Jindřich Nečas Center for Mathematical Modeling
Lecture notes

Volume 4

Topics in mathematical modeling

Volume edited by M. Beneš and E. Feireisl

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Topics in mathematical modeling

Masato Kimura
Faculty of Mathematics
Kyushu University
Hakozaki 6-10-1, Higashi-ku, Fukuoka 812-8581
Japan

Philippe Laurençot
Institut de Mathématiques de Toulouse, CNRS UMR 5219
Université de Toulouse
F–31062 Toulouse cedex 9
France

Shigetoshi Yazaki
Faculty of Engineering
University of Miyazaki
1-1 Gakuen Kibanadai Nishi, Miyazaki 889-2192
Japan

Volume edited by M. Beneš and E. Feireisl

2000 *Mathematics Subject Classification.* *35-06, 49Q10, 53C44, 35B40, 34A34*

Key words and phrases. mathematical modeling, shape derivative, moving interfaces, diffusive Hamilton–Jacobi eqautions, crystalline curvature flow equations

Abstract. The volume provides a record of lectures given by the visitors of the Jindřich Nečas Center for Mathematical Modeling in academic year 2007/2008.

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Charles University in Prague, 2008

ISBN 978-80-7378-060-9

Preface

This volume of the Lecture Notes contains texts prepared by Masato Kimura, Philippe Laurençot and Shigetoshi Yazaki. They were long term visiting scientists at the Nečas Center for Mathematical Modeling in the years 2007 and 2008, and their topics are related each to other.

The Center creates a fruitful and productive environment for its members and for its visitors. The visitors are invited to prepare mini-courses focused on their research field. The texts covering the contents of mini-courses is published in the Lecture Notes of the Nečas Center for Mathematical Modeling.

First part of the volume of the Lecture Notes covers the lecture series of Masato Kimura on dynamics of hyperplanes in $\mathbb{R}^n$. The text discusses aspects of formulation for the problems with moving interfaces including the shape derivatives of energy functionals. The author then derives the motion laws known in this domain by means of differential and variational calculus on hypersurfaces. Finally, the gradient-flow approach is used to put known examples of the hyperplane dynamics into a general framework.

Second part of the volume contains the lecture notes for the course of Philippe Laurençot. This course was devoted to the aspects of diffusive Hamilton–Jacobi equations which play a key role in several recently studied domains of application including the dynamics of hyperplanes, phase transitions and computer image processing. The text presents results related to the long-term behavior of the solution of diffusive Hamilton–Jacobi equations as well as results for the solution extinction in finite time.

Third part of the volume is devoted to the lecture notes for the course of Shigetoshi Yazaki, which analyzed the motion of closed planar curves by crystalline curvature with preservation of area. The text explains key issues related to the given domain and provides insight into the physical application. The author discusses the extent of numerical solution of the given problem as well.

The presented three lecture notes are suitable for the students wishing to learn recent advances in the given field of applied mathematics as well as they can serve as reviews for general mathematical audience. They also can be seen as vivid examples of a cooperation established within the Nečas Center for Mathematical Modeling.

October 2008

M. Beneš
E. Feireisl

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Part 1

Shape derivative of minimum potential energy: abstract theory and applications

Masato Kimura

2000 *Mathematics Subject Classification.* 49-01, 49Q10

Key words and phrases. shape derivative, fracture mechanics, lipschitz deformation on domains

Abstract. The text provides an elementary introduction to mathematical foundation of shape derivative of potential energy and provides several applications of the theory to some elliptic problems. Among many important applications the text focus on the energy release rate in fracture mechanics.

Contents

Preface

The principal aim of this short lecture note is to present a mathematical foundation of shape derivatives of potential energies with applications to some elliptic problems. Among many important applications, in particular, we will focus on the energy release rate in fracture mechanics. I hope that this work will contribute to further mathematical understanding of many applications of the shape derivative including fracture mechanics.

This note was originally prepared for a series of lectures at Graduate School of Mathematics, Kyushu University in 2005-2006. I also had occasions to give a lecture at Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague in March 2006, and an intensive lecture at Graduate School of Informatics, Kyoto University in July 2006. I stayed at Czech Technical University in Prague for one year from April 2007 as a visiting researcher of the Nečas Center for Mathematical Modeling LC06052 financed by MSMT. During the stay, I again had a chance to present a part of this work in the workshop 'Matematika na Vysokých Školách, Variační Principy v Matematice a Fyzice' held at Herbertov in September 2007.

Based on the above lectures, this lecture note was completed during my stay in Prague with a support of NCMM. I would like to express my sincere thanks to all participants in the above lectures and to the organizers who gave me such opportunities, in particular, Professor Yuusuke Iso and Dr. Masayoshi Kubo of Kyoto University. I also would like to be grateful to Professor Michal Beneš and his colleagues in the Mathematical Modelling Group of Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, for their kind hospitalities and the stimulating research environment during my stay in Prague.

The mathematical results in this work have been obtained in a joint work [13, 14] with Dr. Isao Wakano of Kyoto University. I wish to acknowledge his cooperation in this research. Professor Kohji Ohtsuka of Hiroshima Kokusai Gakuin University introduced me to his preceding work in this research area with fruitful discussions. I would like to appreciate his kind encouragement during the research.

Masato Kimura
Prague, February 2008

CHAPTER 1

Shape derivative of minimum potential energy: abstract theory and applications

1. Introduction

The shape derivative (or shape variation) is one of the fundamental techniques in the theories of shape optimization, shape sensitivity analysis, and many free boundary problems. The importance of these research fields has been greatly increasing for the last few decades, because of their many applications to numerical simulations of industrial problems. The expansion of such applications of the shape derivative has been also supplying interesting and challenging problems to mathematicians.

We present a mathematical foundation of shape derivatives of potential energies related to some elliptic problems. A central application which we keep in mind is the energy release rate in a cracked domain, which plays an important role in fracture mechanics. The application of the shape derivative to the energy release rate was originally studied in a series of works by Ohtsuka [16, 17, 18], and mathematical justification of the existence and derivation of some formulas in a general geometric situation was carried out there. To the works by Ohtsuka, we will give an alternative aspect with a new mathematical framework, which enables us to have refined results.

We treat a class of shape derivatives, which includes the energy release rate, as a variational problem with a parameter in Banach spaces. A new parameter variational principle and the classical implicit function theorem in Banach spaces will play key roles in our abstract theory. Similar mathematical approaches to the shape derivative for various problems are found in Sokolowski and Zolesio [22] and [15] etc.

We briefly give an explanation of the energy release rate in fracture mechanics. Scientific investigation to understand crack evolution process in elastic body was originated by Griffith [9] and has been studied from various viewpoints in engineering, physics and mathematics since then. Griffith's idea in the fracture mechanics is even now the fundamental theory in modeling and analysis of the crack behaviour. We here make reference to only very few extended studies from mathematical point of view, Cherepanov [4], Rice [20], Ohtsuka [16], [17], [18], Ohtsuka-Khludnev [19], and Francfort-Marigo [6]. For more complete list of crack problems and fracture mechanics, please see the references in the above papers.

The energy release rate G is a central concept in Griffith's theory and its various extended theories such as [6]. According to such theories, we treat crack evolutions in brittle materials with linear elasticity under a quasi-static situation, in which applied boundary loading is supposed to change slowly and any inertial effect can be ignored. The elastic energy at a fixed moment is supposed to be given by minimization of an elastostatic potential energy. According to the Griffith's theory, the surface energy which is required in the crack evolution is supplied by relaxation of the potential energy along crack growth.

Roughly speaking, the energy release rate is defined as follows. Let Ω_* be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, which corresponds to the un-cracked material under consideration. We assume that a crack Σ exists in Ω_*, where Σ is a closure of an $n-1$ dimensional hypersurface. The cracked elastic body is represented by $\Omega_* \setminus \Sigma$. We consider a virtual crack extension $\Sigma(t)$ with parameter $t \in [0, T)$, where $\Sigma = \Sigma(0) \subset \Sigma(t_1) \subset \Sigma(t_2)$ $(0 \leq^\forall t_1 \leq^\forall t_2 < T)$.

Under the quasi-static assumption, the elastic potential energy $E(t)$ in $\Omega(t) := \Omega_* \setminus \Sigma(t)$ is given by

$$E(t) := \min_v \int_{\Omega_* \setminus \Sigma(t)} W(x, v(x), \nabla v(x)) dx, \tag{1.1}$$

where v is a possible displacement field in $\Omega_* \setminus \Sigma(t)$ with a given boundary condition and $W(x, v(x), \nabla v(x))$ represents a potential energy density including a body force. A given displacement field is imposed on a part of $\partial\Omega_*$, and no boundary condition is imposed for the admissible displacement fields on the other part of $\partial\Omega(t)$ including both sides of $\Sigma(t)$. In other words, the normal stress free condition is imposed for the minimizer on $\partial\Omega(t)$ implicitly.

The energy release rate G at $t = 0$ along the virtual crack extension $\{\Sigma(t)\}_{0 \leq t < T}$ is given by

$$G := \lim_{t \to +0} \frac{E(0) - E(t)}{|\Sigma(t) \setminus \Sigma|}. \tag{1.2}$$

Since $E(t) \leq E(0)$, $G \geq 0$ follows if the limit exists. The Griffith's criterion for the brittle crack extension is given by $G \geq G_c$, where G_c is an energy required to create a new crack per unit $(n-1)$-dimensional volume (i.e., length in 2d and area in 3d) and it is a constant depending on the material property and the position.

Cherepanov [4] and Rice [20] studied the so-called J-integral for straight crack in two dimensional linear elasticity, which is a path-independent integral expressions of the energy release rate. Since these works, theoretical and practical studies of crack evolutions have been much developed by means of such useful mathematical expression of G in two dimensional case.

While most of these mathematically rigorous results have been restricted to two dimensional linear elasticity (and often only for straight cracks), Ohtsuka [16]–[18] and Ohtsuka-Khludnev [19] developed a mathematical formulation of the energy release rate for general curved cracks in multi-dimensional linear or semi-linear elliptic systems. They proved existence of the energy release rate, and obtained its expression by a domain integral and by generalized J-integral.

Based on the idea in [17], we shall give a new mathematical framework for shape derivative of potential energy including the energy release rate. Adopting domain perturbation φ of Lipschitz class, the shape derivative is treated in an abstract parameter variation problem in Banach spaces, where φ is considered as a parameter belonging to Lipschitz class. Instead of estimating the limit (1.2) directly as in [17] and [19], we treat it by means of the Fréchet derivative.

In our approach, the shape derivative of minimum potential energy is derived as a Fréchet derivative in a Banach space within an abstract parameter variation formulas and it is given in a domain integral expression. The key tools in the abstract parameter variation setting are the implicit function theorem and the Lax–Milgram theorem.

The organization of this lecture note is as follows. In Sections 2–5, some fundamental results from nonlinear functional analysis are systematically described, particularly on the Fréchet derivative and the implicit function theorem. Their proofs are also given in most cases and some relatively simple proofs are left to the readers. One of the primary sources of these sections is the introduction of the book [11]. For more details of the nonlinear functional analysis, see [3] etc.

Based on the tools prepared in the previous sections, several abstract parameter variation formulas are established in Section 6. A framework of Lipschitz deformation of domains, which includes crack extensions, is established in Section 7, Minimization problems with a general potential energy in deformed domains are studied in Section 8 as applications of the abstract parameter variation formulas of Section 6. Quadratic energy functionals corresponding to second order linear elliptic equations are treated in Section 9.

We remark that the theorems obtained in Sections 8–9 include the results in [17] and [19] under a weaker assumption for regularity of domain perturbation. In [19], they assumed that the domain perturbation $t \to \varphi(t)$ belongs to $C^2([0,T], W^{2,\infty}(\mathbb{R}^n)^n)$ and derived the domain integral expression of G, whereas we prove it in Theorem 1.56 under a weaker assumption $\varphi \in C^1([0,T], W^{1,\infty}(\mathbb{R}^n)^n)$.

In this note, we use the following notation. For an integer $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space over $\mathbb{R}$. Each point $x \in \mathbb{R}^n$ is expressed by a column vector $x = (x_1, \cdots, x_n)^{\mathrm{T}}$, where $^{\mathrm{T}}$ stands for the transpose of the vector or matrix.

For a real-valued function $f = f(x)$, its gradient is given by a column vector

$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x), \ \cdots, \ \frac{\partial f}{\partial x_n}(x) \right)^{\mathrm{T}} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix},$$

and its Hessian matrix is denoted by

$$\nabla^2 f := \nabla^{\mathrm{T}}(\nabla f) = \nabla(\nabla^{\mathrm{T}} f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \in \mathbb{R}^{n \times n},$$

where we note that ∇^2 does not mean the Laplacian operator in this lecture note. We denote the Laplacian of f by

$$\Delta f := \mathrm{div} \nabla f = \mathrm{tr} \nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

2. Multilinear map and Fréchet derivative

Our mathematical treatment for the shape derivative of the potential energy is based on infinite dimensional differential calculus, i.e. the Fréchet derivative in Banach spaces. Several fundamental facts on multilinear maps and the Fréchet derivative in Banach spaces are collected in this section. For the readers' convenience, their definitions and basic theorems are stated with some exercises.

DEFINITION 1.1 (Multilinear map). We suppose that X_k $(k = 1, \cdots, n,\ n \in \mathbb{N})$, X and Y are real Banach spaces.

(1) A map g is called a *multilinear map* from $\prod_{k=1}^n X_k$ to Y, if $g : \prod_{k=1}^n X_k \to Y$ and the map $[x_k \mapsto g(x_1, \cdots, x_n)]$ is linear from X_k to Y for each $k = 1, \cdots, n$. In the case that $X_1 = \cdots = X_n = X$, g is called a *n-linear map* from X to Y.[1]

(2) A multilinear map g is called *bounded*, if there exists $M > 0$ such that

$$\|g(x_1, \cdots, x_n)\|_Y \leq M\|x_1\|_{X_1} \cdots \|x_n\|_{X_n} \qquad \left({}^\forall(x_1, \cdots, x_n) \in \prod_{k=1}^n X_k\right).$$

The set of bounded multilinear maps $g : \prod_{k=1}^n X_k \to Y$ is denoted by $B(X_1, \cdots, X_n; Y)$. It will be shown (see Exercise 1.2) that $B(X_1, \cdots, X_n; Y)$ is a Banach space with the norm

$$\|g\|_{B(X_1,\cdots,X_n;Y)} := \sup_{x_k \in X_k} \frac{\|g(x_1, \cdots, x_n)\|_Y}{\|x_1\|_{X_1} \cdots \|x_n\|_{X_n}} \qquad (g \in B(X_1, \cdots, X_n; Y)).$$

The set of bounded n-linear maps from X to Y also becomes a Banach space and it is denoted by $B_n(X, Y)$.

(3) An n-linear map g from X to Y is called *symmetric*, if $g(x_1, \cdots, x_n)$ is invariant under all permutations of $x_1, \cdots, x_n$. We write $B_n^{\mathrm{sym}}(X, Y) := \{g \in B_n(X, Y);\ g : \text{symmetric}\}$.

EXERCISE 1.2. *Prove that $B(X_1, \cdots, X_n; Y)$ is a Banach space by using the natural identification $B(X_1, \cdots, X_n; Y) \cong B(X_1, B(X_2, \cdots, X_n; Y))$.*

Similar to the case of linear operators (1-linear maps), the boundedness of an n-linear map is equivalent to its continuity as shown in the next proposition.

PROPOSITION 1.3. *Let X_k $(k = 1, \cdots, n,\ n \in \mathbb{N})$, X and Y be Banach spaces. Then the following three conditions are equivalent.*

(1) *g is a continuous multilinear map from $\prod_{k=1}^n X_k$ to Y.*
(2) *g is a multilinear map from $\prod_{k=1}^n X_k$ to Y, and is continuous at the origin of $\prod_{k=1}^n X_k$.*
(3) *$g \in B(X_1, \cdots, X_n; Y)$.*

PROOF. 1.⇒2. This is trivial.
2.⇒3. If not, for $m \in \mathbb{N}$, there exist $x_k^m \in X_k$ such that $\|g(x_1^m, \cdots, x_n^m)\|_Y \geq m^n \|x_1^m\|_{X_1} \cdots \|x_n^m\|_{X_n}$. Let us define $\xi_k^m := (m\|x_k^m\|_{X_k})^{-1} x_k^m$. Then $\|g(\xi_1^m, \cdots, \xi_n^m)\|_Y \geq 1$. Since $\xi_k^m \to 0$ in X_k as $m \to \infty$, taking the limit as

[1] 'Multilinear' and 'n-linear' are usually called, linear $(n = 1)$, bilinear $(n = 2)$, trilinear $(n = 3)$, etc.

$m \to \infty$, it follows that $\|g(0, \cdots, 0)\|_Y \geq 1$ from the continuity of g at the origin. This contradicts the fact $g(0, \cdots, 0) = 0$.
3.⇒1. This is shown by a recursive argument with respect to $n = 1, 2, \cdots$. □

Some basic properties and typical examples of the n-linear map will be found in the following exercises.

EXERCISE 1.4. *If $n \geq 2$, prove that $B_n^{\mathrm{sym}}(X, Y)$ is a closed subspace of $B_n(X, Y)$.*

EXERCISE 1.5. *Let us consider a bounded bilinear map $f_0 \in B(X, Y; Z)$ and bounded multilinear maps $g_1 \in B(X_1, \cdots, X_n; X)$ and $g_2 \in B(Y_1, \cdots, Y_m; Y)$, and we define*

$$f(x_1, \cdots, x_n, y_1, \cdots, y_m) := f_0(g_1(x_1, \cdots, x_n), g_2(y_1, \cdots, y_m)).$$

Then prove that $f \in B(X_1, \cdots, X_n, Y_1, \cdots, Y_m; Z)$ and

$$\|f\|_{B(X_1,\cdots,X_n,Y_1,\cdots,Y_m;Z)} \leq \|f_0\|_{B(X,Y;Z)} \|g_1\|_{B(X_1,\cdots,X_nY)} \|g_2\|_{B(Y_1,\cdots,Y_m;Y)}.$$

EXERCISE 1.6. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $p, q, r \in [1, \infty]$. For $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we define $g(u, v)(x) := u(x)v(x)$ $(x \in \Omega)$. Prove that $g \in B(L^p(\Omega), L^q(\Omega); L^r(\Omega))$ for $r \leq \mu(p, q)$, where*

$$\mu(p, q) = \begin{cases} \dfrac{pq}{p+q} & (p, q \in [1, \infty)), \\ \min\{p, q\} & (p = \infty \text{ or } q = \infty). \end{cases}$$

EXERCISE 1.7. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $k,\ m \in \mathbb{N} \cup \{0\}$. For $(u_1, \cdots, u_k) \in C^m(\overline{\Omega})^k$, we define*

$$g(u_1, \cdots, u_k)(x) := u_1(x) \cdots u_k(x) \qquad (x \in \overline{\Omega}).$$

Prove that $g \in B_k^{\mathrm{sym}}(C^m(\overline{\Omega}), C^m(\overline{\Omega}))$.

Let us define the Fréchet derivative in Banach spaces together with the Gâteaux derivative.

DEFINITION 1.8 (Fréchet derivative). Let X, Y be real Banach spaces, and let U be an open set in X. A mapping $f : U \to Y$ is *Fréchet differentiable* at $x_0 \in U$ if there is $A \in B(X, Y)$ such that

$$\|f(x_0 + h) - f(x_0) - Ah\|_Y = o(\|h\|_X) \quad \text{as } \|h\|_X \to 0.$$

In this case, we write $A = f'(x_0)$ or $f_x(x_0)$, and $f'(x_0)$ is called the *Fréchet derivative* of f at x_0. If f is Fréchet differentiable at any $x \in U$ and the mapping $U \ni x \mapsto f'(x) \in B(X, Y)$ is continuous, we say $f \in C^1(U, Y)$.

If $f' \in C^1(U, B(X, Y))$ then $f''(x) = (f')'(x) \in B(X, B(X, Y)) = B_2(X, Y)$. If $f'' \in C^0(U, B_2(X, Y))$, we say $f \in C^2(U, Y)$. In the same manner, we can define $C^n(U, Y)$ if the nth Fréchet derivative $f^{(n)} \in C^0(U, B_n(X, Y))$, and define $C^\infty(U, Y) := \cap_{n \in \mathbb{N}} C^n(U, Y)$. In case that $Y = \mathbb{R}$, we simply write $C^n(U) = C^n(U, \mathbb{R})$.

DEFINITION 1.9 (Gâteaux derivative). Under the condition of Definition 1.8, for $x_0 \in U$ and for $(h_1, h_2, \cdots, h_n) \in X^n$ $(n \in \mathbb{N})$, if there is $\mathcal{O}$ an open neighbourhood of the origin of $\mathbb{R}^n$ and $[(t_1, \cdots, t_n) \mapsto f(x_0 + t_1 h_1 + \cdots + t_n h_n)] \in C^n(\mathcal{O}, Y)$, then we define *nth Gâteaux derivative* at x_0 as

$$d^n f(x_0, h_1, h_2, \cdots, h_n) := \frac{\partial^n}{\partial t_1 \cdots \partial t_n} f(x_0 + t_1 h_1 + \cdots + t_n h_n)\Big|_{t_1 = \cdots = t_n = 0}.$$

It is not difficult to show the Gâteaux differentiability from the Fréchet differentiability.

Exercise 1.10. *For $f \in C^n(U, Y)$ $(n \geq 2)$, prove that there exists any nth Gâteaux derivative $d^n f(x, h_1, h_2 \cdots, h_n)$ for all $x \in U$ and it is given by*

$$d^n f(x, h_1, h_2, \cdots, h_n) = f^{(n)}(x)[h_1, \cdots, h_n], \tag{1.3}$$

and that $f^{(n)}(x) \in B_n^{\mathrm{sym}}(X, Y)$ for $x \in U$.

On the other hand, a converse proposition is also valid. A proof of the next theorem is found in § 2.1E of [3], for example.

Theorem 1.11. *Let X, Y be real Banach spaces and U be an open set in X. For $n \in \mathbb{N}$, a mapping $f : U \to Y$ belongs to $C^n(U, Y)$, if and only if the nth Gâteaux derivative $d^n f(x, h_1, h_2, \cdots, h_n)$ exists for all $x \in U$ and $(h_1, \cdots, h_n) \in X^n$ and*

$$[(h_1, \cdots, h_n) \mapsto d^n f(x, h_1, h_2, \cdots, h_n)] \in B_n(X, Y) \qquad ({}^\forall x \in U),$$

and $d^n f \in C^0(U, B_n(X, Y))$. Furthermore, (1.3) holds in this case.

We conclude this section with some examples of Fréchet derivatives in exercises below. These examples will help the readers to understand the notion of higher Fréchet and Gâteaux derivatives.

Exercise 1.12. *Let X, Y be real Banach spaces. For $g \in B_n(X, Y)$ $(n \in \mathbb{N})$, we define $f(x) := g(x, \cdots, x) \in Y$ $(x \in X)$. Prove that $f \in C^\infty(X, Y)$ and $f^{(k)} = 0$ for $k \geq n + 1$. Moreover, in the case that $g \in B_n^{\mathrm{sym}}(X, Y)$, find the kth derivative $f^{(k)} \in B_k(X, Y)$ for $k = 1, \cdots, n$.*

Exercise 1.13. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\varphi \in C^k(\mathbb{R})$ $(k \in \mathbb{N} \cup \{0\})$. For $u \in C^m(\overline{\Omega})$ $(m \in \mathbb{N} \cup \{0\})$, we define $f(u)(x) := \varphi(u(x))$ $(x \in \overline{\Omega})$. Prove that $f \in C^{k-m}(C^m(\overline{\Omega}), C^m(\overline{\Omega}))$ if $k \geq m$.*

Exercise 1.14. *Let X be a real infinite dimensional Hilbert space and let $\{e_k\}_{k=1}^\infty \subset X$ be an orthonormal system. We define*

$$J(u) := \sum_{k=1}^\infty a_k^2 (k^{-2} - a_k^2), \quad a_k := (u, e_k)_X \qquad (u \in U := \{w \in X;\ \|w\|_X < 1\}).$$

Prove that $J \in C^\infty(U)$ and calculate its Fréchet derivatives.

3. Minimization problems

This section is devoted to abstract minimization problems in Banach spaces in connection with the Lax–Milgram theorem. We start our discussion with definitions of global and local minimizers.

Definition 1.15. Let (S, d) be a metric space and J be a real valued functional defined on S.

(1) $u_0 \in S$ is called a *global minimizer* of J in S, if $J(u_0) \leq J(u)$ for all $u \in S$.
(2) $u_0 \in S$ is called a *local minimizer* of J, if there exists an open neighbourhood $\mathcal{O} \subset S$ of u_0 such that u_0 is a global minimizer of J in $\mathcal{O}$.
(3) A local minimizer u_0 of J is called *isolated*, if there exists $\rho > 0$ and $J(u_0) < J(u)$ for all $u \in S$ with $0 < d(u, u_0) < \rho$.

Fundamental variational principles in Banach spaces are stated as follows. Its proof will be left for the readers' exercise.

PROPOSITION 1.16. *Let X be a Banach space and U be an open subset of X. We consider a functional $J : U \to \mathbb{R}$.*

(1) *If $J \in C^1(U)$ and $u_0 \in U$ is a local minimizer of J, we have $J'(u_0) = 0 \in X'$.*
(2) *If $J \in C^2(U)$ and $u_0 \in U$ is a local minimizer of J, we have $J''(u_0)[w, w] \geq 0$ for all $w \in X$.*
(3) *Suppose that $J \in C^2(U)$ and $J''(u)[w, w] \geq 0$ for all $w \in X$ and $u \in U$. If $u_0 \in U$ with $J'(u_0) = 0 \in X'$ and U is star-shaped[2] with respect to u_0 then u_0 is a global minimizer of J. Moreover if $J''(u_0)[w, w] = 0$ implies $w = 0$, then u_0 is a unique global minimizer.*
(4) *Let $J \in C^1(U)$ and $u_0 \in U$ with $J'(u_0) = 0 \in X'$. If $J''(u_0) \in B_2(X, \mathbb{R})$ exists and if there exists $\alpha > 0$ and $J''(u_0)[w, w] \geq \alpha \|w\|_X^2$ for all $w \in X$, then u_0 is an isolated local minimizer of J.*

EXERCISE 1.17. *Prove Proposition 1.16.*

Contrary to analogy with finite dimensional cases, the notion of the Gâteaux derivative is not sufficient to describe the variational principles in infinite dimensional spaces like Proposition 1.16. Validity of the Fréchet derivative is apparent also from the next example.

EXERCISE 1.18. *Let X be a Banach space and U be an open subset of X. We assume that a functional $J : U \to \mathbb{R}$ and $u_0 \in U$ satisfy the condition;*

For each $v \in X$, $t = 0$ is a local minimizer of $f(t) := J(u_0 + tv)$ in $\mathbb{R}$.

(1) *Prove that $J'(u_0) = 0 \in X'$ if $J \in C^1(U)$.*
(2) *Prove that $J''(u_0)[w, w] \geq 0$ for all $w \in X$ if $J \in C^2(U)$.*
(3) *Is u_0 a local minimizer of J? (Hint: Study the functional in Exercise 1.14 with $u_0 = 0$.)*

A unique existence of a global minimizer of a convex functional is obtained if we assume the coercivity condition (1.4) below.

