies homogeneous Dirichlet boundary conditions. Let $-A$ denote the Dirichlet Laplacian in Ω. Then the right-hand side of the first equation in (1.46) can be written as $-Bv$, where $B := (d_1-d_2)A-1$ is a positive self-adjoint operator in $L^2(\Omega)$ with compact resolvent. Hence, denoting $K := B^{-1}$, the first equation in (1.46) can be written as

$$v = K[d_1\Delta w - w_t], \tag{1.47}$$

where K is a positive self-adjoint compact operator in $L^2(\Omega)$. Set $\|\varphi\|_{-1} := \|K^{1/2}\varphi\|_{L^2(\Omega)}$ and

$$L(t) := \frac{1}{2}\|w_t(t)\|_{-1}^2 + \frac{d_1d_2}{2}\|Aw(t)\|_{-1}^2 + \frac{d_1}{2}\|A^{1/2}w(t)\|_{-1}^2 + \frac{1}{p+1}\int_\Omega |w|^{p+1}(t)\,dx.$$

If we substitute the right-hand side in (1.47) for v in the second equation in (1.46), multiply the equation by w and integrate over Ω then we obtain

$$\frac{d}{dt}L(t) = -(d_1+d_2)\|A^{1/2}w_t\|_{-1}^2 - \|w_t\|_{-1}^2 \le 0,$$

hence $w(t)$ stays bounded in $L^{p+1}(\Omega)$. Now the second equation in (1.46) guarantees that $v(t)$ stays bounded in $W^{2-\varepsilon,(p+1)/p}(\Omega)$ for any $\varepsilon > 0$. Since this space is embedded in $L^{p+1}(\Omega)$ for ε small, we see that both u and v are bounded in $L^{p+1}(\Omega)$, hence the solution (u,v) exists globally. Now using the Lyapunov function L it is not difficult to show that the ω-limit set of the trajectory (u,v) equals $\{(0,0)\}$.

(ii) First let us prove the global existence for (1.45). Denote $W = U - V$. Then $W_t = -W$, hence $W(t) = W(0)e^{-t}$ and $U_t = f(W(0)e^{-t}) - U$, which shows $U(t) \to 0$ as $t \to \infty$, and, consequently, $V(t) \to 0$ as $t \to \infty$.

The proof of blow-up for (1.42),(1.44) is somewhat technical and we only give its main idea. Let $\overline{u}$ denote the spatial average of u and $w := u - v$. Assuming that $(u_0 - \overline{u_0}) - (v_0 - \overline{v_0})$ is "suitably large" and setting $y(t) := \int_0^t\int_0^s \|w(\tau) - \overline{w}(\tau)\|_2^{p+1}\,d\tau\,ds$, one can derive the inequality

$$y'' + t^{(p-1)/2}y' > c_1y^{(p+1)/2} + c_2$$

with some $c_1, c_2 > 0$, which guarantees that y cannot exist globally. □

The diffusion induced blow-up in Theorem 1.28(ii) required unequal diffusion coefficients d_1, d_2. In what follows we present another example of diffusion induced blow-up with $d_1 = d_2 = 1$. This example is due to H. Weinberger.

We will consider the system

$$\left.\begin{aligned}
u_t - u_{xx} &= uv(u-v)(1+u) - \delta u, && x \in (-1,1),\ t > 0,\\
v_t - v_{xx} &= -uv(u-v)(1+v) - \delta v, && x \in (-1,1),\ t > 0,\\
u_x = v_x &= 0, && x = \pm 1,\ t > 0,\\
u(x,0) &= u_0(x), && x \in (-1,1),\\
v(x,0) &= v_0(x), && x \in (-1,1),
\end{aligned}\right\} \tag{1.48}$$

where $\delta \ge 0$. We will assume

$$u_0, v_0 \in C^1([-1,1]), \qquad u_0, v_0 > 0 \text{ in } [-1,1], \qquad (u_0)_x, (v_0)_x = 0, \quad x = \pm 1,$$
$$v_0(x) = u_0(-x),\ x \in [-1,1], \qquad u_0 \ge v_0 \text{ in } [0,1].$$

We also set

$$\varphi(x) := \frac{\pi}{4}\sin\frac{\pi x}{2}.$$

Then we have the following theorem.

THEOREM 1.29. *There exists $C > 0$ such that if*

$$\int_0^1 (u_0 - v_0)\varphi\,dx \geq C \qquad \text{and} \qquad \int_{-1}^1 \log(1+u_0)\,dx \geq C, \tag{1.49}$$

then the solution of (1.48) *blows up in finite time.*

All nonnegative solutions of the corresponding system of ODEs are global (and converge to $(0,0)$ *if* $\delta > 0$*).*

PROOF. The global existence for the system of ODEs follows from

$$(U + V + UV)_t = U_t(1+V) + V_t(1+U) = -\delta(U + V + 2UV).$$

In the proof of blow-up for (1.48) we will assume $\delta = 0$ for simplicity.

Since $\tilde u(x,t) := v(-x,t)$ and $\tilde v(x,t) := u(-x,t)$ also solve (1.48), the uniqueness of solutions of (1.48) implies $v(x,t) = u(-x,t)$. Consequently,

$$w := u - v \quad \text{is odd in the } x \text{ variable.} \tag{1.50}$$

Denote

$$m(t) := \min_{[-1,1]} u(t), \qquad M(t) := \max_{[-1,1]} u(t),$$

and notice that $m(t) = \min_{[-1,1]} v(t)$ and similarly for $M(t)$. We will show that

$$M(t) > 1, \quad t > 0. \tag{1.51}$$

Indeed, by adding the equations for u and v, we get

$$(u+v)_t - (u+v)_{xx} = uv(u-v)^2 \geq 0.$$

Integrating and using the boundary conditions, we deduce that

$$\frac{d}{dt}\int_{-1}^1 (u+v)\,dx \geq 0,$$

hence

$$M(t) \geq \frac{1}{4}\int_{-1}^1 2u\,dx = \frac{1}{4}\int_{-1}^1 (u+v)\,dx \geq \frac{1}{4}\int_{-1}^1 (u_0+v_0)\,dx > 1$$

provided the constant C in (1.49) is large enough.

Next denote $\lambda := \pi^2/4$ and assume that the maximal existence time of the solution (u,v) is greater than $T := 1/\lambda$. Set

$$\Phi(t) := e^{\lambda t}\int_{-1}^1 w\varphi\,dx, \qquad \Psi(t) := \int_{-1}^1 \log\Big(\frac{1+u}{2}\Big)\,dx,$$

and

$$E := \{t \in (0,T) : m(t) \geq 1\}, \qquad F := (0,T) \setminus E.$$

We will prove that

$$\Phi, \Psi > 0, \qquad \Phi' \geq \frac{1}{e}\Phi^2\chi_E, \qquad \Psi' \geq \frac{1}{4}\Psi^2\chi_F. \tag{1.52}$$

Integrating $-(1/\Phi)'$ and $-(1/\Psi)'$ and denoting by $|E|$ and $|F|$ the measure of the sets E and F, respectively, we obtain

$$\Phi^{-1}(0) \geq \int_0^T \frac{\Phi'}{\Phi^2}\,dt \geq \frac{|E|}{e}, \qquad \Psi^{-1}(0) \geq \int_0^T \frac{\Psi'}{\Psi^2}\,dt \geq \frac{|F|}{4},$$

hence (1.49) with C large implies

$$\frac{1}{\lambda} = T = |E| + |F| \leq e\Phi^{-1}(0) + 4\Psi^{-1}(0) < \frac{1}{\lambda}$$

which yields a contradiction.

Let us first prove the part of assertion (1.52) concerning Φ. The function w satisfies the equation

$$w_t - w_{xx} = uvw(2+u+v) \quad \text{in } (0,1)\times(0,T)$$

and boundary conditions $w(0,t) = w_x(1,t) = 0$. In addition,$w_0 := u_0 - v_0 \geq 0$ in $[0,1]$ and $w_0 \not\equiv 0$. Hence, the maximum principle guarantees $w > 0$ in $(0,1)\times(0,T)$, thus $\Phi > 0$. Next

$$\frac{d}{dt}\int_{-1}^1 w\varphi\,dx = \underbrace{(w_x\varphi - \varphi_x w)|_{-1}^1}_{=0} + \int_{-1}^1 uv\,\underbrace{w(2+u+v)\varphi}_{\geq w^2|\varphi|}\,dx - \lambda\int_{-1}^1 w\varphi\,dx.$$

If $t \in E$ then $uv \geq 1$ hence multiplying the last identity by $e^{\lambda t}$ and using Jensen's inequality yields

$$\Phi'(t) \geq e^{\lambda t}\int_{-1}^1 w^2|\varphi|\,dx \geq e^{-\lambda t}\Phi^2(t) \geq \frac{1}{e}\Phi^2(t), \quad t \in E.$$

Obviously, $\Phi'(t) \geq 0$ if $t \notin E$, which concludes the proof of (1.52) for Φ.

In order to prove the second part of assertion (1.52) (concerning Ψ), set $z := \log((1+u)/2)$. Then z satisfies

$$z_t - z_{xx} = uv(u-v) + (z_x)^2,$$

consequently

$$\Psi'(t) = \frac{d}{dt}\int_{-1}^1 z\,dx = \int_{-1}^1 (z_x)^2\,dx \geq 0.$$

Assume $t \in F$. Then $m(t) < 1 < M(t)$, hence there exists $\xi(t) \in [-1,1]$ such that $u(\xi(t),t) = 1$, thus $z(\xi(t),t) = 0$ and

$$|z(x,t)| \leq \Big|\int_{\xi(t)}^x z_x\,dx\Big| \leq \Big(\int_{-1}^1 (z_x)^2\,dx\Big)^{1/2}\sqrt{|x-\xi(t)|}.$$

Since $\int_{-1}^1 \sqrt{|x-\xi(t)|}\,dx < 2$, the last estimate implies

$$\Psi^2(t) = \Big(\int_{-1}^1 z\,dx\Big)^2 \leq 4\int_{-1}^1 (z_x)^2\,dx = 4\Psi'(t),$$

which concludes the proof. □

There exist many other examples on diffusion induced blow-up. Let us mention just few of them.

J. Bebernes and A.A. Lacey proved blow-up of some positive solutions of the system

$$\begin{aligned} u_t - u_{xx} &= -uv^p, && x \in (-1,1),\ t > 0, \\ v_t - v_{xx} &= uv^p, && x \in (-1,1),\ t > 0, \\ u(\pm 1, t) &= 1, && t > 0, \\ v_x(\pm 1, t) &= 0, && t > 0, \end{aligned}$$

where $p > 2$. The initial data are chosen in such a way that, in particular, $v_t \geq 0$ and $(u+v)_t \geq 0$. Since there are no positive steady states, (the maxima of) functions v and $u+v$ must tend to infinity as $t \to \infty$. Using this information one can show that $\Phi(t) := \int_{-1}^1 v(t)\,dx$ satisfies $\Phi'(t) \geq c\Phi^{p/2}(t)$ for t large.

The function u in the previous example satisfies $u \leq 1$ and $\partial_\nu u > 0$ on the boundary so that $\int_{-1}^1 (u+v)_t\,dx > 0$. A diffusion induced blow-up in problems with *mass dissipation* (that is $\int_\Omega (u+v)_t\,dx \leq 0$) was found by M. Pierre and D. Schmitt. They considered the system

$$\begin{aligned} u_t - a\Delta u &= f(u,v), && x \in \Omega,\ t > 0, \\ v_t - b\Delta v &= g(u,v), && x \in \Omega,\ t > 0, \\ u_\nu &= \alpha_1(t), && x \in \partial\Omega,\ t > 0, \\ v_\nu &= \alpha_2(t), && x \in \partial\Omega,\ t > 0, \end{aligned}$$

where Ω is the unit ball in $\mathbb{R}^n$, $a, b > 0$, f, g, α_1, α_2 are smooth, $f + g \leq 0$, $\alpha_1, \alpha_2 \leq 0$ and they showed the existence of positive solutions which blow-up in finite time. For example, if $n = 10$ then the solutions have the (self-similar) form

$$u(x,t) = \frac{A(T-t) + B|x|^2}{(T - t + |x|^2)^{5/4}}, \qquad v(x,t) = \frac{C(T-t) + D|x|^2}{(T - t + |x|^2)^{5/4}},$$

where $A, B, C, D > 0$.

There exist various sufficient conditions guaranteeing global existence for systems with mass dissipation and the above example shows the necessity of (some of) such conditions.

Finally, let us mention that one of the first examples on diffusion induced blow-up is due to V. Churbanov. His system is of the form

$$u_t - u_{xx} = f(u,v), \qquad v_t - v_{xx} = 0,$$

considered either as a Cauchy problem or with Neumann boundary conditions. He found a smooth function f such that $f(\cdot, v)$ is bounded for any v (so that the solutions of the corresponding system of ODEs are global) but suitable positive solution of the system of PDEs (starting from bounded initial data) blows up in finite time.

8. Borderline between global existence and blow-up

Consider again the problem

$$\left.\begin{aligned} u_t - \Delta u &= |u|^{p-1}u, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}\right\} \tag{1.53}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $p > 1$ and $u_0 \in L^\infty(\Omega)$. We know from Theorem 1.23 that $T_{\max}(u_0) = \infty$ if u_0 is small (in $L^\infty(\Omega)$, for example) and $T_{\max}(u_0) < \infty$ if u_0 is suitably large. Let us fix $\Phi \in L^\infty(\Omega)$ nonnegative, $\Phi \not\equiv 0$, and consider initial data in the form $u_0 = \alpha\Phi$, $\alpha > 0$. Denote by u_α the corresponding solution of (1.53), $u_\alpha(t) = u(t;\alpha\Phi)$. Then the above arguments show $T_{\max}(\alpha\Phi) = \infty$ for α small and $T_{\max}(\alpha\Phi) < \infty$ for α large. Consequently, the number

$$\alpha^* := \sup\{\alpha \in (0,\infty) : T_{\max}(\alpha\Phi) = \infty\}$$

is positive and finite. In addition, the comparison principle guarantees that $T_{\max}(\alpha\Phi) = \infty$ for all $\alpha < \alpha^*$ and $T_{\max}(\alpha\Phi) < \infty$ for all $\alpha > \alpha^*$. A natural question is what is the asymptotic behavior of the threshold solution

$$u^*(t) := u_{\alpha^*}(t) = u(t;\alpha^*\Phi).$$

Let $p_S := (n+2)/(n-2)_+$ be the critical Sobolev exponent. Then we have the following proposition.

PROPOSITION 1.30. *(i) If $p < p_S$ then $T_{\max}(\alpha^*\Phi) = \infty$ and*

$$\limsup_{t\to\infty} \|u^*(t)\|_\infty < \infty.$$

The ω-limit set of u^* is nonempty and consists of nontrivial equilibria.

(ii) Let $p \geq p_S$, Ω be a ball and Φ be radially symmetric. If $p = p_S$ then $T_{\max}(\alpha^*\Phi) = \infty$ and $\|u^*(t)\|_\infty \to \infty$ as $t \to \infty$. If $p > p_S$ then $T_{\max}(\alpha^*\Phi) < \infty$ (but the solution u^* can be continued beyond $T_{\max}(\alpha^*\Phi)$ in a weak sense).

Concerning assertion (ii) in the preceeding proposition we will only show the following lemma.

LEMMA 1.31. *Let Ω be starshaped (bounded) and $p \geq p_S$. Then u^* cannot be global and bounded.*

PROOF. Assume on the contrary that u^* is global and bounded. Then smoothing estimates based on the variation-of-constants formula easily imply a uniform bound on $u^*(t)$, $t \geq t_0 > 0$, in an interpolation space X_θ which is compactly embedded into $L^\infty(\Omega)$. Consequently, the trajectory $\{u^*(t) : t \geq t_0\}$ is relatively compact in $L^\infty(\Omega)$ and its ω-limit set $\omega(\alpha^*\Phi)$ is nonempty and compact. Since problem (1.53) possesses a strict Lyapunov functional, $\omega(\alpha^*\Phi)$ consists of equilibria. The maximum principle implies that these equilibria are nonnegative. Assume $0 \in \omega(\alpha^*\Phi)$. Then the stability of zero implies that $u(t;\alpha\Phi)$ is global and tends to zero as $t \to \infty$ for all α close to α^*, which contradicts the definition of α^*. Consequently, $\omega(\alpha^*\Phi)$ consists of positive equilibria. But the Pohozaev identity guarantees that there are no positive equilibria of (1.53) if $p \geq p_S$ which yields a contradiction. □

Concerning the proof of (i) in Proposition 1.30, let us first prove the following lemma.