THEOREM 1.19. *Let X be a real reflexive Banach space and let V be a closed subspace of X. Assume that a real valued functional $J \in C^2(X)$ satisfies the coercivity condition:*

$$J''(w)[v, v] \geq \alpha \|v\|_X^2 \qquad ({}^\forall v \in V, \ {}^\forall w \in X), \tag{1.4}$$

where α is a positive constant. Then the functional J admits a unique global minimizer over $V(g) := \{v \in X;\ v - g \in V\}$ for arbitrary $g \in X$. Furthermore, u is

[2] U is called *star-shaped with respect to u_0*, if $(1 - s)u_0 + su \in U$ for all $u \in U$ and $s \in [0, 1]$.

the minimizer of J in $V(g)$ if and only if $u \in V(g)$ satisfies

$$J'(u)[v] = 0 \qquad ({}^\forall v \in V). \tag{1.5}$$

Proof. For arbitrary $v,\ w \in V(g)$, we define $f(t) := J((1-t)w + tv)$, $(0 \le t \le 1)$. Then we have

$$\begin{aligned} f'(t) &= J'((1-t)w + tv)[v-w], \\ f''(t) &= J''((1-t)w + tv)[v-w, v-w] \ge \alpha \|v-w\|_X^2, \end{aligned}$$

and

$$J(v) - J(w) = \int_0^1 \left(f'(0) + \int_0^s f''(t)dt \right) ds \ge J'(w)[v-w] + \frac{\alpha}{2}\|v-w\|_X^2. \tag{1.6}$$

Putting $w = g$, we have

$$J(v) - J(g) \ge -\|J'(g)\|_{X'}\|v-g\|_X + \frac{\alpha}{2}\|v-g\|_X^2.$$

Solving this quadratic inequality, we have

$$J(v) \ge J(g) - \frac{\|J'(g)\|_{X'}^2}{2\alpha}, \tag{1.7}$$

$$\|v\|_X \le \|g\|_X + \frac{1}{\alpha}\left(2\|J'(g)\|_{X'} + \sqrt{2\alpha(|J(g)| + |J(v)|)}\right). \tag{1.8}$$

From (1.7), the functional J is bounded from below. Let $\{u_n\} \subset V(g)$ be a minimizing sequence which attains the infimum of J in $V(g)$. From (1.8), $\{u_n\}$ is bounded in X and there exists a subsequence $\{u_n\}$ which converges weakly in X.[3] The subsequence is again denoted by $\{u_n\}$ and the weak limit is denoted by u. Since $V(g)$ is weakly closed [4], $u \in V(g)$ follows.

From (1.6) with $v = u_n$ and $w = u$,

$$J(u_n) \ge J(u) + J'(u)[u_n - u],$$

follows, and taking limit as $n \to \infty$,

$$\inf_{V(g)} J = \lim_{n\to\infty} J(u_n) \ge J(u) + J'(w)[u-u] = J(u).$$

Thus, u is a global minimizer of J in $V(g)$. Since (1.5) holds, from (1.6), we have

$$J(v) - J(u) \ge \frac{\alpha}{2}\|v-u\|_X^2 \qquad ({}^\forall v \in V(g)), \tag{1.9}$$

and the uniqueness of the minimizer follows. Conversely, if (1.5) holds, from (1.9), it follows that u is the minimizer of J. □

Theorem 1.19 is related to the well-known Lax–Milgram theorem (see [21], for example).

[3] A bounded closed convex subset of a reflexive Banach space is sequentially compact with respect to the weak topology. See [24] § V.2 Theorem 1.

[4] A convex subset of a Banach space is strongly closed if and only if it is weakly closed. See Mazur's theorem [24] § V.1 Theorem 2.

THEOREM 1.20 (Lax–Milgram theorem). *For a reflexive Banach space X, let $A \in B(X, X') = B(X, B(X, \mathbb{R})) \cong B_2(X, \mathbb{R})$ be selfadjoint, i.e., $A' = A$. We assume the coercivity condition:*

$$^{\exists}\alpha > 0 \quad s.t. \quad {}_{X'}\langle Aw, w\rangle_X \geq \alpha \|w\|_X^2 \qquad (^{\forall} w \in X).$$

Then A becomes a linear topological isomorphism from X to X', i.e., A is a bijective bounded linear operator from X to X' and $A^{-1} \in B(X', X)$.

EXERCISE 1.21. *In case that X is a real Hilbert space, prove the Lax–Milgram theorem, applying Theorem 1.19 to the functional $J(u) := \frac{1}{2}{}_{X'}\langle Au, u\rangle_X - {}_{X'}\langle f, u\rangle_X$ for arbitrary fixed $f \in X'$.*

Theorem 1.19 and the Lax–Milgram theorem have many important applications in elliptic partial differential equations. This is an example of a mixed boundary value problem.

EXERCISE 1.22. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ. We suppose that Γ_{D} is a relatively open subportion of Γ, and $\Gamma_{\mathrm{N}} := \Gamma \setminus \Gamma_{\mathrm{D}}$ with $\Gamma_{\mathrm{D}} \neq \emptyset$. For fixed $f \in L^2(\Omega)$ and $g_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}})$, we consider*

$$J(v, a) := \int_\Omega \left(\frac{1}{2} a(x) |\nabla v(x)|^2 - f(x) v(x) \right) dx - \int_{\Gamma_{\mathrm{N}}} a(x) g_{\mathrm{N}}(x) v(x) d\sigma$$
$$(v \in H^1(\Omega),\ a \in C^0(\overline{\Omega})).$$

For $g \in H^1(\Omega)$, we define closed affine subspaces of $H^1(\Omega)$

$$V(g) := \{ v \in H^1(\Omega);\ v = g \text{ on } \Gamma_{\mathrm{D}} \}, \quad V := V(0).$$

(1) *Prove that $J \in C^\infty(H^1(\Omega) \times C^0(\overline{\Omega}))$.*
(2) *We define $L \in B(V, V')$ by ${}_{V'}\langle Lv, z\rangle_V := \partial_X^2 J(w, a)[v, z]$ $(v, z \in V)$. Show that L does not depend on $w \in H^1(\Omega)$ and write down the operator L as a differential operator formally.*
(3) *We fix $a \in C^0(\overline{\Omega})$ with $a(x) \geq \alpha > 0$ $(x \in \overline{\Omega})$ and fix $g \in H^1(\Omega)$. Applying Theorem 1.19 to $J(\cdot, a)$, prove the unique existence of a global minimizer of $J(\cdot, a)$ over $V(g)$, and obtain the Euler equation (1.5) for it.*
(4) *Under the assumption of sufficient regularity for the minimizer and the other functions, show that the Euler equation (1.5) gives the following boundary value problem of an elliptic partial differential equation[5]:*

$$\begin{cases} -\mathrm{div}\,(a(x)\nabla u) = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_{\mathrm{D}} \\ \dfrac{\partial u}{\partial \nu} = g_{\mathrm{N}} & \text{on } \Gamma_{\mathrm{N}}. \end{cases} \tag{1.10}$$

4. Banach contraction mapping theorem

We discuss the continuity of fixed points of contraction mappings with respect to a parameter in an abstract metric space. The results in this section will be used for a proof of the implicit function theorem in Section 5.

[5]The unique minimizer of J is called a *weak* (or H^1) *solution* of the mixed boundary value problem (1.10).

THEOREM 1.23 (Banach contraction mapping theorem). *Let (S,d) be a complete metric space. We suppose that $T : S \to S$ be a contraction, i.e. there is $\theta \in (0,1)$ such that*

$$d(T(x),T(y)) \le \theta d(x,y) \qquad ({}^\forall x,y \in S).$$

Then there uniquely exists $a \in S$ such that $T(a)=a$. Moreover we have

$$d(a,T^n(b)) \le \theta^n d(a,b) \qquad ({}^\forall b \in S,\ n \in \mathbb{N}). \tag{1.11}$$

PROOF. For arbitrary $b \in S$ and $m,n \in \mathbb{N}$ $(n<m)$, since

$$d(T^m(b),T^n(b)) \le \sum_{k=n}^{m-1} d(T^{k+1}(b),T^k(b)) \le \sum_{k=n}^{m-1} \theta^k d(T(b),b) \le \frac{\theta^n}{1-\theta} d(T(b),b),$$

$\{T^n(b)\}_n$ is a Cauchy sequence in S. The limit of $T^n(b)$ is denoted by $a \in S$. Then $a = T(a)$ follows from

$$\begin{aligned} d(a,T(a)) &\le d(a,T^n(b)) + d(T^n(b),T(a)) \\ &\le d(a,T^n(b)) + \theta d(T^{n-1}(b),a) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

The inequality (1.11) is shown as

$$d(a,T^n(b)) = d(T^n(a),T^n(b)) \le \theta^n d(a,b) \qquad ({}^\forall b \in S,\ n \in \mathbb{N}),$$

and the uniqueness of the fixed point a immediately follows from it. □

We consider a family of "uniform" contraction mappings. We find that, if it is continuous with respect to a parameter, the continuity of the fixed points follows.

THEOREM 1.24. *Let (S,d) be a complete metric space and let (Λ,l) be a metric space. We suppose that $T : S \times \Lambda \to S$ be a uniform contraction, i.e. there is $\theta \in (0,1)$ such that*

$$d(T(x,\lambda),T(y,\lambda)) \le \theta d(x,y) \qquad ({}^\forall x,y \in S,\ \forall \lambda \in \Lambda). \tag{1.12}$$

Then the unique fixed point $g(\lambda) \in S$, $T(g(\lambda),\lambda) = g(\lambda)$ exists for all $\lambda \in \Lambda$. If $T(x,\cdot) \in C^0(\Lambda,S)$ for all $x \in S$, then $g \in C^0(\Lambda,S)$.

PROOF. We fix $\lambda_0 \in \Lambda$. For arbitrary $\lambda \in \Lambda$, we have

$$\begin{aligned} d(g(\lambda),g(\lambda_0)) &= d(T(g(\lambda),\lambda),T(g(\lambda_0),\lambda_0)) \\ &\le d(T(g(\lambda),\lambda),T(g(\lambda_0),\lambda)) + d(T(g(\lambda_0),\lambda),T(g(\lambda_0),\lambda_0)). \end{aligned}$$

Applying (1.12) to the first term, we have

$$d(g(\lambda),g(\lambda_0)) \le \frac{1}{1-\theta} d(T(g(\lambda_0),\lambda),T(g(\lambda_0),\lambda_0)), \tag{1.13}$$

and $g(\lambda)$ converges to $g(\lambda_0))$ as $\lambda \to \lambda_0$ from the continuity of T. □

5. Implicit function theorem

In this section, we prove the implicit function theorem in Banach spaces. We first present some results on the differentiability of inverse operators and fixed points of contraction mappings.

THEOREM 1.25. *Let X, Y, Z be real Banach spaces, and let U be an open set in X and $A \in C^k(U, B(Y,Z))$ for some $k \in \mathbb{N} \cup \{0\}$. We suppose that, for $x \in U$, there exists $\Lambda(x) \in B(Z,Y)$ such that $\Lambda(x) = [A(x)]^{-1}$. Then $\Lambda \in C^k(U, B(Z,Y))$ follows, and if $k \geq 1$ we have*

$$\Lambda'(x)\xi = -\Lambda(x)(A'(x)\xi)\Lambda(x) \in B(Z,Y) \qquad (^\forall x \in U,\ ^\forall \xi \in X).$$

PROOF. First, we assume $k = 0$. For arbitrary $x_0, x \in U$, we define $Q := \|\Lambda(x) - \Lambda(x_0)\|_{B(Z,Y)}$ and $Q_0 := \|\Lambda(x_0)\|_{B(Z,Y)}$. Since $\|\Lambda(x)\|_{B(Z,Y)} \leq Q + Q_0$, for arbitrary $z \in Z$, we have

$$\begin{aligned}\|(\Lambda(x) - \Lambda(x_0))z\|_Y &= \|\Lambda(x)(A(x_0) - A(x))\Lambda(x_0)z\|_Y \\ &\leq (Q + Q_0)\|A(x_0) - A(x)\|_{B(Y,Z)}Q_0\|z\|_Z,\end{aligned}$$

Hence, $Q \leq (Q + Q_0)Q_0\|A(x_0) - A(x)\|_{B(Y,Z)}$ follows. Let $x_0 \in U$ be fixed and let $\|x - x_0\|_X$ be sufficiently small so that $\|A(x) - A(x_0)\|_{B(Y,Z)} \leq (2Q_0)^{-1}$. Then we have

$$\frac{1}{2}Q \leq Q_0^2\|A(x) - A(x_0)\|_{B(Y,Z)} \to 0 \quad \text{as } x \to x_0 \text{ in } X.$$

Hence, Λ is continuous at x_0.

Next, we assume $k = 1$. We fix $x_0 \in U$ and define $R(x) := A(x) - A(x_0) - A'(x_0)[x - x_0]$ for $x \in U$. Since $R(x) = o(\|x - x_0\|_X)$ as $x \to x_0$, we have

$$\begin{aligned}&\Lambda(x) - \Lambda(x_0) + \Lambda(x)(A'(x_0)[x - x_0])\Lambda(x_0) \\ &\qquad = (\Lambda(x) - \Lambda(x_0))(A'(x_0)[x - x_0])\Lambda(x_0) - \Lambda(x)R(x)\Lambda(x_0) \\ &\qquad\qquad = o(\|x - x_0\|_X) \text{ as } x \to x_0.\end{aligned}$$

It follows that there exists $\Lambda'(x_0) \in B(X, B(Z,Y))$, and furthermore we have $\Lambda'(x_0)\xi = -\Lambda(x_0)(A'(x_0)\xi)\Lambda(x_0)$ for $\xi \in X$. Since $\Lambda' \in C^0(U, B(X, B(Z,Y)))$, we have proved $\Lambda \in C^1(U, B(Z,Y))$.

If $k = 2$, since $\Lambda \in C^1$ and $A' \in C^1$, we have $\Lambda' \in C^1$ and $\Lambda \in C^2$ follows. For $k = 3, 4, \cdots$, the assertion follows recursively. $\square$

THEOREM 1.26 ([11]). *Let X, Y be real Banach spaces, and let U, V be open sets in X and Y, respectively. We assume that $T \in C^k(U \times \overline{V}, \overline{V})$ and $\overline{V} \ni y \mapsto T(x,y) \in \overline{V}$ is a contraction on $\overline{V}$ uniformly in $x \in U$, i.e., there exists $\theta \in (0,1)$ such that*

$$\|T(x,y_1) - T(x,y_2)\|_Y \leq \theta\|y_1 - y_2\|_Y \qquad (^\forall x \in U, \quad ^\forall y_1,\ y_2 \in \overline{V}).$$

For $x \in U$, let $f(x) \in \overline{V}$ be the unique fixed point of $T(x,\cdot)$ in $\overline{V}$, i.e. $T(x, f(x)) = f(x)$. Then $f \in C^k(U,Y)$ follows.

PROOF. If $k = 0$, the assertion follows from Theorem 1.12. In the case $k = 1$, we define $K(x) := T_y(x, f(x)) \in B(Y)$. Since $\|K(x)\|_{B(Y)} \le \theta < 1$, there exists

$$(I - K(x))^{-1} = \sum_{m=0}^{\infty} K(x)^m \in B(Y), \qquad \|(I - K(x))^{-1}\|_{B(Y)} \le \frac{1}{1-\theta}.$$

For fixed $x_0 \in U$, we define $\alpha(x) := f(x) - f(x_0)$ $(x \in U)$ and

$$A_0 := (I - K(x_0))^{-1} T_x(x_0, f(x_0)) \in B(X, Y),$$
$$R(x) := (I - K(x_0))\alpha(x) - T_x(x_0, f(x_0))[x - x_0] \qquad (x \in U).$$

Then we have

$$f(x) - f(x_0) = \alpha(x) = A_0(x - x_0) + (I - K(x_0))^{-1} R(x) \qquad (x \in U). \tag{1.14}$$

Since

$$\begin{aligned}\alpha(x) &= T(x, f(x_0) + \alpha(x)) - T(x_0, f(x_0)) \\ &= T_x(x_0, f(x_0))[x - x_0] + T_y(x_0, f(x_0))[\alpha(x)] + R(x),\end{aligned}$$

it follows that $R(x) = o(\|x - x_0\|_X + \|\alpha(x)\|_Y)$ as $\|x - x_0\|_X \to 0$. From (1.14), we also have $\alpha(x) = O(\|x - x_0\|_X)$ as $\|x - x_0\|_X \to 0$. From these estimates, there exists the Fréchet derivative of f at x_0 and $f'(x_0) = A_0$. Hence, we have proved

$$f'(x) = (I - K(x))^{-1} T_x(x, f(x)) \in B(X, Y) \qquad (x \in U).$$

From Theorem 1.25, $(I - K(\cdot))^{-1} \in C^0(U, B(Y))$ holds, and we have $f \in C^1(U, Y)$.

For the case $k \ge 2$, we can prove $f \in C^k(U, Y)$ by a recursive argument for $k = 1, 2, \cdots$. □

COROLLARY 1.27. *Under the condition of Theorem 1.26, we have*

$$\|f(x) - f(x_0)\|_Y \le \frac{1}{1-\theta} \|T(x, f(x_0)) - f(x_0)\|_Y \qquad ({}^\forall x_0,\ x \in U_0).$$

PROOF. This immediately follows from the inequality (1.13). □

The implicit function theorem in Banach spaces is stated as follows.

THEOREM 1.28 (Implicit function theorem). *Let X, Y, Z be real Banach spaces and U, V be open sets in X and Y, respectively. We suppose that $F : U \times V \to Z$ and $(x_0, y_0) \in U \times V$ satisfy the conditions;*

(1) $F(x_0, y_0) = 0$.
(2) $F \in C^0(U \times V, Z)$.
(3) $F(x, \cdot) \in C^1(V, Z)$ *for* $x \in U$ *and* F_y *is continuous at* $(x, y) = (x_0, y_0)$.
(4) $(F_y(x_0, y_0))^{-1} \in B(Z, Y)$.

Then there exist a convex open neighbourhood of (x_0, y_0), $U_0 \times V_0 \subset U \times V$ and $f \in C^0(U_0, V_0)$, such that, for $(x, y) \in U_0 \times V_0$, $F(x, y) = 0$ if and only if $y = f(x)$. Moreover, if $F \in C^k(U \times V, Z)$ $(k \in \mathbb{N})$, then $f \in C^k(U_0, V_0)$.

PROOF. Let us define

$$T(x, y) := y - F_y(x_0, y_0)^{-1} F(x, y) \in Y \qquad (x, y) \in U \times V.$$

Then $F(x, y) = 0$ is equivalent to $y = T(x, y)$.

For $\rho > 0$, we define $U(\rho) := \{x \in U;\ \|x - x_0\|_X < \rho\}$ and $V(\rho) := \{y \in V;\ \|y - y_0\|_Y < \rho\}$. We fix $\rho_0 > 0$ such that $\{y \in Y;\ \|y - y_0\|_Y \le \rho_0\} \subset V$. Since F_y is continuous at (x_0, y_0), for arbitrary $\varepsilon > 0$, there exist $\delta \in (0, \rho_0]$ such that

$$\|F_y(x, y) - F_y(x_0, y_0)\|_{B(Y,Z)} < \varepsilon \qquad ({}^\forall(x, y) \in U(\delta) \times V(\delta)).$$

For $x \in U(\delta)$ and $y_1,\ y_2 \in V(\delta)$, from the equality

$$T(x, y_1) - T(x, y_2) = F_y(x_0, y_0)^{-1} \int_0^1 \{F_y(x_0, y_0) - F_y(x, y_1 + s(y_2 - y_1))\}[y_1 - y_2]ds,$$

we have

$$\|T(x, y_1) - T(x, y_2)\|_Y \le L\varepsilon\|y_1 - y_2\|_Y,$$

where $L := \|F_y(x_0, y_0)^{-1}\|_{B(Z,Y)}$.

Since $F \in C^0(U \times V, Z)$, for arbitrary $\varepsilon_0 > 0$, there exists $\delta_0 \in (0, \delta]$ such that, for $x \in U(\delta_0)$, we have $\|F(x, y_0)\|_Z < \varepsilon_0$. For $(x, y) \in U(\delta_0) \times V(\delta)$, we have

$$\|T(x, y) - y_0\|_Y = \|T(x, y) - T(x, y_0) - F_y(x_0, y_0)^{-1}F(x, y_0)\|_Y < L\varepsilon\|y - y_0\|_Y + L\varepsilon_0.$$

Hence, let $\varepsilon := (2L)^{-1}$ and we choose $\delta \in (0, \rho_0]$ as above and fix it, and, for $\varepsilon_0 := \delta(4L)^{-1}$, let us choose $\delta_0 \in (0, \delta]$ as above. Then, for $U_0 := U(\delta_0)$ and $V_0 := V(\delta/2)$, we have $T(x, y) \in V_0$ for $(x, y) \in U_0 \times \overline{V_0}$ and

$$\|T(x, y_1) - T(x, y_2)\|_Y \le \frac{1}{2}\|y_1 - y_2\|_Y \qquad ({}^\forall x \in U_0,\ {}^\forall y_1, y_2 \in \overline{V_0}).$$

Since $T \in C^0(U_0 \times \overline{V_0}, \overline{V_0})$, from Theorem 1.26, there exists $f \in C^0(U_0, V_0)$ and the assertion follows.

In the case that $F \in C^k(U \times V, Z)$ $(k \in \mathbb{N})$, $T \in C^k(U_0 \times \overline{V_0}, \overline{V_0})$ holds and $f \in C^k(U_0, V_0)$ also follows from Theorem 1.26. □

Corollary 1.29. *Under the condition of the proof of Theorem 1.28, we have*

$$\|f(x) - f(x_0)\|_Y \le 2\|F_y(x_0, y_0)^{-1}\|_{B(Z,Y)}\|F(x, y_0)\|_Z \qquad ({}^\forall x \in U_0).$$

Proof. This immediately follows from Corollary 1.27. □

6. Parameter variation formulas

We establish a theory of parameter variation of abstract potential energy in Banach spaces in this section. The implicit function theorem and the Lax-Milgram theorem play essential roles in the arguments. The obtained results will provide a fundamental framework for our analysis of shape derivative of elliptic potential energies.

As in the next theorem, we consider a family of minimizers $u(\mu)$ of a functional $J(u, \mu)$ in a Banach space X, where μ is a parameter in another Banach space M. Our aim is to study the Fréchet derivatives of $u(\mu)$ and $J_*(\mu) = \min_u J(u, \mu)$ with respect to the parameter $\mu \in M$. A simple but basic observation in our analysis is the relation (1.15) below.

Theorem 1.30. *Let X and M be real Banach spaces. For $\mathcal{U}_0 \subset X$ and an open subset $\mathcal{O}_0 \subset M$, we consider a real valued functional $J : \mathcal{U}_0 \times \mathcal{O}_0 \to \mathbb{R}$ and a map $u : \mathcal{O}_0 \to \mathcal{U}_0$. We define $J_*(\mu) := J(u(\mu), \mu)$ for $\mu \in \mathcal{O}_0$. We suppose the following conditions.*

(1) $J \in C^0(\mathcal{U}_0 \times \mathcal{O}_0)$, $J(w,\cdot) \in C^1(\mathcal{O}_0)$ *for* $w \in \mathcal{U}_0$, *and* $\partial_M J \in C^0(\mathcal{U}_0 \times \mathcal{O}_0, M')$.

(2) $u \in C^0(\mathcal{O}_0, X)$ *and* $u(\mu)$ *is a global minimizer of* $J(\cdot,\mu)$ *in* $\mathcal{U}_0$ *for each* $\mu \in \mathcal{O}_0$.

Then we have $J_* \in C^1(\mathcal{O}_0)$ *and*

$$J_*'(\mu) = \partial_M J(u(\mu),\mu) \qquad (\mu \in \mathcal{O}_0). \tag{1.15}$$

REMARK 1.31. *If* $\mathcal{U}_0$ *is an open set and* $J \in C^1(\mathcal{U}_0, \mathcal{O}_0)$ *and* $u \in C^1(\mathcal{O}_0, X)$, *the formula (1.15) is easily obtained as follows;*

$$J_*'(\mu) = D_\mu[J(u(\mu),\mu)] = \partial_X J(u(\mu),\mu)[u'(\mu)] + \partial_M J(u(\mu),\mu) = \partial_M J(u(\mu),\mu),$$

where the D_μ *denotes the Fréchet differential operator with respect to* $\mu \in M$ *and the last equality follows from* $\partial_X J(u(\mu),\mu) = 0 \in X'$.

PROOF OF THEOREM 1.30. We fix $\mu_0 \in \mathcal{O}_0$ and we define $u_0 := u(\mu_0)$ and

$$r(\mu) := J_*(\mu) - J_*(\mu_0) - \partial_M J(u_0,\mu_0)[\mu - \mu_0] \qquad (\mu \in \mathcal{O}_0).$$

Since $u(\mu)$ is a global minimizer and $u \in C^0(\mathcal{O}_0, X)$, if μ is close to μ_0, we have

$$r(\mu) \le J(u_0,\mu) - J(u_0,\mu_0) - \partial_M J(u_0,\mu_0)[\mu - \mu_0] = o(\|\mu - \mu_0\|_M),$$

and

$$\begin{aligned} r(\mu) &\ge J(u(\mu),\mu) - J(u(\mu),\mu_0) - \partial_M J(u_0,\mu_0)[\mu - \mu_0] \\ &= \int_0^1 \{\partial_M J(u(\mu), \mu_0 + s(\mu - \mu_0)) - \partial_M J(u_0,\mu_0)\}[\mu - \mu_0] ds \\ &\qquad = o(\|\mu - \mu_0\|_M). \end{aligned}$$

It follows that $r(\mu) = o(\|\mu - \mu_0\|_M)$, and we have (1.15) and $J_*' \in C^0(\mathcal{O}_0, M')$. □

Theorem 1.30 is extended to the following C^k-version.

COROLLARY 1.32. *Under the condition of Theorem 1.30, we assume that* $\mathcal{U}_0$ *is open. Let* $k \in \mathbb{N}$. *If* $\partial_M J \in C^k(\mathcal{U}_0 \times \mathcal{O}_0, M')$ *and* $u \in C^k(\mathcal{O}_0, X)$, *then* $J_* \in C^{k+1}(\mathcal{O}_0)$.

PROOF. This immediately follows from the formula (1.15). □

In the above theorem, we have assumed the existence and the regularity of the minimizer $u(\mu)$, whereas we can derive them from the Lax–Milgram theorem if we assume the coercivity of the functional J.

THEOREM 1.33. *Let* X *and* M *be real Banach spaces and* $\mathcal{U}$ *and* $\mathcal{O}$ *be open subsets of* X *and* M, *respectively. We consider a real valued functional* $J : \mathcal{U} \times \mathcal{O} \to \mathbb{R}$ *and fix* $\mu_0 \in \mathcal{O}$. *We assume*

(1) $J(\cdot,\mu) \in C^2(\mathcal{U})$ *for* $\mu \in \mathcal{O}$ *and* $\partial_X J \in C^0(\mathcal{U} \times \mathcal{O}, X')$.

(2) $u_0 \in \mathcal{U}$ *satisfies* $\partial_X J(u_0,\mu_0) = 0$.

(3) $\partial_X^2 J$ *is continuous at* $(w,\mu) = (u_0,\mu_0)$.

(4) *There exists* $\alpha > 0$ *such that* $\partial_X^2 J(u_0,\mu_0)[w,w] \ge \alpha \|w\|_X^2$ *for* $w \in X$.

Then there exist a convex open neighbourhood of (u_0,μ_0), $\mathcal{U}_0 \times \mathcal{O}_0 \subset \mathcal{U} \times \mathcal{O}$ *and* $u \in C^0(\mathcal{O}_0, \mathcal{U}_0)$, *such that, for* $\mu \in \mathcal{O}_0$, *the following three conditions are equivalent.*

(a) $w \in \mathcal{U}_0$ *is a local minimizer of* $J(\cdot, \mu)$.
(b) $w \in \mathcal{U}_0$ *satisfies* $\partial_X J(w, \mu) = 0$.
(c) $w = u(\mu)$.

In this case, $u(\mu)$ *is a global minimizer of* $J(\cdot, \mu)$ *on* $\mathcal{U}_0$.

Proof. We define a map $F := \partial_X J$ from $\mathcal{U} \times \mathcal{O}$ to X' and apply Theorem 1.28 at $(w, \mu) = (u_0, \mu_0)$. From the Lax–Milgram theorem (Theorem 1.20), $\partial_X F(u_0, \mu_0) = \partial_X^2 J(u_0, \mu_0)$ becomes a linear topological isomorphism from X to X'. Then, from the implicit function theorem (Theorem 1.28), there exist a convex open neighbourhood of (u_0, μ_0), $\mathcal{U}_0 \times \mathcal{O}_0 \subset \mathcal{U} \times \mathcal{O}$ and $u \in C^0(\mathcal{O}_0, U_0)$, such that, for $\mu \in \mathcal{O}_0$, $w \in \mathcal{U}_0$ satisfies $\partial_X J(w, \mu) = 0$ if and only if $w = u(\mu)$.

From the continuity of $\partial_X^2 J$ at (u_0, μ_0), without loss of generality, (after replacing $\mathcal{U}_0$ and $\mathcal{O}_0$ with smaller ones if we need) we can assume that

$$\partial_X^2 J(v, \mu)[w, w] \geq \frac{\alpha}{2} \|w\|_X^2 \qquad ({}^\forall w \in X,\ {}^\forall (v, \mu) \in \mathcal{U}_0 \times \mathcal{O}_0). \tag{1.16}$$

For $\mu \in \mathcal{O}_0$, if $w \in \mathcal{U}_0$ is a local minimizer of $J(\cdot, \mu)$ in $\mathcal{U}_0$, the $\partial_X J(w, \mu) = 0$ follows. Conversely, if $w \in \mathcal{U}_0$ satisfies $\partial_X J(w, \mu) = 0$, w is a local minimizer in $\mathcal{U}_0$ from the condition (1.16). It also follows from (1.16) that $u(\mu)$ is a global minimizer of $J(\cdot, \mu)$ on $\mathcal{U}_0$ (Proposition 1.16 (3)). □

Higher regularity of $u(\mu)$ also follows from the implicit function theorem.

Theorem 1.34. *Under the condition of Theorem 1.33, we additionally assume that* $\partial_X J \in C^k(\mathcal{U} \times \mathcal{O}, X')$ *for some* $k \in \mathbb{N}$. *Then* $u \in C^k(\mathcal{O}_0, \mathcal{U}_0)$.

Proof. The assertion follows from the implicit function theorem (Theorem 1.28). □

Under the condition of Theorem 1.33, we define

$$J_*(\mu) := J(u(\mu), \mu) \qquad (\mu \in \mathcal{O}_0).$$

As a consequence of Theorem 1.34, a sufficient condition for $J_* \in C^1(\mathcal{O}_0)$ is $J \in C^1(\mathcal{U} \times \mathcal{O})$ and $\partial_X J \in C^1(\mathcal{U} \times \mathcal{O}, X')$. Due to Theorem 1.30, however, the condition $\partial_X J \in C^1(\mathcal{U} \times \mathcal{O}, X')$ is not necessary as shown in the next theorem.

Theorem 1.35. *Under the condition of Theorem 1.33, we additionally assume that* $J \in C^k(\mathcal{U} \times \mathcal{O})$ *for some* $k \in \mathbb{N}$, *then* $J_* \in C^k(\mathcal{O}_0)$ *and it satisfies (1.15).*

Proof. From Theorem 1.30, $J_* \in C^1(\mathcal{O}_0)$ and (1.15) immediately follows. Since $u \in C^{k-1}(\mathcal{O}_0, X)$ follows from Theorem 1.34, $J_* \in C^k(\mathcal{O}_0)$ is obtained from the formula (1.15). □

In Theorem 1.35 with $k = 1$, we have derived $J_* \in C^1(\mathcal{O}_0)$ without assuming any differentiability of $u(\mu)$ more than $u \in C^0(\mathcal{O}_0, X)$ (this is from Theorem 1.33). Actually, we have a Hölder regularity of u.

Proposition 1.36. *Under the condition of Theorem 1.33, we additionally assume that* $J \in C^1(\mathcal{U} \times \mathcal{O})$, *then we have*

$$\|u(\mu) - u_0\|_X = o\left(\|\mu - \mu_0\|_M^{1/2}\right) \qquad \text{as} \ \ \|\mu - \mu_0\|_M \to 0.$$

PROOF. Let $\rho_0 > 0$ with $\{v \in X;\ \|v - u_0\|_X \le \rho_0\} \subset \mathcal{U}_0$. For $h \in X$ with $\|h\|_X = 1$, $\mu \in \mathcal{O}_0$ and $\rho \in (0, \rho_0]$, we have

$$J(u_0 + \rho h, \mu) = J(u_0, \mu) + \rho \partial_X J(u_0, \mu)[h] + \rho^2 \int_0^1 (1 - s)\partial_X^2 J(u_0 + s\rho h, \mu)[h, h]ds.$$

$$\begin{aligned}
&\partial_X J(u_0, \mu)[h] = \partial_X J(u_0, \mu)[h] - \partial_X J(u_0, \mu_0)[h] \\
&\quad = \left\{\rho^{-1}(J(u_0 + \rho h, \mu) - J(u_0, \mu)) - \rho \int_0^1 (1 - s)\partial_X^2 J(u_0 + s\rho h, \mu)[h, h]ds\right\} \\
&\quad - \left\{\rho^{-1}(J(u_0 + \rho h, \mu_0) - J(u_0, \mu_0)) - \rho \int_0^1 (1 - s)\partial_X^2 J(u_0 + s\rho h, \mu_0)[h, h]ds\right\} \\
&\quad = \rho^{-1} \int_0^1 \{\partial_M J(u_0 + \rho h, \mu_0 + t(\mu - \mu_0)) - \partial_M J(u_0, \mu_0 + t(\mu - \mu_0))\}[\mu - \mu_0]dt \\
&\qquad\qquad - \rho \int_0^1 (1 - s)\{\partial_X^2 J(u_0 + s\rho h, \mu) - \partial_X^2 J(u_0 + s\rho h, \mu_0)\}[h, h]ds
\end{aligned}$$

Choosing $\rho := \|\mu - \mu_0\|_M^{1/2}$, we have

$$\|\partial_X J(u_0, \mu)\|_{X'} \le 2\omega(\rho)\rho \qquad (\mu \in \mathcal{O}_0,\ \|\mu - \mu_0\|_M \le \rho_0^2),$$

where, for $r > 0$,

$$S(r) := \left\{(w, \lambda) \in X \times M;\ \|w - u_0\|_X \le r, \|\lambda - \mu_0\|_M \le r^2\right\},$$

$$\begin{aligned}
\omega(r) := \sup_{(w,\lambda)\in S(r)} &\|\partial_M J(w, \lambda) - \partial_M J(u_0, \mu_0)\|_{M'} \\
&+ \sup_{(w,\lambda)\in S(r)} \|\partial_X^2 J(w, \lambda) - \partial_X^2 J(u_0, \mu_0)\|_{B_2(X,\mathbb{R})}.
\end{aligned}$$

We remark that $\omega(r) \to 0$ as $r \to +0$. Hence, from Corollary 1.29, we have $\|u(\mu) - u_0\|_X \le 2\alpha^{-1}\|\partial_X J(u_0, \mu)\|_{X'} \le 4\alpha^{-1}\omega(\rho)\rho = o(\rho)$. □

Under the conditions of Theorem 1.33, $\partial_X^2 J(u(\mu), \mu)$ can be regarded as a linear topological isomorphism from X to X' from the Lax–Milgram theorem. Therefore, we can define $\Lambda(\mu) \in B(X', X)$ which satisfies

$$\partial_X^2 J(u(\mu), \mu)[\Lambda(\mu)h, w] = h[w] \qquad ({}^\forall w \in X,\ {}^\forall h \in X').$$

THEOREM 1.37. *Under the condition of Theorem 1.34 with $k = 1$,*

$$u'(\mu) = -\Lambda(\mu)h_0(\mu) \qquad (\mu \in \mathcal{O}_0), \tag{1.17}$$

holds, where $h_0(\mu) := \partial_M \partial_X J(u(\mu), \mu) \in B(M, X')$.

PROOF. Differentiating $\partial_X J(u(\mu), \mu) = 0 \in X'$ by μ, we have

$$\partial_X^2 J(u(\mu), \mu)[u'(\mu)] + \partial_M \partial_X J(u(\mu), \mu) = 0 \in B(M, X').$$

This is equivalent to (1.17) from the Lax–Milgram theorem. □

The second Fréchet derivative of J_* is given by the next formula.

THEOREM 1.38. *Under the condition of Theorem 1.33, we additionally assume that* $J \in C^2(\mathcal{U} \times \mathcal{O})$ *then* $J_* \in C^2(\mathcal{O}_0)$ *and it satisfies*

$$J_*''(\mu)[\mu_1, \mu_2] = \partial_M^2 J(u(\mu), \mu)[\mu_1, \mu_2] - {}_X\langle \Lambda(\mu) h_0(\mu)[\mu_1],\ h_0(\mu)[\mu_2] \rangle_{X'} \qquad (\mu \in \mathcal{O}_0,\ \mu_1, \mu_2 \in M).$$

PROOF. Differentiating the formula (1.15) by μ and substituting (1.17), we obtain the formula. □

7. Lipschitz deformation of domains

A systematic investigation of domain deformation with Lipschitz domain mappings is carried out in this section. The domain mapping method is one of the important techniques of the shape sensitivity analysis and optimal shape design theory.

We consider a domain deformation with Lipschitz transform $\varphi : \Omega \to \varphi(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$) and φ is $\mathbb{R}^n$-valued $W^{1,\infty}$ function. It is known that a function in $W^{1,\infty}$ is Lipschitz in the following sense. For a function $u : \Omega \to \mathbb{R}^k$, we define

$$|u|_{\mathrm{Lip},\Omega} := \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|}.$$

If $|u|_{\mathrm{Lip},\Omega} < \infty$, u is called uniformly Lipschitz continuous on Ω.[6]

PROPOSITION 1.39. *Let* Ω *be a domain in* $\mathbb{R}^n$*. For* $u \in W^{1,\infty}(\Omega)$*, there is* $\tilde{u} \in C^0(\Omega)$ *such that* $\tilde{u}(x) = u(x)$ *for a.e.* $x \in \Omega$*, in other words, we can regard* $W^{1,\infty}(\Omega) \subset C^0(\Omega)$*. If* Ω *is convex,* $W^{1,\infty}(\Omega) = C^{0,1}(\overline{\Omega})$ *as a subset of* $C^0(\Omega)$*. Moreover, we have*

$$\|\nabla u\|_{L^\infty(\Omega)} = |u|_{\mathrm{Lip},\Omega} \qquad (u \in W^{1,\infty}(\Omega) \cap C^0(\Omega)).$$

EXERCISE 1.40. *Prove Proposition 1.39. (Hint:* $W^{1,\infty} \subset C^{0,1}$ *is shown by using the Friedrichs' mollifier. The converse is obtained by direct calculation of the distributional derivative as a limit of a finite difference.)*

In the following arguments, we fix a bounded convex domain $\Omega_0 \subset \mathbb{R}^n$ and we identify $W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ with $C^{0,1}(\overline{\Omega_0}, \mathbb{R}^n)$. We denote by φ_0 the identity map on $\mathbb{R}^n$, i.e., $\varphi_0(x) = x$ ($x \in \mathbb{R}^n$).

PROPOSITION 1.41. *Suppose that* $\varphi \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ *satisfies* $|\varphi - \varphi_0|_{\mathrm{Lip},\Omega_0} < 1$*. Then* φ *is a bi-Lipschitz transform from* Ω_0 *to* $\varphi(\Omega_0)$*, i.e.* φ *is bijective from* Ω_0 *onto an open set and* φ *and* φ^{-1} *are both uniformly Lipschitz continuous.*

PROOF. Let $\mu := \varphi - \varphi_0$ and $\theta := |\mu|_{\mathrm{Lip},\Omega_0} \in (0,1)$. First, we show that $\varphi(\Omega_0)$ is open. We fix arbitrary $y_0 \in \varphi(\Omega_0)$ with $y_0 = \varphi(x_0)$, $x_0 \in \Omega_0$. Let $\delta > 0$ such that $\overline{B_\delta(x_0)} \subset \Omega_0$, where $B_\delta(x_0) := \{x \in \mathbb{R}^n;\ |x - x_0| < \delta\}$. For $y \in B_{(1-\theta)\delta}(y_0)$, we show that $y \in \varphi(\Omega_0)$. It is easily checked that $T(\xi) := y - \mu(\xi)$ is a uniform contraction on $\overline{B_\delta(x_0)}$. From the contraction mapping theorem (Theorem 1.23),

[6] For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $\alpha \in (0,1]$, we define $C^{0,\alpha}(\overline{\Omega}) := \{u \in C^0(\Omega) \cap L^\infty(\Omega);\ {}^\exists K > 0,\ |u(x) - u(y)| \le K|x - y|^\alpha\ ({}^\forall x, y \in \Omega)\}$. $u \in C^{0,\alpha}(\overline{\Omega})$ is called an uniform α-Hölder function if $\alpha \in (0,1)$, and $u \in C^{0,1}(\overline{\Omega})$ is called a uniform Lipschitz function.

there is a fixed point $x = T(x) = y - \mu(x)$ in $\overline{B_\delta(x_0)}$, that is $y = \varphi(x)$. Hence, $\varphi(\Omega_0)$ is an open set. Since $|\varphi(x_1) - \varphi(x_2)| \geq |x_1 - x_2| - |\mu(x_1) - \mu(x_2)| \geq (1-\theta)|x_1 - x_2|$ for $x_1, x_2 \in \Omega_0$, it follows that φ is injective and φ^{-1} satisfies uniform Lipschitz condition on $\varphi(\Omega_0)$. □

For an arbitrary open set Ω with $\overline{\Omega} \subset \Omega_0$, we define

$$\mathcal{O}(\Omega) := \left\{ \varphi \in W^{1,\infty}(\Omega_0, \mathbb{R}^n);\ |\varphi - \varphi_0|_{\mathrm{Lip},\Omega_0} < 1,\ \overline{\varphi(\Omega)} \subset \Omega_0 \right\}. \tag{1.18}$$

We note that $\mathcal{O}(\Omega)$ is an open subset of $W^{1,\infty}(\Omega_0, \mathbb{R}^n)$. For $\varphi \in \mathcal{O}(\Omega)$, we denote by $\Omega(\varphi)$ the deformed domain, i.e., $\Omega(\varphi) := \varphi(\Omega)$, hereafter.

A virtual crack extension in a cracked elastic body for the energy release rate (1.2) is treated as follows. Let Ω_* be an un-cracked elastic body in $\mathbb{R}^n$ with $\overline{\Omega_*} \subset \Omega_0$, and let $\Sigma \subset \overline{\Omega_*}$ be an initial crack. We assume that Σ is a closure of an $(n-1)$-dimensional smooth hypersurface. Then, we consider the energy release rate of a cracked domain $\Omega := \Omega_* \setminus \Sigma$ along a virtual crack extension $\Sigma(t)$ parametrized by $0 \leq t < T$, where $\Sigma(t)$ is closed and satisfies

$$\Sigma = \Sigma(0) \subset \Sigma(t_1) \subset \Sigma(t_2) \quad (0 \leq^{\forall} t_1 \leq^{\forall} t_2 < T).$$

We suppose that the virtual crack extension is expressed by a parametrized domain mappings $\varphi(t) \in \mathcal{O}(\Omega)$ with $\varphi(0) = \varphi_0$, $\varphi(t)(\Sigma) = \Sigma(t)$, $\Omega(\varphi(t)) = \Omega_* \setminus \Sigma(t)$, and $\varphi(t)(x) = x$ in a neighbourhood of $\partial\Omega_*$. Without loss of generality, we choose a parameter t by $t = \mathcal{H}^{n-1}(\Sigma(t) \setminus \Sigma)$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

If the elastic potential energy in the deformed domain $\Omega(\varphi)$ is denoted by $E_*(\varphi)$, the energy release rate G along $\{\Sigma(t)\}_{0 \leq t < T}$ at $t = 0$ is expressed as

$$G = E'_*(\varphi_0)[\dot{\varphi}(0)],$$

where E'_* is the Fréchet derivative of E_* with respect to φ and $\dot{\varphi}$ is the t derivative of φ. Namely, the calculation and the mathematical justification of the energy release rate G are reduced to those of the Fréchet derivative of E_*.

Example 1.42. *Let Ω_* be a bounded domain in $\mathbb{R}^2$. We define*

$$\Sigma(t) := \left\{ (x_1, 0)^{\mathrm{T}} \in \mathbb{R}^2;\ c_0 \leq x_1 \leq c_1 + t \right\} \qquad (0 \leq t \leq T),$$

and assume that $\Sigma(T) \subset \Omega_*$. We consider a virtual crack extension $\{\Sigma(t)\}_{0 \leq t < T}$ in a cracked domain $\Omega := \Omega_* \setminus \Sigma(0)$. We suppose that Ω_0 be a bounded convex domain with $\overline{\Omega_*} \subset \Omega_0$ and that a cut off function $q \in W^{1,\infty}(\Omega_0)$ with $\mathrm{supp}(q) \subset$ *$\{x \in \Omega_*;\ x_1 > c_0\}$ and $q(x) = 1$ at $x = (c_1, 0)^{\mathrm{T}}$. Then we define*

$$\varphi(t)(x) := x + t\, q(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (x \in \Omega_0,\ t \geq 0).$$

If $|\varphi(t) - \varphi_0|_{\mathrm{Lip},\Omega_0} = t\,|q|_{\mathrm{Lip},\Omega_0} < 1$, then $\varphi(t)$ is a bi-Lipschitz transform from Ω_* onto itself and satisfies $\varphi(t)(\Sigma(0)) = \Sigma(t)$ and $\Omega(\varphi(t)) = \Omega_* \setminus \Sigma(t)$.

Example 1.43. *Let $\Omega = \Omega' \times (0,1)$ be a bounded cylindrical domain in $\mathbb{R}^n$, where Ω' is a bounded Lipschitz domain in $\mathbb{R}^{n-1}$. For fixed $h \in W^{1,\infty}(\Omega')$, we consider a domain perturbation of Ω:*

$$\Omega(t) := \{(x', x_n) \in \Omega' \times \mathbb{R};\ t\,h(x') < x_n < 1\},$$

for small $|t| << 1$. We fix a bounded convex domain Ω_* with $\overline{\Omega} \subset \Omega_*$. Then, this Lipschitz boundary deformation is expressed by $\varphi(t) \in \mathcal{O}(\Omega)$ as follows. Let $q \in W^{1,\infty}(\Omega_0)$ be a cut off function with $\operatorname{supp}(q) \subset \{(x', x_n) \in \Omega_0;\ x_n < 1\}$ and $q(x', 0) = 1$ for $x' \in \Omega'$. Then we define

$$\varphi(t)(x) := \begin{pmatrix} x' \\ x_n + tq(x)h(x') \end{pmatrix} \qquad (x = (x', x_n)^{\mathrm{T}} \in \Omega_0,\ t \geq 0).$$

If $|\varphi(t) - \varphi_0|_{\mathrm{Lip},\Omega_0} = t\,|qh|_{\mathrm{Lip},\Omega_0} < 1$, then $\varphi(t)$ is a bi-Lipschitz transform from Ω_* onto itself and satisfies $\Omega(\varphi(t)) = \Omega(t)$.

For each deformation $\varphi \in \mathcal{O}(\Omega)$, we define a pushforward operator φ_* which transforms a function v on Ω to a function $\varphi_* v := v \circ \varphi^{-1}$ on $\Omega(\varphi)$, if φ satisfies Proposition 1.41. We define

$$\nabla\varphi^{\mathrm{T}}(x) := \left(\frac{\partial \varphi_j}{\partial x_i}(x)\right)_{i\downarrow, j\rightarrow} \in \mathbb{R}^{n\times n}, \qquad (x \in \Omega_0),$$

$$A(\varphi) := \left(\nabla\varphi^{\mathrm{T}}\right)^{-1} \in L^\infty(\Omega_0, \mathbb{R}^{n\times n}), \quad \kappa(\varphi) := \det \nabla\varphi^{\mathrm{T}} \in L^\infty(\Omega_0, \mathbb{R}).$$

These Jacobi matrices and Jacobian appear in the pullback of differentiation and integration on $\Omega(\varphi)$ to Ω. For a function v on Ω, we have

$$[\nabla(\varphi_* v)] \circ \varphi = A(\varphi)\nabla v \quad \text{a.e in } \Omega \quad (v \in W^{1,1}(\Omega)),$$

$$\int_{\Omega(\varphi)} (\varphi_* v)(y)dy = \int_\Omega v(x)\kappa(\varphi)(x)dx \qquad (v \in L^1(\Omega)).$$

These equalities are well known in the case $\varphi \in C^1$. However, for $\varphi \in W^{1,\infty} = C^{0,1}$, these are not so trivial. See [5] and [25] etc. for details.

EXERCISE 1.44. *Under the condition of Proposition 1.41, prove that*

$$\operatorname{ess-}\inf_{\Omega_0} \kappa(\varphi) > 0.$$

Fréchet derivatives of $\kappa(\varphi)$ and $A(\varphi)$ with respect to φ are obtained as follows.

THEOREM 1.45. *Let Ω be an open subset of Ω_0.*

(1) *It holds that $\kappa \in C^\infty(W^{1,\infty}(\Omega, \mathbb{R}^n), L^\infty(\Omega))$, and the $(n+1)$-th Fréchet derivative of κ vanishes, i.e., $\kappa^{(n+1)} = 0 \in B_{n+1}(W^{1,\infty}(\Omega, \mathbb{R}^n), L^\infty(\Omega))$. In particular, we have*

$$\kappa'(\varphi_0)[\mu] = \operatorname{div}\mu \quad \textit{for} \quad \mu \in W^{1,\infty}(\Omega, \mathbb{R}^n).$$

(2) *We define $\mathcal{O}_0(\Omega) := \{\varphi \in W^{1,\infty}(\Omega, \mathbb{R}^n);\ \operatorname{ess-inf}_\Omega \kappa(\varphi) > 0\}$, which is an open subset of $W^{1,\infty}(\Omega, \mathbb{R}^n)$. Then $A \in C^\infty(\mathcal{O}_0(\Omega), L^\infty(\Omega, \mathbb{R}^{n\times n}))$ holds and, in particular, we have the formula:*

$$A'(\varphi_0)[\mu] = -\nabla\mu^{\mathrm{T}} \quad \textit{for} \quad \mu \in W^{1,\infty}(\Omega, \mathbb{R}^n).$$

PROOF. Since the determinant is a polynomial of degree n, the proposition $\kappa \in C^\infty(W^{1,\infty}(\Omega, \mathbb{R}^n), L^\infty(\Omega))$ is clear. For fixed $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, we define

$$m_{ij}(t) := \delta_{ij} + t\frac{\partial \mu_j}{\partial x_i} \in L^\infty(\Omega) \qquad (i, j = 1, \cdots, n,\ t \in \mathbb{R}),$$

where δ_{ij} is Kronecker's delta. Then we have

$$\begin{aligned}
\kappa'(\varphi_0)[\mu] &= \left.\frac{d}{dt}\right|_{t=0} \kappa(\varphi_0 + t\mu) \\
&= \left.\frac{d}{dt}\right|_{t=0} \det\left(m_{ij}(t)\right) = \left.\frac{d}{dt}\right|_{t=0} \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) m_{1\sigma(1)}(t) \cdots m_{n\sigma(n)}(t) \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(m_{11}(t) \cdots m_{nn}(t)\right) = \sum_{i=1}^{n} m_{11}(0) \cdots m'_{ii}(0) \cdots m_{nn}(0) = \operatorname{div}\mu.
\end{aligned}$$

Let the (i,j) component of $A(\varphi)$ be denoted by $a_{ij}(\varphi) \in L^\infty(\Omega)$. Then we have $a_{ij}(\varphi) = \alpha_{ij}(\varphi)/\kappa(\varphi)$, where $\alpha_{ij}(\varphi)$ is the (i,j) cofactor of $\nabla\varphi^{\mathrm{T}}$, which is a polynomial of $\frac{\partial \varphi_k}{\partial x_l}$ of degree $n-1$. Since $\operatorname{ess\text{-}inf}_\Omega \kappa(\varphi) > 0$ for $\varphi \in \mathcal{O}_0(\Omega)$, $a_{ij} \in C^\infty(\mathcal{O}_0(\Omega), L^\infty(\Omega))$ follows. For fixed $\mu \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, differentiating the identity

$$A(\varphi_0 + t\mu)(I + t\nabla\mu^{\mathrm{T}}) = I \qquad (I\text{: identity matrix of degree } n),$$

by $t \in \mathbb{R}$ at $t = 0$, we have

$$A'(\varphi_0)[\mu] + A(\varphi_0)\nabla\mu^{\mathrm{T}} = O.$$

Since $A(\varphi_0) = I$, we have $A'(\varphi_0)[\mu] = -\nabla\mu^{\mathrm{T}}$. □

We conclude this section with three basic propositions for the pushforward operator φ_*.

Proposition 1.46. *Under the condition of Proposition 1.41, for $p \in [1, \infty]$, φ_* is a linear topological isomorphism from $L^p(\Omega)$ onto $L^p(\Omega(\varphi))$, and a linear topological isomorphism from $W^{1,p}(\Omega)$ onto $W^{1,p}(\Omega(\varphi))$. More precisely, we have*

$$m(\varphi)\|v\|_{L^p(\Omega)} \le \|\varphi_* v\|_{L^p(\Omega(\varphi))} \le M(\varphi)^n \|v\|_{L^p(\Omega)} \qquad (v \in L^p(\Omega), 1 \le p \le \infty),$$

$$\frac{m(\varphi)}{M(\varphi)}\|\nabla v\|_{L^p(\Omega)} \le \|\nabla(\varphi_* v)\|_{L^p(\Omega(\varphi))} \le \frac{M(\varphi)^{n-1}}{m(\varphi)}\|\nabla v\|_{L^p(\Omega)} \qquad (v \in W^{1,p}(\Omega), 1 \le p \le \infty),$$

where

$$\begin{aligned}
m(\varphi) &:= \min\left(1, \quad \operatorname*{ess\text{-}inf}_{\Omega} \kappa(\varphi)\right), \\
M(\varphi) &:= \max\left(1, \quad n \max_{1 \le i,j \le n} \left\| \frac{\partial \varphi_j}{\partial x_i} \right\|_{L^\infty(\Omega)}\right).
\end{aligned}$$

Proof. Since we have

$$m(\varphi) \le \kappa(\varphi)(x) \le M(\varphi)^n \qquad \text{a.e. } x \in \Omega,$$

the estimates for L^p norms are obtained. We also have

$$|\nabla v(x)| \le M(\varphi)|\nabla(\varphi_* v)(\varphi(x))| \qquad \text{a.e. } x \in \Omega,$$

from the equality $\nabla v = (\nabla\varphi^{\mathrm{T}})\nabla(\varphi_* v)$.[7] Note that α_{ij} is given by a cofactor of $\nabla\varphi^{\mathrm{T}}$ and is estimated as $|\alpha_{ij}| \le M(\varphi)^{n-1}$ a.e. in Ω. Hence, we have

$$|\nabla(\varphi_* v)(\varphi(x))| \le \frac{M(\varphi)^{n-1}}{m(\varphi)}|\nabla v(x)| \qquad \text{a.e. } x \in \Omega.$$

From these estimates, we obtain the L^p estimates for the gradients. □

PROPOSITION 1.47. *Let $k \in \mathbb{N}\cup\{0\}$, $l \in \{0,1\}$, and $p \in [1,\infty]$. We suppose that $f \in W^{k+l,p}(\Omega_0)$ if $p \in [1,\infty)$, and $f \in C^{k+l}(\Omega_0)$ if $p = \infty$. Then the mapping $[\varphi \mapsto f\circ\varphi]$ belongs to $C^k(\mathcal{O}(\Omega), W^{l,p}(\Omega))$.*

PROOF. We fix a domain Ω_1 with a smooth boundary such that $\overline{\Omega} \subset \Omega_1 \subset \overline{\Omega_1} \subset \Omega_0$. Let η_ε be a Friedrichs' mollifier and define $f_\varepsilon := \eta_\varepsilon * \tilde{f}$, where $\tilde{f}$ is the zero extension of f to $\mathbb{R}^n$. Then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and f_ε tends to f strongly in $W^{k+l,p}(\Omega_1)$ as $\varepsilon \to +0$.