LEMMA 1.32. *Assume that all global solutions of* (1.53) *satisfy the following estimate*

$$\|u(t)\|_\infty \leq C(\|u(0)\|_\infty), \tag{1.54}$$

where $C : [0,\infty) \to [0,\infty)$ *is nondecreasing. Then* u^* *is global and bounded.*

PROOF. Let $\{\alpha_k\}$ be an increasing sequence of positive reals converging to α^*. Then $T_{\max}(\alpha_k\Phi) = \infty$, hence (1.54) implies

$$\|u(t;\alpha_k\Phi)\|_\infty \leq C(\|\alpha_k\Phi\|_\infty) \leq C(\|\alpha^*\Phi\|_\infty) =: C^*.$$

Now the continuous dependence of solutions on initial data guarantees

$$\|u(t;\alpha^*\Phi)\|_\infty = \lim_{k\to\infty} \|u(t;\alpha_k\Phi)\|_\infty \leq C^*,$$

which concludes the proof. □

Notice that Lemma 1.32 and the proof of Lemma 1.31 guarantee Proposition 1.30(i) provided we can prove the following theorem.

THEOREM 1.33. *Let* $1 < p < p_S$ *and* Ω *be bounded. Then* (1.54) *is true for all global solutions of* (1.53).

Notice also that Lemma 1.32 and the proof of Lemma 1.31 show that the assumption $p < p_S$ in Theorem 1.33 is optimal (at least if Ω is starshaped). We will prove Theorem 1.33 only under more restrictive assumption $p < 1+4/n$. More sophisticated energy and interpolation arguments were used by T. Cazenave and P.L. Lions in 1984 in the case $p(3n-4) < 3n+8$ and by the author in 1999 in the optimal case $p < p_S$. Another proof based on scaling arguments is due to Y. Giga but his result applies to nonnegative solutions only. Let us also mention that estimate (1.54) and its modifications have many other important applications.

PROOF OF THEOREM 1.33 FOR $p < 1+4/n$. Let the solution $u(t)$ be global. We will denote by $\delta, C_1, C_2, \dots$ positive constants which depend only on $\|u_0\|_\infty$. Straightforward estimates based on the variation-of-constants formula show that there exist δ and C_1, C_2 such that

$$\|u(t)\|_\infty \leq C_1 \quad \text{for} \quad t \in [0, 2\delta] \tag{1.55}$$

and $\|u(\delta)\|_{H^1(\Omega)} \leq C_2$, hence $E(u(\delta)) \leq C_3$, where E is the corresponding energy function, see (1.30). Now estimate (1.34) (with u_0 replaced by $u(\delta)$) shows that

$$y(t) := \|u(t)\|_2^2 \leq \Big(\frac{2E(u(\delta))}{\tilde{C}c_p}\Big)^{2/(p+1)}, \quad t \geq \delta,$$

since otherwise y blows up in finite time. This estimate guarantees $\|u(t)\|_2 \leq C_4$ for some C_4 and $t \geq \delta$. Now the well-posedness of (1.53) in $L^2(\Omega)$ (which is due to $p < 1+4/n$) and standard smoothing estimates guarantee that $\|u(t)\|_\infty \leq C_5$ for $t \geq 2\delta$. This estimate and (1.55) conclude the proof. □

9. Universal bounds and blow-up rates

In Theorem 1.33 we proved a uniform bound for global solutions by using energy arguments. In this section we also derive some uniform a priori estimates of solutions. These estimates will not even depend on the initial data and therefore we will call them *universal bounds*. Our method will not use energy arguments but will heavily rely on scaling arguments and positivity of the solutions.

Consider the problem

$$\left.\begin{aligned} u_t - \Delta u = f(u), &\qquad x \in \Omega,\ t \in (0,T) \\ u = 0, &\qquad x \in \partial\Omega,\ t \in (0,T), \\ u \geq 0, &\qquad x \in \Omega,\ t \in (0,T), \end{aligned}\right\} \tag{1.56}$$

where $\Omega \subset \mathbb{R}^n$, $0 < T \leq \infty$,

$$f \in C(\mathbb{R}^+, \mathbb{R}), \quad \lim_{u\to\infty} \frac{f(u)}{u^p} = \ell \in (0,\infty), \tag{1.57}$$

$p > 1$ and either $p < p_B := n(n+2)/(n-1)^2$ or

$$p < p_S, \quad \Omega \text{ and } u \text{ are radially symmetric.}$$

Theorem 1.34. *Under the above assumptions there exists $C = C(f,\Omega)$ such that any classical solution u of* (1.56) *satisfies*

$$u(x,t) \leq C\big(1 + t^{-1/(p-1)} + (T-t)^{-1/(p-1)}\big).$$

If $f(u) = u^p$ and $\Omega = \mathbb{R}^n$ then

$$u(x,t) \leq C(n,p)\big(t^{-1/(p-1)} + (T-t)^{-1/(p-1)}\big).$$

Here $(T-t)^{-1/(p-1)} := 0$ if $T = \infty$.

Corollary 1.35. *(i) (Blow-up rate estimate) If $\|u(t)\|_\infty \to \infty$ as $t \to T < \infty$ then $\|u(t)\|_\infty \leq C(T-t)^{-1/(p-1)}$ as $t \to T$.*

(ii) (Initial blow-up rate estimate) We have $\|u(t)\|_\infty \leq Ct^{-1/(p-1)}$ as $t \to 0$.

(iii) (Decay rate estimate) Let $\Omega = \mathbb{R}^n$ and $T = \infty$. Then $\|u(t)\|_\infty \leq Ct^{-1/(p-1)}$ as $t \to \infty$.

The blow-up rate and decay rate estimates in Corollary 1.35 are optimal; the initial blow-up rate estimate is optimal if $p \geq 1 + 2/n$. The proof of Theorem 1.34 is based on scaling and the following Liouville-type theorem which is due to M.F. Bidaut-Véron (case (i)) and P. Poláčik, Ph. Souplet and the author (case (ii)).

Proposition 1.36. *Consider nonnegative (classical) solutions of the problem*

$$u_t - \Delta u = u^p, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{1.58}$$

(i) If $1 < p < p_B$ then the only nonnegative solution of (1.58) *is the trivial solution $u \equiv 0$.*

(ii) If $1 < p < p_S$ then the only nonnegative radially symmetric solution of (1.58) *is the trivial solution $u \equiv 0$.*

In addition to Proposition 1.36 we will need the following simple "doubling" lemma.

LEMMA 1.37. *Let* (X,d) *be a complete metric space,* $Q \subset X$ *open,* $M : Q \to (0,\infty)$ *bounded on compact sets,* $k > 0$ *and*

$$Q_k := \{x \in Q : M(x) > 2k/dist(x, \partial Q)\}.$$

Let $z \in Q_k$*. Then there exists* $w \in Q_k$ *such that* $M(w) \geq M(z)$ *and* $M(y) \leq 2M(w)$ *whenever* $d(y,w) \leq k/M(w)$*.*

PROOF. Assume on the contrary that the assertion fails. Set $w_0 := z$. Then by our assumption there exists $w_1 \in Q$ such that

$$M(w_1) > 2M(w_0) \quad \text{and} \quad d(w_1, w_0) \leq \frac{k}{M(w_0)}.$$

Notice that these conditions guarantee $w_1 \in Q_k$. Hence, by our assumption there exists $w_2 \in Q$ such that

$$M(w_2) > 2M(w_1) \quad \text{and} \quad d(w_2, w_1) \leq \frac{k}{M(w_1)} < \frac{k}{2M(w_0)}.$$

Again $w_2 \in Q_k$. By induction we obtain a sequence $w_j \in Q_k$, $j = 1, 2, \dots$, such that $M(w_j) > 2^j M(w_0)$ and $d(w_j, w_{j+1}) < k/(2^j M(w_0))$. The last estimate guarantees that $w_j \to w \in Q$ which contradicts the boundedness of M on compact sets. □

PROOF OF THEOREM 1.34. For simplicity we will assume $\Omega = \mathbb{R}^n$, $f(u) = u^p$ and $p < p_B$. Assume on the contrary that there exist solutions u_k of (1.56) on $Q_k := \mathbb{R}^n \times (0, T_k)$ and $(x_k, t_k) \in Q_k$ such that

$$u_k(x_k, t_k) > (2k)^{2/(p-1)} \big(t_k^{-1/(p-1)} + (T_k - t_k)^{-1/(p-1)}\big).$$

Set $M_k(x,t) := u_k^{(p-1)/2}(x,t)$. Then

$$M_k(x_k, t_k) > 2k \max\big(t_k^{-1/2}, (T_k - t_k)^{-1/2}\big) = 2k/\text{dist}((x_k, t_k), \partial Q_k),$$

where $Q_k \subset \mathbb{R}^n \times \mathbb{R}$ is endowed with the parabolic distance $d((x,t),(y,s)) = |x - y| + |t-s|^{1/2}$. By Lemma 1.37 we may assume that

$$M_k(x,t) \leq 2M_k(x_k, t_k) \quad \text{whenever} \quad \text{dist}((x,t),(x_k,t_k)) \leq \frac{k}{M_k(x_k, t_k)}.$$

Set $\lambda_k := 1/M_k(x_k, t_k)$ and

$$v_k(y,s) := \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s).$$

Then v_k solves the equation $\partial_t v_k - \Delta v_k = v_k^p$ in a rescaled domain containing the set $B_k := \{(y,s) : |y| < k/2,\ |s| < k^2/4\}$ and $0 \leq v_k \leq 2^{2/(p-1)}$ in B_k, $v_k(0,0) = 1$. Standard parabolic regularity estimates guarantee that we may pass to the limit $v_k \to v$, where v is a nontrivial nonnegative (bounded) solution of (1.58). This contradicts Proposition 1.36. □

Appendix

Here we recall the definition of real and complex interpolation spaces. We will assume that $(X_0, |\cdot|_0), (X_1, |\cdot|_1)$ are Banach spaces and there exists a locally convex space Z such that $X_j \hookrightarrow Z$, $j = 0, 1$.

Given $x \in X_0 + X_1$ and $t > 0$, set

$$K(t, x) := \inf\{|x_0|_0 + t|x_1|_1 : x = x_0 + x_1\}$$

and

$$|x|_{\theta,q} := \|t^{-\theta-1/q} K(t, x)\|_{L^q(0,\infty)}, \qquad 0 < \theta < 1,\ 1 \le q \le \infty,$$

where $1/\infty := 0$. Then the real interpolation space $(X_0, X_1)_{\theta,q}$ is defined as follows:

$$(X_0, X_1)_{\theta,q} := \big(\{x \in X_0 + X_1 : |x|_{\theta,q} < \infty\}, |\cdot|_{\theta,q}\big).$$

Next assume that the spaces X_0, X_1 are complex. Denote $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and $S_j := \{z \in \mathbb{C} : \operatorname{Re} z = j\}$. Let C_0 denote the space of continuous functions vanishing at infinity and let $\mathcal{F}(X_0, X_1)$ be the set of all bounded and continuous functions from $\overline{S}$ into $X_0 + X_1$, such that $f|_S$ is holomorphic and $f|_{S_j} \in C_0(S_j, X_j)$, $j = 0, 1$. Then $\mathcal{F}(X_0, X_1)$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{F}} := \max\big(\sup_{t\in\mathbb{R}} |f(it)|_0, \sup_{t\in\mathbb{R}} |f(1 + it)|_1\big).$$

Given $\theta \in (0, 1)$, set

$$|x|_\theta := \inf\{\|f\|_{\mathcal{F}} : f(\theta) = x\}.$$

The complex interpolation space $[X_0, X_1]_\theta$ is defined as follows:

$$[X_0, X_1]_\theta := \big(\{x \in X_0 + X_1 : f(\theta) = x \ \text{ for some } f \in \mathcal{F}(X_0, X_1)\}, |\cdot|_\theta\big).$$

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Part 5

Nonlinear Fredholm operators and PDEs

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Abstract. Somewhat expanded notes of five lectures delivered in May 2007 at the Czech Academy of Sciences and Charles University in Prague are presented. The general purpose of these lectures was to call the attention to the importance of Fredholm operators in nonlinear partial differential equations. As usual, "nonlinear" should be understood as "not necessarily linear" as opposed to "definitely not linear".

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Preface

These are the somewhat expanded notes of five lectures delivered in May 2007 at the Czech Academy of Sciences and Charles University in Prague. The general purpose of these lectures was to call the attention to the importance of Fredholm operators in nonlinear partial differential equations. As usual, "nonlinear" should be understood as "not necessarily linear" as opposed to "definitely not linear".

Everyone is well aware of the key role of the Leray–Schauder degree in many existence and bifurcation questions for nonlinear PDEs. However, the Leray–Schauder degree is only available for mappings of the form $I - K$ where K is a (nonlinear) compact map. Even though various generalizations have been obtained in the literature (e.g., condensing maps), the assumptions about K always involve some remnant of compactness, because compactness must hide somewhere in any theory built up from the Leray–Schauder degree. These compactness-related assumptions are usually not satisfied when $I - K$ arises from a nonlinear PDE on an unbounded domain. On the other hand, it is not uncommon that such PDEs can be accounted for by a nonlinear operator F acting between possibly different spaces, which is *Fredholm of index* 0.

As it turns out, a degree theory exists for such mappings, which generalizes Leray–Schauder's but is not derived from it. This means that the construction of the degree is completely different -it does not go by finite dimensional approximation and reduction to Brouwer's degree- and it requires only the properness of the operator.

These notes are divided in two parts, with the first part being devoted to the aforementioned degree theory for Fredholm mappings. This theory, which in its primitive form is almost as old as Leray–Schauder's, is nevertheless much less known, in spite of a 1965 major contribution by Smale in one of his most famous papers. The reason is that the properness property, which comes for free in the Leray–Schauder theory because of the special "$I - K$" structure, is a highly nontrivial question with no known general answer for mappings that are not compact perturbations of the identity (or reducible to that form by diffeomorphism).

Quite naturally, the second part addresses the Fredholm and properness properties of differential operators on unbounded domains, the two ingredients needed to use the degree discussed in the first part. The resolution of the properness question relies on features of PDE problems which have no analog in a completely abstract framework. The Fredholmness issue, which is a purely linear matter, is of independent interest (Fredholm alternative).

The proofs are only sketched, often in very vague terms merely describing the general line of argument, but all the necessary references to the literature are given.

CHAPTER 1

Degree for Fredholm mappings of index 0

1. Fredholm operators

Let X and Y be Banach spaces. A *linear* operator $L \in \mathcal{L}(X,Y)$ is *Fredholm* if

$$\dim \ker L < \infty \text{ and } \operatorname{codim} \operatorname{rge} L < \infty.$$

If so, the index of L is defined by

$$\dim \ker L - \operatorname{codim} \operatorname{rge} L \in \mathbb{Z}.$$

Some of the fundamental properties of (linear) Fredholm operators are

1) The set of Fredholm operators of index $\nu \in \mathbb{Z}$ is open in $\mathcal{L}(X,Y)$ ("local constancy of the index").

2) If $L \in \mathcal{L}(X,Y)$ is Fredholm of index ν and $K \in \mathcal{L}(X,Y)$ is *compact,* then $L+K$ is Fredholm of index ν.

3) If $L \in \mathcal{L}(X,Y)$ is Fredholm, then $\operatorname{rge} L$ is *closed* in Y.

4) Invertible operators are norm-dense in the set of Fredholm operators of index 0.

REMARK 1.1. *If $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, then every $L \in \mathcal{L}(X,Y)$ is Fredholm of index $m-n$.*

A typical example of a linear Fredholm operator of index 0 is given by $X = Y$ and

$$L = I - K,$$

where $K \in \mathcal{L}(X)$ is *compact.* More generally, if

$$L = A - K,$$

where $A \in \mathcal{L}(X,Y)$ is invertible and $K \in \mathcal{L}(X,Y)$ is compact, then L is Fredholm of index 0.