First, we prove the theorem for $k = 0$. We fix $\varphi_1 \in \mathcal{O}(\Omega)$, and suppose that $\varphi \in \mathcal{O}(\Omega)$ satisfies $\|\varphi - \varphi_1\|_{W^{1,\infty}(\Omega_0)} \le \rho$ for sufficiently small fixed $\rho > 0$. Without loss of generality, we assume that $\overline{\Omega(\varphi_1)} \subset \Omega_1$ and $\overline{\Omega(\varphi)} \subset \Omega_1$. We have

$$\begin{aligned}\|f\circ\varphi_1 - f\circ\varphi\|_{W^{l,p}(\Omega)} &\le \|f\circ\varphi_1 - f_\varepsilon\circ\varphi_1\|_{W^{l,p}(\Omega)} \\ &\quad + \|f_\varepsilon\circ\varphi_1 - f_\varepsilon\circ\varphi\|_{W^{l,p}(\Omega)} + \|f_\varepsilon\circ\varphi - f\circ\varphi\|_{W^{l,p}(\Omega)}.\end{aligned}$$

The last term (and the first term as its special case) is estimated as

$$\|f_\varepsilon\circ\varphi - f\circ\varphi\|_{W^{l,p}(\Omega)} \le C(\rho,p)\|f_\varepsilon - f\|_{W^{l,p}(\Omega(\varphi))} \le C(\rho,p)\|f_\varepsilon - f\|_{W^{l,p}(\Omega_1)},$$

where $C(\rho,p)$ is a positive constant depending only on ρ and p. For the second term, if $p \in [1,\infty)$, we have the estimates

$$\|f_\varepsilon\circ\varphi_1 - f_\varepsilon\circ\varphi\|^p_{L^p(\Omega)} = \int_\Omega |f_\varepsilon(\varphi_1(x)) - f_\varepsilon(\varphi(x))|^p\,dx$$

$$\le \int_\Omega \left(\|\nabla f_\varepsilon\|_{L^\infty(\Omega_0)}\|\varphi_1 - \varphi\|_{L^\infty(\Omega_0)}\right)^p dx = |\Omega|\|\nabla f_\varepsilon\|^p_{L^\infty(\Omega_0)}\|\varphi_1 - \varphi\|^p_{L^\infty(\Omega_0)},$$

and

$$\begin{aligned}\|\nabla(f_\varepsilon\circ\varphi_1 - f_\varepsilon\circ\varphi)\|^p_{L^p(\Omega)} &= \int_\Omega \left|[\nabla\varphi_1^{\mathrm{T}}(x)]\nabla f_\varepsilon(\varphi_1(x)) - [\nabla\varphi^{\mathrm{T}}(x)]\nabla f_\varepsilon(\varphi(x))\right|^p dx \\ &\le \int_\Omega \Big\{\left|[\nabla\varphi_1^{\mathrm{T}}(x)]\left(\nabla f_\varepsilon(\varphi_1(x)) - \nabla f_\varepsilon(\varphi(x))\right)\right| \\ &\qquad + \left|\left([\nabla\varphi_1^{\mathrm{T}}(x)] - [\nabla\varphi^{\mathrm{T}}(x)]\right)\nabla f_\varepsilon(\varphi(x))\right|\Big\}^p dx \\ &\le C(\rho,\varepsilon,p)\|\varphi_1 - \varphi\|^p_{W^{1,\infty}(\Omega_0)},\end{aligned}$$

where $C(\rho,\varepsilon,p)$ is a positive constant depending only on ρ, ε and p. For the case $p = \infty$, similarly we have

$$\|f_\varepsilon\circ\varphi_1 - f_\varepsilon\circ\varphi\|_{W^{1,\infty}(\Omega)} \le C'(\rho,\varepsilon)\|\varphi_1 - \varphi\|_{W^{1,\infty}(\Omega_0)}.$$

[7] For $x \in \mathbb{R}^n$ and $n\times n$ matrix $A = (a_{ij})$, $|Ax| \le (\sum_{i,j} a_{ij}^2)^{1/2}|x|$ holds, where $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$.

From these estimates, it follows that the mapping $[\varphi \mapsto f \circ \varphi]$ is continuous at $\varphi = \varphi_1$. Since φ_1 is arbitrary in $\mathcal{O}(\Omega)$, it belongs to $C^0(\mathcal{O}(\Omega), W^{l,p}(\Omega))$.

Let us proceed to the case $k = 1$. For an arbitrary $\mu \in W^{1,\infty}(\Omega_0)$, we consider the Gâteaux derivative $\frac{d}{dt}|_{t=0} f \circ (\varphi + t\mu)$ in $W^{l,p}(\Omega)$. We define

$$F(t) := f \circ (\varphi + t\mu) \in W^{l,p}(\Omega), \qquad F_\varepsilon(t) := f_\varepsilon \circ (\varphi + t\mu) \in W^{l,p}(\Omega).$$

Then it is easy to see that there exists $a > 0$ such that

$$\lim_{\varepsilon \to +0} \|F_\varepsilon - F\|_{C^1([-a,a], W^{l,p}(\Omega))} = 0,$$

and that

$$\frac{d}{dt} f \circ (\varphi + t\mu) = \lim_{\varepsilon \to +0} \frac{d}{dt} f_\varepsilon \circ (\varphi + t\mu) = \lim_{\varepsilon \to +0} (\nabla f_\varepsilon \circ (\varphi + t\mu)) \cdot \mu = (\nabla f \circ (\varphi + t\mu)) \cdot \mu.$$

Hence $[\varphi \mapsto f \circ \varphi]$ is Gâteaux differentiable and the derivative is given by $(\nabla f \circ \varphi) \cdot \mu \in W^{l,p}(\Omega)$. Since $[\varphi \mapsto \nabla f \circ \varphi] \in C^0(\mathcal{O}(\Omega), W^{l,p}(\Omega))$, in other words, we have $[\varphi \mapsto \nabla f \circ \varphi] \in C^0(\mathcal{O}(\Omega), B(W^{1,\infty}(\Omega_0), W^{l,p}(\Omega)))$. From Theorem 1.11, it follows that $[\varphi \mapsto f \circ \varphi]$ belongs to $C^1(\mathcal{O}(\Omega), W^{l,p}(\Omega))$.

For the case $k \geq 2$, we can prove the assertion in the same way. □

Proposition 1.48. *We assume $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ with $\mathrm{supp}(\mu) \subset \Omega$.*

(1) *If $|\mu|_{\mathrm{Lip},\Omega_0} < 1$, then $\varphi = \varphi_0 + \mu$ is a bi-Lipschitz transform from Ω onto itself.*
(2) *For $t \in \mathbb{R}$ with $|t\mu|_{\mathrm{Lip},\Omega_0} < 1$, we define a bi-Lipschitz transform $\varphi(t) = \varphi_0 + t\mu$ from Ω to itself. Let $l \in \{0, 1\}$ and $p \in [1, \infty]$. Suppose that $f \in W^{l,p}(\Omega)$ if $p \in [1, \infty)$, and $f \in C^l(\Omega) \cap W^{l,\infty}(\Omega)$ if $p = \infty$. Then $\varphi(t)_* f \to f$ strongly in $W^{l,p}(\Omega)$ as $t \to 0$.*

Proof. From Proposition 1.41, the claim 1 is clear. For the claim 2, let us fix $t_0 > 0$ with $|t_0\mu|_{\mathrm{Lip},\Omega_0} < 1$. Then, from Proposition 1.46, there exist $C > 0$ such that the following inequalities hold for $|t| \leq t_0$,

$$\|\varphi(t)_* f - f\|_{W^{l,p}(\Omega)} = \|\varphi(t)_*(f - f \circ \varphi(t))\|_{W^{l,p}(\Omega)} \leq C\|f - f \circ \varphi(t)\|_{W^{l,p}(\Omega)}.$$

Since $[\varphi \mapsto f \circ \varphi] \in C^0(\mathcal{O}(\Omega), W^{l,p}(\Omega))$ from Proposition 1.47, we obtain

$$\|f - f \circ \varphi(t)\|_{W^{l,p}(\Omega)} = \|f \circ \varphi_0 - f \circ \varphi(t)\|_{W^{l,p}(\Omega)} \to 0,$$

as $t \to 0$. □

8. Potential energy in deformed domains

Let Ω_0 be a fixed bounded convex open set of $\mathbb{R}^n$ $(n \geq 2)$. We consider an open set Ω whose closure is contained in Ω_0. We suppose that $v \in H^1(\Omega)$ be a scalar valued function which describes a physical state in $\Omega \subset \mathbb{R}^n$. For such $v(x)$, we introduce the following energy functional:

$$E(v, \Omega) := \int_\Omega W(x, v(x), \nabla v(x)) dx,$$

where

$$W(\xi, \eta, \zeta) \in \mathbb{R} \quad \text{for} \quad (\xi, \eta, \zeta) \in \Omega_0 \times \mathbb{R} \times \mathbb{R}^n,$$

is a given energy density function. We assume some suitable regularity conditions and boundedness of its derivatives in the following argument. For simplicity, the partial derivatives of W with respect to ξ, η and ζ will be denoted by $\nabla_\xi W = (\frac{\partial W}{\partial \xi_1}, \cdots, \frac{\partial W}{\partial \xi_n})^{\mathrm{T}}$, $W_\eta = \frac{\partial W}{\partial \eta}$, and $\nabla_\zeta W = (\frac{\partial W}{\partial \zeta_1}, \cdots, \frac{\partial W}{\partial \zeta_n})^{\mathrm{T}}$, respectively. Moreover, for $v \in H^1(\Omega)$, we often write $W(v(x)) = W(x, v(x), \nabla v(x))$, $\nabla_\xi W(v(x)) = \nabla_\xi W(x, v(x), \nabla v(x))$, etc.

We consider the following minimization problem.

Problem 1.49. *Let V be a closed subspace of $H^1(\Omega)$ with $H^1_0(\Omega) \subset V \subset H^1(\Omega)$, and let $V(g) := \{v \in H^1(\Omega);\ v - g \in V\}$ for $g \in H^1(\Omega)$. For given $g \in H^1(\Omega)$, find a local minimizer u of $E(\cdot, \Omega)$ in $V(g)$, i.e. $u \in V(g)$ and there exists $\rho > 0$ such that*

$$E(u, \Omega) \leq E(w, \Omega) \qquad ({}^\forall w \in V(g) \text{ with } \|w - u\|_{H^1(\Omega)} < \rho). \tag{1.19}$$

If u is a local minimizer, under suitable regularity conditions for W, formally we obtain the variation formulas;

$$\int_\Omega \{W_\eta(u(x))v(x) + \nabla_\zeta W(u(x)) \cdot \nabla v(x)\}\, dx = 0 \qquad ({}^\forall v \in V), \tag{1.20}$$

$$-\mathrm{div}\,[\nabla_\zeta W(u(x))] + W_\eta(u(x)) = 0 \qquad \text{in } \Omega$$

For fixed Ω and $V \subset H^1(\Omega)$ as Problem 1.49, we consider a family of minimization problems which is parametrized by $\varphi \in \mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega)$ is defined by (1.18).

We define

$$V(\varphi, g) := \varphi_*(V(g)) = \{v \in H^1(\Omega(\varphi));\ \varphi_*^{-1}(v) - g \in V\}$$
$$(\varphi \in \mathcal{O}(\Omega),\ g \in H^1(\Omega)).$$

Problem 1.50. *For given $\varphi \in \mathcal{O}(\Omega)$ and $g \in H^1(\Omega)$, find a local minimizer $u(\varphi)$ of $E(\cdot, \Omega(\varphi))$ in $V(\varphi, g)$, i.e. $u(\varphi) \in V(\varphi, g)$ and there exists $\rho > 0$ such that*

$$E(u(\varphi), \Omega(\varphi)) \leq E(w, \Omega(\varphi)) \qquad ({}^\forall w \in V(\varphi, g) \text{ with } \|w - u(\varphi)\|_{H^1(\Omega(\varphi))} < \rho). \tag{1.21}$$

In many mathematical models of actual (quasi-)static systems, the potential energy in a deformed domain should be a local minimum. So, we define

$$E_*(\varphi) := E(u(\varphi), \Omega(\varphi)), \tag{1.22}$$

for local minimizers $u(\varphi)$.

We note that $v \in V$ is equivalent to $\varphi_*(v + g) \in V(\varphi, g)$. Using the formulas in § 7, we have

$$E(\varphi_*(v + g), \Omega(\varphi)) = \mathcal{E}(v, \varphi) \qquad (v \in V),$$

where

$$\mathcal{E}(v, \varphi) := \tilde{\mathcal{E}}(v + g, \varphi) \qquad (v \in H^1(\Omega)),$$

$$\tilde{\mathcal{E}}(v, \varphi) := \int_\Omega W\left(\varphi(x),\ v(x),\ [A(\varphi)(x)]\nabla v(x)\right) \kappa(\varphi)(x) dx \qquad (v \in H^1(\Omega)).$$

We define $v(\varphi) := \varphi_*^{-1}(u(\varphi)) - g \in V$. Then $u(\varphi)$ is a local minimizer of $E(\cdot, \Omega(\varphi))$ in $V(\varphi, g)$, if and only if $v(\varphi)$ is a local minimizer of $\mathcal{E}(\cdot, \varphi)$ in V. The following theorem is a direct consequence of Theorem 1.30.

THEOREM 1.51. *Suppose that $\tilde{\mathcal{E}} \in C^1(H^1(\Omega) \times \mathcal{O}(\Omega))$. Let $\mathcal{U}_0 \subset V$ and let $\mathcal{O}_0$ be an open subset of $\mathcal{O}(\Omega)$ with $\varphi_0 \in \mathcal{O}_0$. If $v \in C^0(\mathcal{O}_0, V)$ and $v(\varphi)$ is a global minimizer of $\mathcal{E}(\cdot,\varphi)$ in $\mathcal{U}_0$, then we have $E_* \in C^1(\mathcal{O}_0)$ and*

$$E'_*(\varphi) = \mathcal{E}_\varphi(v(\varphi), \varphi),$$

where E_* is defined by (1.22) *with $u(\varphi) = \varphi_*(v(\varphi) + g)$.*

We remark that

$$E'_*(\varphi_0) = \mathcal{E}_\varphi(v(\varphi_0), \varphi_0) = \tilde{\mathcal{E}}_\varphi(u(\varphi_0), \varphi_0).$$

Under the suitable regularity conditions for $W(\xi, \eta, \zeta)$, we have the following formula. For $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$,

$$\begin{aligned}
&\tilde{\mathcal{E}}_\varphi(u, \varphi_0)[\mu] \\
&\quad = \frac{d}{dt} \int_\Omega W\left(x + t\mu(x),\ u(x),\ [A(\varphi_0 + t\mu)(x)]\nabla u(x)\right) \kappa(\varphi_0 + t\mu)(x)dx \Big|_{t=0} \\
&\qquad\qquad = \int_\Omega \left(\nabla_\xi W(u) \cdot \mu - (\nabla_\zeta W(u))^{\mathrm{T}}(\nabla \mu^{\mathrm{T}})\nabla u + W(u)\mathrm{div}\mu\right) dx. \quad (1.23)
\end{aligned}$$

By means of the above domain mappings, we obtain another type of variation formula for Problem 1.49, so called 'interior variation' which is different from (1.20).

THEOREM 1.52. *Suppose that $\tilde{\mathcal{E}} \in C^1(H^1(\Omega) \times \mathcal{O}(\Omega))$. If u is a local minimizer of $E(\cdot, \Omega)$ in $V(g)$ as in Problem 1.49, then we have*

$$\tilde{\mathcal{E}}_\varphi(u, \varphi_0)[\mu] = 0 \qquad (\mu \in W^{1,\infty}(\Omega),\ \mathrm{supp}(\mu) \subset \Omega).$$

PROOF. For $\mu \in W^{1,\infty}(\Omega)$ with $\mathrm{supp}(\mu) \subset \Omega$, we define $\varphi(t) = \varphi_0 \pm t\mu$ for $t \geq 0$. From Proposition 1.46 and 1.48, if $|t\mu|_{\mathrm{Lip},\Omega_0} < 1$, the corresponding pushforward operator $\varphi(t)_*$ is a linear topological isomorphism from $H^1(\Omega)$ onto itself, and

$$\lim_{t \to 0} \|\varphi(t)_* u - u\|_{H^1(\Omega)} = 0.$$

Since

$$E(\varphi(t)_* u, \Omega) \geq E(u, \Omega) = E(\varphi(0)_* u, \Omega),$$

for $0 \leq t << 1$, we obtain

$$\pm\, \tilde{\mathcal{E}}_\varphi(u, \varphi_0)[\mu] = \frac{d}{dt}\tilde{\mathcal{E}}(u, \varphi(t))\Big|_{t=0} = \frac{d}{dt}E(\varphi(t)_* u, \Omega)\Big|_{t=0} \geq 0.$$

Hence, we have $\tilde{\mathcal{E}}_\varphi(u, \varphi_0)[\mu] = 0$. □

9. Applications

Based on the discussions in previous sections, we study the shape derivative of quadratic elliptic potential energies which correspond to scalar valued linear elliptic problems of second order. However, our framework is applicable even to vector valued elliptic systems such as linear elasticity problems and to some semilinear problems as shown in [17].

EXAMPLE 1.53. *We consider the following potential energy.*

$$W(\xi,\eta,\zeta) = \frac{1}{2}\left(\zeta^{\mathrm{T}}B(\xi)\zeta + b(\xi)\eta^2\right) - f(\xi)\eta, \tag{1.24}$$

$$E(v,\Omega) = \int_\Omega \left\{\frac{1}{2}\left((\nabla v)^{\mathrm{T}}B(x)\nabla v + b(x)v^2\right) - f(x)v\right\}dx,$$

where $k \in \mathbb{N}\cup\{0\}$ *and* $B(\xi)$ *is an* $n\times n$ *symmetric matrix (which is denoted by* $\mathbb{R}^{n\times n}_{\mathrm{sym}}$*) and it satisfies* $B = (b_{ij}) \in C^k(\Omega_0, \mathbb{R}^{n\times n}_{\mathrm{sym}})$ *and* $^\exists\beta_0 > 0$ *such that* $\zeta^{\mathrm{T}}B(\xi)\zeta \geq \beta_0|\zeta|^2$ $({}^\forall\xi \in \Omega_0,\ {}^\forall\zeta \in \mathbb{R}^n)$, *and*

$$b \in C^k(\Omega_0,\mathbb{R}), \qquad f \in W^{k,2}(\Omega_0,\mathbb{R}), \tag{1.25}$$

with the condition $b(\xi) \geq 0$ *for* $\xi \in \Omega_0$. *We remark that*

$$\begin{aligned}
\nabla_\xi W(\xi,\eta,\zeta) &= \frac{1}{2}\left(\sum_{i,j=1}^n \nabla b_{ij}(\xi)\zeta_i\zeta_j + \nabla b(\xi)\eta^2\right) - \nabla f(\xi)\eta,\\
W_\eta(\xi,\eta,\zeta) &= b(\xi)\eta - f(\xi),\\
\nabla_\zeta W(\xi,\eta,\zeta) &= B(\xi)\zeta.
\end{aligned}$$

We suppose that Γ_D *is a nonempty Lipschitz portion of* $\partial\Omega$ *and that a bounded trace operator* $\gamma_0 :\ H^1(\Omega) \to L^2(\Gamma_D)$ *is defined and*

$$V := \{v \in H^1(\Omega);\ \gamma_0(v) = 0\} \neq \{0\}, \tag{1.26}$$

holds.[8] *Then the minimization problem 1.21 corresponds to the following linear elliptic boundary value problem.*

$$\begin{cases}
-\mathrm{div}(B(x)\nabla u) + b(x)u = f(x) & \text{in } \Omega(\varphi),\\
u = g & \text{on } \Gamma_D,\\
(B\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega\setminus\Gamma_D,
\end{cases}$$

where ν *is a unit normal vector on* $\partial\Omega$.

From the Poincaré inequality and the Lax–Milgram theorem, there exists a unique global minimizer $u(\varphi) \in V(g,\varphi)$ *for* $\varphi \in \mathcal{O}(\Omega)$.

The Poisson equation is a special case of Example 1.53.

EXAMPLE 1.54. *We consider the following potential energy which corresponds to the Poisson equation.*

$$W(\xi,\eta,\zeta) = \frac{1}{2}|\zeta|^2 - f(\xi)\eta,$$

$$E(v,\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - f(x)v\right)dx,$$

where $k \in \mathbb{N}\cup\{0\}$ *and* $f \in W^{k,2}(\Omega_0,\mathbb{R})$. *This is a special case of Example 1.53. We remark that*

$$\nabla_\xi W(\xi,\eta,\zeta) = -\nabla f(\xi)\eta, \qquad W_\eta(\xi,\eta,\zeta) = -f(\xi), \qquad \nabla_\zeta W(\xi,\eta,\zeta) = \zeta.$$

[8] Under the condition (1.26), from the well known Poincaré inequality, the coercivity of this functional follows. We remark that, if $b(\xi) > 0$ for $\xi \in \Omega_0$, we do not need the condition (1.26).

Under the same boundary condition, the minimization problem 1.21 corresponds to the following boundary value problem of the Poisson equation:

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega(\varphi), \\ u = g & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \Gamma_D. \end{cases}$$

LEMMA 1.55. *For the potential energy of Example 1.53, $\tilde{\mathcal{E}} \in C^k(H^1(\Omega) \times \mathcal{O}(\Omega))$ and (1.23) holds.*

PROOF. In the case of Example 1.53, $\tilde{\mathcal{E}}$ becomes

$$\begin{aligned} &\tilde{\mathcal{E}}(v, \varphi) \\ &= \int_\Omega \left\{ \frac{1}{2} \left(A(\varphi)\nabla v\right)^{\mathrm{T}} B(\varphi(x)) \left(A(\varphi)\nabla v\right) + \frac{1}{2} b(\varphi(x)) v^2 - f(\varphi(x)) v \right\} \kappa(\varphi) dx \\ &\qquad = \int_\Omega \left\{ \frac{1}{2}\Psi_1(v,\varphi)(x) + \frac{1}{2}\Psi_2(v,\varphi)(x) - \Psi_3(v,\varphi)(x) \right\} \kappa(\varphi)(x) dx \end{aligned}$$

where $\Psi_1(v,\varphi) := \left(A(\varphi)\nabla v\right)^{\mathrm{T}} (B \circ \varphi) \left(A(\varphi)\nabla v\right)$, $\Psi_2(v,\varphi) := (b \circ \varphi) v^2$, $\Psi_3(v,\varphi) := (f \circ \varphi) v$. Under the assumptions, from Theorem 1.45 and Proposition 1.47, we have

$$\begin{aligned} [(v,\varphi) \mapsto A(\varphi)\nabla v] &\in C^\infty(H^1(\Omega) \times \mathcal{O}(\Omega), L^2(\Omega)^n), \\ [\varphi \mapsto B \circ \varphi] &\in C^k(\mathcal{O}(\Omega), L^\infty(\Omega, \mathbb{R}^{n\times n})), \\ [(w,B) \mapsto w^{\mathrm{T}} B w] &\in C^\infty(L^2(\Omega)^n \times L^\infty(\Omega, \mathbb{R}^{n\times n}), L^1(\Omega)). \end{aligned}$$

From these regularities, it follows that $\Psi_1 \in C^k(H^1(\Omega) \times \mathcal{O}(\Omega), L^1(\Omega))$. Similarly, from the following regularities

$$\begin{aligned} [\varphi \mapsto b \circ \varphi] &\in C^k(\mathcal{O}(\Omega), L^\infty(\Omega)), \\ [(b,v) \mapsto bv^2] &\in C^\infty(L^\infty(\Omega) \times H^1(\Omega), L^1(\Omega)), \\ [\varphi \mapsto f \circ \varphi] &\in C^k(\mathcal{O}(\Omega), L^2(\Omega)) \\ [(f,v) \mapsto fv] &\in C^\infty(L^2(\Omega) \times H^1(\Omega), L^1(\Omega)), \end{aligned}$$

Ψ_2 and Ψ_3 also belong to $C^k(H^1(\Omega) \times \mathcal{O}(\Omega), L^1(\Omega))$. Since $\kappa \in C^\infty(\mathcal{O}(\Omega), L^\infty(\Omega))$ from Theorem 1.45, we conclude that $\tilde{\mathcal{E}} \in C^k(H^1(\Omega) \times \mathcal{O}(\Omega))$. □

We obtain the following theorem from Lemma 1.55 and Theorem 1.35.

THEOREM 1.56. *For the potential energy of Example 1.53, $E_* \in C^k(\mathcal{O}(\Omega))$ holds, and if $k \geq 1$ we have*

$$E'_*(\varphi_0)[\mu] = \int_\Omega \left(\nabla_\xi W(u) \cdot \mu - (\nabla_\zeta W(u))^{\mathrm{T}} (\nabla \mu^{\mathrm{T}}) \nabla u + W(u) \mathrm{div}\mu \right) dx \qquad (\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)),$$

where u is the global minimizer of $E(\cdot,\Omega)$ in $V(g)$. In particular, in the case of Example 1.54, we have

$$E'_*(\varphi_0)[\mu] = \int_\Omega \left\{ -(\nabla f \cdot \mu)u - (\nabla^{\mathrm{T}} u)(\nabla \mu^{\mathrm{T}})(\nabla u) + \left(\frac{1}{2}|\nabla u|^2 - fu\right) \mathrm{div}\mu \right\} dx$$
$$(\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)).$$

We consider the case of Example 1.53. The global minimizer u belongs to $H^1(\Omega)$, but not to $H^2(\Omega)$ in general. The regularity loss is caused by geometrical singularities such as a crack tip and a corner of the domain boundary, or by an incompatible point of a mixed boundary condition.

Theorem 1.57. *We consider Example 1.53 with $k = 1$. Let $\mathcal{G}$ be a nonempty open set in $\mathbb{R}^n$ with a Lipschitz boundary. If there exists a sequence of subdomains $\{\Omega_l\}_l$ in which the Gauss–Green formula holds and*

$$\Omega_1 \subset \Omega_2 \subset \cdots \Omega \textit{ with } \bigcup_{l=1}^{\infty} \Omega_l = \Omega,$$

and if the global minimizer u belongs to $H^2(\Omega_l \cap \mathcal{G})$ for each l, then we have

$$E'_*(\varphi_0)[\mu] = \lim_{l\to\infty} \int_{\partial\Omega_l} \{W(u)\ \mu \cdot \nu - (\nabla_\zeta W(u) \cdot \nu)\,(\nabla u \cdot \mu)\}\, d\mathcal{H}_x^{n-1}, \tag{1.27}$$

for $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ with $\mathrm{supp}(\mu) \subset \mathcal{G}$.