If now $\mathcal{O} \subset X$ is an open subset and $F : \mathcal{O} \to Y$ is C^1, then F is said to be Fredholm if

$$DF(x) \in \mathcal{L}(X,Y) \text{ is Fredholm for every } x \in \mathcal{O}.$$

If F is Fredholm, then $\operatorname{index} DF(x)$ is constant on every connected component of $\mathcal{O}$. In particular, if $\mathcal{O}$ is connected, then $\operatorname{index} F$ is well defined by $\operatorname{index} F = \operatorname{index} DF(x)$ for every $x \in X$. More generally, $\operatorname{index} F$ is defined whenever $\operatorname{index} DF(x)$ is independent of $x \in \mathcal{O}$.

A typical example of (nonlinear) Fredholm mapping of index 0 is given by

$$F(x) = Ax - K(x),$$

where $A \in \mathcal{L}(X,Y)$ is invertible and $DK(x) \in \mathcal{L}(X,Y)$ is compact for every $x \in \mathcal{O}$. This is the case if $K : \mathcal{O} \to Y$ is C^1 and (nonlinear) compact, for it is a simple exercise to check that $DK(x)$ is compact for every $x \in \mathcal{O}$. However, the latter is a strict special case of the former. For instance, a number of examples from PDEs on *unbounded* domains can be written in the form $Ax - K(x)$ with $DK(x)$ compact while K *is not* compact. More generally, $F(x) = \Phi(x) - K(x)$ with Φ a *local* diffeomorphism and $DK(x)$ compact is also Fredholm of index 0.

2. Classical degree theories

Before discussing degree theory for nonlinear Fredholm mappings (of index 0), it is helpful to begin with a brief review of the Brouwer and the Leray–Schauder degrees.

2.1. Brouwer degree. Let $\Omega \subset \mathbb{R}^n$ be an open subset and let

$$F : \overline{\Omega} \to \mathbb{R}^n$$

be continuous and *proper* (i.e., $F^{-1}(Q)$ is compact whenever Q is compact).

If also F is C^1 on Ω, a point $y \in \mathbb{R}^n \backslash F(\partial\Omega)$ is called a regular value of F if $DF(x)$ is invertible for every $x \in F^{-1}(y)$. Since F is proper and $y \notin F(\partial\Omega)$, $F^{-1}(y)$ is a *finite* subset of Ω whenever y is a regular value of F, so that

$$d(F, \Omega, y) := \sum_{x \in F^{-1}(y)} \operatorname{sign} \det DF(x) \in \mathbb{Z},$$

is well defined and called the (Brouwer) degree of F at y (relative Ω). As usual, the above sum is understood to be 0 if $F^{-1}(y) = \emptyset$.

Let now $y \in \mathbb{R}^n \backslash F(\partial\Omega)$ be arbitrary (regular value or not). By the properness of F, it follows that $F(\partial\Omega)$ is closed in $\mathbb{R}^n$ and so there is an open ball $B(y, \varepsilon)$ such that $B(y, \varepsilon) \cap F(\partial\Omega) = \emptyset$. By Sard's theorem, there are plenty of regular values of F in $B(y, \varepsilon)$. Let y_0 and y_1 be any two such regular values and call H the homotopy:

$$H(t, x) := F(x) - ty_1 - (1 - t)y_0.$$

To say that y_0 and y_1 are regular values of F means that 0 is a regular value of both $H(0, \cdot)$ and $H(1, \cdot)$ and this property is unaffected by small enough perturbations of y_0 and y_1. Yet, with F being only C^1, it may not be possible to ascertain that 0 is *also* a regular value of H (i.e., that the derivative of H w.r.t. (t, x) is onto $\mathbb{R}^n$ at every point of $H^{-1}(0)$). Nonetheless, it follows once again from Sard's theorem that if F is C^2, then it may be assumed with no loss of generality that 0 is a regular value of $H(0, \cdot)$ and $H(1, \cdot)$ as well as a regular value of H. Then, it can be shown that

$$d(H(0, \cdot), \Omega, 0) = d(H(1, \cdot), \Omega, 0), \tag{1.1}$$

which amounts to the relation

$$d(F, \Omega, y_0) = d(F, \Omega, y_1).$$

(It should be noted that the proof of (1.1) is essentially the only really nontrivial part of the Brouwer degree theory.)

From the above, it then makes sense to define (when F is C^2)

$$d(F, \Omega, y) := d(F, \Omega, z),$$

where z is any regular value of F close enough to y.

Even though the above definition requires F to be C^2, it is easy to extend the definition of the degree when F is only continuous by noticing that (i) F can be uniformly approximated by C^2 functions and (ii) that $d(G, \Omega, y)$ is well defined and independent of G of class C^2 uniformly close enough to F. When Ω is bounded, polynomials can be used to resolve (i); the general case is slightly more delicate but still elementary (use a partition of unity). Thus, it is possible to define

$$d(F, \Omega, y) := d(G, \Omega, y)$$

with G as above when F is only continuous (and $y \notin F(\partial\Omega)$).

The main two properties of Brouwer's degree are

1) If $y \notin F(\partial\Omega)$ and $d(F, \Omega, y) \neq 0$, then $F^{-1}(y) \neq \emptyset$

and

2) If $H : [0,1] \times \overline{\Omega}$ is continuous and proper and $y \notin H([0,1] \times \partial\Omega)$, then $d(H(0,\cdot), \Omega, 0) = d(H(1,\cdot), \Omega, 0)$.

In practice, the first property is often used to show that the equation $F(x) = y$ is solvable and the second one is used to reduce the verification that $d(F, \Omega, y) \neq 0$ (needed in the first) to a case when F is “simple enough” that the degree is easily calculable.

Remark 1.2. *Recent new aspects of the Brouwer degree, or at least closely related to the Brouwer degree, can be found in Rabier [68].*

2.2. Leray–Schauder degree. The Leray–Schauder degree is a generalization of the Brouwer degree to a special class of infinite dimensional mappings. Specifically, if X is a Banach space and Ω is a *bounded* open subset of X, and if $K : \overline{\Omega} \to X$ is a (nonlinear) compact map -which incorporates continuity- then the (Leray–Schauder) degree

$$d(I - K, \Omega, y) \in \mathbb{Z}$$

can be defined for every $y \in X \backslash (I - K)(\partial\Omega)$ by an approximation procedure and reduction to the finite dimensional case when Brouwer's degree can be used. This is possible because a nonlinear compact mapping is “almost” finite dimensional valued, at least when restricted to a bounded subset. This is one, but not the only, reason why Ω is assumed to be bounded. Another reason is that the compactness of $K : \overline{\Omega} \to X$ and the boundedness of Ω imply that $I - K$ is *proper* (and hence closed). Directly or indirectly, properness is behind the justification of several arguments in the construction of the Leray–Schauder degree and therefore a crucial property. That it is automatically satisfied by the “Leray–Schauder maps” $I - K$ above is the main reason for the popularity (and usefulness) of this degree, since indeed properness is, as a rule, a much more delicate issue in infinite dimension than it is in finite dimension (where it is a mere matter of growth at infinity and, in particular, always true on closed bounded subsets).

For future use, we note that in the simple case when K is *linear* (and compact) and if $0 \in \Omega$, then $d(I - K, \Omega, 0)$ is defined if and only if $I - K$ is invertible and, if so,

$$d(I - K, \Omega, 0) = (-1)^m \tag{1.2}$$

where m is the sum of the (algebraic) multiplicities of the eigenvalues of K lying in the interval $(1, \infty)$.

2.3. From Leray–Schauder to Fredholm mappings of index 0. After its introduction by Leray and Schauder [49] in 1934, the Leray–Schauder degree was subsequently extended to various classes of mappings beyond the compact perturbations of I. In particular, there have been several attempts, none fully satisfactory, to generalize this degree to Fredholm mappings of index 0. We only record three of those below:

- A mod 2 degree for C^1 proper Fredholm mappings of index 0 was developed by Caccioppoli [16] as early as 1936. Unlike the integer-valued Leray–Schauder degree, Caccioppoli's degree takes only the values 0 and 1 and, as a result, must be interpreted -roughly speaking- as a mod 2 count of the number of points in the pre-image (even though this number need not actually be finite). Such a degree is still useful to prove the existence of solutions to functional equations, but, since it does not even incorporate a *sign* (as integers do), it has no value in bifurcation problems, where indeed the relevant degree arguments are based on a sign change. Furthermore, Caccioppoli's treatment is at best vague in places, notably regarding homotopy invariance.
- A mod 2 degree for proper C^2 Fredholm mappings of index 0 was discussed by Smale [81] in 1965, where the ambiguities in Caccioppoli's approach were removed by making use of what is now known as the Sard-Smale theorem (from the same paper). Note the C^2 versus C^1 assumption by Caccioppoli. Naturally, Smale's degree has the same applications and limitations as Caccioppoli's.
- An integer-valued degree for proper C^2 mappings of index 0 on Banach manifolds was introduced by Elworthy and Tromba [25], [26] in 1970. However, the definition of this degree assumes the existence of an "oriented Fredholm structure" and some compatibility of the Fredholm mapping with the Fredholm structure. The technicalities associated with Fredholm structures make this degree little user-friendly and hence difficult to use in concrete applications.

3. A $\mathbb{Z}$-valued degree for Fredholm mappings of index 0

Let X and Y be Banach spaces and let $\Omega \subset X$ be an open subset. Given $F : \overline{\Omega} \to Y$ continuous and proper and such that $F : \Omega \to Y$ is C^1 and Fredholm of index 0, we shall now explain how to define a $\mathbb{Z}$-valued degree $d(F, \Omega, y)$ for every $y \in Y \backslash F(\partial\Omega)$ without any recourse to Fredholm structures. The very definition of this degree makes it easy to use in practice – as easy as the Leray–Schauder degree – even though the actual construction is technically quite involved in the C^1 case (it is much less demanding for C^2 mappings).

When compared with the Leray–Schauder theory, the main new difficulty is to verify the properness hypothesis, which is no longer a trivial issue for mappings other than the Leray–Schauder mappings $I - K$ with K compact and Ω bounded (for which the Leray–Schauder degree is available). Fortunately, this verification, while sometimes delicate, is not out of reach in important applications, notably PDEs. This will be discussed in Part 2.

Before describing the idea of the construction of the degree, it is worth mentioning two major issues.

- In the case of Brouwer's degree, the definition of the degree at regular values is
$$d(F, \Omega, y) := \sum_{x \in F^{-1}(y)} \operatorname{sign} \det DF(x)$$
and there is no such thing as a *globally defined* determinant function for linear mappings acting between Banach spaces, even if attention is confined to Fredholm mappings of index 0. It is not difficult to define determinants for the linear mappings in the vicinity some given linear Fredholm mapping of index 0 ("local" determinants), but no such determinant will make sense for *all* Fredholm mappings of index 0, nor can it be defined in intrinsic terms. As a result, there is no coherent way to compare the signs of $\det DF(x)$ when x runs over the set $F^{-1}(y)$ (because this sign can be made $+1$ or -1 by modifying the choice of "det" in the vicinity of any such x).
- A 1965 theorem of Kuiper [46] asserts that the general linear group of an *infinite dimensional* Hilbert space is contractible and therefore (path) connected. A simple by-product of this theorem is that *no* degree for Fredholm mappings of index 0 can comply with the (fundamental, in practice) homotopy invariance property *and* generalize the Leray–Schauder degree. To see this, let X be an infinite dimensional Hilbert space and choose Ω such that $0 \in \Omega$. Let $K_0, K_1 \in \mathcal{L}(X)$ be compact and such that $d(I - K_0, \Omega, 0) = 1$ and $d(I - K_1, \Omega, 0) = -1$ (Leray–Schauder degree; see (1.2)). Then, it is impossible to continuously deform $I - K_0$ into $I - K_1$ within the set of invertible *compact perturbations* of I (otherwise, they would have the same degree since the Leray–Schauder degree is homotopy invariant). However, by Kuiper's theorem, it *is* possible to continuously deform $I - K_0$ into $I - K_1$ within the set of *all* the invertible linear operators on X. Since every such operator is obviously Fredholm of index 0 and proper, the existence of a homotopy invariant degree for such operators that also generalizes Leray–Schauder's shows that $1 = d(I - K_0, \Omega, 0) = d(I - K_1, \Omega, 0) = -1$, which is absurd.

3.1. The parity. The resolution of the first issue -the lack of a globally defined determinant function- is not very technical but certainly the most subtle part of the construction. A key ingredient is the so-called *parity* of a path of Fredholm operators of index 0 that we now discuss.

First, let $L \in \mathcal{L}(X, Y)$ be (linear) and Fredholm of index 0. It is standard and easily seen that there is an invertible $N \in \mathcal{L}(Y, X)$ such that
$$NL = I - K,$$
where $K \in \mathcal{L}(X, Y)$ is *compact*.

Now, instead of a single operator L as above, consider a continuous path $L(t) \in \mathcal{L}(X, Y)$ of Fredholm operators of index 0, where $t \in [a, b]$. Then, generalizing the above remark, it is not difficult to show that there is a continuous path $N(t) \in$

$\mathcal{L}(Y, X)$ of invertible operators such that

$$N(t)L(t) = I - K(t),$$

where $K(t) \in \mathcal{L}(X, Y)$ is compact for every $t \in [a, b]$. Such a path $N(\cdot)$ is often called a *parametrix* for the path $L(\cdot)$. It is by no means unique.

Suppose now that the endpoints $L(a)$ and $L(b)$ are invertible, so that $I - K(a)$ and $I - K(b)$ above are invertible. Furthermore, $I - K(a)$ and $I - K(b)$ have a finite number of negative eigenvalues (corresponding to the eigenvalues of $K(a)$ and $K(b)$ in $(1, \infty)$). Set

$$m(a) := \text{sum of the algebraic multiplicities of the eigenvalues of } K(a) \text{ in } (1, \infty),$$

$$m(b) := \text{sum of the algebraic multiplicities of the eigenvalues of } K(b) \text{ in } (1, \infty).$$

Naturally, $m(a)$ and $m(b)$ depend upon the parametrix $N(\cdot)$, but it turns out that the sum $m(a) + m(b)$ is *independent of the parametrix* $N(\cdot)$, so that

$$\sigma(L, [a, b]) := (-1)^{m(a)}(-1)^{m(b)} = (-1)^{m(a)+m(b)} \in \{-1, 1\}$$

is well defined and depends only upon $L(\cdot)$. It is called the *parity* of the path $L(\cdot)$.

REMARK 1.3. *If either $L(a)$ or $L(b)$ is not invertible, then $\sigma(L, [a, b])$ is not defined: This concept makes sense only for paths of linear Fredholm operators of index* 0 *with* invertible endpoints.

To understand the significance of the parity, consider the case when $X = Y = \mathbb{R}^n$. If so, it is easily checked that

$$\sigma(L, [a, b]) = \operatorname{sign} \det L(a) \operatorname{sign} \det L(b),$$

so that $\sigma(L, [a, b]) = 1$ if $\det L(a)$ and $\det L(b)$ have the same sign while $\sigma(L, [a, b]) = -1$ otherwise. Since $\sigma(L, [a, b])$ continues to make sense when X (and Y) are infinite dimensional, the parity being ± 1 can be interpreted as the property that the determinants of $L(a)$ and $L(b)$ have the same or opposite signs, even though there is no such thing as a determinant for $L(a)$ or $L(b)$ in this case!

The parity has several fundamental properties, the most important of which is its *homotopy invariance*. This means that $\sigma(L, [a, b])$ is unchanged if $L(\cdot)$ is continuously deformed into another path of Fredholm operators of index 0, as long as the endpoints remain invertible during the deformation. Also, the parity has several multiplicative properties (notably relative to composition of operators and to juxtaposition of paths) and the parity of a path of invertible operators is always 1.

3.2. Definition of the degree. Let $\mathcal{O} \subset X$ be an open *connected and simply connected* subset and let $F : \mathcal{O} \to Y$ be C^1 and Fredholm of index 0. Also, let $\Omega \subset \mathcal{O}$ be *any* open subset and suppose that $F : \overline{\Omega} \to Y$ is continuous (which of course is not an extra assumption if $\overline{\Omega} \subset \mathcal{O}$, but this need not be the case) and *proper*.

Choose a point $p \in \mathcal{O}$ such that $DF(p)$ is invertible (if any such point exists). Call p a *base-point* of F and let $y \in Y \backslash F(\partial\Omega)$ be a regular value of $F_{|\Omega}$. Note that $F(\partial\Omega)$ is closed in Y and that $F^{-1}(y)$ is a finite subset of Ω because F is proper on $\overline{\Omega}$.

If $F^{-1}(y) \neq \emptyset$, then $F^{-1}(y) = \{x_1, ..., x_k\}$ for some $k \in \mathbb{N}$. For $1 \leq j \leq k$, chose a continuous path $\gamma_j : [0,1] \to \mathcal{O}$ joining p to x_j. Such a path exists since $\mathcal{O}$ is (path-) connected.

Since F is Fredholm of index 0 and both $DF(p)$ and $DF(x_j)$ are invertible (the former by the choice of p and the latter since y is a regular value of $F_{|\Omega}$) the parity $\sigma(DF \circ \gamma_j, [0,1])$ is unambiguously defined. Furthermore, if $\tilde{\gamma}_j$ is another path like γ_j, then γ_j and $\tilde{\gamma}_j$ are homotopic (since $\mathcal{O}$ is simply connected) and so $DF \circ \gamma_j$ and $DF \circ \tilde{\gamma}_j$ are homotopic as paths of Fredholm operators of index 0. By the homotopy invariance of the parity,

$$\sigma(DF \circ \tilde{\gamma}_j, [0,1]) = \sigma(DF \circ \gamma_j, [0,1]).$$

Therefore, if we set

$$d_p(F, \Omega, y) := \sum_{j=1}^{k} \sigma(DF \circ \gamma_j, [0,1]) \in \mathbb{Z},$$

then the right-hand side is independent of the choice of the paths $\gamma_1, ..., \gamma_k$ and so depends only upon p, F, y and Ω. The notation d_p emphasizes the fact that the degree defined above depends upon the choice of the base-point p. If $F^{-1}(y) = \emptyset$, we simply define

$$d_p(F, \Omega, y) := 0.$$

In light of the interpretation of the parity given in the previous subsection, this definition of the degree at regular values is obviously reminiscent of the definition of the Brouwer degree at regular values.

Two natural questions arise: What if the base-point is changed and what if there is no base-point? Both answers are rather simple.

First, assume that $q \in \mathcal{O}$ is another base-point. The degree $d_q(F, \Omega, y)$ can be calculated by first going from q to p by a fixed continuous path τ in $\mathcal{O}$ and then going from p to x_j by following γ_j used in the definition of $d_p(F, \Omega, y)$. Then by using the multiplicative property of the parity with respect to juxtaposition of paths and assuming that τ is parametrized by $t \in [0,1]$, we obtain

$$d_q(F, \Omega, y) = \sigma(DF \circ \tau, [0,1]) d_p(F, \Omega, y).$$

In particular, this shows that changing the base-point only changes the degree up to sign. As a result, the "absolute" degree

$$|d|(F, \Omega, y) := |d_p(F, \Omega, y)|$$

is independent of the base-point and thus does not require the existence of base-points. Therefore, we may define

$$|d|(F, \Omega, y) = 0$$

when F has no base-point (which, not surprisingly, turns out to be the only definition consistent with the remainder of the theory).

The next step consists in defining the degree $d_p(F, \Omega, y)$ when $y \in Y \backslash F(\partial\Omega)$. Exactly as in the case of Brouwer's degree, this definition is

$$d_p(F, \Omega, y) := d_p(F, \Omega, z),$$

where z is any regular value of $F_{|\Omega}$ close enough to y. The existence of such z is ensured by the Sard-Smale theorem (instead of Sard's theorem in the case of Brouwer's degree), but it must first be proved that $d_p(F, \Omega, z)$ is independent of the regular value z close enough to y. If F is C^2, this can be done by following step by step the approach described earlier for the Brouwer degree, just using the Sard-Smale theorem (for homotopies) instead of Sard's theorem. If F is only C^1, this justification is more delicate. In that regard, it must be pointed out that, in infinite dimension, the approximation theorems (by smoother mappings) used in the construction of Brouwer's degree are not available (in spite of partial results by Bonic and Frampton [8] which are not directly relevant here). Therefore, the problem cannot be resolved by approximating a C^1 mapping F by C^2 mappings (not to mention the properness issue).

Remark 1.4. *The properness of F makes it possible to remove the separability assumption in Smale's original proof of the Sard-Smale theorem. This remark is due to Quinn and a proof is given in Quinn and Sard [57]. This is why no separability assumption was made in the above discussion.*

Remark 1.5. *If F above has the form $F = I - K$ with K compact and $d(I - K, \Omega, y)$ denotes the Leray–Schauder degree, then $d_p(I - K, \Omega, y)$ coincides with $d(I - K, \Omega, y)$ for every $y \notin (I - K)(\partial\Omega)$ or with $-d(I - K, \Omega, y)$ for every $y \notin (I-K)(\partial\Omega)$ (and whether $d_p = d$ or $d_p = -d$ depends in general upon the base-point p).*

3.3. Homotopy variance of the degree. As pointed out earlier, Kuiper's theorem implies that no degree theory for Fredholm mappings of index 0 can be homotopy invariant if it generalizes the Leray–Schauder degree (at least when $X = Y$ is a Hilbert space, but of course this difficulty cannot disappear in the more general Banach space setting). This has been a major issue in earlier theories which, of necessity, had to be restricted to subclasses of Fredholm mappings of index 0 which are narrow enough that homotopy invariance is not in contradiction with Kuiper's theorem. The use of the base-point degree eliminates all restrictions without contradicting Kuiper's theorem, as we now explain.

To begin with, it must be emphasized that the homotopy invariance property is crucial in degree theory only because it says how the degree changes under homotopy (it does not!). As a result, the failure of the homotopy invariance property is not a shortcoming of a degree theory provided that the change -or lack thereof- can be monitored during any given homotopy. Once again, the parity is the tool needed to monitor such changes with the base-point degree.

More precisely, let $H : [0,1] \times \mathcal{O} \to Y$ be C^1 and such that $H(t, \cdot)$ is Fredholm of index 0 for every $t \in [0,1]$ and such that $H : [0,1] \times \overline{\Omega} \to Y$ is continuous and proper. Now, let $p_0 \in \mathcal{O}$ be a base-point for $H(0, \cdot)$ and let $p_1 \in \mathcal{O}$ be a base-point for $H(1, \cdot)$. If $y \in Y \backslash H([0,1] \times \partial\Omega)$, then obviously $y \notin H(0, \partial\Omega)$ and $y \notin H(1, \partial\Omega)$, so that $d_{p_0}(H(0, \cdot), \Omega, y)$ and $d_{p_1}(H(1, \cdot), \Omega, y)$ are defined. Furthermore,

$$d_{p_0}(H(0, \cdot), \Omega, y) = \sigma(D_x H \circ \Gamma, [0,1]) d_{p_1}(H(1, \cdot), \Omega, y),$$

where $\Gamma : [0,1] \to [0,1] \times \mathcal{O}$ is any continuous path joining $(0, p_0)$ to $(1, p_1)$. Thus, the degree is either preserved (when $\sigma(D_x H \circ \Gamma, [0,1]) = 1$) or changed into its negative (when $\sigma(D_x H \circ \Gamma, [0,1]) = -1$), which is consistent with Kuiper's theorem.

REMARK 1.6. *If p is a base-point of both $H(0,\cdot)$ and $H(1,\cdot)$, it may still happen that $d_p(H(0,\cdot),\Omega,y) = -d_p(H(1,\cdot),\Omega,y)$. However, if p is a base-point of $H(t,\cdot)$ for every $t \in [0,1]$, then $d_p(H(0,\cdot),\Omega,y) = d_p(H(1,\cdot),\Omega,y)$.*

REMARK 1.7. *If either $H(0,\cdot)$ or $H(1,\cdot)$ has no base-point, then $|d|(H(t,\cdot),\Omega,y) = 0$ for every $t \in [0,1]$. More generally, this remains true if $H(t_0,\cdot)$ has no base-point for some $t_0 \in [0,1]$. Thus, if (say) $H(0,\cdot)$ has no base-point but $H(1,\cdot)$ does, then $d_{p_1}(H(1,\cdot),\Omega,y) = 0$ for every base-point p_1 of $H(1,\cdot)$.*

3.4. An application: Global bifurcation. Let $K : X \to X$ be a (nonlinear) compact mapping such that $K(0) = 0$ and K is Fréchet differentiable at 0. Consider the problem of finding $(\lambda, x) \in \mathbb{R} \times X$ such that

$$x - \lambda K(x) = 0. \tag{1.3}$$

Obviously, $(\lambda, 0)$ is a solution for every $\lambda \in \mathbb{R}$ (trivial branch of solutions). Suppose now that λ_0^{-1} is an eigenvalue of $DK(0)$ of odd algebraic multiplicity ($DK(0) \in \mathcal{L}(X)$ is compact since K is compact). It is an old result of Krasnosell'skii [45] that $(\lambda_0, 0)$ is a point of bifurcation for (1.3), i.e., that (1.3) has solutions (λ, x) with $x \neq 0$ arbitrarily close to $(\lambda_0, 0)$. While this result is only local, Rabinowitz proved the following much stronger property: Call $\mathcal{S}$ the closure in $\mathbb{R} \times X$ of the nontrivial solutions of (1.3) and $\mathcal{C}$ the connected component of $\mathcal{S} \cup \{(\lambda_0, 0)\}$ containing $(\lambda_0, 0)$. Then, either

(i) $\mathcal{C}$ is unbounded

or

(ii) $\mathcal{C}$ contains a point $(\lambda_1, 0)$ with $\lambda_1 \neq \lambda_0$.

The proof of this result eventually relies on the fact that, otherwise, there are $\lambda_- < \lambda_0 < \lambda_+$ with $\lambda_\pm$ arbitrarily close to λ_0 such that $d(I - \lambda_- K, B_\varepsilon, 0) = d(I - \lambda_+ K, B_\varepsilon, 0)$ (Leray–Schauder degree) for every $\varepsilon > 0$ small enough, where $B_\varepsilon \subset X$ is the open ball with center 0 and radius ε. This cannot be true because, for small ε, $x = 0$ is the only solution of $x - \lambda_\pm K(x) = 0$ in B_ε, so that $d(I - \lambda_\pm K, B_\varepsilon, 0) = (-1)^{m_\pm}$ where $m_\pm$ is the sum of the algebraic multiplicities of the eigenvalues of $DK(0)$ greater than $\lambda_\pm^{-1}$. Thus, $m_+ - m_-$ is the (odd) multiplicity of λ_0^{-1}, so that $(-1)^{m_+} = -(-1)^{m_-} \neq (-1)^{m_-}$.

Above, an important point of the proof is that it relies on the change of sign of some Leray–Schauder degree, which cannot be captured if this degree is replaced by a mere mod 2 degree. By using the $\mathbb{Z}$-valued degree for Fredholm mappings of index 0, a broad generalization of this global bifurcation result can be obtained: Let $F(= F(\lambda, x)) : \mathbb{R} \times X \to Y$ be C^1 Fredholm of index 1 (so that $F(\lambda,\cdot) : X \to X$ is C^1 Fredholm of index 0 for every λ) and suppose that

$$F(\lambda, 0) = 0, \qquad \forall \lambda \in \mathbb{R}.$$

Suppose that there are $\lambda_- < \lambda_+$ such that $D_x F(\lambda_\pm, 0) \in \mathcal{L}(X, Y)$ is invertible and that

$$\sigma(D_x F(\cdot, 0), [\lambda_-, \lambda_+]) = -1.$$

If $\mathcal{S}$ denotes the closure in $\mathbb{R} \times X$ of the nontrivial solutions of

$$F(\lambda, x) = 0$$

and $\mathcal{C}$ is the connected component of $S \cup [\lambda_-, \lambda_+] \times \{0\}$ containing $[\lambda_-, \lambda_+] \times \{0\}$. Then, either

(i) $\mathcal{C}$ is noncompact

or

(ii) $\mathcal{C}$ contains a point $(\lambda_1, 0)$ with $\lambda_1 \notin [\lambda_-, \lambda_+]$.

REMARK 1.8. *If also F is proper on the closed bounded subsets of $\mathbb{R} \times X$, which is the case when $F(\lambda, x) = x - \lambda K(x)$ with K compact, as in Rabinowitz' theorem, then "noncompact" above is equivalent to "unbounded".*

Above, the condition $\sigma(D_x F(\cdot, 0), [\lambda_-, \lambda_+]) = -1$ replaces and generalizes the assumption that λ_0^{-1} is an eigenvalue of $DK(0)$ of odd multiplicity in Rabinowitz' theorem. Although the proof is based on the base-point degree defined earlier (instead of Leray–Schauder's), no properness assumption is needed. Indeed, while Fredholm mappings need not be proper, not even proper on arbitrary closed bounded subsets, they are *locally proper*, i.e., proper when restricted to any small enough closed neighborhood of a point (this is true regardless of the index). This property turns out to be sufficient for the proof of the global bifurcation theorem.

4. Comments and references

There are of course multiple sources for the Brouwer and Leray–Schauder degrees. The most popular seems to be Lloyd [51] but others include Röthe [76] or Fonseca and Gangbo [34]. Deimling [21] also contains an introduction to various generalizations of the Leray–Schauder degree (but not to Fredholm mappings). Not every exposition of the Brouwer degree can be used as a guideline for the construction of the base-point degree for Fredholm mappings of index 0. For instance, an integral representation due to Heinz is sometimes deemed convenient (as in [21]) to justify the definition of the degree at singular (i.e., not regular) values. This argument has no infinite dimensional variant.

The concept of parity was introduced by Fitzpatrick and Pejsachowicz [29] in connection with semilinear PDEs and investigated further in [30]. The base-point degree was developed by Fitzpatrick, Pejsachowicz and Rabier [31] for C^2 mappings and subsequently by Pejsachowicz and Rabier [56] in the C^1 case. Equivariant properties (i.e., Borsuk's theorem and generalizations) are discussed in Fitzpatrick, Pejsachowicz and Rabier [33].

For simplicity, the exposition given here was limited to the case when the domain $\mathcal{O}$ of F is connected and simply connected. This suffices in many applications when $\mathcal{O} = X$, but in fact a degree can be defined for every *orientable* C^1 (proper) Fredholm mapping F of index 0 irrespective of its domain. Here, "orientable" means that the parity of $DF \circ \gamma$ along any path $\gamma : [a, b] \to \mathcal{O}$ joining two regular points p and q of F (base-points) depends only upon the endpoints p and q of the path. This is equivalent to the existence of a function ε defined on the set of base-points of F with values in $\{-1, 1\}$ such that

$$\sigma(DF \circ \gamma, [a, b]) = \varepsilon(p)\varepsilon(q).$$

The function ε is not unique (for instance, it may be replaced by $-\varepsilon$) and every choice of ε is called an orientation of F. If every connected component of $\mathcal{O}$ is simply connected, then every C^1 Fredholm mapping defined on $\mathcal{O}$ is orientable. In fact, this remains true if "simply connected" is replaced by the weaker requirement that $H^1(\mathcal{O}) = 0$, where $H^1(\mathcal{O})$ is the first cohomology group of $\mathcal{O}$ with coefficients in $\mathbb{Z}$. On the other hand, some classes of mappings are orientable irrespective of their domain. For instance, this is the case when $X = Y$ is finite dimensional (and then $\varepsilon(x) = \operatorname{sign}\det DF(x)$ is a possible orientation) or if $X = Y$ and $F = I - K$ is a Leray–Schauder map.

This more general approach to the degree is taken in Fitzpatrick, Pejsachowicz and Rabier [32] (where the assumption that F is C^2 can be replaced by C^1 now that a C^1 theory is available). This reference also contains a generalization of the degree to mappings between Banach *manifolds* and an explanation of the connections with several earlier works, notably that of Elworthy and Tromba [25], [26].

A different definition of orientation and correspondingly different definition of the degree have subsequently been given by Benevieri and Furi [6], without using the concept of parity. Instead, an orientation is defined by the choice of an equivalence class of finite rank perturbations of $DF(x)$ for every $x \in \mathcal{O}$. This leads to a definition of the degree which is less explicit than the base-point degree (though essentially equivalent to it), but allows for a simpler treatment of the C^1 case.

More recently, the degree theory was extended to compact perturbations of C^1 Fredholm mappings by Rabier and Salter [69] and (independently) Benevieri, Calamai and Furi [5], who also discuss more general perturbations.

The work by the Russian school (Borisovich, Zvyagin and Sapronov [9] and the references therein, Zvyagin [86]) with similar goals should also be acknowledged, in which the general idea is to combine the Elworthy-Tromba approach via Fredholm structures with the arguments of Caccioppoli [16]. The use of Fredholm structures is (notoriously) inconvenient in practice and the technicalities are not always clear, but some of these works also develop degree theories for Fredholm mappings of positive index; see for instance Zvyagin and Ratiner [87]. If so, the degree is no longer an integer but an element of a more abstract group.