Proof. We define $\widetilde{\Omega}_l := \Omega_l \cap \mathcal{G}$. Under the conditions, we have

$$-\mathrm{div}\,[\nabla_\zeta W(u)] + W_\eta(u) = 0 \qquad \text{in } L^2(\widetilde{\Omega}_l),$$

which is equivalent to

$$-\mathrm{div}(B\nabla u) + bu = f \qquad \text{in } L^2(\widetilde{\Omega}_l).$$

From the following equalities:[9]

$$\int_{\widetilde{\Omega}_l} W(u(x))\mathrm{div}\mu(x)dx = \int_{\partial\widetilde{\Omega}_l} W(u(x))\mu(x) \cdot \nu d\mathcal{H}_x^{n-1} - \int_{\widetilde{\Omega}_l} \nabla_x[W(u(x))] \cdot \mu(x)dx,$$

$$\nabla_x[W(u(x))] = \nabla_\xi W(u(x)) + W_\eta(u(x))\nabla u(x) + [\nabla^2 u(x)]\nabla_\zeta W(u(x)) \quad \text{in } \widetilde{\Omega}_l,$$

$$\begin{aligned}
&\int_{\widetilde{\Omega}_l} W_\eta(u(x))\nabla u(x) \cdot \mu(x)dx = \int_{\widetilde{\Omega}_l} \mathrm{div}\,[\nabla_\zeta W(u)]\,\nabla u(x) \cdot \mu(x)dx \\
&= \int_{\partial\widetilde{\Omega}_l} (\nabla_\zeta W(u) \cdot \nu)\,(\nabla u(x) \cdot \mu(x))\, d\mathcal{H}_x^{n-1} - \int_{\widetilde{\Omega}_l} \nabla_\zeta^{\mathrm{T}} W(u)\nabla\,(\nabla u(x) \cdot \mu(x))\, dx \\
&= \int_{\partial\Omega_l} (\nabla_\zeta W(u) \cdot \nu)\,(\nabla u(x) \cdot \mu(x))\, d\mathcal{H}_x^{n-1} \\
&\qquad - \int_{\Omega_l} \nabla_\zeta^{\mathrm{T}} W(u)\left\{(\nabla^2 u(x))\mu(x) + (\nabla\mu^{\mathrm{T}})\nabla u\right\} dx,
\end{aligned}$$

[9] In this note, $\nabla u(x)$ denotes the gradient of u as a column vector and $\nabla^2 u(x)$ denotes the Hessian matrix.

$$
\begin{aligned}
\int_{\widetilde{\Omega}_l} & \nabla_x[W(u(x))] \cdot \mu(x) dx \\
&= \int_{\Omega_l} \left\{\nabla_\xi W(u(x)) + [\nabla^2 u(x)] \nabla_\zeta W(u(x))\right\} \cdot \mu(x) dx \\
&= \int_{\partial\Omega_l} (\nabla_\zeta W(u) \cdot \nu) (\nabla u(x) \cdot \mu(x))\, d\mathcal{H}_x^{n-1} \\
&\quad - \int_{\Omega_l} \nabla_\zeta^{\mathrm{T}} W(u) \left\{(\nabla^2 u(x))\mu(x) + (\nabla \mu^{\mathrm{T}})\nabla u\right\} dx \\
&= \int_{\partial\Omega_l} (\nabla_\zeta W(u) \cdot \nu) (\nabla u(x) \cdot \mu(x))\, d\mathcal{H}_x^{n-1} \\
&\quad + \int_{\Omega_l} \left\{\nabla_\xi W(u) \cdot \mu dx - \nabla_\zeta^{\mathrm{T}} W(u) (\nabla \mu^{\mathrm{T}}) \nabla u\right\} dx,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
E'_*(\varphi_0)[\mu] &= \int_\Omega \left(\nabla_\xi W(u) \cdot \mu - (\nabla_\zeta W(u))^{\mathrm{T}} (\nabla \mu^{\mathrm{T}}) \nabla u + W(u) \mathrm{div} \mu\right) dx \\
&= \int_{\Omega\setminus\Omega_l} \left(\nabla_\xi W(u) \cdot \mu - (\nabla_\zeta W(u))^{\mathrm{T}} (\nabla \mu^{\mathrm{T}}) \nabla u + W(u) \mathrm{div} \mu\right) dx \\
&\quad + \int_{\Omega_l} \left(\nabla_\xi W(u) \cdot \mu - (\nabla_\zeta W(u))^{\mathrm{T}} (\nabla \mu^{\mathrm{T}}) \nabla u\right) dx \\
&\quad + \int_{\partial\Omega_l} W(u)\ \mu \cdot \nu\ d\mathcal{H}_x^{n-1} - \int_{\Omega_l} \nabla_x[W(u)] \cdot \mu dx \\
&= \int_{\Omega\setminus\Omega_l} \left(\nabla_\xi W(u) \cdot \mu - (\nabla_\zeta W(u))^{\mathrm{T}} (\nabla \mu^{\mathrm{T}}) \nabla u + W(u) \mathrm{div} \mu\right) dx \\
&\quad + \int_{\partial\Omega_l} \left\{W(u)\ \mu \cdot \nu - (\nabla_\zeta W(u) \cdot \nu) (\nabla u \cdot \mu)\right\} d\mathcal{H}_x^{n-1},
\end{aligned}
$$

where ν denotes the outward unit normal of $\partial\Omega_l$. The first term tends to 0 if $l \to \infty$. Hence, we have the formula (1.27). □

COROLLARY 1.58. *In particular, for the case of Example 1.54, we have*

$$
E'_*(\varphi_0)[\mu] = \lim_{l\to\infty} \int_{\partial\Omega_l} \left\{\left(\frac{1}{2}|\nabla u|^2 - fu\right)\ \mu \cdot \nu - (\nabla u \cdot \nu) (\nabla u \cdot \mu)\right\} d\mathcal{H}_x^{n-1}. \quad (1.28)
$$

In the absence of singularity, Corollary 1.58 has the following form.

THEOREM 1.59. *Let Ω be a bounded domain with C^2-boundary. Under the condition of Example 1.54, we assume that $\Gamma_D = \partial\Omega$ with $g \equiv 0$. Then the following formula holds.*

$$
E'_*(\varphi_0)[\mu] = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \mu \cdot \nu\ d\mathcal{H}_x^{n-1}.
$$

PROOF. From the regularity theorem of elliptic boundary value problems, we have $u \in H^2(\Omega)$. So, we can choose $\Omega_l = \Omega$. We remark that

$$
\nabla u \cdot \mu = (\nabla u \cdot \nu)(\mu \cdot \nu), \qquad |\nabla u \cdot \nu| = |\nabla u| \qquad \text{on } \partial\Omega,
$$

since u and its tangential derivatives vanish on the boundary. Hence we have

$$E_*'(\varphi_0)[\mu] = \int_{\partial\Omega} \left\{ \left(\frac{1}{2}|\nabla u|^2 - fu \right) \ \mu \cdot \nu - (\nabla u \cdot \nu)\,(\nabla u \cdot \mu) \right\} d\mathcal{H}_x^{n-1}$$

$$= \int_{\partial\Omega} \left\{ \left(\frac{1}{2}|\nabla u|^2 \right) \ \mu \cdot \nu - \left(|\nabla u|^2 \right) \ \mu \cdot \nu \right\} d\mathcal{H}_x^{n-1} = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \ \mu \cdot \nu \ d\mathcal{H}_x^{n-1}.$$

□

This theorem can be generalized as follows.

THEOREM 1.60. *Under the condition of Example 1.54, we assume that* $\mu \in W^{1,\infty}(\Omega_0, \mathbb{R}^n)$ *satisfies*

$$\operatorname{supp}(\mu) \cap \operatorname{supp}(\ g\ |_{\overline{\Gamma_D}}) = \emptyset, \qquad \operatorname{supp}(\mu) \cap \overline{\Gamma_D} \cap \overline{\partial\Omega \setminus \Gamma_D} = \emptyset,$$

$$^\exists \mathcal{G} : \textit{an open set of } \mathbb{R}^n \textit{ s.t. } \mathcal{G} \supset \operatorname{supp}(\mu) \textit{ and } \partial\Omega \cap \mathcal{G} \textit{ is } C^2\textit{-class.}$$

Then the following formula holds.

$$E_*'(\varphi_0)[\mu] = -\frac{1}{2} \int_{\Gamma_D} |\nabla u|^2 \mu \cdot \nu \ d\mathcal{H}_x^{n-1} - \int_{\partial\Omega \setminus \Gamma_D} f\, u \ \mu \cdot \nu \ d\mathcal{H}_x^{n-1}.$$

Bibliography

[1] R. A. Adams, Sobolev Spaces. Pure and Applied Mathematics, Vol.65, Academic Press, New York-London (1975).

[2] G. I. Barenblatt, The mathematical theory of equilibrium of cracks in brittle fracture. Advances in Appl. Mech., Vol.7 (1962) 55-129.

[3] M. S. Berger, Nonlinearity and Functional Analysis. Lectures on nonlinear problems in mathematical analysis, Pure and Applied Mathematics. Academic Press, New York-London (1977).

[4] G. P. Cherepanov, *On Crack propagation in continuous media.* Prikl. Math. Mekh. Vol.31, No.3 (1967), 476-493.

[5] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1992).

[6] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids, Vol.46 (1998), 1319-1342.

[7] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order. Second edition, Grundlehren der Mathematischen Wissenschaften, 224. Springer-Verlag, Berlin (1983).

[8] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, (1986).

[9] A. A. Griffith, *The phenomenon of rupture and flow in solids.* Phil. Trans. Royal Soc. London, A221 (1920), 163-198.

[10] P. Grisvard, Elliptic Problems in Nonsmooth Domains. Monographs and Studies in Mathematics, 24, Pitman (Advanced Publishing Program), Boston, MA, (1985).

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, vol. 840, Springer Verlag (1981).

[12] G. R. Irwin, Fracture dynamics. Fracturing of Metals, American Society of Metals, (1948) 147-166.

[13] M. Kimura and I. Wakano, *New mathematical approach to the energy release rate in crack extension.* Trans. Japan Soc. Indust. Appl. Math. Vol.16, No.3 (2006), 345-358. (in Japanese)

[14] M. Kimura and I. Wakano, *Shape derivative of potential energy and energy release rate in fracture mechanics.* (in preparation)

[15] A. Novruzi and M. Pierre, *Structure of shape derivatives.* J. Evol. Equ. Vol.2, No.3 (2002), 365-382.

[16] K. Ohtsuka, *Generalized J-integral and three-dimensional fracture mechanics I.* Hiroshima Math. J., Vol.11 (1981), 21-52.

[17] K. Ohtsuka, *Generalized J-integral and its applications I – Basic theory –.* Japan J. Appl. Math. Vol.2, No.2 (1985), 329-350.

[18] K. Ohtsuka, *Generalized J-integral and three-dimensional fracture mechanics II.* Hiroshima Math. J., Vol.16 (1986), 327-352.

[19] K. Ohtsuka and A. Khludnev, *Generalized J-integral method for sensitivity analysis of static shape design.* Control & Cybernetics, Vol.29 (2000), 513-533.

[20] J. R. Rice, *A path-independent integral and the approximate analysis of strain concentration by notches and cracks.* J. Appl. Mech., Vol.35 (1968), 379-386.

[21] M. Schechter, Principles of Functional Analysis. second edition, Graduate Studies in Mathematics, Vol.36, American Mathematical Society, Providence, RI, (2002).

[22] J. Sokolowski and J.-P. Zolesio, Introduction to Shape Optimization. Shape Sensitivity Analysis. Springer Series in Computational Mathematics, Vol.16, Springer-Verlag, Berlin, (1992).

[23] I. Wakano, *Analysis for stress intensity factors in two dimensional elasticity.* Proc. Japan Acad., Vol.73, Ser.A, No.5 (1997), 86-88.

[24] K. Yosida, Functional Analysis. sixth edition, Springer-Verlag, Berlin-New York (1980).

[25] W. P. Ziemer, Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation. Graduate Texts in Mathematics, 120, Springer-Verlag, New York (1989).

Part 2

Geometry of hypersurfaces and moving hypersurfaces in $\mathbb{R}^m$ for the study of moving boundary problems

Masato Kimura

2000 *Mathematics Subject Classification.* 14Q10, 35R35, 53C44

Key words and phrases. hypersurfaces, moving hypersurfaces, gradient flows

ABSTRACT. The text provides a record of an intensive lecture course on the moving boundary problems. Keeping the applications to moving boundary problems in $\mathbb{R}^2$ or $\mathbb{R}^3$ in mind, the text systematically constructs the mathematical foundations of hypersurfaces and moving hypersurfaces in $\mathbb{R}^m$ for $m \geq 2$. Main basic concepts treated in the text are differential and integral formulas for hypersurfaces, geometric quantities and the signed distance functions for moving hypersurfaces, the variational formulas and the transport identities, and the gradient structure of moving boundary problems.

Contents

Preface

From the end of March 2007, I had an occasion to spend one year at Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague as a visiting researcher of the Nečas Center for Mathematical Modeling. One of the common research subjects in the Mathematical Modelling Group of Professor Michal Beneš in Czech Technical University, where I visited, is the moving boundary problems related to physics, chemistry, biology and engineering. This is a note for my intensive lectures on the moving boundary problems which I gave at Czech Technical University in November 2007.

The principal object of my intensive lectures was to construct a basic mathematical framework for the moving boundary problems and to derive several useful formulas without assuming any knowledge on the differential geometry. Most of the topics in this note are basically included in the course of the classical differential geometry. However, the standard description of the differential geometry is not always suitable for the applied mathematicians who intend to study moving boundary problems.

Keeping the applications to moving boundary problems in $\mathbb{R}^2$ or $\mathbb{R}^3$ in mind, we systematically construct the mathematical foundations of hypersurfaces and moving hypersurfaces in $\mathbb{R}^m$ for $m \geq 2$. Main basic concepts treated in this note are differential and integral formulas for hypersurfaces, geometric quantities and the signed distance functions for moving hypersurfaces, the variational formulas and the transport identities, and the gradient structure of moving boundary problems. By the word ”geometric”, we just mean that the quantity or the operator does not depend on the choice of the Cartesian coordinate in $\mathbb{R}^m$.

On the other hand, we had to omit many important topics, such as the construction of the partition of unity, the hight function and the domain mapping method, the Sobolev spaces on hypersurfaces, the well-posedness of some moving boundary problems, etc. We are not able to touch on these issues except for some examples and exercises without proofs.

Finally, I would like to thank Professor Michal Beneš and his colleagues for giving me this opportunity and for attending the lectures. Especially, I am grateful to Mr. Tomáš Oberhuber who carefully read the first manuscript and gave me several useful comments on this work.

Masato Kimura
Fukuoka, July 2008

CHAPTER 1

Preliminaries

In this chapter, we collect our notation in this lecture note and several basic facts on the geometries of hypersurfaces in the standard style of geometrical treatment. We start from the well-known Frenet's formula for plane curves. In general dimension, we introduce the standard local parametric representation of a hypersurface and define the surface integral on the hypersurface. In the last section, we give a definition of the principal curvatures based on the graph representation of the hypersurface in general dimension.

1. Notation

We use the following notation throughout this note. For an integer $m \in \mathbb{N}$ with $m \geq 2$, $\mathbb{R}^m$ denotes the m-dimensional Euclidean space over $\mathbb{R}$. Each point $\boldsymbol{x} \in \mathbb{R}^m$ is expressed by a column vector $\boldsymbol{x} = (x_1, \cdots, x_m)^{\mathrm{T}}$, where $^{\mathrm{T}}$ stands for the transpose of the vector or matrix.

For two column vectors $\boldsymbol{a} = (a_1, \cdots, a_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $\boldsymbol{b} = (b_1, \cdots, b_m)^{\mathrm{T}} \in \mathbb{R}^m$, their inner product is denoted by

$$\boldsymbol{a} \cdot \boldsymbol{b} := \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b} = \sum_{i=1}^{m} a_i b_i,$$

and $|\boldsymbol{x}| := \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$. For two square matrices $A = (a_{ij}) \in \mathbb{R}^{m \times m}$ and $B = (b_{ij}) \in \mathbb{R}^{m \times m}$, their componentwise inner product is also denoted by

$$A : B := \operatorname{tr}\left(A^{\mathrm{T}} B\right) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} b_{ij},$$

where tr stands for the trace of a square matrix.

For a real-valued function $f = f(\boldsymbol{x})$, its gradient is given by a column vector

$$\nabla f(\boldsymbol{x}) = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}) := \left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}),\ \cdots,\ \frac{\partial f}{\partial x_m}(\boldsymbol{x})\right)^{\mathrm{T}} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\boldsymbol{x}) \\ \vdots \\ \frac{\partial f}{\partial x_m}(\boldsymbol{x}) \end{pmatrix},$$

and its transpose is denoted by ∇^{T}

$$\nabla^{\mathrm{T}} f(\boldsymbol{x}) = \left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}),\ \cdots,\ \frac{\partial f}{\partial x_m}(\boldsymbol{x})\right).$$

For a (column) vector-valued function $\boldsymbol{h}(\boldsymbol{x}) = (h_1(\boldsymbol{x}), \cdots, h_m(\boldsymbol{x}))^{\mathrm{T}} \in \mathbb{R}^m$, its transpose is denoted by $\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})^{\mathrm{T}} = (h_1(\boldsymbol{x}), \cdots, h_m(\boldsymbol{x}))$. The above gradient operator acts on $\boldsymbol{h}$ as follows.

$$\nabla \boldsymbol{h}^{\mathrm{T}}(\boldsymbol{x}) := (\nabla h_1(\boldsymbol{x}),\ \cdots,\ \nabla h_m(\boldsymbol{x})) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial h_m}{\partial x_1}(\boldsymbol{x}) \\ \vdots & & \vdots \\ \frac{\partial h_1}{\partial x_m}(\boldsymbol{x}) & \cdots & \frac{\partial h_m}{\partial x_m}(\boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

$$\nabla^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{x}) := (\nabla \boldsymbol{h}^{\mathrm{T}}(\boldsymbol{x}))^{\mathrm{T}} = \begin{pmatrix} \nabla^{\mathrm{T}} h_1(\boldsymbol{x}) \\ \vdots \\ \nabla^{\mathrm{T}} h_m(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial h_1}{\partial x_m}(\boldsymbol{x}) \\ \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1}(\boldsymbol{x}) & \cdots & \frac{\partial h_m}{\partial x_m}(\boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

We note that ∇^{T} does not mean the divergence operator in this paper. The divergence operator is denoted by

$$\mathrm{div}_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}) = \mathrm{div} \boldsymbol{h}(\boldsymbol{x}) := \sum_{i=1}^{m} \frac{\partial h_i}{\partial x_i}(\boldsymbol{x}) = \mathrm{tr}(\nabla \boldsymbol{h}^{\mathrm{T}}) = \mathrm{tr}(\nabla^{\mathrm{T}} \boldsymbol{h}).$$

This div acts on a column vector.

For a real-valued function $f = f(\boldsymbol{x})$, its Hessian matrix is denoted by

$$\nabla^2 f := \nabla^{\mathrm{T}}(\nabla f) = \nabla(\nabla^{\mathrm{T}} f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_m^2} \end{pmatrix} \in \mathbb{R}^{m\times m},$$

where we note that ∇^2 does not mean the Laplacian operator in this paper. We denote the Laplacian of f by

$$\Delta f := \mathrm{div} \nabla f = \mathrm{tr} \nabla^2 f = \sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2}.$$

2. Plane curves

We start from the plane curve. Let Γ be a C^2-class curve in $\mathbb{R}^2$ parametrized by a length parameter $s \in \mathcal{I}$, where $\mathcal{I}$ is an interval:

$$\Gamma = \{\boldsymbol{\gamma}(s) = (\gamma_1(s), \gamma_2(s))^{\mathrm{T}} \in \mathbb{R}^2;\ s \in \mathcal{I}\}, \quad \boldsymbol{\gamma} \in C^2(\mathcal{I}, \mathbb{R}^2), \quad |\boldsymbol{\gamma}'(s)| = 1.$$

The tangential and normal unit vectors are given by

$$\boldsymbol{\tau}(s) := \boldsymbol{\gamma}'(s), \qquad \boldsymbol{\nu}(s) := (-\gamma_2'(s), \gamma_1'(s))^{\mathrm{T}}.$$

EXERCISE 1.1. *Prove that* $\boldsymbol{\tau}'(s) \parallel \boldsymbol{\nu}(s)$ *and* $\boldsymbol{\nu}'(s) \parallel \boldsymbol{\tau}(s)$.

The signed curvature $\kappa(s)$ is defined by the formulas

$$\boldsymbol{\tau}'(s) = \kappa(s)\boldsymbol{\nu}(s), \qquad \boldsymbol{\nu}'(s) = -\kappa(s)\boldsymbol{\tau}(s).$$

These are known as the Frenet formula for plane curves. We draw the readers' attention to the above sign convention of $\boldsymbol{\nu}$ and κ, which we use throughout this note.

EXAMPLE 1.2. *We consider a circle of radius* $r > 0$*:* $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^2;\ |\boldsymbol{x}| = r\}$. *Then we have the following relations:*

$$\boldsymbol{\gamma}(s) = \left(r\cos\left(\frac{s}{r}\right),\ r\sin\left(\frac{s}{r}\right)\right)^{\mathrm{T}} \quad (0 \le s < 2\pi r),$$

$$\boldsymbol{\tau}(s) = \left(-\sin\left(\frac{s}{r}\right),\ \cos\left(\frac{s}{r}\right)\right)^{\mathrm{T}}, \qquad \boldsymbol{\nu}(s) = -\left(\cos\left(\frac{s}{r}\right),\ \sin\left(\frac{s}{r}\right)\right)^{\mathrm{T}}$$

$$\boldsymbol{\tau}'(s) = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right),\ -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right)^{\mathrm{T}} = \frac{1}{r}\boldsymbol{\nu}(s), \qquad \kappa(s) = \frac{1}{r}.$$

EXERCISE 1.3. *For* Γ *given by a graph* $\eta = u(\xi)$ $(\xi \in \mathcal{I}_1 \subset \mathbb{R})$ *in* ξ-η *plane with* $u \in C^2(\mathcal{I}_1)$ *and* $\frac{ds}{d\xi} > 0$*, prove that*

$$\kappa = \frac{u''(\xi)}{(1 + |u'(\xi)|^2)^{\frac{3}{2}}}.$$

EXERCISE 1.4. *For* $\Gamma = \{\boldsymbol{\varphi}(\xi) = (\varphi_1(\xi), \varphi_2(\xi))^{\mathrm{T}} \in \mathbb{R}^2;\ \xi \in \mathcal{I}_2 \subset \mathbb{R}\}$ *with* $\boldsymbol{\varphi} \in C^2(\mathcal{I}_2)$ *and* $\frac{ds}{d\xi} > 0$*, prove that*

$$\kappa = \frac{\varphi_1'(\xi)\varphi_2''(\xi) - \varphi_1''(\xi)\varphi_2'(\xi)}{|\boldsymbol{\varphi}'(\xi)|^3}.$$

3. Parametric representation

Let $m \in \mathbb{N}$ and $m \ge 2$. We consider $(m-1)$-dimensional hypersurfaces embedded in $\mathbb{R}^m$.

DEFINITION 1.5. Let $k \in \mathbb{N}$. A subset $\Gamma \subset \mathbb{R}^m$ is called *a* C^k*-class hypersurface in* $\mathbb{R}^m$, if, for each $\boldsymbol{x} \in \Gamma$, there exist,

$$\begin{cases} \mathcal{O}: \ \text{a nonempty bounded domain in } \mathbb{R}^m, \\ \mathcal{U}: \ \text{a nonempty bounded domain in } \mathbb{R}^{m-1}, \\ \boldsymbol{\varphi} \in C^k(\mathcal{U}, \mathbb{R}^m) \end{cases} \tag{1.1}$$

such that $\boldsymbol{x} \in \mathcal{O}$ and they satisfy the conditions:

$$\boldsymbol{\varphi} : \mathcal{U} \to \Gamma \cap \mathcal{O} \text{ is bijective, and } \operatorname{rank}(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi})) = m - 1 \ ({}^{\forall}\boldsymbol{\xi} \in \mathcal{U}). \tag{1.2}$$

The triplet $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi})$ gives a kind of local embedding C^k-coordinate of the hypersurface Γ. We define

$$\mathsf{C}_\mathsf{k}(\Gamma) := \{(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi});\ (\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi}) \text{ satisfies (1.1) and (1.2)}\}.$$

If a subset $\{(\mathcal{O}_\mu, \mathcal{U}_\mu, \boldsymbol{\varphi}_\mu)\}_{\mu \in M} \subset \mathsf{C}_\mathsf{k}(\Gamma)$ satisfies $\Gamma \subset \cup_{\mu \in M}\mathcal{O}_\mu$, then we call $\{(\mathcal{O}_\mu, \mathcal{U}_\mu, \boldsymbol{\varphi}_\mu)\}_{\mu \in M}$ a local embedding C^k-coordinate system of Γ. Moreover, if Γ is compact, we can choose its finite subset $\{(\mathcal{O}_{\mu_j}, \mathcal{U}_{\mu_j}, \boldsymbol{\varphi}_{\mu_j})\}_{j=1,2,\cdots,J}$ such that it is a local C^k-embedding coordinate system of Γ, too.

EXERCISE 1.6. *Let* $k \in \mathbb{N}$*. For a* C^k*-class hypersurface* Γ *in* $\mathbb{R}^m$*, suppose that* $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi})$ *and* $(\tilde{\mathcal{O}}, \tilde{\mathcal{U}}, \tilde{\boldsymbol{\varphi}})$ *are two triplets belonging to* $\mathsf{C}_\mathsf{k}(\Gamma)$ *with* $\Sigma := \Gamma \cap \mathcal{O} \cap \tilde{\mathcal{O}} \ne \emptyset$*. Then, prove that* $\boldsymbol{\varphi}^{-1} \circ \tilde{\boldsymbol{\varphi}} \in C^k(\tilde{\boldsymbol{\varphi}}^{-1}(\Sigma), \boldsymbol{\varphi}^{-1}(\Sigma))$.

To represent a hypersurface, there are mainly four ways;

(i) parametric representation by local coordinate (Definition 1.5),

(ii) local graph representation (next section),
(iii) level set approach,
(iv) domain mapping method.

Although the domain mapping method is important and widely used, we omit it in this note except for Section 1.

The level set approach enables us to treat a hypersurface globally in a fixed orthogonal coordinate system. It is useful especially to represent moving hypersurfaces.

EXERCISE 1.7. *Let $\mathcal{O}$ be an open set in $\mathbb{R}^m$, and let $w \in C^k(\mathcal{O})$. Then, for a fixed $c \in \mathbb{R}$, we define a (c-)level set Γ of w by*

$$\Gamma := \{\boldsymbol{x} \in \mathcal{O};\ w(\boldsymbol{x}) = c\}.$$

Prove that Γ is a C^k-class hypersurface in $\mathbb{R}^m$ provided $\Gamma \neq \emptyset$ and $\nabla w(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \Gamma$. (Conversely, any C^k-class hypersurface can be represented as a level set of a C^k-class function. An example of such functions is given by the signed distance function (Section 2).)

DEFINITION 1.8. Let $k \in \mathbb{N}$ and let l be an integer with $0 \leq l \leq k$. For a C^k-class hypersurface Γ in $\mathbb{R}^m$ and a function f defined on Γ, f is called of *C^l-class* if $f \circ \boldsymbol{\varphi} \in C^l(\mathcal{U})$ for any $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi}) \in \mathsf{C}_\mathsf{k}(\Gamma)$. The set of all C^l-class functions on Γ is denoted by $C^l(\Gamma)$.