For existence results beyond degree theory but related to regular/critical values, see Rabier [59], [65] and the background material in [58].

The first (and still main) global bifurcation theorem is due to Rabinowitz [73]. Its generalization to Fredholm mappings of index 0 discussed earlier can be found in Pejsachowicz and Rabier [56]. Incidentally, by simply rescaling the λ variable, this theorem yields a global bifurcation result when $F(\lambda, x)$ is only defined on $J \times X$ where $J \subset \mathbb{R}$ is an open interval. If $\mathcal{S}$ is the closure of the nontrivial solutions relative $J \times X$, then the alternative is unchanged. However, if J is a closed interval, then the third option must be added that $\mathcal{C}$ is compact and intersects $\partial J \times X$.

CHAPTER 2

Fredholmness and properness of differential operators

To use the degree theory developed in Part 1 when $F : X \to Y$ represents a nonlinear differential operator and X and Y are function spaces, criteria are needed to decide that F is Fredholm of index 0 and that F is proper, or at least proper on some closed subset of X of interest.

In spite of the fact that F is nonlinear, the Fredholmness properties are "only" linear ones since, by the very definition of Fredholmness, they depend solely upon the derivatives $DF(u) \in \mathcal{L}(X, Y)$. (From now on, we call u instead of x the generic variable of X and x will refer to a point of the space $\mathbb{R}^N$.) In contrast, the properness issue is entirely nonlinear.

As a result, the questions reduce to finding sufficient conditions for linear differential operators to be Fredholm (of index 0) and sufficient conditions for nonlinear differential operators to be proper, at least on closed bounded subsets. Naturally, the functional setting (choice of X and Y) plays a role in the answers to these questions as well.

The possibilities are so varied (type of operator, geometry of the domain, boundary/initial conditions, function spaces) that it is of course impossible to give a nearly complete answer. However, as a rule, elliptic problems on bounded domains and smooth boundaries with "standard" homogeneous boundary conditions are often Fredholm of index 0 and proper on closed bounded subsets. Actually, such problems are often (but of course not always) accounted for by a Leray–Schauder mapping $F = I - K$ with K compact, so that the Leray–Schauder degree is available and there is no need for the Fredholm degree theory. In contrast, even in the simplest cases, the Leray–Schauder theory is typically *not* available for problems over *unbounded* domains.

Accordingly, our subsequent discussion will be mostly limited to problems on the entire space (with the exception of the application to Navier-Stokes), so that no additional boundary condition needs to be taken into account. Furthermore, we shall confine attention to function spaces that are Sobolev spaces. For simplicity of notation, $W^{m,p}$ denotes the Sobolev space $W^{m,p}(\mathbb{R}^N)$ (L^p if $m = 0$) and likewise $C^k = C^k(\mathbb{R}^N)$, etc. Many nonlinear mappings on Sobolev spaces are only defined when $W^{m,p}$ is a Banach algebra, i.e., when $mp > N$. Since the natural choice of m is often dictated by the order of the operator, it is thus of primary importance *not* to limit the discussion to $p = 2$ and to allow arbitrarily large values of p.

One surprising by-product of the Fredholmness of linear and nonlinear PDE operators on unbounded domains, not discussed in these notes, is its strong impact on the asymptotic behavior of solutions. For such matters, see [62], [66].

1. Fredholmness of elliptic operators on $\mathbb{R}^N$

In this section, $P(x,\partial)$ denotes the (scalar) differential operator

$$P(x,\partial) := \sum_{|\beta|\le m} a_\beta(x)\partial^\beta, \tag{2.1}$$

where $m \ge 0$ is an integer, $x \in \mathbb{R}^N, \beta = (\beta_1, ..., \beta_N) \in (\mathbb{N} \cup \{0\})^N$ is a multi-index and a_β is a complex-valued function defined (at least) a.e. on $\mathbb{R}^N$ for the Lebesgue measure.

1.1. The Cordes–Illner theory. A function $a \in C^0$ is said to have *vanishing oscillation* at infinity if

$$\lim_{|x|\to\infty} \sup_{|y-x|\le 1} |a(x) - a(y)| = 0.$$

Above, $\sup_{|y-x|\le 1}$ can be replaced by $\sup_{|y-x|\le\delta}$ for any $\delta > 0$. For instance, if a is C^1, then a has vanishing oscillation at infinity if $\lim_{|x|\to\infty} |\nabla a(x)| = 0$. This condition is by no means necessary, for if $a(x)$ has a limit a_∞ when $|x| \to \infty$, then a has also vanishing oscillation at infinity regardless of the behavior of ∇a (which need not even exist).

The following result, due to Cordes [19] and Illner [42], gives a necessary and sufficient condition for $P(x,\partial)$ in (2.1) to be Fredholm from $W^{m,p}$ to L^p for every $p \in (1,\infty)$.

THEOREM 2.1. *Suppose that the coefficients a_β are continuous and bounded on $\mathbb{R}^N$ and have vanishing oscillation at infinity. Then, given $p \in (1,\infty)$, $P(x,\partial) : W^{m,p} \to L^p$ is Fredholm if and only if there are constants $c > 0$ and $\rho > 0$ such that*

$$\left|\sum_{|\beta|\le m} a_\beta(x) i^{|\beta|-m}\eta^\beta\right| \ge c(1+|\eta|^2)^{\frac{m}{2}}, \quad (x,\eta) \in \mathbb{R}^N \times \mathbb{R}^N : |x| + |\eta| \ge \rho. \tag{2.2}$$

Note that, in Theorem 2.1, the boundedness of the coefficients ensures that $P(x,\partial)$ does map $W^{m,p}$ to L^p. It is helpful to notice that condition (2.2) is equivalent to both the conditions

(i) $\left|\sum_{|\beta|=m} a_\beta(x)\eta^\beta\right| \ge c|\eta|^m, \qquad \forall (x,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$

and

(ii) $\underline{\lim}_{|x|\to\infty} \left|\sum_{|\beta|\le m} a_\beta(x) i^{|\beta|-m}\eta^\beta\right| \ge c(1+|\eta|^2)^{\frac{m}{2}}, \qquad \forall \eta \in \mathbb{R}^N,$

for some constant $c > 0$.

Of course, (i) is just the uniform ellipticity of $P(x,\partial)$, but (ii) depends upon all the coefficients a_β, not merely the leading coefficients with $|\beta| = m$. For example $P(x,\partial) = -\Delta+1$ satisfies the hypotheses of Theorem 2.1 with $m = 2$, but $P(x,\partial) = -\Delta$ does not ((ii) fails since there is no $c > 0$ such that $|\eta|^2 \ge c(1+|\eta|^2)^{\frac{1}{2}}$ for every $\eta \in \mathbb{R}^N$).

This elementary example also shows where the difference with the bounded domain case lies: There is no contradiction with $-\Delta + 1$ being Fredholm (and in fact even an isomorphism) and $-\Delta$ not being Fredholm because the embedding $W^{2,p} \hookrightarrow L^p$ is not compact and so neither operator is a compact perturbation of the other This is a general phenomenon: It is primarily the lack of compactness in the Sobolev embedding theorems which is responsible for the possible failure of Fredholmness in elliptic problems over unbounded domains.

The Cordes–Illner theorem is much simpler to prove when $p = 2$ (but still nontrivial). In this special case, it was proved earlier by Taylor [84] and another proof can be found in Hörmander [40], where systems are also discussed. For $p \neq 2$, it is a remarkable application of the theory of *commutative Banach algebras*, but C^*-algebra arguments (as in [84]) cannot be used when $p \neq 2$ and major technical difficulties have to be resolved. Anecdotally, it may be pointed out that the Cordes-Illner theorem is surprisingly little known and has subsequently been rediscovered in special cases by pseudo-differential operator methods long after its original publication ([27]).

By using the Banach algebra machinery, it is a fairly simple matter to extend Theorem 2.1 to elliptic systems

$$P(x,\partial) := \sum_{|\beta|\leq m} A_\beta(x)\partial^\beta, \tag{2.3}$$

where now $A_\beta(x)$ are $r \times r$ matrices. When $p = 2$, this was done by Cordes and Herman [20] and Hörmander (loc. cit). The general case is discussed by Sun [83] and can be summarized as follows.

THEOREM 2.2. *Suppose that the matrix coefficients A_β are continuous and bounded on $\mathbb{R}^N$ and have vanishing oscillation at infinity. Then, given $p \in (1,\infty)$, $P(x,\partial) : (W^{m,p})^r \to (L^p)^r$ is Fredholm if and only if there are constants $c > 0$ and $\rho > 0$ such that*

$$\left|\det\left(\sum_{|\beta|\leq m} A_\beta(x) i^{|\beta|-m}\eta^\beta\right)\right| \geq c(1+|\eta|^2)^{\frac{mr}{2}}, \quad (x,\eta) \in \mathbb{R}^N \times \mathbb{R}^N : |x| + |\eta| \geq \rho. \tag{2.4}$$

By letting $|\eta| \to \infty$ it follows that (2.4) implies the uniform Petrovsky ellipticity condition

$$\left|\det\left(\sum_{|\beta|=m} A_\beta(x)\eta^\beta\right)\right| \geq c|\eta|^{mr}, \quad \forall (x,\eta) \in \mathbb{R}^N \times \mathbb{R}^N,$$

but, just like (2.2), it also incorporates a condition on the lower order coefficients.

When $p = 2$ and under a stronger assumption about the coefficients A_β (or a_β in (2.1)), namely, $A_\beta \in C^\infty$ and

$$|\partial^\gamma A_\beta(x)| = O(|x|^{-|\gamma|}) \text{ as } |x| \to \infty, \tag{2.5}$$

for all multi-indices γ, the index of $P(x,\partial)$ is given by the so-called Fedosov-Hörmander formula (Fedosov [28], Hörmander [40])

$$\text{index } P(x,\partial) = -\left(\frac{i}{2\pi}\right)^N \frac{(N-1)!}{(2N-1)!} \int_{\partial B} Tr((\sigma^{-1}d\sigma)^{\wedge 2N-1}),$$

where $\sigma(x,\eta) := \sigma_{P(x,\partial)}(x,\eta)$ is the symbol of $P(x,\partial)$ and B is any open ball in $\mathbb{R}^N \times \mathbb{R}^N$ such that $\sigma(x,\eta)$ is invertible on the exterior of B (recall condition (2.4)).

It is noteworthy that the right-hand side vanishes when $r < N$ (so that the index is 0). This is observed in Bott and Seeley [10] and further explained in Rabier [67]. In particular, if $r = 1$ (scalar case), the index is 0. It is a conjecture of Bott and Seeley -apparently still unresolved, even when $r = 1$- that the Fedosov-Hörmander formula remains valid without the decay assumption (2.5). (Case in point: Coefficients with vanishing oscillation at infinity need not have C^∞ approximations satisfying (2.5); however, they do if the coefficients have limits as $|x| \to \infty$.)

REMARK 2.3. *It is easily checked, for instance with $N = 1$ and $r = 1$, that the index need not be 0 if $r \geq N$. Therefore, the value of the index becomes a significant issue in systems in which the number of unknown functions equals or exceeds the number of variables on which they depend. Of course, the index may still be 0 in this case.*

The necessary and sufficient conditions for Fredholmness in the Cordes–Illner theorem are independent of $p \in (1, \infty)$, so that Fredholmness is simultaneously true or false for all such values of p. This suggests, but does not prove, that the index, when defined, is independent of p. This issue has recently been resolved in the affirmative in Rabier [67]. The difficulty is that the proof of the Cordes–Illner theorem is based on the characterization of the Fredholm operators on a Banach space X as those operators which are invertible in $\mathcal{L}(X)/\mathcal{K}(X)$ (quotient by the two-sided ideal of compact operators). While this conveniently identifies all the Fredholm operators at once, the drawback is that all information about the index is lost.

From the p-independence of the index, it follows that the Fedosov-Hörmander formula is actually valid for all $p \in (1, \infty)$.

REMARK 2.4. *There are much earlier results about the p-independence of the index of singular integral operators on L^p. For instance, this was established by Seeley [79] in a class for which ellipticity is equivalent to Fredholmness. This assumption is not satisfied by the operators arising from PDEs (as the Cordes-Illner theorem shows).*

1.2. Other results. If the coefficients a_β in (2.1) are bounded but do not have vanishing oscillation at infinity, nothing general is known about the Fredholmness of $P(x, \partial)$, but partial results are available. For example, the *Lax–Milgram theorem* plus elliptic regularity yields sufficient conditions about the coefficients for $P(x, \partial)$ to be an isomorphism from $W^{m,2}$ to L^2. (These conditions do not include the vanishing oscillation assumption of the Cordes–Illner theorem.) More generally, $P(x, \partial) : W^{m,2} \to L^2$ will be Fredholm of index 0 if it is a compact perturbation of such an isomorphism. In that regard, it is useful to keep in mind that the multiplication by a L^∞ function tending (essentially) to 0 at infinity is a compact operator from $W^{m,p}$ to L^p for every $p \in [1, \infty)$.

If, in addition, $m = 2$ (second order elliptic problems), then more is true: Under general assumptions about the coefficients, the index of $P(x, \partial) : W^{2,p} \to L^p$ is independent of $p \in (1, \infty)$. Thus, in this case, the fact that $P(x, \partial) : W^{2,2} \to L^2$ is Fredholm of index 0 ensures that $P(x, \partial) : W^{2,p} \to L^p$ is Fredholm of index 0 for

$p \in (1, \infty)$. For example, if

$$P(x, \partial)u := -\sum_{i,j=1}^{N} \partial_j (a_{ij}(x)\partial_i u) + \sum_{i=1}^{N} b_i(x)\partial_i u + c(x)u, \tag{2.6}$$

with *real* coefficients satisfying

$$a_{ij} \in W^{1,\infty}, \qquad b_i, c \in L^\infty,$$

as well as the uniform ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2, \qquad \forall \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N,$$

then

THEOREM 2.5. *The set* $\{\lambda \in \mathbf{C} : P(x, \partial) + \lambda : W^{2,p} \to L^p$ *is Fredholm of index* $0\}$ *is independent of* $p \in (1, \infty)$.

Theorem 2.5 is proved in Rabier [60]. If the sesquilinear form $a(\cdot, \cdot)$ associated with $P(x, \partial)$ is coercive on $W^{1,2}$, then $P(x, \partial)$ is an isomorphism of $W^{2,2}$ to L^2 (see for instance [60], but this has been known for long). Since this form is coercive upon replacing $P(x, \partial)$ by $P(x, \partial) + \lambda$ with $\lambda > 0$ large enough, it follows that the set $\{\lambda \in \mathbf{C} : P(x, \partial) + \lambda : W^{2,p} \to L^p$ is Fredholm of index $0\}$ is nonempty.

Several ingredients in the proof of Theorem 2.5 are rather specific to second order *scalar* equations and are not valid for general higher order problems or systems. One is that the semigroup on L^2 generated by $P(x, \partial)$ satisfies a Gaussian estimate. From results by Arendt [2], this is needed to show that the maximal domain of $P(x, \partial)$ on L^p is actually $W^{2,p}$. A second and equally important feature is the unique continuation property (Hörmander [41], Garofalo and Lin [37]). The unique continuation property is only true for some higher order scalar operators or systems (even of second order).

In special cases of (2.6), the boundedness requirements of the coefficients may be relaxed. For example, it is shown in Rabier and Stuart [70] that

$$P(x, \partial) := -\Delta + V(x) : W^{2,p} \to L^p$$

is Fredholm of index 0 for every $p \in (1, \infty)$ when V is a Kato-Rellich potential (such potentials need not be locally bounded).

REMARK 2.6. *In spite of scattered partial results, the Fredholm properties of differential operators on unbounded domains with other geometries (and boundary conditions) have been studied little, even in the case of exterior domains or unbounded cylinders.*

2. Properness of nonlinear elliptic operators on $\mathbb{R}^N$

Let X and Y be Banach spaces and let $C \subset X$ be a closed subset. If $F : C \to Y$ is continuous, when is F proper? Unless C is compact, the only *general* answer known to date is that F is proper when $Y = X, C$ is bounded and $F = I - K$ with $K : C \to X$ compact. The proof is trivial. As pointed out earlier, this structure is too restrictive for most PDE problems over unbounded domains. However, properness results for nonlinear second order elliptic operators have been obtained by Rabier and Stuart [71], by using some of the specific features of such operators. A summary of this work is given below.

In what follows,

$$F(u) := -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(\cdot, u, \nabla u)\partial^2_{\alpha\beta}u + b(\cdot, u, \nabla u), \tag{2.7}$$

where $x \in \mathbb{R}^N$ and $a_{\alpha\beta} = a_{\alpha\beta}(x, \xi)$ and $b = b(x, \xi)$ satisfy appropriate conditions, described below. (As in [71], we assume that $a_{\alpha\beta}$ and b are real-valued, although this is not essential to the arguments used in that paper.)

Let $f(= f(x, \xi)) : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ be a function. Whenever we need to display the components of x and ξ, we shall always use $x = (x_1, ..., x_N)$ and $\xi = (\xi_0, ..., \xi_N)$. By viewing $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$ and $\mathbb{R}^N \times \mathbb{R}$ as bundles over $\mathbb{R}^N$, f can be identified with the "bundle" map

$$(x, \xi) \in \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to (x, f(x, \xi)) \in \mathbb{R}^N \times \mathbb{R}.$$

The terminology "bundle map" is convenient to refer to properties of f in which the "base" variable x and the "fiber" variable ξ play markedly different roles, but bundle theory is not involved in any of the subsequent considerations.

Definition 2.7. We shall say that f is an equicontinuous C^0 bundle map if f is continuous and the collection $(f(x, \cdot))_{x\in\mathbb{R}^N}$ is equicontinuous at every point of $\mathbb{R} \times \mathbb{R}^N$. If $k \geq 0$ is an integer, we shall say that f is an equicontinuous C^k_ξ bundle map if the partial derivatives $D^\kappa_\xi f$, $|\kappa| \leq k$, exist and are equicontinuous C^0 bundle maps.

The above definition is now used to formulate the assumptions about $a_{\alpha\beta}$ and b in (2.7):

$$a_{\alpha\beta} \text{ is an equicontinuous } C^1_\xi \text{bundle map}, \quad 1 \leq \alpha, \beta \leq N,$$

$$a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N),\ \partial_{\xi_i} a_{\alpha\beta}(\cdot, 0) \in L^\infty(\mathbb{R}^N),\ 1 \leq \alpha, \beta \leq N,\ 0 \leq i \leq N,$$

$$\begin{cases} \sum_{\alpha,\beta=1}^N a_{\alpha\beta}(x, \xi)\eta_\alpha\eta_\beta \geq \gamma(x, \xi)|\eta|^2, \\ \forall \eta = (\eta_1, ..., \eta_N) \in \mathbb{R}^N, \quad \forall (x, \xi) \in \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N), \end{cases}$$

where $\gamma : \mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N) \to (0, \infty)$ is bounded from below by a positive constant $\gamma_{\tilde{K}}$ on every compact subset $\tilde{K}$ of $\mathbb{R}^N \times (\mathbb{R} \times \mathbb{R}^N)$ (for instance, γ lower semicontinuous),

$$b \text{ is an equicontinuous } C^1_\xi \text{ bundle map},$$

$$b(\cdot, 0) \in L^p(\mathbb{R}^N) \text{ for some } p \in (1, \infty) \text{ and } \partial_{\xi\alpha} b(\cdot, 0) \in L^\infty(\mathbb{R}^N), \quad 0 \leq \alpha \leq N.$$

The first result ensures the differentiability and Fredholmness of F.

THEOREM 2.8. *If $p > N$ and the above assumptions hold, then F in (2.7) is C^1 from $W^{2,p}$ to L^p and*

$$DF(0)h = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,0)\partial^2_{\alpha\beta}h + \sum_{\alpha=1}^{N} \frac{\partial b}{\partial \xi_\alpha}(x,0)\partial_\alpha h + \frac{\partial b}{\partial \xi_0}(x,0)h.$$

Furthermore, F is Fredholm if and only if $DF(0) \in \mathcal{L}(W^{2,p}, L^p)$ is Fredholm and its index is well defined and equal to the index of $DF(0)$.

Since F is Fredholm if and only if $DF(u)$ is Fredholm for every $u \in W^{2,p}$, the above theorem shows that $DF(0)$ being Fredholm suffices for this property to hold. This is not trivial since, in general, $DF(u)$ is *not* a compact perturbation of $DF(0)$ (it is one in the *semilinear* case when $a_{\alpha\beta}$ depends only upon x, but not on u or ∇u). That the index of F is well defined and equals the index of $DF(0)$ follows at once from the local constancy of the index and the connectedness of $W^{2,p}$. Of course, the results of the previous section can be used to decide whether the linear differential operator $DF(0)$ is Fredholm and to determine its index.

The hypotheses made in Theorem 2.8 do not suffice to ensure that F is proper on the closed and bounded subsets of $W^{2,p}$. This can be obtained under additional assumptions about the behavior of the "coefficients" at infinity. The simplest case is when these coefficients have limits when $|x| \to \infty$. More precisely, in addition to the previous assumptions about $a_{\alpha\beta}$ and b, assume that there are continuous mappings $a^\infty_{\alpha\beta} = a^\infty_{\alpha\beta}(\xi), 1 \leq \alpha, \beta \leq N$ and $c^\infty_\alpha = c^\infty_\alpha(\xi), 0 \leq \alpha \leq N$ (thus all independent of x) such that

$$\lim_{|x|\to\infty} |a_{\alpha\beta}(x,\xi) - a^\infty_{\alpha\beta}(\xi)| = 0, \qquad 1 \leq \alpha, \beta \leq N,$$

and

$$\lim_{|x|\to\infty} \left| \frac{\partial b}{\partial \xi_\alpha}(x,\xi) - c^\infty_\alpha(\xi) \right| = 0, \qquad 0 \leq \alpha \leq N,$$

where the limits are uniform in ξ on bounded subsets of $\mathbb{R}^{N+1}$.

Now, if $p > N$, define the limiting operator $F^\infty : W^{2,p} \to L^p$ by

$$F^\infty(u) := -\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(u,\nabla u)\partial^2_{\alpha\beta}u + \sum_{\alpha=1}^{N} c^\infty_\alpha(u,\nabla u)\partial_\alpha u + c^\infty_0(u,\nabla u)u.$$

(The assumption $p > N$ ensures that $a^\infty_{\alpha\beta}(u,\nabla u), c^\infty_\alpha(u,\nabla u) \in L^\infty$ for every $u \in W^{2,p}$, so that F^∞ does map $W^{2,p}$ to L^p.) Then,

THEOREM 2.9. *F is proper on the closed bounded subsets of $W^{2,p}$ ($p > N$) if and only if the equation $F^\infty(u) = 0$ has no nonzero solution in $W^{2,p}$.*

Some general lines of argument to prove that $F^\infty(u) = 0$ has no nontrivial solution, as required in Theorem 2.9, can be found in [72] but the issue is far from having been exhausted. Theorem 2.9 can also be used to establish the properness of F on $W^{2,p}$ (not merely on the closed bounded subsets of $W^{2,p}$) since this is equivalent to properness on closed bounded subsets plus boundedness of the solutions for bounded right-hand sides. In practice, the latter property amounts to finding (*a priori*) norm-estimates for the solutions.

Even though some kind of asymptotic property seems to be needed for properness on closed bounded subsets, Theorem 2.9 can be generalized in various ways. For instance, it remains valid when the coefficients are asymptotically N-periodic (i.e., $a_{\alpha\beta}^{\infty}$ and c_{α}^{∞} above are N-periodic in x rather than just x -independent; see [71] for precise statements).

Another more general variant requires only $a_{\alpha\beta}$ and $\frac{\partial b}{\partial \xi_\alpha}$ to have limits when x tends to ∞ in each direction $s \in \mathbb{S}^{N-1}$. This gives rise to a family $(F_s^{\infty})_{s\in\mathbb{S}^{N-1}}$ of limiting operators and the properness of F on the closed bounded subsets of $W^{2,p}$ is *equivalent* to $F_s^{\infty}(u) = 0$ not having any nonzero solution for any $s \in \mathbb{S}^{N-1}$. Both generalizations (N-periodic/directional limits at infinity) can be combined to get an even more general necessary and sufficient condition for properness. Ultimately, the proof of Theorem 2.9 and other variants relies on a generalization of Ascoli's theorem [63], although this is only implicit in the treatment given in [71].

Applications of Theorems 2.8 and 2.9 to bifurcation for quasilinear elliptic equations (based on the abstract results of Subsection 3.4) are given in Rabier and Stuart [72]. On the other hand, the main results of this section have been extended to elliptic systems by Gebran and Stuart [38].

3. The Navier–Stokes problem on exterior domains

This section discusses an application of the degree theory for Fredholm mappings of index 0 to the Navier-Stokes problem. Since this is neither a scalar equation nor a system which is elliptic in the sense of Petrovsky (and since the domain of interest is not the entire space), this problem does not fit into the general framework discussed earlier. The subsequent discussion follows Galdi and Rabier [36].

3.1. The problem. Let $U \subset \mathbb{R}^3$ denote an exterior domain (i.e., $\mathbb{R}^3\backslash U$ is compact) with Lipschitz continuous boundary. The problem is to find $\mathbf{v} = (v_1, v_2, v_3)$ and p such that

$$\begin{cases} -\Delta \mathbf{v} + \lambda \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = f \text{ in } U, \\ \nabla\cdot\mathbf{v} = 0 \text{ in } U, \\ \mathbf{v} = \mathbf{v}_* \text{ on } \partial U, \\ \lim_{|x|\to\infty} \mathbf{v}(x) = \mathbf{e}_1 := (1,0,0). \end{cases} \tag{2.8}$$

Above, $\lambda > 0$ is the Reynolds number and f and $\mathbf{v}_*$ (given) represent the body forces and the boundary velocity, respectively. The condition $\lim_{|x|\to\infty} \mathbf{v}(x) = \mathbf{e}_1$ simply means that the velocity $\mathbf{v}$ has a constant *nonzero* limit at infinity (after rescaling, this constant may indeed be chosen the unit vector $\mathbf{e}_1$).

The functions $\mathbf{v}$ satisfying the required boundary and asymptotic conditions do not form a vector space. This can be fixed by introducing a vector-valued function $\mathbf{W}_*$ such that

$$\begin{cases} \nabla\cdot\mathbf{W}_* = 0 \text{ in } U, \\ \mathbf{W}_* = \mathbf{v}_* - \mathbf{e}_1 \text{ on } \partial U, \\ \lim_{|x|\to\infty} \mathbf{W}_*(x) = \mathbf{0}, \end{cases} \tag{2.9}$$

and by setting

$$\mathbf{u} := \mathbf{v} - \mathbf{e}_1 - \mathbf{W}_*.$$

With this change of variable, (2.8) becomes

$$\begin{cases} -\Delta\mathbf{u} + \lambda\partial_1\mathbf{u} + \lambda\mathbf{W}_*\cdot\nabla\mathbf{u} + \lambda\mathbf{u}\cdot\nabla\mathbf{W}_* + \lambda\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = g \text{ in } U, \\ \nabla\cdot\mathbf{u} = 0 \text{ in } U, \\ \mathbf{u} = \mathbf{0} \text{ on } \partial U, \\ \lim_{|x|\to\infty}\mathbf{u}(x) = \mathbf{0}, \end{cases} \tag{2.10}$$

where $\partial_1 := \frac{\partial}{\partial x_1}$ and

$$g := f + \Delta\mathbf{W}_* - \lambda\partial_1\mathbf{W}_* - \lambda\mathbf{W}_*\cdot\nabla\mathbf{W}_*.$$

The formulation (2.10) may be viewed as a perturbation of the (linear) *Oseen problem*

$$\begin{cases} -\Delta\mathbf{u} + \lambda\partial_1\mathbf{u} + \nabla p = g \text{ in } U, \\ \nabla\cdot\mathbf{u} = 0 \text{ in } U, \\ \mathbf{u} = \mathbf{0} \text{ on } \partial U, \\ \lim_{|x|\to\infty}\mathbf{u}(x) = \mathbf{0}, \end{cases} \tag{2.11}$$

and the idea will be to try to solve (2.10) in a functional setting in which the Oseen operator with homogeneous boundary and asymptotic conditions (left-hand side of (2.11)) is known to have good properties.

3.2. The functional setting. The Oseen operator has no particularly interesting property in classical Sobolev spaces. However, it was found by Galdi [35] that such properties are recovered in suitable "anisotropic" Sobolev spaces allowing for different integrability conditions for different derivatives. These spaces are described below.

Given $q \in [1,2)$, set

$$q^* := \frac{3q}{3-q}, \quad s_1(q) := \frac{4q}{4-q}, \quad s_2(q) := \frac{2q}{2-q},$$

so that

$$q < s_1(q) < q^* < s_2(q).$$

Now, set

$$\mathbf{X}^{1,q}(U) := \{\mathbf{v} \in \left(L^{s_2(q)}(U)\right)^3 : \partial_1\mathbf{v} \in (L^q(U))^3, \\ \partial_i\mathbf{v} \in \left(L^{s_1(q)}(U)\right)^3, 1 \le i \le 3, \nabla\cdot\mathbf{v} = 0\}.$$

Clearly, $\mathbf{X}^{1,q}(U) \subset \left(W^{s_1(q)}_{loc}(U)\right)^3$, so that the functions in $\mathbf{X}^{1,q}(U)$ have a well defined trace on ∂U, which makes it possible to define the space

$$\mathbf{X}^{1,q}_0(U) := \{\mathbf{v} \in \mathbf{X}^{1,q}(U) : \mathbf{v} = \mathbf{0} \text{ on } \partial U\}.$$

Next, let

$$\mathbf{X}^{2,q}(U) := \{\mathbf{v} \in \mathbf{X}^{1,q}(U) : \partial^2_{ij}\mathbf{v} \in (L^q(U))^3, 1 \le i,j \le 3\}$$

and

$$Y^{1,q}(U) := \{p \in L^{q^*}(U) : \nabla p \in L^q(U)\}.$$

Then (see primarily [35, Theorem 7.1, p. 418] but also [36] for more details)

THEOREM 2.10. *For every* $\lambda > 0$, *the (Oseen) operator*

$$(\mathbf{u}, p) \in \left(\mathbf{X}^{2,q}(U) \cap \mathbf{X}_0^{1,q}(U)\right) \times Y^{1,q}(U) \mapsto L_\lambda(\mathbf{u}, p) := -\Delta\mathbf{u} + \lambda\partial_1\mathbf{u} + \nabla p \in L^q(U), \quad (2.12)$$

is an isomorphism.

Essentially, the above theorem ensures that, given $g \in (L^q(U))^3$, the problem (2.11) has a unique solution $(\mathbf{u}, p) \in \left(\mathbf{X}^{2,q}(U) \cap \mathbf{X}_0^{1,q}(U)\right) \times Y^{1,q}(U)$, except that the reason why $\lim_{|x|\to\infty} \mathbf{u}(x) = \mathbf{0}$ (required in (2.11)) for such solutions is not clear. We shall return to this point later on. For the time being, it suffices to point out that the functions of the space $\mathbf{X}^{2,q}(U)$ do possess decay properties at infinity, even though not necessarily pointwise. For example, if $q = 4/3$ (a value soon to be seen of special relevance), then ([35, Lemma 5.2, p. 62])

$$\lim_{R\to\infty} \int_{\mathbb{S}^2} |\mathbf{u}(R\omega)|^{\frac{12}{5}} d\omega = 0,$$

for every $\mathbf{u} \in \mathbf{X}^{2,4/3}(U)$.

3.3. Existence of solutions. Using the functional setting of the previous subsection, the nonlinear problem (2.10) of interest may now be rephrased as finding $(\mathbf{u}, p) \in \left(\mathbf{X}^{2,q}(U) \cap \mathbf{X}_0^{1,q}(U)\right) \times Y^{1,q}(U)$ such that

$$F_\lambda(\mathbf{u}, p) = g, \quad (2.13)$$

where

$$F_\lambda(\mathbf{u}, p) := L_\lambda(\mathbf{u}, p) + \lambda\mathbf{W}_* \cdot \nabla\mathbf{u} + \lambda\mathbf{u} \cdot \nabla\mathbf{W}_* + \lambda\mathbf{u} \cdot \nabla\mathbf{u} \quad (2.14)$$

and L_λ is the Oseen operator (2.12).