EXERCISE 1.9. *Let $k \in \mathbb{N}$ and let l be an integer with $0 \leq l \leq k$. Let $\{(\mathcal{O}_\mu, \mathcal{U}_\mu, \boldsymbol{\varphi}_\mu)\}_{\mu \in M}$ be a local embedding C^k-coordinate system of Γ. Suppose that a function f defined on Γ satisfies $f \circ \boldsymbol{\varphi}_\mu \in C^l(\mathcal{U}_\mu)$ for all $\mu \in M$. Then, prove that $f \in C^l(\Gamma)$.*

Let $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi}) \in \mathsf{C}_\mathsf{k}(\Gamma)$. Then, for each $\boldsymbol{\xi} \in \mathcal{U}$, $\frac{\partial \boldsymbol{\varphi}}{\partial \xi_1}(\boldsymbol{\xi}), \cdots, \frac{\partial \boldsymbol{\varphi}}{\partial \xi_{m-1}}(\boldsymbol{\xi})$ are $m-1$ independent $\mathbb{R}^m$ vectors, which are tangent to Γ at $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}) \in \Gamma$. We denote by $\mathrm{T}_{\boldsymbol{x}}(\Gamma)$ the tangent space at $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}) \in \Gamma$:

$$\mathrm{T}_{\boldsymbol{x}}(\Gamma) := \mathrm{span}\left\langle \frac{\partial \boldsymbol{\varphi}}{\partial \xi_1}(\boldsymbol{\xi}),\ \cdots,\ \frac{\partial \boldsymbol{\varphi}}{\partial \xi_{m-1}}(\boldsymbol{\xi}) \right\rangle.$$

This is an $m-1$ dimensional subspace of $\mathbb{R}^m$.

EXERCISE 1.10. *Prove that $\mathrm{T}_{\boldsymbol{x}}(\Gamma)$ does not depend on the choice of local C^k-embedding coordinate of Γ which covers $\boldsymbol{x}$.*

We define the following functions of $\boldsymbol{\xi} \in \mathcal{U}$:

$$G(\boldsymbol{\xi}) := \left(\nabla_{\boldsymbol{\xi}} \boldsymbol{\varphi}^{\mathrm{T}}(\boldsymbol{\xi})\right) \left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi})\right) \in \mathbb{R}_{\mathrm{sym}}^{(m-1)\times(m-1)},$$

$$\text{i.e.,} \quad G(\boldsymbol{\xi}) = (g_{ij}(\boldsymbol{\xi})), \quad g_{ij} = \frac{\partial \boldsymbol{\varphi}}{\partial \xi_i} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial \xi_j}\ ,$$

$$g(\boldsymbol{\xi}) := \det G(\boldsymbol{\xi}).$$

$$\check{\boldsymbol{\varphi}}^i(\boldsymbol{\xi}) := (\varphi_1(\boldsymbol{\xi}), \cdots, \varphi_{i-1}(\boldsymbol{\xi}), \varphi_{i+1}(\boldsymbol{\xi}), \cdots, \varphi_m(\boldsymbol{\xi}))^{\mathrm{T}} \in \mathbb{R}^{m-1},$$

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) := (\alpha_1(\boldsymbol{\xi}), \cdots, \alpha_m(\boldsymbol{\xi}))^{\mathrm{T}} \in \mathbb{R}^m, \quad \alpha_i(\boldsymbol{\xi}) := (-1)^{m+i} \det\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \check{\boldsymbol{\varphi}}^i(\boldsymbol{\xi})\right).$$

THEOREM 1.11. *Under the above conditions, for* $\boldsymbol{\xi} \in \mathcal{U}$, *we have* $\boldsymbol{\alpha}(\boldsymbol{\xi}) \in \mathrm{T}_{\boldsymbol{\varphi}(\boldsymbol{\xi})}(\Gamma)^{\perp}$ *and*

$$\det\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi}),\ \boldsymbol{\alpha}(\boldsymbol{\xi})\right) = g(\boldsymbol{\xi}) = |\boldsymbol{\alpha}(\boldsymbol{\xi})|^2 > 0.$$

PROOF. Since $\operatorname{rank}(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi})) = m-1$, at least one of α_i is not zero, i.e., $\boldsymbol{\alpha}(\boldsymbol{\xi}) \neq \mathbf{0}$. From the equality

$$\begin{aligned} \boldsymbol{\alpha}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial\xi_j} &= \sum_{i=1}^{m}\alpha_i\frac{\partial\varphi_i}{\partial\xi_j} = \sum_{i=1}^{m}(-1)^{m+i}\det\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\check{\boldsymbol{\varphi}}^i\right)\frac{\partial\varphi_i}{\partial\xi_j} \\ &= \det\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi}),\ \frac{\partial\boldsymbol{\varphi}}{\partial\xi_j}\right) = 0, \end{aligned}$$

for each $j = 1, \cdots, m-1$, we obtain $\boldsymbol{\alpha}(\boldsymbol{\xi}) \in \mathrm{T}_{\boldsymbol{\varphi}(\boldsymbol{\xi})}(\Gamma)^{\perp}$.

We define $J(\boldsymbol{\xi}) := \left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi}),\ \boldsymbol{\alpha}(\boldsymbol{\xi})\right)$. Then we have

$$\det J = \sum_{i=1}^{m}(-1)^{m+i}\det\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\check{\boldsymbol{\varphi}}^i\right)\alpha_i = \sum_{i=1}^{m}|\alpha_i|^2 = |\boldsymbol{\alpha}|^2,$$

and

$$\begin{aligned} (\det J)^2 &= \det(J^{\mathrm{T}}J) = \det\left(\begin{pmatrix} \nabla_{\boldsymbol{\xi}}\boldsymbol{\varphi}^{\mathrm{T}} \\ \boldsymbol{\alpha}^{\mathrm{T}} \end{pmatrix}\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi},\ \boldsymbol{\alpha}\right)\right) \\ &= \det\begin{pmatrix} G & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & |\boldsymbol{\alpha}|^2 \end{pmatrix} = (\det G)|\boldsymbol{\alpha}|^2. \end{aligned}$$

Hence, we obtain the assertions. □

COROLLARY 1.12. *We can define a unit normal vector field* $\boldsymbol{\nu} \in C^{k-1}(\Gamma\cap\mathcal{O}, \mathbb{R}^m)$ *by*

$$\boldsymbol{\nu}(\boldsymbol{x}) := \frac{\boldsymbol{\alpha}(\boldsymbol{\xi})}{|\boldsymbol{\alpha}(\boldsymbol{\xi})|} \qquad (\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}) \in \Gamma\cap\mathcal{O}),$$

and we have

$$\sqrt{g(\boldsymbol{\xi})} = \det\left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi}),\ \boldsymbol{\nu}(\boldsymbol{\varphi}(\boldsymbol{\xi}))\right).$$

We remark that the tangent space is expressed as

$$\mathrm{T}_{\boldsymbol{x}}(\Gamma) = \{\boldsymbol{y}\in\mathbb{R}^m;\ \boldsymbol{y}\cdot\boldsymbol{\nu}(\boldsymbol{x}) = 0\}.$$

DEFINITION 1.13. A C^k-class hypersurface Γ in $\mathbb{R}^m$ is called *oriented*, if there exists a (single-valued) unit normal vector field on Γ, i.e., $\boldsymbol{\nu} \in C^{k-1}(\Gamma, \mathbb{R}^m)$ with $\boldsymbol{\nu}(\boldsymbol{x}) \in \mathrm{T}_{\boldsymbol{x}}(\Gamma)^{\perp}$ and $|\boldsymbol{\nu}(\boldsymbol{x})| = 1$ $(\boldsymbol{x}\in\Gamma)$.

Under the conditions of Exercise 1.7, the unit normal vector field is expressed as

$$\boldsymbol{\nu}(\boldsymbol{x}) = \frac{\nabla w(\boldsymbol{x})}{|\nabla w(\boldsymbol{x})|} \quad \text{or} \quad -\frac{\nabla w(\boldsymbol{x})}{|\nabla w(\boldsymbol{x})|} \qquad (\boldsymbol{x}\in\Gamma),$$

and Γ becomes an oriented hypersurface (see Lemma 2.1).

We define the surface integral on Γ as follows. Let $k\in\mathbb{N}$ and let Γ be a C^k-class hypersurface in $\mathbb{R}^m$. For a function f defined on Γ, we consider the surface integral:

$$\int_\Gamma f\,d\mathcal{H}^{m-1} = \int_\Gamma f(\boldsymbol{x})\,d\mathcal{H}_{\boldsymbol{x}}^{m-1},$$

where $\mathcal{H}^{m-1}_{\boldsymbol{x}}$ is the $m-1$ dimensional Hausdorff measure with respect to $\boldsymbol{x}$ (see [5], [20] etc.), provided f is $\mathcal{H}^{m-1}$-integrable. In particular, if $\mathrm{supp}(f) \subset (\Gamma \cap \mathcal{O})$ and $f \circ \boldsymbol{\varphi} \in L^1(\mathcal{U})$ for $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi}) \in \mathsf{C}_{\mathsf{k}}(\Gamma)$, we have

$$\int_\Gamma f(\boldsymbol{x})\, d\mathcal{H}^{m-1}_{\boldsymbol{x}} = \int_{\mathcal{U}} f(\boldsymbol{\varphi}(\boldsymbol{\xi}))\sqrt{g(\boldsymbol{\xi})}\, d\boldsymbol{\xi}.$$

4. Graph representation and principal curvatures

Let Γ be a C^k-class hypersurface in $\mathbb{R}^m$ ($k \in \mathbb{N}$). We fix an arbitrary point on Γ and assume that it is the origin $\mathbf{0} \in \Gamma \subset \mathbb{R}^m$ without loss of generality. We choose a suitable orthogonal coordinate $(\xi_1, \xi_2, \cdots, \xi_{m-1}, \eta)^{\mathrm{T}}$ of $\mathbb{R}^m$ such that the tangent space and the unit normal vector at $\mathbf{0} \in \Gamma$ are given by

$$\mathrm{T}_{\mathbf{0}}(\Gamma) = \left\{(\boldsymbol{\xi}, 0)^{\mathrm{T}} \in \mathbb{R}^m;\ \boldsymbol{\xi} = (\xi_1, \cdots, \xi_{m-1})^{\mathrm{T}} \in \mathbb{R}^{m-1}\right\},$$

$$\boldsymbol{\nu}(\mathbf{0}) = (0, \cdots, 0, 1)^{\mathrm{T}} \in \mathbb{R}^m.$$

In a neighborhood of the origin, from the implicit function theorem, Γ is locally expressed by a graph $\eta = u(\boldsymbol{\xi})$ ($\boldsymbol{\xi} \in \mathcal{U}$), where $\mathcal{U}$ is a bounded domain of $\mathbb{R}^{m-1}$ with $\mathbf{0} \in \mathcal{U}$, and

$$u \in C^k(\mathcal{U}), \quad \nabla_{\boldsymbol{\xi}} u(\mathbf{0}) = \mathbf{0} \in \mathbb{R}^{m-1}.$$

The unit normal vector of Γ in the neighborhood of the origin is given by

$$\boldsymbol{\nu}(\boldsymbol{x}) = \frac{1}{\sqrt{1 + |\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi})|^2}} \begin{pmatrix} -\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) \\ 1 \end{pmatrix} \qquad \left(\boldsymbol{x} = \begin{pmatrix} \boldsymbol{\xi} \\ u(\boldsymbol{\xi}) \end{pmatrix} \in \Gamma\right). \tag{1.3}$$

Proposition 1.14. *Let Γ be a bounded C^k-class hypersurface ($k \geq 1$). Suppose that $\overline{\Gamma}$ is contained in a C^k-class hypersurface $\tilde{\Gamma}$. Then, for $f \in C^l(\Gamma)$ ($0 \leq l \leq k$), there exists an open set $\mathcal{O} \subset \mathbb{R}^m$ with $\Gamma \subset \mathcal{O}$ and $\tilde{f} \in C^l(\mathcal{O})$ such that $\tilde{f}|_\Gamma = f$.*

Proof. First, we assume that

$$\mathrm{supp}(f) \subset \Gamma' := \{(\boldsymbol{\xi}, u(\boldsymbol{\xi}))^{\mathrm{T}} \in \mathbb{R}^m;\ \boldsymbol{\xi} \in \mathcal{U}\} \tag{1.4}$$

in the above local graph representation. Then we can construct $\tilde{f}$ by $\tilde{f}(\boldsymbol{\xi}, \eta) := f(\boldsymbol{\xi}, u(\boldsymbol{\xi}))$ in a neighborhood of Γ'. For general f, applying a partition of unity, we decompose $f = \sum_{j=1}^N f_j$, where each f_j satisfies the condition (1.4), and we define $\tilde{f} := \sum_{j=1}^N \tilde{f}_j$. □

We assume that $k \geq 2$ hereafter, and denote by $\nabla^2_{\boldsymbol{\xi}} u$ the Hessian matrix of $u(\boldsymbol{\xi})$

$$\nabla^2_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(\boldsymbol{\xi})\right)_{i,j=1,\cdots,m-1} \in \mathbb{R}^{(m-1)\times(m-1)}_{\mathrm{sym}}.$$

Definition 1.15 (principal curvatures and directions). Let the eigenvalues and the eigenvectors of the symmetric matrix $\nabla^2_{\boldsymbol{\xi}} u(\mathbf{0})$ be denoted by $\kappa_i \in \mathbb{R}$ and $\boldsymbol{e}'_i \in \mathbb{R}^{m-1}$ ($i = 1, \cdots, m-1$) with

$$\nabla^2_{\boldsymbol{\xi}} u(\mathbf{0})\boldsymbol{e}'_i = \kappa_i \boldsymbol{e}'_i, \qquad \boldsymbol{e}'_i \cdot \boldsymbol{e}'_j = \delta_{ij}.$$

Then κ_i ($i = 1, \cdots, m-1$) are called *principal curvatures of* Γ *at* $\mathbf{0} \in \Gamma$, and $\boldsymbol{e}_i := (\boldsymbol{e}'_i, 0)^{\mathrm{T}} \in \mathrm{T}_{\mathbf{0}}(\Gamma) \subset \mathbb{R}^m$ is called *a principal direction with respect to* κ_i. The sum of the principal curvatures is denoted by $\kappa := \sum_{i=1}^{m-1} \kappa_i$, and call κ *mean* *curvature* in stead of $\kappa/(m-1)$ in this lecture note. *The Gauss-Kronecker curvature* is also denoted by $\kappa_g := \Pi_{i=1}^{m-1}\kappa_i$.

The meaning of the principal curvatures is clarified in the next proposition.

PROPOSITION 1.16. *Under the conditions of Definition 1.15, for* $\boldsymbol{\tau} = (\boldsymbol{\tau}', 0)^{\mathrm{T}} \in \mathrm{T}_{\mathbf{0}}(\Gamma)$ *with* $\boldsymbol{\tau}' \in \mathbb{R}^{m-1}$ *and* $|\boldsymbol{\tau}| = |\boldsymbol{\tau}'| = 1$, *we define a plane curve in* $\boldsymbol{\tau}$-η *plane with a coordinate* $(\sigma, \eta)^{\mathrm{T}} \in \mathbb{R}^2$ *by*

$$\Gamma_\tau := \left\{(\sigma, u(\sigma\boldsymbol{\tau}'))^{\mathrm{T}} \in \mathbb{R}^2;\ \sigma \in \mathcal{I}\right\},$$

where $\mathcal{I}$ *is an open interval with* $0 \in \mathcal{I}$. *Let the curvature of* Γ_τ *at* $(\sigma, \eta) = (0, 0)$ *be denoted by* κ_τ. *Then, it is given as*

$$\kappa_\tau = (\boldsymbol{\tau}')^{\mathrm{T}} \left(\nabla^2_{\boldsymbol{\xi}} u(\mathbf{0})\right) \boldsymbol{\tau}' = \sum_{i=1}^{m-1} \kappa_i |\boldsymbol{\tau} \cdot \boldsymbol{e}_i|^2.$$

In particular, $\kappa_{e_i} = \kappa_i$ *holds.*

PROOF. We obtain the assertions from the equality:

$$\kappa_\tau = \left.\frac{d^2}{d\sigma^2} u(\sigma\boldsymbol{\tau}')\right|_{\sigma=0} = (\boldsymbol{\tau}')^{\mathrm{T}} \left(\nabla^2_{\boldsymbol{\xi}} u(\mathbf{0})\right) \boldsymbol{\tau}'.$$

□

The following lemma will be used in proving Theorem 2.10.

LEMMA 1.17. *Under the conditions of Definition 1.15, let* $\boldsymbol{y} \in C^1(\mathcal{I}, \Gamma)$ *with* $\boldsymbol{y}(0) = \mathbf{0}$ *and* $\boldsymbol{y}'(0) = \boldsymbol{e}_i$, *where* $\mathcal{I}$ *is an open interval with* $0 \in \mathcal{I}$. *Then, we have*

$$\left.\frac{d}{ds}\boldsymbol{\nu}(\boldsymbol{y}(s))\right|_{s=0} = -\kappa_i \boldsymbol{e}_i.$$

PROOF. We denote by $\nu_m(\boldsymbol{x})$ the m-th component of $\boldsymbol{\nu}(\boldsymbol{x})$. At $\boldsymbol{x} = (\boldsymbol{\xi}, u(\boldsymbol{\xi}))^{\mathrm{T}}$, we have

$$\boldsymbol{\nu}(\boldsymbol{x}) = \nu_m(\boldsymbol{x}) \begin{pmatrix} -\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) \\ 1 \end{pmatrix}.$$

We note that

$$\left.\frac{d}{ds}\nu_m(\boldsymbol{y}(s))\right|_{s=0} = \boldsymbol{\nu}(\mathbf{0}) \cdot \left.\frac{d}{ds}\boldsymbol{\nu}(\boldsymbol{y}(s))\right|_{s=0} = \left.\frac{1}{2}\,\frac{d}{ds}|\boldsymbol{\nu}(\boldsymbol{y}(s))|^2\right|_{s=0} = 0.$$

We define $\boldsymbol{\xi}(s) \in \mathbb{R}^{m-1}$ for small s such that $\boldsymbol{y}(s) = (\boldsymbol{\xi}(s), u(\boldsymbol{\xi}(s)))^{\mathrm{T}}$. Since $\frac{d\boldsymbol{\xi}}{ds}(0) = \boldsymbol{e}_i'$, we obtain

$$\begin{aligned}
\left.\frac{d}{ds}\boldsymbol{\nu}(\boldsymbol{y}(s))\right|_{s=0} &= \left.\frac{d}{ds}\left(\nu_m(\boldsymbol{y}(s)) \begin{pmatrix} -\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}(s)) \\ 1 \end{pmatrix}\right)\right|_{s=0} \\
&= \nu_m(\mathbf{0}) \begin{pmatrix} -\left.\dfrac{d}{ds}\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}(s))\right|_{s=0} \\ 0 \end{pmatrix} \\
&= -\begin{pmatrix} \nabla^2_{\boldsymbol{\xi}} u(\mathbf{0}) \boldsymbol{e}_i' \\ 0 \end{pmatrix} \\
&= -\kappa_i \boldsymbol{e}_i.
\end{aligned}$$

□

CHAPTER 2

Differential calculus on hypersurfaces

In this chapter, we introduce some differential operators on hypersurfaces with their basic formulas and define the Weingarten map, which will play an essential role in our study of principal curvatures on the hypersurfaces.

Throughout this chapter, we suppose that Γ is an oriented C^k-class hypersurface in $\mathbb{R}^m$ with $k \geq 1$ and $m \geq 2$, and that $\mathcal{O}$ is an open set of $\mathbb{R}^m$ with $\Gamma \subset \mathcal{O}$.

1. Differential operators on Γ

We derive several reduction formulas from the differential operators in $\mathbb{R}^m$ to the ones on hypersurfaces in this section.

Lemma 2.1. *If $f \in C^1(\mathcal{O})$ and $f|_\Gamma = 0$, then $\nabla f(\boldsymbol{x}) \in \mathrm{T}_{\boldsymbol{x}}(\Gamma)^\perp$.*

Proof. We assume that Γ is locally parametrized as $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}) \in \Gamma \in \mathbb{R}^m$ by $\boldsymbol{\xi} \in \mathbb{R}^{m-1}$. Since $\mathrm{T}_{\boldsymbol{x}}(\Gamma) = \langle \frac{\partial \boldsymbol{\varphi}}{\partial \xi_1}(\boldsymbol{\xi}), \cdots, \frac{\partial \boldsymbol{\varphi}}{\partial \xi_{m-1}}(\boldsymbol{\xi}) \rangle$ for $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi})$, the assertion follows from

$$0 = \frac{\partial}{\partial \xi_i} f(\boldsymbol{\varphi}(\boldsymbol{\xi})) = (\nabla f(\boldsymbol{x}))^{\mathrm{T}} \ \frac{\partial \boldsymbol{\varphi}}{\partial \xi_i}(\boldsymbol{\xi}) \qquad (i = 1, \cdots, m-1).$$

$\square$

Definition 2.2 (Gradient on Γ). For $f \in C^1(\Gamma)$, we define

$$\nabla_\Gamma f(\boldsymbol{x}) := \Pi_{\boldsymbol{x}} \nabla \tilde{f}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma),$$

where $\Pi_{\boldsymbol{x}} := (I - \boldsymbol{\nu}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})^{\mathrm{T}})$ is the orthogonal projection from $\mathbb{R}^m$ to $\mathrm{T}_{\boldsymbol{x}}(\Gamma)$, and $\tilde{f} \in C^1(\mathcal{O})$ is arbitrary C^1-extension of f to an open neighborhood of Γ with $\tilde{f}|_\Gamma = f$. From Lemma 2.1, $\nabla_\Gamma f$ does not depend on the choice of $\tilde{f}$.

Definition 2.3 (Divergence on Γ). For $\boldsymbol{h} \in C^1(\Gamma, \mathbb{R}^m)$, we define

$$\mathrm{div}_\Gamma \boldsymbol{h} := \mathrm{tr} \nabla_\Gamma \boldsymbol{h}^{\mathrm{T}}.$$

Definition 2.4 (Laplace–Beltrami operator). Let $k \geq 2$. For $f \in C^2(\Gamma)$, we define

$$\Delta_\Gamma f := \mathrm{div}_\Gamma \nabla_\Gamma f.$$

This Δ_Γ is called *the Laplace–Beltrami operator on* Γ.

Differential rules for these differential operators on Γ are collected in the following three propositions.

Proposition 2.5. *We suppose that $f \in C^k(\Gamma)$ and $\boldsymbol{h} \in C^k(\Gamma, \mathbb{R}^m)$.*

(i) $\nabla_\Gamma f \in C^{k-1}(\Gamma, \mathbb{R}^m)$.

(ii) $\mathrm{div}_\Gamma \boldsymbol{h} \in C^{k-1}(\Gamma)$.
(iii) $\Delta_\Gamma f \in C^{k-2}(\Gamma, \mathbb{R}^m)$ *(if $k \geq 2$).*

PROPOSITION 2.6 (Product rules). *Suppose that $f,\ g \in C^1(\Gamma)$ and that $\boldsymbol{h} \in C^1(\Gamma, \mathbb{R}^m)$. Then we have the following formulas on Γ.*

(i) $\nabla_\Gamma(fg) = (\nabla_\Gamma f)g + (\nabla_\Gamma g)f$.
(ii) $\mathrm{div}_\Gamma(f\boldsymbol{h}) = f\,\mathrm{div}_\Gamma \boldsymbol{h} + \nabla_\Gamma f \cdot \boldsymbol{h}$.
(iii) $\Delta_\Gamma(fg) = (\Delta_\Gamma f)\, g + 2(\nabla_\Gamma f) \cdot (\nabla_\Gamma g) + f\,(\Delta_\Gamma g)$ *(if $k \geq 2$ and $f,\ g \in C^2(\Gamma)$).*

PROPOSITION 2.7 (Chain rules). *Suppose that $f \in C^1(\Gamma)$ and $g \in C^1(\mathbb{R})$, and that $\boldsymbol{y} \in C^1(\mathbb{R}, \mathbb{R}^m)$ with $\boldsymbol{y}(s) \in \Gamma$ for $s \in \mathbb{R}$. Then we have the following formulas on Γ.*

(i) $\dfrac{d}{ds}\,(f \circ \boldsymbol{y}(s)) = \nabla_\Gamma^{\mathrm{T}} f(\boldsymbol{y}(s))\ \boldsymbol{y}'(s) \quad (s \in \mathbb{R})$.
(ii) $\nabla_\Gamma(g \circ f)(\boldsymbol{x}) = (\nabla_\Gamma f(\boldsymbol{x}))\,(g' \circ f(\boldsymbol{x}))) \quad (\boldsymbol{x} \in \Gamma)$.

EXERCISE 2.8. *Prove the above three propositions.*

The reduction formula for div_Γ is given as follows.

PROPOSITION 2.9. *For $\boldsymbol{h} \in C^1(\mathcal{O}, \mathbb{R}^m)$, we have*

$$\mathrm{div}_\Gamma \boldsymbol{h} = \mathrm{div}\boldsymbol{h} - \boldsymbol{\nu} \cdot \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\nu}} \qquad \text{on} \quad \Gamma.$$

PROOF. From the definition of div_Γ, we obtain

$$\mathrm{div}_\Gamma \boldsymbol{h} = \mathrm{tr}\left\{(I - \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}})\nabla \boldsymbol{h}^{\mathrm{T}}\right\} = \mathrm{div}\boldsymbol{h} - \mathrm{tr}\left\{\boldsymbol{\nu}\frac{\partial \boldsymbol{h}^{\mathrm{T}}}{\partial \boldsymbol{\nu}}\right\} = \mathrm{div}\boldsymbol{h} - \boldsymbol{\nu} \cdot \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\nu}}\ ,$$

on Γ. □

2. Weingarten map and principal curvatures

We assume that $k \geq 2$ in this section.

THEOREM 2.10. *We define $W \in C^{k-2}(\Gamma, \mathbb{R}^{m \times m})$ by*

$$W(\boldsymbol{x}) := -\nabla_\Gamma^{\mathrm{T}} \boldsymbol{\nu}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Gamma.$$

Then, $W(\boldsymbol{x})$ is symmetric and

$$\begin{cases} W(\boldsymbol{x})\boldsymbol{e}_i = \kappa_i \boldsymbol{e}_i \quad (i = 1, \cdots, m-1) \\[2ex] W(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{0} \end{cases}$$

holds, where κ_i and $\boldsymbol{e}_i$ are the principal curvatures and the corresponding principal directions at $\boldsymbol{x} \in \Gamma$ with $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$.

PROOF. We fix $\boldsymbol{x} \in \Gamma$. The equality $W(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) = 0$ is clear from the definition of W. For each principal direction $\boldsymbol{e}_i$, there exists $\boldsymbol{y} \in C^1(\mathcal{I}, \Gamma)$ such that $\boldsymbol{y}(0) = \boldsymbol{x}$ and $\boldsymbol{y}'(0) = \boldsymbol{e}_i$, where $\mathcal{I}$ is an open interval with $0 \in \mathcal{I}$. Then, from Lemma 1.17, we have

$$\kappa_i \boldsymbol{e}_i = -\frac{d}{ds}[\boldsymbol{\nu}(\boldsymbol{y}(s))]|_{s=0} = -\left(\nabla_\Gamma^{\mathrm{T}} \boldsymbol{\nu}(\boldsymbol{y}(0))\right) \boldsymbol{y}'(0) = W(\boldsymbol{x})\boldsymbol{e}_i. \tag{2.1}$$

Since the eigenvectors of matrix $W(\boldsymbol{x})$ consists of the orthonormal basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{m-1}, \boldsymbol{\nu}(\boldsymbol{x})\}$, it follows that the matrix $W(\boldsymbol{x})$ is symmetric. □

DEFINITION 2.11 (Weingarten map). We consider the linear mapping corresponding to the matrix $W(\boldsymbol{x})$

$$\begin{array}{cccccl} W(\boldsymbol{x}) & : & \mathbb{R}^m & \longrightarrow & \mathrm{T}_{\boldsymbol{x}}(\Gamma) \subset \mathbb{R}^m \\ & & \Cup & & \Cup \\ & & \boldsymbol{p} & \longmapsto & W(\boldsymbol{x})\boldsymbol{p} = \displaystyle\sum_{i=1}^{m-1} \kappa_i (\boldsymbol{p} \cdot \boldsymbol{e}_i)\boldsymbol{e}_i \end{array}$$

which does not depend on the choice of the orthogonal coordinate system. We call it *the (extended) Weingarten map at* $\boldsymbol{x} \in \Gamma$. If we restrict the map $W(\boldsymbol{x})$ to the tangent space $\mathrm{T}_{\boldsymbol{x}}(\Gamma)$, this is usually called the Weingarten map or the second fundamental tensor.