To prove the existence of solutions of (2.13), it must first be verified that the problem is well posed, i.e., that F_λ in (2.14) does map $\left(\mathbf{X}^{2,q}(U) \cap \mathbf{X}_0^{1,q}(U)\right) \times Y^{1,q}(U)$ into $(L^q(U))^3$. This is not true for all $q \in [1, 2)$, but it is true if $q = 4/3$ (this uses various embedding properties spelled out in [36]). Furthermore, F_λ is polynomial in $(\mathbf{u}, p)$, hence C^∞, and its linearization $DF_\lambda(\mathbf{u}, p)$ is a *compact* perturbation of L_λ for every $(\mathbf{u}, p) \in \left(\mathbf{X}^{2,4/3}(U) \cap \mathbf{X}_0^{1,4/3}(U)\right) \times Y^{1,4/3}(U)$. Thus, it follows from Theorem 2.10 with $q = 4/3$ that F_λ is Fredholm of index 0. (However, F_λ is *not* a nonlinear compact perturbation of L_λ.)

In order to use the degree theory for Fredholm mappings of index 0 developed in Part 1 with $\Omega = \mathcal{O} = \left(\mathbf{X}^{2,4/3}(U) \cap \mathbf{X}_0^{1,4/3}(U)\right) \times Y^{1,4/3}(U)$, the mapping F_λ must also be proper. Rather unexpectedly, this depends crucially upon the boundary condition $\mathbf{v}_*$ in (2.8), because the possible choices for $\mathbf{W}_*$ in (2.9) depend on the properties of $\mathbf{v}_*$. In turn, this has an impact on whether F_λ in (2.14) is proper.

At any rate, $F_\lambda : \left(\mathbf{X}^{2,4/3}(U) \cap \mathbf{X}_0^{1,4/3}(U)\right) \times Y^{1,4/3}(U) \to \left(L^{4/3}(U)\right)^3$ is proper if ∂U is C^2, $\mathbf{v}_* \in \left(W^{5/4,4/3}(\partial U)\right)^3$ and

$$\int_{\Gamma_i} \mathbf{v}_* \cdot \mathbf{n} = 0, \quad 1 \le i \le k,$$

where $\Gamma_1, ..., \Gamma_k$ are the connected components of ∂U.

By using a homotopy from F_λ to L_λ and a degree argument, it follows that (2.13) (i.e. (2.10) and hence the original problem (2.8)) is solvable for every $g \in \left(L^{4/3}(U)\right)^3$ ([36, Theorem 5.1]). The "full" $\mathbb{Z}$-valued base-point degree is not needed for this result: The mod 2 Smale degree (or, alternatively, the "absolute" degree discussed in Part 1) suffices.

Some refinements given in [36] are that the number of solutions is *generically* odd (even though it need not be finite for some g) and that $\mathbf{v}(x)$ tends *uniformly* to $\mathbf{e}_1$ as $|x| \to \infty$ for every solution $\mathbf{v}$ of (2.8) obtained above if, in addition, $f \in \left(L^{4/3}(U) \cap L^q(U)\right)^3$ for some $q \in (3/2, 2)$. (If $f \in \left(L^{4/3}(U)\right)^3$, the convergence is only in the generalized sense mentioned earlier.)

REMARK 2.11. *The problem (2.8) can also be formulated in dimension $N \neq 3$, but the approach described in this section goes through* only *if $N = 3$. Furthermore, when $N = 3$, it works* only *with the choice $q = 4/3$.*

4. Fredholmness of evolution operators

Although relatively little known, elliptic operators are not the only differential operators to be Fredholm when acting between suitable function spaces and indeed the same thing is true for many evolution operators. The story begins with isomorphism theorems for problems with constant (i.e., time-independent) coefficients. Such results are discussed in the next two subsections

4.1. Constant coefficients: Hilbert space case. Let X be a Hilbert space and let A be a closed unbounded linear operator on X with domain W. Then, W is a Hilbert space for the graph norm

$$||x||_W = \left(||x||_X^2 + ||Ax||_X^2\right)^{\frac{1}{2}}$$

and $A \in \mathcal{L}(W, X)$.

Now, consider the problem of finding $u : \mathbb{R} \to W$ such that

$$\frac{du}{dt} = Au + f \tag{2.15}$$

where $f : \mathbb{R} \to X$ is given. Note that this problem is on the entire line and does not come along with any initial condition for u.

If $f \in L^2(\mathbb{R}, X)$, this problem can be easily resolved by Fourier transform: Since X is a Hilbert space, the Fourier transform

$$f \in L^2(\mathbb{R}, X) \mapsto \widehat{f} \in L^2(\mathbb{R}, X)$$

is an isomorphism (this can be seen exactly as in the scalar case, by first considering the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R}, X)$). Then, taking (formally) the Fourier transform of both sides of (2.15), we obtain

$$i\xi\widehat{u}(\xi) = A\widehat{u}(\xi) + \widehat{f}(\xi),$$

which yields

$$\widehat{u}(\xi) = (i\xi I - A)^{-1}\widehat{f}(\xi),$$

provided that $i\xi I - A : W \to X$ is invertible for every $\xi \in \mathbb{R}$. In fact, if also $||(i\xi I - A)^{-1}||_{\mathcal{L}(X)}$ is bounded independently of $\xi \in \mathbb{R}$, the above formula defines a unique function $\widehat{u} \in L^2(\mathbb{R}, X)$ and hence a unique function $u \in L^2(\mathbb{R}, X)$.

Now, suppose not only that $(i\xi I - A)^{-1}$ is bounded in $\mathcal{L}(X)$ but also that (A is invertible and) that $\xi(i\xi I - A)^{-1}$ is bounded in $\mathcal{L}(X)$. Then $u \in W^{1,2}(\mathbb{R}, X)$ and $(i\xi I - A)^{-1}$ is bounded not only in $\mathcal{L}(X)$ but also in $\mathcal{L}(X, W)$ since W is equipped with the graph norm of A and since $A(i\xi I - A)^{-1} = -I + i\xi(i\xi I - A)^{-1}$ is bounded in $\mathcal{L}(X)$. It follows that $\widehat{u} \in L^2(\mathbb{R}, W)$ and hence that $u \in L^2(\mathbb{R}, W)$. Thus, $u \in W^{1,2}(\mathbb{R}, X) \cap L^2(\mathbb{R}, W)$ and it is readily checked that u solves (2.15). Furthermore, if $u \in W^{1,2}(\mathbb{R}, X) \cap L^2(\mathbb{R}, W)$ solves (2.15), then the above Fourier transform arguments are legitimate (not merely formal). In particular, $u = 0$ if $f = 0$, so that the solution of (2.15) in $W^{1,2}(\mathbb{R}, X) \cap L^2(\mathbb{R}, W)$ is unique.

With suitable modifications, the above approach can be extended to the L^p setting, $p \in (1, \infty)$. If $\dim X < \infty$ (and $W = X$) the existence of a solution in $W^{1,p}(\mathbb{R}, X)$ for $f \in L^p(\mathbb{R}, X)$ (surjectivity of $\frac{d}{dt} - A$) follows from Mikhlin's L^p multiplier theorem. The same surjectivity result, but for $\frac{d}{dt} + A^*$ and with p replaced by $p' = p/(p-1)$ yields the uniqueness. In general (but still when X is a Hilbert space), the same approach works since Mikhlin's theorem can be generalized to this case (see Schwartz [77]). Thus, altogether,

THEOREM 2.12. *Let X be a Hilbert space and A a closed unbounded operator on X with domain W such that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that $||\xi(i\xi I - A)^{-1}||_{\mathcal{L}(X)} \leq C$ for some constant $C > 0$ and every $\xi \in \mathbb{R}$. Then, given $p \in (1, \infty)$, the operator $\frac{d}{dt} - A$ is an isomorphism of $W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ onto $L^p(\mathbb{R}, X)$ (i.e., the problem (2.15) has a unique solution $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ for every $f \in L^p(\mathbb{R}, X)$).*

Theorem 2.12 is usually credited to Mielke [53], but results of this sort were obtained much earlier for the initial value problem on the half-line

$$\begin{cases} \frac{du}{dt} = Au + f \text{ on } \mathbb{R}_+, \\ u(0) = 0, \end{cases} \tag{2.16}$$

when, in addition to the hypotheses of Theorem 2.12, A generates a *bounded* holomorphic semigroup. Such results can actually be derived from Theorem 2.12: Given $f \in L^p(\mathbb{R}_+, X)$, call $\widetilde{f}$ the extension of f by 0 for $t < 0$. By Theorem 2.12, there is a unique $\widetilde{u} \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ such that $\frac{d\widetilde{u}}{dt} = A\widetilde{u} + \widetilde{f}$ on $\mathbb{R}$ and it is not difficult to see that, necessarily, $\widetilde{u} = 0$ on $(-\infty, 0]$, so that the restriction of $\widetilde{u}$ to $[0, \infty)$ solves (2.16). Conversely, every solution u of (2.16) in $W_0^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, W)$ extended by 0 for $t < 0$ solves $\frac{d\widetilde{u}}{dt} = A\widetilde{u} + \widetilde{f}$ on $\mathbb{R}$ and so must be unique. A direct proof of this result (not using Theorem 2.12 but still based in part on Schwartz' generalization of Mikhlin's theorem) was given by de Simon [80] as early as 1964. This can also be deduced from the special case $p = 2$ by a result of Sobolevskii [82].

REMARK 2.13. *In general, the hypotheses required of A in Theorem 2.12 in no way imply that A generates a semigroup. In fact, these hypotheses are unchanged when A is changed into $-A$. Based on this remark, it is easy to find operators A satisfying the hypotheses of Theorem 2.12 for which the Cauchy problem is ill-posed in both positive and negative time.*

4.2. Constant coefficients: UMD space case. It is of course a very natural question whether Theorem 2.12 remains true when X is a Banach space. A striking result of Pisier hints that the answer is negative: If it is assumed that Mikhlin's multiplier theorem holds with X being a Banach space, then X is necessarily isomorphic to a Hilbert space.

Pisier's result was never published, but the argument can be found for instance in Arendt and Bu [3]. While this only proves that the argument for the proof of Theorem 2.12 cannot be repeated when X is not a Hilbert space, a definitive result showing that Theorem 2.12 is not true in arbitrary Banach spaces can be found in Kalton and Lancien [43]. Nevertheless, a generalization of Mikhlin's multiplier theorem in suitable Banach spaces was obtained by Weis [85], complementing earlier partial results by Bourgain [12] and Clément et al. [18].

Two basic ingredients are needed in Weis' multiplier theorem. First, X must be a UMD Banach space. There are two equivalent definitions for such spaces, introduced by Burkholder [13] in 1966. The original definition refers to an unconditionality property of X-valued martingales (UMD stands for "unconditionality of martingale differences") that we shall not spell out here. The second definition, more analytical, is that X is UMD if the Hilbert transform is a bounded operator on $L^p(\mathbb{R}, X)$ for some $p \in (1, \infty)$. Recall that the Hilbert transform is the convolution by $pv\left(\frac{1}{t}\right)$, unambiguously defined on $\mathcal{S}(\mathbb{R}, X)$ by

$$\int_{\mathbb{R}} \frac{\varphi(s)}{t-s} ds := \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{t-\varepsilon} \frac{\varphi(s)}{t-s} ds + \int_{t+\varepsilon}^{\infty} \frac{\varphi(s)}{t-s} ds \right).$$

That the two definitions are equivalent was shown by Bourgain [11] and Burkholder [15]. Furthermore, by a geometric characterization of Burkholder [14], if X is UMD, then the Hilbert transform is a bounded operator on $L^p(\mathbb{R}, X)$ for *every* $p \in (1, \infty)$.

REMARK 2.14. *Historically, things have been much more complicated. For instance, when* $X = \mathbb{R}$, *the boundedness of the Hilbert transform is essentially trivial when* $p = 2$, *but was a major result of M. Riesz [74] for* $p \neq 2$.

Every Hilbert space X is UMD since the Fourier transform is an isomorphism of $L^2(\mathbb{R}, X)$ and the Fourier transform of $pv\left(\frac{1}{t}\right)$ is (a multiple of) the function $\operatorname{sign} \xi$ (hence, the Hilbert transform is a bounded operator on $L^2(\mathbb{R}, X)$). Other UMD spaces include $L^q(\Omega)$ when (Ω, μ) is a σ-finite measure space and $q \in (1, \infty)$ or, more generally, $L^{q_1}(\Omega_1, L^{q_2}(\Omega_2))$ under similar conditions (see [17]).

Some of the main properties of UMD spaces include

- Every UMD space is (super) reflexive.
- X is UMD if and only if X^* is UMD.
- If X is UMD and Y is a closed subspace of X, then Y is UMD.
- If X is UMD and Y is a closed subspace of X, then X/Y is UMD.
- The product of two UMD spaces is UMD.

Except for the super reflexivity (Maurey [52], Aldous [1]), the above properties are essentially elementary. The (super) reflexivity of UMD spaces (which is not even sufficient) places severe limitations on those function spaces that are UMD. For example, Hölder spaces are not UMD. On the other hand, Sobolev spaces $W^{m,q}$

with $q \in (1,\infty)$ or products thereof are UMD spaces (being isomorphic to closed subspaces of products of L^q spaces).

The second ingredient in Weis' multiplier theorem is the use of a *stronger concept of boundedness* than ordinary boundedness in $\mathcal{L}(X)$, customarily referred to as r-boundedness, where “r” stands for “randomized” (or “Rademacher”, since the Rademacher functions are used in the definition). This concept, which seems to have first been implicitly used by Bourgain in [12], was explicitly identified by Berkson and Gillepsie [7] and further studied in [18]. We only give the definition, which is somewhat technical and certainly not natural on a first reading, and refer to [3], [7] , [17], [22], [85] for various comments, equivalent formulations and properties.

In what follows, $r_k(t) := \operatorname{sign}\sin 2^{k-1}\pi t, k \in \mathbb{N}$, denotes the sequence of Rademacher functions on $[0,1]$.

DEFINITION 2.15. Let X and Y be Banach spaces. The subset $\mathcal{T} \subset \mathcal{L}(X,Y)$ is said to be r-bounded if there is a constant $C \geq 0$ such that

$$\int_0^1 \left\|\sum_{k=1}^{\kappa} r_k(\tau)T_k x_k\right\|_Y d\tau \leq C\int_0^1 \left\|\sum_{k=1}^{\kappa} r_k(\tau)x_k\right\|_X d\tau, \tag{2.17}$$

for every finite collections $T_1,...,T_\kappa \in \mathcal{T}$ and $x_1,...,x_\kappa \in X$. The smallest constant C for which (2.17) holds is called the r-bound of $\mathcal{T}$, denoted by $r(\mathcal{T})$ or by $r_{\mathcal{L}(X,Y)}(\mathcal{T})$ if it is important to specify that $\mathcal{T}$ is viewed as a subset of $\mathcal{L}(X,Y)$.

By letting $\kappa = 1$ in (2.17), every r-bounded subset $\mathcal{T} \subset \mathcal{L}(X,Y)$ is bounded and $\sup_{T\in\mathcal{T}} ||T|| \leq r(\mathcal{T})$. For $p,q \in [1,\infty)$, the Khintchin-Kahane inequality (Lindenstrauss and Tzafriri [50, Part II, p. 74]) ensures the existence of a constant $A_{p,q} > 0$ such that

$$\left(\int_0^1 \left\|\sum_{k=1}^{\kappa} r_k(\tau)x_k\right\|_X^p d\tau\right)^{\frac{1}{p}} \leq A_{p,q}\left(\int_0^1 \left\|\sum_{k=1}^{\kappa} r_k(\tau)x_k\right\|_X^q d\tau\right)^{\frac{1}{q}}.$$

in any Banach space X. As a result, the definition of r-boundedness is unaffected if $\int_0^1 ||\cdot||d\tau$ is replaced by $\left(\int_0^1 ||\cdot||^p d\tau\right)^{\frac{1}{p}}$ in (2.17), for any $p \in [1,\infty)$ (but of course the r-bound changes). In particular, if $p = 2$, the orthonormality of the Rademacher functions shows that r-boundedness is *the same* as boundedness if X and Y are *Hilbert* spaces. In general Banach spaces (even UMD), r-boundedness is a stronger requirement than ordinary boundedness.

We are now in a position to quote Weis' generalization of the Mikhlin multiplier theorem ([22], [85]).