The following corollary is clear from the fact that $\{\kappa_1(\boldsymbol{x}), \cdots, \kappa_{m-1}(\boldsymbol{x}), 0\}$ are the eigenvalues of $W(\boldsymbol{x})$.

COROLLARY 2.12.

$$\kappa(\boldsymbol{x}) = \mathrm{tr}\ W(\boldsymbol{x}) = -\mathrm{div}_\Gamma \boldsymbol{\nu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma),$$

$$\kappa_g(\boldsymbol{x}) = \det\left(W(\boldsymbol{x}) + \boldsymbol{\nu}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})^{\mathrm{T}}\right) \qquad (\boldsymbol{x} \in \Gamma).$$

The following proposition is useful to calculate the mean curvature.

PROPOSITION 2.13. *If an extension* $\tilde{\boldsymbol{\nu}}$ *of* $\boldsymbol{\nu}$ *satisfies*

$$\tilde{\boldsymbol{\nu}} \in C^1(\mathcal{O}, \mathbb{R}^m), \qquad |\tilde{\boldsymbol{\nu}}(\boldsymbol{x})| = 1 \quad (\boldsymbol{x} \in \mathcal{O}), \qquad \tilde{\boldsymbol{\nu}}|_\Gamma = \boldsymbol{\nu},$$

then $\kappa = -\mathrm{div}\tilde{\boldsymbol{\nu}}$ *holds on* Γ.

PROOF. We fix $\boldsymbol{x} \in \Gamma$ and $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x})$. For $\rho \in \mathbb{R}$, if $|\rho| << 1$, then

$$|\tilde{\boldsymbol{\nu}}(\boldsymbol{x} - \rho\boldsymbol{\nu})|^2 = 1.$$

Differentiating this equality by ρ at $\rho = 0$, we obtain

$$0 = \frac{d}{d\rho}|\tilde{\boldsymbol{\nu}}(\boldsymbol{x} - \rho\boldsymbol{\nu})|^2\bigg|_{\rho=0} = 2\tilde{\boldsymbol{\nu}}(\boldsymbol{x})^{\mathrm{T}}\ \left\{\nabla^{\mathrm{T}}\tilde{\boldsymbol{\nu}}(\boldsymbol{x})(-\boldsymbol{\nu})\right\} = -2\boldsymbol{\nu}^{\mathrm{T}}\ \frac{\partial\tilde{\boldsymbol{\nu}}}{\partial\boldsymbol{\nu}}(\boldsymbol{x}).$$

Hence, from Proposition 2.9, we have

$$\mathrm{div}\tilde{\boldsymbol{\nu}}(\boldsymbol{x}) = \mathrm{div}_\Gamma\tilde{\boldsymbol{\nu}}(\boldsymbol{x}) + \boldsymbol{\nu} \cdot \frac{\partial\tilde{\boldsymbol{\nu}}}{\partial\boldsymbol{\nu}}(\boldsymbol{x}) = \mathrm{div}_\Gamma\boldsymbol{\nu}(\boldsymbol{x}) = -\kappa(\boldsymbol{x}).$$

□

EXAMPLE 2.14. *Under the conditions of Exercise 1.7, if* $k \geq 2$ *and* $\boldsymbol{\nu}$ *is in the direction from the domain* $\{w > c\}$ *to* $\{w < c\}$, *then the mean curvature of the level set* Γ *is given as*

$$\kappa(\boldsymbol{x}) = \mathrm{div}\left(\frac{\nabla w(\boldsymbol{x})}{|\nabla w(\boldsymbol{x})|}\right) \qquad (\boldsymbol{x} \in \Gamma = \{\boldsymbol{x} \in \mathcal{O};\ w(\boldsymbol{x}) = c\}).$$

EXERCISE 2.15. *If Γ is given by a graph $\eta = u(\boldsymbol{\xi})$ with $u \in C^2(\mathbb{R}^{m-1})$ and mth component of $\boldsymbol{\nu}$ is positive, i.e., $\nu_m > 0$, then prove that*

$$\kappa(\boldsymbol{x}) = \operatorname{div}_{\boldsymbol{\xi}} \left(\frac{\nabla_{\boldsymbol{\xi}}\, u(\boldsymbol{\xi})}{\sqrt{1 + |\nabla_{\boldsymbol{\xi}}\, u(\boldsymbol{\xi})|^2}} \right) \qquad \textit{for} \quad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{\xi} \\ u(\boldsymbol{\xi}) \end{pmatrix} \in \Gamma.$$

PROPOSITION 2.16. *For $f \in C^2(\mathcal{O})$, we have*

$$\Delta_\Gamma f = \Delta f + \kappa \frac{\partial f}{\partial \boldsymbol{\nu}} - \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} \qquad \text{on} \quad \Gamma.$$

PROOF. At a point $\boldsymbol{x} \in \Gamma$, we have

$$\begin{aligned}
\Delta f &= \operatorname{div}(\nabla f) = \operatorname{div}_\Gamma(\nabla f) + \boldsymbol{\nu}^{\mathrm{T}}(\nabla^2 f)\boldsymbol{\nu} \\
&= \operatorname{div}_\Gamma \left(\nabla_\Gamma f + \boldsymbol{\nu} \frac{\partial f}{\partial \boldsymbol{\nu}} \right) + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} \\
&= \Delta_\Gamma f + (\operatorname{div}_\Gamma \boldsymbol{\nu}) \frac{\partial f}{\partial \boldsymbol{\nu}} + \boldsymbol{\nu}^{\mathrm{T}} \nabla_\Gamma \left(\frac{\partial f}{\partial \boldsymbol{\nu}} \right) + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2} \\
&= \Delta_\Gamma f - \kappa \frac{\partial f}{\partial \boldsymbol{\nu}} + \frac{\partial^2 f}{\partial \boldsymbol{\nu}^2}.
\end{aligned}$$

□

The next theorem is an extension of the Frenet's formula for plane curves to $\mathbb{R}^m$.

THEOREM 2.17. *Let $\boldsymbol{\gamma}$ be the identity map on Γ, i.e., $\boldsymbol{\gamma}(\boldsymbol{x}) := \boldsymbol{x}$ for $\boldsymbol{x} \in \Gamma$. Then we have the following formulas:*

$$\nabla_\Gamma \boldsymbol{\gamma}^{\mathrm{T}}(\boldsymbol{x}) = I - \boldsymbol{\nu}(\boldsymbol{x})\boldsymbol{\nu}^{\mathrm{T}}(\boldsymbol{x}) = \Pi_{\boldsymbol{x}}(\Gamma) \qquad (\boldsymbol{x} \in \Gamma),$$
$$\Delta_\Gamma \boldsymbol{\gamma}(\boldsymbol{x}) = \kappa(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma).$$

PROOF. We extend $\boldsymbol{\gamma}$ to whole $\mathbb{R}^m$ as $\boldsymbol{\gamma}(\boldsymbol{x}) = (\gamma_1(\boldsymbol{x}), \cdots, \gamma_m(\boldsymbol{x}))^{\mathrm{T}} := \boldsymbol{x}$. Then we have

$$\begin{aligned}
\nabla_\Gamma \boldsymbol{\gamma}^{\mathrm{T}} &= (I - \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}})\nabla \boldsymbol{\gamma}^{\mathrm{T}} \;=\; (I - \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}) \;=\; I - (\boldsymbol{\nu}\nu_1, \cdots, \boldsymbol{\nu}\nu_m), \\
\Delta_\Gamma \boldsymbol{\gamma}^{\mathrm{T}} &= -(\operatorname{div}_\Gamma(\boldsymbol{\nu}\nu_1), \cdots, \operatorname{div}_\Gamma(\boldsymbol{\nu}\nu_m)) \\
&= -(\operatorname{div}_\Gamma \boldsymbol{\nu})\boldsymbol{\nu}^{\mathrm{T}} - \boldsymbol{\nu}^{\mathrm{T}}(\nabla_\Gamma \boldsymbol{\nu}^{\mathrm{T}}) \;=\; -(\operatorname{div}_\Gamma \boldsymbol{\nu})\boldsymbol{\nu}^{\mathrm{T}} \;=\; \kappa \boldsymbol{\nu}^{\mathrm{T}}.
\end{aligned}$$

□

One of the most important tools in our analysis is the following Gauss–Green formula on Γ.

THEOREM 2.18 (Gauss–Green formula on Γ). *Let $\boldsymbol{h} \in C^1(\Gamma, \mathbb{R}^m)$ and $f \in C^1(\Gamma)$. We assume that* $\operatorname{supp}(\boldsymbol{h} f)$ *is compact in Γ.* [1] *Then we have*

$$\int_\Gamma \boldsymbol{h} \cdot \nabla_\Gamma f \, d\mathcal{H}^{m-1} = -\int_\Gamma (\operatorname{div}_\Gamma \boldsymbol{h} + \kappa \boldsymbol{\nu} \cdot \boldsymbol{h}) f \, d\mathcal{H}^{m-1}.$$

[1] If Γ is compact, this condition is always satisfied.

A proof of this theorem will be given in Section 1.

COROLLARY 2.19. *For* $f \in C^1(\Gamma)$ *and* $g \in C^2(\Gamma)$ *with compact* $\operatorname{supp}(fg)$, *we have*

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g \; d\mathcal{H}^{m-1} = -\int_\Gamma f \; \Delta_\Gamma g \; d\mathcal{H}^{m-1}.$$

CHAPTER 3

Signed distance function

We study the signed distance function for a hypersurface in this chapter. In the first section, we show the Lipschitz property of the signed distance function for a general closed set. In the second section, we give various differential formulas for smooth hypersurfaces. The signed distance function has many good properties and it is a useful mathematical tool in mathematical and numerical analysis for free boundary problems (see [12, 14, 15, 16] etc.).

1. Signed distance function in general

We begin from the situation without any regularity assumption on Γ.

We suppose that there are open subsets Ω_+ and Ω_+ of $\mathbb{R}^m$ with $\Omega_+ \cap \Omega_- = \emptyset$, and we define a closed set $\Gamma := \mathbb{R}^m \setminus (\Omega_+ \cup \Omega_-)$. Then we define

$$d(\boldsymbol{x}) = \begin{cases} \operatorname{dist}(\boldsymbol{x}, \Gamma) & (\boldsymbol{x} \in \Omega_+), \\ 0 & (\boldsymbol{x} \in \Gamma), \\ -\operatorname{dist}(\boldsymbol{x}, \Gamma) & (\boldsymbol{x} \in \Omega_-), \end{cases}$$

where

$$\operatorname{dist}(\boldsymbol{x}, \Gamma) := \inf_{\boldsymbol{y} \in \Gamma} |\boldsymbol{x} - \boldsymbol{y}|.$$

This $d(\boldsymbol{x})$ is called the signed distance function for Γ.

Exercise 3.1. *Prove that, for a closed set $\Gamma \subset \mathbb{R}^m$ and $\boldsymbol{x} \in \mathbb{R}^m$, there exists $\bar{\boldsymbol{x}} \in \Gamma$ such that*

$$|\boldsymbol{x} - \bar{\boldsymbol{x}}| = \min_{\boldsymbol{y} \in \Gamma} |\boldsymbol{x} - \boldsymbol{y}| = \operatorname{dist}(\boldsymbol{x}, \Gamma).$$

Theorem 3.2. *The signed distance function is Lipschitz continuous with Lipschitz constant 1, i.e.,*

$$|d(\boldsymbol{x}) - d(\boldsymbol{y})| \leq |\boldsymbol{x} - \boldsymbol{y}| \qquad (\boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^m).$$

Proof. We consider the following two cases depending on the value of $d(\boldsymbol{x})d(\boldsymbol{y})$. If $d(\boldsymbol{x})d(\boldsymbol{y}) \geq 0$, without loss of generality, we can assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ are both in $\Omega_+ \cup \Gamma$. In case that they are both in $\Omega_- \cup \Gamma$, we can prove the assertion in the same way. Let $\bar{\boldsymbol{y}} \in \Gamma$ satisfy $d(\boldsymbol{y}) = |\boldsymbol{y} - \bar{\boldsymbol{y}}|$. Since $d(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \Gamma)$, we have

$$d(\boldsymbol{x}) - d(\boldsymbol{y}) = d(\boldsymbol{x}) - |\boldsymbol{y} - \bar{\boldsymbol{y}}| \leq |\boldsymbol{x} - \bar{\boldsymbol{y}}| - |\boldsymbol{y} - \bar{\boldsymbol{y}}| \leq |\boldsymbol{x} - \boldsymbol{y}|.$$

The other inequality $d(\boldsymbol{y}) - d(\boldsymbol{x}) \leq |\boldsymbol{x} - \boldsymbol{y}|$ is obtained in the same way. Hence we obtain $|d(\boldsymbol{x}) - d(\boldsymbol{y})| \leq |\boldsymbol{x} - \boldsymbol{y}|$.

If $d(\boldsymbol{x})d(\boldsymbol{y}) < 0$, without loss of generality, we can assume that $\boldsymbol{x} \in \Omega_+$ and $\boldsymbol{y} \in \Omega_-$. Since Ω_+ and Ω_- are open sets, there exists $\theta \in (0,1)$ such that $\boldsymbol{z} := (1-\theta)\boldsymbol{x} + \theta\boldsymbol{y} \in \Gamma$. Then we have

$$|d(\boldsymbol{x}) - d(\boldsymbol{y})| = d(\boldsymbol{x}) - d(\boldsymbol{y}) = \mathrm{dist}(\boldsymbol{x}, \Gamma) + \mathrm{dist}(\boldsymbol{y}, \Gamma) \leq |\boldsymbol{x} - \boldsymbol{z}| + |\boldsymbol{z} - \boldsymbol{y}| = |\boldsymbol{x} - \boldsymbol{y}|.$$

□

COROLLARY 3.3. *The signed distance function d is differentiable a.e. in $\mathbb{R}^m$, and its gradient ∇d coincides with the distribution sense gradient of d. Moreover, $d \in W^{1,\infty}(\mathbb{R}^m)$ and it satisfies*

$$|\nabla d(\boldsymbol{x})| \leq 1 \quad \textit{a.e. in } \ \mathbb{R}^m.$$

PROOF. This corollary follows from the Rademacher's Theorem (see [5] etc.). □

Let $\tilde{d}$ be an another signed distance function for a closed set $\tilde{\Gamma} = \mathbb{R}^m \setminus (\tilde{\Omega}_+ \cup \tilde{\Omega}_-)$. We assume that Γ and $\tilde{\Gamma}$ are both compact (bounded and closed). The Hausdorff distance between two compact sets Γ and $\tilde{\Gamma}$ is defined by

$$\mathrm{dist}_{\mathrm{H}}(\Gamma,\ \tilde{\Gamma}) := \max\left(\max_{\boldsymbol{x}\in\Gamma} \mathrm{dist}(\boldsymbol{x}, \tilde{\Gamma}),\ \max_{\tilde{\boldsymbol{x}}\in\tilde{\Gamma}} \mathrm{dist}(\tilde{\boldsymbol{x}}, \Gamma) \right).$$

THEOREM 3.4. *Under the assumptions, we have* $\mathrm{dist}_{\mathrm{H}}(\Gamma,\ \tilde{\Gamma}) \leq \|d - \tilde{d}\|_{L^\infty(\mathbb{R}^m)}$.

PROOF. For $\boldsymbol{x} \in \Gamma$, since $d(\boldsymbol{x}) = 0$, we obtain

$$\mathrm{dist}(\boldsymbol{x}, \tilde{\Gamma}) = |\tilde{d}(\boldsymbol{x})| = |\tilde{d}(\boldsymbol{x}) - d(\boldsymbol{x})| \leq \|d - \tilde{d}\|_{L^\infty(\mathbb{R}^m)},$$

and

$$\max_{\boldsymbol{x}\in\Gamma} \mathrm{dist}(\boldsymbol{x}, \tilde{\Gamma}) \leq \|d - \tilde{d}\|_{L^\infty(\mathbb{R}^m)}.$$

In the same way, we obtain the other inequality. □

2. Signed distance function for hypersurface

In this section, we assume that Γ is an oriented $m-1$ dimensional hypersurface embedded in $\mathbb{R}^m$ $(m \geq 2)$ of C^k-class $(k \in \mathbb{N},\ k \geq 2)$. Then, $\boldsymbol{\nu} \in C^{k-1}(\Gamma, \mathbb{R}^m)$ follows. We define

$$X(\boldsymbol{y}, r) := \boldsymbol{y} - r\boldsymbol{\nu}(\boldsymbol{y}) \in \mathbb{R}^m \qquad (\boldsymbol{y} \in \Gamma,\ r \in \mathbb{R}), \tag{3.1}$$

$$\mathcal{N}^\varepsilon(\Gamma_0) \ := \ \{X(\boldsymbol{y}, r);\ \boldsymbol{y} \in \Gamma_0,\ |r| < \varepsilon\} \qquad (\Gamma_0 \subset \Gamma,\ \varepsilon > 0).$$

$$\mathcal{N}^\varepsilon_\pm(\Gamma_0) \ := \ \{X(\boldsymbol{y}, r);\ \boldsymbol{y} \in \Gamma_0,\ 0 < \pm r < \varepsilon\} \qquad (\Gamma_0 \subset \Gamma,\ \varepsilon > 0).$$

From Theorem 2.10 and 2.17, we have

$$\nabla_\Gamma^{\mathrm{T}} X(\boldsymbol{y}, r) = I - \boldsymbol{\nu}(\boldsymbol{y})\boldsymbol{\nu}^{\mathrm{T}}(\boldsymbol{y}) + rW(\boldsymbol{y}) \qquad (\boldsymbol{y} \in \Gamma,\ r \in \mathbb{R}). \tag{3.2}$$

LEMMA 3.5. *We fix a point $\boldsymbol{x}^* \in \Gamma$ and denote by κ_i $(i = 1, \cdots, m-1)$ the principal curvatures of Γ at $\boldsymbol{x}^*$. We define*

$$\kappa_+ := \max_{1\leq i\leq m-1} \kappa_i, \qquad \kappa_- := \min_{1\leq i\leq m-1} \kappa_i,$$

$$\varepsilon_-^* := \begin{cases} -\dfrac{1}{\kappa_+} & \text{if } \kappa_+ > 0 \\ -\infty & \text{if } \kappa_+ \le 0, \end{cases} \qquad \varepsilon_+^* := \begin{cases} -\dfrac{1}{\kappa_-} & \text{if } \kappa_- < 0 \\ +\infty & \text{if } \kappa_- \ge 0. \end{cases}$$

For any $\varepsilon_-^* <^\forall \varepsilon_- <^\forall \varepsilon_+ < \varepsilon_+^*$, there exists a bounded domain $\mathcal{O}^* \subset \mathbb{R}^m$ with $\boldsymbol{x}^* \in \mathcal{O}^*$ such that X is C^{k-1}-diffeomorphism from $(\Gamma \cap \mathcal{O}^*) \times [\varepsilon_-, \varepsilon_+]$ to its image in $\mathbb{R}^m$.

Proof. For $r \in [\varepsilon_-, \varepsilon_+]$, $1 + r\kappa_i > 0$ $(i = 1, \cdots, m-1)$ holds. It follows that $\operatorname{rank}(I + rW(\boldsymbol{x}^*)) = m - 1$ and $I + rW(\boldsymbol{x}^*)$ is bijective from $\mathrm{T}_{\boldsymbol{x}^*}(\Gamma)$ onto itself.

We consider a local embedding coordinate $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi}) \in \mathsf{C}_\mathsf{k}(\Gamma)$ with $\boldsymbol{x}^* \in \mathcal{O}$, and we define $\boldsymbol{\xi}^* := \boldsymbol{\varphi}^{-1}(\boldsymbol{x}^*) \in \mathcal{U} \subset \mathbb{R}^{m-1}$. Let us define a new coordinate $\tilde{\boldsymbol{\xi}} = (\boldsymbol{\xi}, r)^{\mathrm{T}} = (\xi_1, \cdots, \xi_{m-1}, r)^{\mathrm{T}} \in \mathcal{U} \times \mathbb{R} \subset \mathbb{R}^m$. We define

$$\Phi(\boldsymbol{\xi}, r) := X(\boldsymbol{\varphi}(\boldsymbol{\xi}), r) \qquad ((\boldsymbol{\xi}, r) \in \mathcal{U} \times \mathbb{R}). \tag{3.3}$$

Then its Jacobian matrix is given by

$$\nabla_{\tilde{\boldsymbol{\xi}}}^{\mathrm{T}} \Phi(\tilde{\boldsymbol{\xi}}) = \left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \Phi, \ \frac{\partial \Phi}{\partial r} \right) = \left(\nabla_\Gamma^{\mathrm{T}} X(\boldsymbol{y}, r) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}), \ -\boldsymbol{\nu}(\boldsymbol{y}) \right),$$

where $\boldsymbol{y} = \boldsymbol{\varphi}(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \mathcal{U}$. From (3.2), we obtain

$$\begin{aligned} \nabla_\Gamma^{\mathrm{T}} X(\boldsymbol{y}, r) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}) &= \left(I - \boldsymbol{\nu}(\boldsymbol{y}) \boldsymbol{\nu}^{\mathrm{T}}(\boldsymbol{y}) + rW(\boldsymbol{y}) \right) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}) \\ &= \left(I + rW(\boldsymbol{y}) \right) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}), \end{aligned}$$

and

$$\nabla_{\tilde{\boldsymbol{\xi}}}^{\mathrm{T}} \Phi(\tilde{\boldsymbol{\xi}}) = \left(\left(I + rW(\boldsymbol{y}) \right) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}), \ -\boldsymbol{\nu}(\boldsymbol{y}) \right). \tag{3.4}$$

Since $\{\frac{\partial}{\partial \xi_i} \boldsymbol{\varphi}(\boldsymbol{\xi}^*)\}_{i=1,\cdots,m-1}$ is a basis of $\mathrm{T}_{\boldsymbol{x}^*}(\Gamma)$, if $r \in [\varepsilon_-, \varepsilon_+]$, so is the set of column vectors of $(I + rW(\boldsymbol{x}^*)) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}^*)$, too. Hence, from

$$\nabla_{\tilde{\boldsymbol{\xi}}}^{\mathrm{T}} \Phi(\boldsymbol{\xi}^*, \xi_m) = \left(\left(I + \xi_m W(\boldsymbol{x}^*) \right) \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{\xi}^*), \ -\boldsymbol{\nu}(\boldsymbol{x}^*) \right),$$

it follows that $\det \nabla_{\tilde{\boldsymbol{\xi}}}^{\mathrm{T}} \Phi(\boldsymbol{\xi}^*, \xi_m) \neq 0$.

We choose a bounded domain $\mathcal{U}^* \subset \mathcal{U}$ such that $\boldsymbol{\xi}^* \in \mathcal{U}^*$ and that Φ becomes a C^{k-1}-diffeomorphism from $\mathcal{U}^* \times [\varepsilon_-, \varepsilon_+]$ onto its image. There also exists a bounded domain $\mathcal{O}^* \subset \mathbb{R}^m$ such that $\Gamma \cap \mathcal{O}^* = \boldsymbol{\varphi}(\mathcal{U}^*)$. Then, it satisfies the assertion. $\square$

Proposition 3.6. *We suppose that Γ_0 is a bounded hypersurface and it satisfies $\overline{\Gamma_0} \subset \Gamma$. Then there exists $\varepsilon_0 > 0$ such that X is C^{k-1}-diffeomorphism from $\overline{\Gamma_0} \times [-\varepsilon_0, \varepsilon_0]$ onto $\overline{\mathcal{N}^{\varepsilon_0}(\Gamma_0)}$.*

Exercise 3.7. *Prove Proposition 3.6.*

In the following argument, for simple description, we suppose the following condition:

$$^\exists \varepsilon > 0 \text{ s.t. } X \text{ is a } C^{k-1}\text{-diffeomorphism from } \Gamma \times (-\varepsilon, \varepsilon) \text{ onto } \mathcal{N}^\varepsilon(\Gamma). \tag{3.5}$$

Then, we can define

$$(\boldsymbol{\gamma}(\boldsymbol{x}), d(\boldsymbol{x})) := X^{-1}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)). \tag{3.6}$$

We remark that $\boldsymbol{\gamma}(\boldsymbol{x}) \in \Gamma$ stands for the perpendicular foot on Γ from $x \in \mathcal{N}^\varepsilon(\Gamma)$. The function $d(\boldsymbol{x})$ is none other than the signed distance function for Γ.

For simplicity, we often write the perpendicular foot from $\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)$ as $\bar{\boldsymbol{x}} := \boldsymbol{\gamma}(\boldsymbol{x}) \in \Gamma$. Moreover, for a function f defined on Γ, we define an extension of f by

$$\bar{f}(\boldsymbol{x}) := f(\bar{\boldsymbol{x}}) \qquad (\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)).$$

We remark that

$$\boldsymbol{x} = \bar{\boldsymbol{x}} - d(\boldsymbol{x})\boldsymbol{\nu}(\bar{\boldsymbol{x}}) \qquad (\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)). \tag{3.7}$$

We have the following theorem.

THEOREM 3.8. *Under the above conditions, $\boldsymbol{\gamma} \in C^{k-1}(\mathcal{N}^\varepsilon(\Gamma), \mathbb{R}^m)$ and $d \in C^k(\mathcal{N}^\varepsilon(\Gamma))$ hold. Furthermore they satisfy the following formulas for $\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)$,*

$$\nabla d(\boldsymbol{x}) = -\boldsymbol{\nu}(\bar{\boldsymbol{x}}), \qquad \boldsymbol{\gamma}(\boldsymbol{x}) = \boldsymbol{x} - d(\boldsymbol{x})\nabla d(\boldsymbol{x}),$$

$$\nabla^2 d(\boldsymbol{x}) = \left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}\bar{W}, \qquad \Delta d(\boldsymbol{x}) = \sum_{i=1}^{m-1} \frac{\kappa_i(\bar{\boldsymbol{x}})}{1 + d(\boldsymbol{x})\kappa_i(\bar{\boldsymbol{x}})},$$

$$\nabla^{\mathrm{T}}\boldsymbol{\gamma}(\boldsymbol{x}) = \left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}\Pi_{\bar{\boldsymbol{x}}} = \Pi_{\bar{\boldsymbol{x}}}\left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}, \tag{3.8}$$

where $\bar{\boldsymbol{x}} := \boldsymbol{\gamma}(\boldsymbol{x})$ and $\bar{W} := W(\bar{\boldsymbol{x}})$.

PROOF. From the inverse function theorem, it follows that $\boldsymbol{\gamma}$ and d belong to C^{k-1}-class.