THEOREM 2.16. *Let X and Y be UMD Banach spaces and let $M \in C^1(\mathbb{R}\backslash\{0\},\mathcal{L}(X,Y))$ be such that both the sets $\{M(\xi) : \xi \in \mathbb{R}\backslash\{0\}\}$ and $\{\xi M'(\xi) : \xi \in \mathbb{R}\backslash\{0\}\}$ are r-bounded. Then, given $p \in (1,\infty)$, the operator (where $\mathcal{F}$ denotes Fourier transform)*

$$f \in L^p(\mathbb{R},X) \mapsto \mathcal{F}^{-1}(M\mathcal{F}f) \in L^p(\mathbb{R},Y),$$

is well defined and bounded.

In turn, this yields the following generalization of Theorem 2.12.

THEOREM 2.17. *Let X be a UMD Banach space and A a closed unbounded operator on X with domain W such that $\sigma(A) \cap i\mathbb{R} = \emptyset$ and that the set $\{\xi(i\xi I - A)^{-1} : \xi \in \mathbb{R}\}$ is r-bounded in $\mathcal{L}(X)$. Then, given $p \in (1, \infty)$, the operator $\frac{d}{dt} - A$ is an isomorphism of $W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ onto $L^p(\mathbb{R}, X)$ (i.e., the problem (2.15) has a unique solution $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ for every $f \in L^p(\mathbb{R}, X)$).*

The existence part follows from Theorem 2.16 with $Y = X$ and $M(\xi) = (i\xi I - A)^{-1}$ and (next) $M(\xi) = \xi(i\xi I - A)^{-1}$. The uniqueness can be handled either by a duality argument (Rabier [61, Theorem 4.1]) under the additional assumption that W is dense in X, or via properties of the Carleman transform (Arendt and Duelli [4, Theorem 2.4], who acknowledge Schweiker [78]), without this extra assumption. (The r-boundedness condition used in [61, Theorem 4.1] is seemingly weaker than in Theorem 2.17, but in fact equivalent to it since the latter is necessary by [4, Theorem 2.4]).

Of course, since every Hilbert space is UMD and since r-boundedness coincides with boundedness in Hilbert space, Theorem 2.17 is a generalization of Theorem 2.12.

REMARK 2.18. *Just like Theorem 2.12, Theorem 2.17 implies the existence and uniqueness of a solution $u \in W_0^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, W)$ of the Cauchy problem (2.16) when $f \in L^p(\mathbb{R}_+, X)$ and A generates a bounded holomorphic semigroup. Note that this existence and uniqueness implies the same when $\mathbb{R}_+$ is replaced by a bounded interval $(0, T)$ (just extend f by 0 for $t > T$).*

REMARK 2.19. *Since W is equipped with the graph norm of A, Theorem 2.17 amounts to saying that there is a constant $C > 0$ such that*

$$||u||_{W^{1,p}(\mathbb{R},X)} + ||Au||_{L^p(\mathbb{R},X)} \leq C||f||_{L^p(\mathbb{R},X)},$$

for every $f \in L^p(\mathbb{R}, X)$ and corresponding solution $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ of (2.15). This type of inequality is often referred to as "L^p maximal regularity" in the literature (mostly in connection with the Cauchy problem (2.16) and hence with $\mathbb{R}$ replaced by $\mathbb{R}_+$ or by a bounded interval $(0, T)$).

4.3. How to check r-boundedness. In concrete applications of Theorem 2.17 when X is not a Hilbert space, the main difficulty is to verify the r-boundedness of the set $\{\xi(i\xi I - A)^{-1} : \xi \in \mathbb{R}\}$ in $\mathcal{L}(X)$. The subsequent comments make this task a little simpler.

First, due to the assumption $\sigma(A) \cap i\mathbb{R} = \emptyset$, the mapping $\lambda(\lambda I - A)^{-1}$ is holomorphic on an open neighborhood of the imaginary axis. As a result, for every $a > 0$, the set

$$\{\xi(i\xi I - A)^{-1} : \xi \in \mathbb{R}, |\xi| \leq a\}$$

is r-bounded ([22, Proposition 3.10, p. 31], [85]). Therefore, the problem is "only" the r-boundedness of $\{\xi(i\xi I - A)^{-1}\}$ for large values of $|\xi|$ (the union of two r-bounded sets is r-bounded). In turn, this may sometimes be established with the help of some perturbation results for r-boundedness.

More precisely, let H be an unbounded linear operator on X with domain $D(H) \supset W$ and suppose that there are constants $\alpha \geq 0$ and $\beta > 0$ such that

$$||Hx||_X \leq \alpha||Ax||_X + \beta||x||_X, \quad \forall x \in W \tag{2.18}$$

(i.e., H is A-bounded). Then, if α is *small enough* and $\{\xi(i\xi I - A)^{-1} : \xi \in \mathbb{R}\}$ is r-bounded, the set $\{\xi(i\xi I-(A+H))^{-1}\}$ is r-bounded for $|\xi|$ large enough (Kuntsmann and Weis [47], Rabier [61, Theorem 3.5]). By exchanging the roles of A and $A+H$ and since (2.18) also reads (assuming $\alpha < 1$)

$$||Hx||_X \leq \frac{\alpha}{1-\alpha}||(A+H)x||_X + \frac{\beta}{1-\alpha}||x||_X, \quad \forall x \in W,$$

it follows that if α is small enough and $\{\xi(i\xi I - (A+H))^{-1} : \xi \in \mathbb{R}\}$ is r-bounded, the set $\{\xi(i\xi I - A)^{-1}\}$ is r-bounded for $|\xi|$ large enough.

Two cases are especially useful: the case when $H \in \mathcal{L}(W,X)$ is compact (or, more generally, A-compact; see Kato [44, p. 194] for the definition), for then $\alpha > 0$ may be chosen arbitrarily small in (2.18) (see Hess [39] since UMD spaces are reflexive), and the case when $H \in \mathcal{L}(X)$ and so $\alpha = 0$ in (2.18). In particular, with $H = \mu I$, it follows that if the set $\{\xi(i\xi I - (A+\mu I))^{-1} : \xi \in \mathbb{R}\}$ is r-bounded for some $\mu \in \mathbb{R}$, then $\{\xi(i\xi I - A)^{-1}\}$ is r-bounded for $|\xi|$ large enough.

In the important case when $(-1)^m A$ is a system of elliptic differential operators of order $2m$ with homogeneous boundary conditions on a domain U of $\mathbb{R}^N$ with compact and smooth enough boundary, a sufficient condition for the r-boundedness of $\{\xi(i\xi I - (A+\mu I))^{-1} : \xi \in \mathbb{R}\}$ for some $\mu > 0$ is given in [22, Theorem 8.2, p. 102] when X is a suitable Sobolev space (this r-boundedness is a by-product of the r-sectoriality statement in that theorem when $\phi_{\mathcal{A}} < \frac{\pi}{2}$ (notation of [22, Theorem 8.2, p. 102]).

REMARK 2.20. *In [22], the solutions of the system have values in a Banach space E which may be infinite dimensional, but must be UMD; in most applications, $E = \mathbb{R}^M$ or $\mathbb{C}^M$ with $M = 1$ being of course the scalar case. The concept of ellipticity used in [22] (parameter-ellipticity) is stronger than Petrovsky ellipticity.*

4.4. Time-dependent coefficients. We now discuss the problem

$$\frac{du}{dt} = A(t)u + f, \tag{2.19}$$

on the line when the coefficients $A(t)$ are no longer constant. We henceforth assume that X is a UMD Banach space, that $(A(t))_{t\in\mathbb{R}}$ is a family of unbounded operators on X with *common domain* W and that the following assumptions hold.

(i) W is a Banach space and the embedding $W \hookrightarrow X$ is compact and dense.

(ii) $A \in C^0(\mathbb{R}, \mathcal{L}(W,X))$.

(iii) There are operators $A_+, A_- \in GL(W,X)$ such that

$$\lim_{t\to\infty} ||A(t) - A_+||_{\mathcal{L}(W,X)} = \lim_{t\to-\infty} ||A(t) - A_-||_{\mathcal{L}(W,X)} = 0.$$

The hypothesis (iii) justifies the notation $A_- = A(-\infty)$, $A_+ = A(\infty)$, convenient to formulate further assumptions. As usual, $\overline{\mathbb{R}} := [-\infty, \infty]$. The operators $A_\pm$ are also viewed as unbounded operators on X with domain W. The next condition controls the behavior of the resolvent of $A(t)$ on the imaginary axis.

(iv) For every $t \in \overline{\mathbb{R}}$, there is $\xi_t \geq 0$ such that the set $\{\xi(i\xi I - A(t))^{-1} : |\xi| \geq \xi_t\}$ is r-bounded in $\mathcal{L}(X)$ (see Definition 2.15).

(v) $\sigma(A_\pm) \cap i\mathbb{R} = \emptyset$.

When $A(t) = A$ is constant, (i) is not required, (ii) and (iii) are vacuous (and $A_\pm = A$) and (iv) to (v) coincide the hypotheses required in Theorem 2.17 (in particular they imply the r-boundedness of $\{\xi(i\xi I - A(t))^{-1} : \xi \in \mathbb{R}\}$, as noted in the previous subsection). These assumptions do not ensure that the operator $\frac{d}{dt} - A(\cdot)$ has any isomorphism property. However:

THEOREM 2.21. *Under the above assumptions, the operator $\frac{d}{dt} - A(\cdot)$ is Fredholm from $W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, W)$ onto $L^p(\mathbb{R}, X)$ for every $p \in (1, \infty)$.*

Theorem 2.21 is proved in Rabier [64]. The proof is based on Theorem 2.17, by freezing the coefficients and using a partition of unity, and by a duality argument. If X is a Hilbert space, $A(t)$ is selfadjoint for every t and $p = 2$ (and $A(\cdot)$ is C^1), a proof following the same line was given earlier by Robbin and Salamon [75] .

In addition, more can be said about the index of $\frac{d}{dt} - A(\cdot)$: Since the embedding $W \hookrightarrow X$ is compact and $\sigma(A(t))$ is not the whole complex plane (by (iv)), it follows that $A(t)$ has compact resolvent for every $t \in \mathbb{R}$. Therefore, the spectrum of $A(t)$ consists entirely of isolated eigenvalues with finite multiplicity. As t runs from $-\infty$ to ∞, some of these eigenvalues cross the imaginary axis. As shown in [64], the index of $\frac{d}{dt} - A(\cdot)$ is just the algebraic count of these crossing eigenvalues, where positive crossing is from left to right (thus negative from right to left) and each eigenvalue is counted with its multiplicity. How to make sense of this possibly ambiguous statement is also explained in [64].

In addition, it is shown in [64, Theorem 8.4 and Corollary 8.5] that Theorem 2.21 can be used to prove that $\frac{d}{dt} - A(\cdot)$ is also Fredholm with index $-\nu \leq 0$ from $W_0^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, W)$ onto $L^p(\mathbb{R}_+, X)$ when $A(t)$ generates a holomorphic semigroup for every $t \geq 0$ and $A(0)$ generates a bounded holomorphic semigroup (if so, $A(t)$ need not be defined for $t < 0$ and is simply chosen to coincide with $A(0)$ for $t < 0$). Furthermore, ν equals the sum of the algebraic multiplicities of A_+ with positive real parts. This has a direct impact on the solvability in $W_0^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, W)$ of the initial value problem

$$\begin{cases} \frac{du}{dt} = A(t)u + f, \\ u(0) = 0, \end{cases}$$

when $f \in L^p(\mathbb{R}_+, X)$. Specifically, ν scalar conditions about f ensures solvability, necessarily unique. In particular, if A_+ generates a bounded holomorphic semigroup, then $\nu = 0$ and $\frac{d}{dt} - A(\cdot) : W_0^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, W) \to L^p(\mathbb{R}_+, X)$ is an isomorphism.

REMARK 2.22. *When $A(t)$ is an elliptic system for every $t \in \mathbb{R}$, the discussion of the previous section shows that [22, Theorem 8.2, p. 102] ensures that condition (iv) holds.*

Various other results in the spirit of Theorem 2.21 but with $D(A(t))$ possibly t-dependent have been obtained by Di Giorgio, Lunardi and Schnaubelt [24] or Latushkin and Tomilov [48], by completely different methods that assume the existence of exponential dichotomies (a nontrivial issue, especially when $D(A(t))$ is indeed t-dependent). See also Di Giorgio and Lunardi [23] for the case of Hölder rather than Sobolev spaces.

4.5. Nonlinear parabolic problems. As mentioned at the end of the previous subsection, Theorem 2.21 implies that $\frac{d}{dt} - A(\cdot) : W_0^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, W) \to L^p(\mathbb{R}_+, X)$ is an isomorphism (hence Fredholm of index 0) under simple conditions about $A(t), 0 \leq t \leq \infty$. Using this together with the degree theory discussed in Part 1, Morris [54] has recently proved the existence of solutions for nonlinear parabolic problems of the form

$$\begin{cases} \frac{\partial u}{\partial t} - A(t,x)u + F(t,x,u) = f(t,x), t \geq 0, x \in \Omega, \\ u(x,0) = g(x), x \in \Omega, \\ u(t,x) = 0, t \geq 0, x \in \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $A(t, \cdot)$ is a second order elliptic differential operator for every $t \geq 0$ and $F : \mathbf{R}_+ \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying suitable conditions.

Solutions are obtained in $L^p(\mathbb{R}_+, W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \cap W^{1,p}(\mathbb{R}_+, L^p(\Omega))$ for arbitrary right-hand side $f \in L^p(\mathbb{R}_+, L^p(\Omega)) = L^p(\mathbb{R}_+ \times \Omega)$ and arbitrary g in the "correct" trace space. In contrast, more classical arguments based on the contraction mapping principle (or the implicit function theorem) can only provide the existence for small enough data f, g.

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Jindřich Nečas

Jindřich Nečas was born in Prague on December 14th, 1929. He studied mathematics at the Faculty of Natural Sciences at the Charles University from 1948 to 1952. After a brief stint as a member of the Faculty of Civil Engineering at the Czech Technical University, he joined the Czechoslovak Academy of Sciences where he served as the Head of the Department of Partial Differential Equations. He held joint appointments at the Czechoslovak Academy of Sciences and the Charles University from 1967 and became a full time member of the Faculty of Mathematics and Physics at the Charles University in 1977. He spent the rest of his life there, a significant portion of it as the Head of the Department of Mathematical Analysis and the Department of Mathematical Modeling.

His initial interest in continuum mechanics led naturally to his abiding passion to various aspects of the applications of mathematics. He can be rightfully considered as the father of modern methods in partial differential equations in the Czech Republic, both through his contributions and through those of his numerous students. He has made significant contributions to both linear and non-linear theories of partial differential equations. That which immediately strikes a person conversant with his contributions is their breadth without the depth being compromised in the least bit. He made seminal contributions to the study of Rellich identities and inequalities, proved an infinite dimensional version of Sard's Theorem for analytic functionals, established important results of the type of Fredholm alternative, and most importantly established a significant body of work concerning the regularity of partial differential equations that had a bearing on both elliptic and parabolic equations. At the same time, Nečas also made important contributions to rigorous studies in mechanics. Notice must be made of his work, with his collaborators, on the linearized elastic and inelastic response of solids, the challenging field of contact mechanics, a variety of aspects of the Navier-Stokes theory that includes regularity issues as well as important results concerning transonic flows, and finally non-linear fluid theories that include fluids with shear-rate dependent viscosities, multi-polar fluids, and finally incompressible fluids with pressure dependent viscosities.

Nečas was a prolific writer. He authored or co-authored eight books. Special mention must be made of his book "Les méthodes directes en théorie des équations elliptiques" which has already had tremendous impact on the progress of the subject and will have a lasting influence in the field. He has written a hundred and forty seven papers in archival journals as well as numerous papers in the proceedings of conferences all of which have had a significant impact in various areas of applications of mathematics and mechanics.

Jindřich Nečas passed away on December 5th, 2002. However, the legacy that Nečas has left behind will be cherished by generations of mathematicians in the Czech Republic in particular, and the world of mathematical analysts in general.

Jindřich Nečas Center for Mathematical Modeling

The Nečas Center for Mathematical Modeling is a collaborative effort between the Faculty of Mathematics and Physics of the Charles University, the Institute of Mathematics of the Academy of Sciences of the Czech Republic and the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University.

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