We differentiate the equation:

$$r = d(X(\boldsymbol{y}, r)), \tag{3.9}$$

by $\boldsymbol{y} \in \Gamma$, then we have

$$\mathbf{0}^{\mathrm{T}} = \nabla^{\mathrm{T}} d(X(\boldsymbol{y}, r))\nabla_\Gamma^{\mathrm{T}} X(\boldsymbol{y}, r). \tag{3.10}$$

The principal direction at $\bar{\boldsymbol{x}} \in \Gamma$ corresponding to $\kappa_i(\bar{\boldsymbol{x}})$ is denoted by $\boldsymbol{e}_i$ for $i = 1, \cdots, m-1$. Substituting $\boldsymbol{y} = \bar{\boldsymbol{x}}$ and $r = d(\boldsymbol{x})$ to (3.10) and multiplying it to $\boldsymbol{e}_i$, we obtain

$$\begin{aligned} 0 &= \nabla^{\mathrm{T}} d(\boldsymbol{x})\nabla_\Gamma^{\mathrm{T}} X(\bar{\boldsymbol{x}}, d(\boldsymbol{x}))\boldsymbol{e}_i \\ &= \nabla^{\mathrm{T}} d(\boldsymbol{x})(I - \boldsymbol{\nu}(\bar{\boldsymbol{x}})\boldsymbol{\nu}^{\mathrm{T}}(\bar{\boldsymbol{x}}) + d(\boldsymbol{x})\bar{W})\boldsymbol{e}_i = \nabla^{\mathrm{T}} d(\boldsymbol{x})(1 + d(\boldsymbol{x})\kappa_i(\bar{\boldsymbol{x}}))\boldsymbol{e}_i. \end{aligned}$$

Since $1 + d(\boldsymbol{x})\kappa_i(\bar{\boldsymbol{x}}) > 0$, we have $\nabla d(\boldsymbol{x}) \cdot \boldsymbol{e}_i = 0$ for $i = 1, \cdots, m-1$. On the other hand, differentiating (3.9) by r and substituting $\boldsymbol{y} = \bar{\boldsymbol{x}}$ and $r = d(\boldsymbol{x})$, we obtain

$$1 = \nabla^{\mathrm{T}} d(X(\boldsymbol{y}, r))\frac{\partial X}{\partial r}(\boldsymbol{y}, r)|_{\boldsymbol{y}=\bar{\boldsymbol{x}}, r=d(\boldsymbol{x})} = -\nabla^{\mathrm{T}} d(\boldsymbol{x})\boldsymbol{\nu}(\bar{\boldsymbol{x}}).$$

Thus we obtain $\nabla d(\boldsymbol{x}) = -\boldsymbol{\nu}(\bar{\boldsymbol{x}}) = -\boldsymbol{\nu}(\boldsymbol{\gamma}(\boldsymbol{x}))$. Since $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ both belong to C^{k-1}-class, $\nabla d \in C^{k-1}(\mathcal{N}^\varepsilon(\Gamma), \mathbb{R}^m)$ and hence $d \in C^k(\mathcal{N}^\varepsilon(\Gamma))$ follows. The equation $\boldsymbol{\gamma}(\boldsymbol{x}) = \boldsymbol{x} - d(\boldsymbol{x})\nabla d(\boldsymbol{x})$ also follows from (3.7).

Applying the gradient operator ∇^{T} to (3.7), we have

$$\begin{aligned} I &= \nabla^{\mathrm{T}}\boldsymbol{\gamma}(\boldsymbol{x}) - \boldsymbol{\nu}(\bar{\boldsymbol{x}})\nabla^{\mathrm{T}} d(\boldsymbol{x}) - d(\boldsymbol{x})\nabla_\Gamma^{\mathrm{T}}\boldsymbol{\nu}(\bar{\boldsymbol{x}})\nabla^{\mathrm{T}}\boldsymbol{\gamma}(\boldsymbol{x}) \\ &= (I + d(\boldsymbol{x})\bar{W})\nabla^{\mathrm{T}}\boldsymbol{\gamma}(\boldsymbol{x}) + \boldsymbol{\nu}(\bar{\boldsymbol{x}})\boldsymbol{\nu}^{\mathrm{T}}(\bar{\boldsymbol{x}}). \end{aligned}$$

Since

$$\det(I + d(\boldsymbol{x})\bar{W}) = \prod_{i=1}^{m-1}(1 + d(\boldsymbol{x})\kappa_i(\bar{\boldsymbol{x}})) > 0,$$

we obtain

$$\nabla^{\mathrm{T}}\boldsymbol{\gamma}(\boldsymbol{x}) = \left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}\Pi_{\bar{\boldsymbol{x}}}.$$

We remark that $(I + d(\boldsymbol{x})\bar{W})^{-1}$ and $\Pi_{\bar{\boldsymbol{x}}}$ are commutative because $\Pi_{\bar{\boldsymbol{x}}}\bar{W} = \bar{W} = \bar{W}\Pi_{\bar{\boldsymbol{x}}}$. Hence finally, we arrive at

$$\begin{aligned}
\nabla^2 d(\boldsymbol{x}) &= \nabla^{\mathrm{T}}(\nabla d(\boldsymbol{x})) = -\nabla^{\mathrm{T}}(\boldsymbol{\nu}(\boldsymbol{\gamma}(\boldsymbol{x}))) = -\nabla_\Gamma^{\mathrm{T}}\boldsymbol{\nu}(\bar{\boldsymbol{x}})\nabla^{\mathrm{T}}\boldsymbol{\gamma}(\boldsymbol{x}) \\
&= \bar{W}\left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}\Pi_{\bar{\boldsymbol{x}}} = \left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}\bar{W}\Pi_{\bar{\boldsymbol{x}}} \\
&= \left(I + d(\boldsymbol{x})\bar{W}\right)^{-1}\bar{W}.
\end{aligned}$$

The Laplacian $\Delta d(\boldsymbol{x}) = \mathrm{tr}\nabla^2 d(\boldsymbol{x})$ is easily calculated from this formula. □

THEOREM 3.9. *If $f \in C^{k-1}(\Gamma)$ then $\bar{f} \in C^{k-1}(\mathcal{N}^\varepsilon(\Gamma))$ and*

$$\nabla\bar{f}(\boldsymbol{x}) = (I + d(\boldsymbol{x})\bar{W}(\boldsymbol{x}))^{-1}\nabla_\Gamma f(\bar{\boldsymbol{x}}) \qquad (\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)),$$

in particular,

$$\nabla\bar{f}(\boldsymbol{x}) = \nabla_\Gamma f(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma). \tag{3.11}$$

Moreover, if $k \geq 3$, then the equation

$$\nabla^2\bar{f}(\boldsymbol{x}) = \nabla_\Gamma(\nabla_\Gamma^{\mathrm{T}} f)(\boldsymbol{x}) + \boldsymbol{\nu}(\boldsymbol{x})\,\nabla_\Gamma^{\mathrm{T}} f(\boldsymbol{x})\,W(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma),$$

holds. In particular,

$$\Delta\bar{f}(\boldsymbol{x}) = \Delta_\Gamma f(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma).$$

PROOF. Since $\bar{f} = f \circ \boldsymbol{\gamma}$, $\bar{f} \in C^{k-1}(\mathcal{N}^\varepsilon(\Gamma))$ is clear. From Theorem 3.8, we have

$$\begin{aligned}
\nabla\bar{f}(\boldsymbol{x}) &= \nabla\boldsymbol{\gamma}^{\mathrm{T}}(\boldsymbol{x})\,\nabla_\Gamma f(\bar{\boldsymbol{x}}) \\
&= (I + d(\boldsymbol{x})\bar{W}(\boldsymbol{x}))^{-1}\Pi_{\bar{\boldsymbol{x}}}\nabla_\Gamma f(\bar{\boldsymbol{x}}) \\
&= (I + d(\boldsymbol{x})\bar{W}(\boldsymbol{x}))^{-1}\nabla_\Gamma f(\bar{\boldsymbol{x}}).
\end{aligned}$$

The latter part is shown as follows. To the equation

$$\begin{aligned}
\overline{\nabla_\Gamma^T f}(\boldsymbol{x}) &= \left\{(I + d(\boldsymbol{x})\bar{W}(\boldsymbol{x}))\nabla\bar{f}(\boldsymbol{x})\right\}^{\mathrm{T}} \\
&= \nabla^{\mathrm{T}}\bar{f}(\boldsymbol{x})(I + d(\boldsymbol{x})\bar{W}(\boldsymbol{x})) \qquad (\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)),
\end{aligned}$$

we multiply ∇ at $\boldsymbol{x} \in \Gamma$, then, from (3.11), we obtain

$$\begin{aligned}
&\nabla_\Gamma(\nabla_\Gamma^{\mathrm{T}} f)(\boldsymbol{x}) = \nabla(\overline{\nabla_\Gamma^{\mathrm{T}} f})(\boldsymbol{x}) \\
&= \nabla(\nabla^{\mathrm{T}}\bar{f}(\boldsymbol{x}) + d(\boldsymbol{x})\nabla^{\mathrm{T}}\bar{f}(\boldsymbol{x})\,\bar{W}(\boldsymbol{x})) \\
&= \nabla^2\bar{f}(\boldsymbol{x}) + \nabla d(\boldsymbol{x})\left(\nabla^{\mathrm{T}}\bar{f}(\boldsymbol{x})\,\bar{W}(\boldsymbol{x})\right) + d(\boldsymbol{x})\nabla\left(\nabla^{\mathrm{T}}\bar{f}(\boldsymbol{x})\,\bar{W}(\boldsymbol{x})\right) \\
&= \nabla^2\bar{f}(\boldsymbol{x}) - \boldsymbol{\nu}(\boldsymbol{x})\nabla_\Gamma^{\mathrm{T}} f(\boldsymbol{x})W(\boldsymbol{x}),
\end{aligned}$$

and

$$\begin{aligned}
\Delta\bar{f}(\boldsymbol{x}) &= \mathrm{tr}\nabla^2\bar{f}(\boldsymbol{x}) = \mathrm{tr}\left(\nabla_\Gamma(\nabla_\Gamma^{\mathrm{T}} f)(\boldsymbol{x}) + \boldsymbol{\nu}(\boldsymbol{x})\,(W(\boldsymbol{x})\nabla_\Gamma f(\boldsymbol{x}))^{\mathrm{T}}\right) \\
&= \mathrm{div}_\Gamma\nabla_\Gamma f(\boldsymbol{x}) + \boldsymbol{\nu}(\boldsymbol{x})\cdot(W(\boldsymbol{x})\nabla_\Gamma f(\boldsymbol{x})) = \Delta_\Gamma f(\boldsymbol{x}).
\end{aligned}$$

□

CHAPTER 4

Curvilinear coordinates

In this chapter, we study various differential and integral formulas in a curvilinear coordinate of $\mathcal{N}^\varepsilon(\Gamma)$. A proof of the Gauss–Green's theorem on Γ is also given.

1. Differential and integral formulas in $\mathcal{N}^\varepsilon(\Gamma)$

Let Γ be a compact C^k-class hypersurface in $\mathbb{R}^m$ with $k \geq 2$. We suppose the condition (3.5) in this section.

For $r \in (-\varepsilon, \varepsilon)$, we define

$$\Gamma_r := \{\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma);\ d(\boldsymbol{x}) = r\}.$$

Then, we have

$$\mathcal{N}^\varepsilon(\Gamma) = \bigcup_{|r|<\varepsilon} \Gamma_r\,.$$

By d_r, $\boldsymbol{\nu}_r$, κ_r and W_r, we denote the signed distance function, the unit normal vector, the mean curvature and the Weingarten map, respectively, for Γ_r. Then we have the following proposition.

Proposition 4.1. *Under the above conditions, for $r \in (-\varepsilon, \varepsilon)$, Γ_r is a C^k-class hypersurface and*

$$\begin{aligned} d_r(\boldsymbol{x}) &= d(\boldsymbol{x}) - r && (\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)) \\ \boldsymbol{\nu}_r(\boldsymbol{x}) &= \boldsymbol{\nu}(\bar{\boldsymbol{x}}) && (\boldsymbol{x} \in \Gamma_r) \\ \kappa_r(\boldsymbol{x}) &= \sum_{i=1}^{m-1} \frac{\kappa_i(\bar{\boldsymbol{x}})}{1 + r\kappa_i(\bar{\boldsymbol{x}})} && (\boldsymbol{x} \in \Gamma_r) \\ W_r(\boldsymbol{x}) &= (I + r\bar{W})^{-1}\bar{W} && (\boldsymbol{x} \in \Gamma_r) \end{aligned}$$

Proof. The equality $d_r(\boldsymbol{x}) = d(\boldsymbol{x}) - r$ is clear, and the other assertions follow from Theorem 3.8. $\square$

Theorem 4.2. *For $f \in C^0(\overline{\Gamma_r})$, we have*

$$\int_{\Gamma_r} f(\boldsymbol{x})\, d\mathcal{H}^{m-1}_{\boldsymbol{x}} = \int_\Gamma f(X(\boldsymbol{y}, r)) \det(I + rW(\boldsymbol{y}))\, d\mathcal{H}^{m-1}_{\boldsymbol{y}}. \tag{4.1}$$

Proof. We fix a local coordinate $(\mathcal{O}, \mathcal{U}, \boldsymbol{\varphi}) \in \mathsf{C}_\mathsf{k}(\Gamma)$. First, we assume that

$$\operatorname{supp}(f \circ X(\cdot, r)) \subset (\Gamma \cap \mathcal{O}). \tag{4.2}$$

Then, for Φ defined by (3.3), we have

$$\nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \Phi(\boldsymbol{\xi}, r) = (I + rW(\boldsymbol{y}))\, \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \varphi(\boldsymbol{\xi}).$$

Similar to Section 3, we define

$$\begin{aligned} G_r(\boldsymbol{\xi}) &:= \left(\nabla_{\boldsymbol{\xi}} \Phi^{\mathrm{T}}(\boldsymbol{\xi}, r)\right) \left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \Phi(\boldsymbol{\xi}, r)\right) \\ &= \left(\nabla_{\boldsymbol{\xi}} \varphi^{\mathrm{T}}(\boldsymbol{\xi})\right) (I + rW(\boldsymbol{y}))^2 \left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \varphi(\boldsymbol{\xi})\right), \end{aligned}$$

$$g_r(\boldsymbol{\xi}) := \det G_r(\boldsymbol{\xi}) = \left(\det\,(I + rW(\boldsymbol{y}))\right)^2 \det G(\boldsymbol{\xi}),$$

$$\sqrt{g_r(\boldsymbol{\xi})} = \det\,(I + rW(\boldsymbol{y}))\, \sqrt{g(\boldsymbol{\xi})}.$$

Hence, the surface integral on Γ_r is given by (4.1) under the condition (4.2). For general f, applying a partition of unity, we decompose $f = \sum_{j=1}^{N} f_j$, where each f_j satisfies the condition (4.2). □

THEOREM 4.3. *For $\boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma)$, we define*

$$\boldsymbol{y} := \boldsymbol{\gamma}(\boldsymbol{x}) \in \Gamma, \qquad r := d(\boldsymbol{x}) \in (-\varepsilon, \varepsilon).$$

Then, we have

$$d\boldsymbol{x} = \det(I + rW(\boldsymbol{y})) d\mathcal{H}_{\boldsymbol{y}}^{m-1} dr \qquad \text{in } \mathcal{N}^\varepsilon(\Gamma).$$

$$\text{i.e.} \quad \int_{\mathcal{N}^\varepsilon(\Gamma)} f(\boldsymbol{x}) d\boldsymbol{x} = \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma} f(X(\boldsymbol{y}, r)) \det(I + rW(\boldsymbol{y}))\, d\mathcal{H}_{\boldsymbol{y}}^{m-1} dr,$$

for $f \in C^0(\overline{\mathcal{N}^\varepsilon(\Gamma)})$. In other words, we have

$$\int_{\mathcal{N}^\varepsilon(\Gamma)} f(\boldsymbol{x}) d\boldsymbol{x} = \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma_r} f(\boldsymbol{x})\, d\mathcal{H}_{\boldsymbol{x}}^{m-1} dr\,.$$

PROOF. From (3.4) and Corollary 1.12, the Jacobian becomes

$$\det \nabla_{\boldsymbol{\xi}}^{\mathrm{T}} \Phi(\boldsymbol{\xi}, r) = -\det\,(I + rW(\boldsymbol{y}))\, \sqrt{g(\boldsymbol{\xi})}.$$

This gives the formulas of the theorem. □

From this theorem, we immediately have the following corollary.

COROLLARY 4.4. *For $f \in C^0(\overline{\mathcal{N}^\varepsilon(\Gamma)})$,*

$$\lim_{r \downarrow 0} \frac{1}{2r} \int_{\mathcal{N}^r(\Gamma)} f(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Gamma} f(\boldsymbol{x})\, d\mathcal{H}_{\boldsymbol{x}}^{m-1},$$

holds.

We shall prove the Gauss–Green theorem on Γ by using Corollary 4.4.

PROOF OF THEOREM 2.18. We decompose the vector field $\boldsymbol{h} = \boldsymbol{h}_0 + \boldsymbol{h}_1$ as follows.

$$\boldsymbol{h}_0(\boldsymbol{x}) := \Pi_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}), \quad \boldsymbol{h}_1(\boldsymbol{x}) := \boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}_0(\boldsymbol{x}) \qquad (\boldsymbol{x} \in \Gamma).$$

We have

$$\begin{aligned}
\int_\Gamma \boldsymbol{h}\cdot\nabla_\Gamma f\, d\mathcal{H}^{m-1} &= \int_\Gamma \boldsymbol{h}_0\cdot\nabla\bar{f}\, d\mathcal{H}^{m-1} \\
&= \lim_{r\downarrow 0}\frac{1}{2r}\int_{\mathcal{N}^r(\Gamma)} \overline{\boldsymbol{h}_0}\cdot\nabla\bar{f}\, d\boldsymbol{x} \\
&= -\lim_{r\downarrow 0}\frac{1}{2r}\int_{\mathcal{N}^r(\Gamma)} (\operatorname{div}\overline{\boldsymbol{h}_0})\,\bar{f}\, d\boldsymbol{x} \\
&= -\int_\Gamma (\operatorname{div}\overline{\boldsymbol{h}_0})\,\bar{f}\, d\mathcal{H}^{m-1} \\
&= -\int_\Gamma (\operatorname{div}_\Gamma\boldsymbol{h}_0)\, f\, d\mathcal{H}^{m-1}
\end{aligned}$$

The term $\operatorname{div}_\Gamma\boldsymbol{h}_0$ is computed as follows:

$$\begin{aligned}
\operatorname{div}_\Gamma\boldsymbol{h}_0 &= \operatorname{div}_\Gamma\big((I-\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}})\boldsymbol{h}\big) \\
&= \operatorname{div}_\Gamma\boldsymbol{h} - (\operatorname{div}_\Gamma\boldsymbol{\nu})\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{h} - \boldsymbol{\nu}^{\mathrm{T}}\nabla_\Gamma(\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{h}) \\
&= \operatorname{div}_\Gamma\boldsymbol{h} - \kappa\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{h}.
\end{aligned}$$

Substituting this to the above equation, we obtain the Gauss–Green's theorem on Γ. $\square$

CHAPTER 5

Moving hypersurfaces

We extend our formulation to moving hypersurfaces, i.e., time dependent hypersurfaces. In the first section, we introduce a new geometric quantity the normal velocity and a new derivative the normal time derivative. We give several formulas for the signed distance function in Section 2, and for the normal derivatives of several geometric quantities in Section 3.

1. Normal time derivatives

Let us consider a time dependent hypersurface $\Gamma(t)$ $(t \in \mathcal{I})$, where $\mathcal{I}$ is an interval. We assume that each $\Gamma(t)$ is a nonempty oriented $(m-1)$-dimensional hypersurface in $\mathbb{R}^m$. We define

$$\mathcal{M} = \bigcup_{t \in \mathcal{I}} (\Gamma(t) \times \{t\}) \subset \mathbb{R}^{m+1}, \tag{5.1}$$

and we call $\mathcal{M}$ a *moving hypersurface* in $\mathbb{R}^m$. The geometric quantities of $\Gamma(t)$ such as $\boldsymbol{\nu}$, κ and W etc. are denoted by $\boldsymbol{\nu}(\boldsymbol{x}, t)$, $\kappa(\boldsymbol{x}, t)$ and $W(\boldsymbol{x}, t)$ etc. for $\boldsymbol{x} \in \Gamma(t)$.

In this note, we assume that the moving hypersurface $\mathcal{M}$ has the following regularity.

Definition 5.1 ($C^{2,1}$-class moving hypersurface). An oriented moving hypersurface $\mathcal{M}$ of the form (5.1) is called of *$C^{2,1}$-class*, if and only if $\mathcal{M}$ is a C^1-class m-dimensional hypersurface in $\mathbb{R}^m \times \mathbb{R}$ and $\boldsymbol{\nu} \in C^1(\mathcal{M}, \mathbb{R}^m)$.

Proposition 5.2. *Let $\mathcal{M}$ be an oriented moving hypersurface of the form (5.1). Then, $\mathcal{M}$ belongs to $C^{2,1}$-class if and only if $\mathcal{M}$ is locally represented by a graph $\eta = u(\boldsymbol{\xi}, t)$ $(\boldsymbol{\xi} \in \mathcal{U} \subset \mathbb{R}^{m-1},\ \mathcal{I}_0 \subset \mathcal{I})$ in a suitable Cartesian coordinate $(\boldsymbol{\xi}, \eta)$ of $\mathbb{R}^m$ with*

$$u \in C^1(\mathcal{U} \times \mathcal{I}_0), \qquad \nabla_{\boldsymbol{\xi}} u \in C^1(\mathcal{U} \times \mathcal{I}_0, \mathbb{R}^{m-1}). \tag{5.2}$$

More precisely, for any $(\boldsymbol{x}, t) \in \mathcal{M}$, there exist a suitable Cartesian coordinate $(\boldsymbol{\xi}, \eta)$ of $\mathbb{R}^m$, and a bounded domain $\mathcal{U} \subset \mathbb{R}^{m-1}$, an open subinterval $\mathcal{I}_0 \subset \mathcal{I}$, a function u satisfying (5.2), and a bounded domain $\mathcal{Q} \subset \mathbb{R}^m \times \mathbb{R}$, such that $(\boldsymbol{x}, t) \in \tilde{\mathcal{O}}$ and

$$\{\boldsymbol{x} \in \Gamma(t);\ (\boldsymbol{x}, t) \in \mathcal{Q}\} = \{(\boldsymbol{\xi}, u(\boldsymbol{\xi}, t));\ \boldsymbol{\xi} \in \mathcal{U}\} \qquad (t \in \mathcal{I}_0).$$

Exercise 5.3. *Prove Proposition 5.2.*

In the following arguments, we always assume that $\mathcal{M}$ is of $C^{2,1}$ class. We call $\boldsymbol{y}$ a C^1-trajectory on $\mathcal{M}$ if $\boldsymbol{y} \in C^1(\mathcal{I}_0, \mathbb{R}^m)$ with $\boldsymbol{y}(t) \in \Gamma(t)$ for $t \in \mathcal{I}_0$, where $\mathcal{I}_0$ is a subinterval of $\mathcal{I}$.

Definition 5.4 (Normal velocity). We define *a normal velocity* at $(\boldsymbol{x}, t) \in \mathcal{M}$ by $v(\boldsymbol{x}, t) := \boldsymbol{y}'(t) \cdot \boldsymbol{\nu}(\boldsymbol{x}, t)$, where $\boldsymbol{y}$ is a C^1-trajectory on $\mathcal{M}$ through $(\boldsymbol{x}, t)$.

THEOREM 5.5. *The normal velocity* $v \in C^0(\mathcal{M})$ *is well-defined. Moreover, it satisfies* $v(\cdot, t) \in C^1(\Gamma(t))$ *and* $\nabla_\Gamma v \in C^0(\mathcal{M}, \mathbb{R}^m)$.

PROOF. Let $\boldsymbol{y}(t)$ be a C^1-trajectory on $\mathcal{M}$. Without loss of generality, we suppose that $\boldsymbol{y}(t) \in \mathcal{M} \cap \mathcal{Q}$ for $t \in \mathcal{I}_0$ under the condition of Proposition 5.2. Then, there exists $\boldsymbol{\zeta} \in C^1(\mathcal{I}_0, \mathcal{U})$ such that $\boldsymbol{y}(t) = (\boldsymbol{\zeta}(t), u(\boldsymbol{\zeta}(t), t))^{\mathrm{T}}$ for $t \in \mathcal{I}_0$. From (1.3) and the equality

$$\boldsymbol{y}'(t) = \begin{pmatrix} \boldsymbol{\zeta}'(t) \\ \nabla_{\boldsymbol{\xi}} u(\boldsymbol{\zeta}(t), t) \cdot \boldsymbol{\zeta}'(t) + u_t(\boldsymbol{\zeta}(t), t) \end{pmatrix}, \tag{5.3}$$

we obtain

$$v(\boldsymbol{y}(t), t) = \boldsymbol{\nu}(\boldsymbol{x}, t) \cdot \boldsymbol{y}'(t) = \frac{u_t(\boldsymbol{\zeta}(t), t)}{\sqrt{1 + |\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\zeta}(t), t)|^2}}.$$

For fixed $t \in \mathcal{I}_0$ and $\boldsymbol{x} = (\boldsymbol{\xi}, \eta)^{\mathrm{T}} = \boldsymbol{y}(t) \in \Gamma(t)$, since $\boldsymbol{\zeta}(t) = \boldsymbol{\xi} \in \mathcal{U}$, it follows that

$$v(\boldsymbol{x}, t) = \frac{u_t(\boldsymbol{\xi}, t)}{\sqrt{1 + |\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}, t)|^2}},$$

and the right hand side does not depend on the choice of the C^1-trajectory. Moreover, the assertions on the regularity follow from this expression. □

PROPOSITION 5.6. *Under the condition of Proposition 5.2, we have*

$$v(\boldsymbol{x}, t) = \frac{u_t(\boldsymbol{\xi}, t)}{\sqrt{1 + |\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}, t)|^2}} \qquad (\boldsymbol{x} = (\boldsymbol{\xi}, u(\boldsymbol{\xi}, t))^{\mathrm{T}} \in \Gamma(t)), \tag{5.4}$$

for $\boldsymbol{\xi} \in \mathcal{U}$ *and* $t \in \mathcal{I}_0$.

DEFINITION 5.7. A C^1-trajectory $\boldsymbol{y}(t)$ $(t \in \mathcal{I}_0)$ on $\mathcal{M}$ is called a normal trajectory on $\mathcal{M}$, if and only if $\boldsymbol{y}'(t) \in \mathrm{T}_{\boldsymbol{y}(t)}(\Gamma(t))^{\perp}$ for $t \in \mathcal{I}_0$.

We remark that $\boldsymbol{y}'(t) = v(\boldsymbol{y}(t), t)\boldsymbol{\nu}(\boldsymbol{y}(t), t)$ holds for any normal trajectory $\boldsymbol{y}(t)$.

PROPOSITION 5.8. *For* $(\boldsymbol{x}_0, t_0) \in \mathcal{M}$*, there uniquely exists a normal trajectory on* $\mathcal{M}$ *through* $(\boldsymbol{x}_0, t_0)$.

PROOF. Under the conditions of the proof of Theorem 5.5, $\boldsymbol{y}(t)$ is a normal trajectory on $\mathcal{M}$ through $(\boldsymbol{x}_0, t_0)$ if and only if

$$\begin{cases} \boldsymbol{y}'(t) &= v(\boldsymbol{y}(t), t)\, \boldsymbol{\nu}(\boldsymbol{y}(t), t) \\ \boldsymbol{y}(t_0) &= \boldsymbol{x}_0. \end{cases} \tag{5.5}$$

Substituting (1.3), (5.3) and (5.4), we obtain

$$\begin{cases} \boldsymbol{\zeta}'(t) &= -\dfrac{u_t(\boldsymbol{\zeta}(t), t)}{1 + |\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\zeta}(t), t)|^2} \nabla_{\boldsymbol{\xi}} u(\boldsymbol{\zeta}(t), t), \\ \boldsymbol{\zeta}(t_0) &= \boldsymbol{\xi}_0, \end{cases} \tag{5.6}$$

where $\boldsymbol{x}_0 = (\boldsymbol{\xi}_0, u(\boldsymbol{\xi}_0, t_0))$. The system of ordinary differential equations (5.6) is equivalent to (5.5) and permits a unique solution since the right hand side of the first equation of (5.6) satisfies the local Lipschitz condition with respect to the $\boldsymbol{\zeta}$-variable. □

DEFINITION 5.9 (Normal time derivative). Let $f \in C^1(\mathcal{M}, \mathbb{R}^k)$. For $(\boldsymbol{x}_0, t_0) \in \mathcal{M}$ and the normal trajectory $\boldsymbol{y}(t)$ on $\mathcal{M}$ through $(\boldsymbol{x}_0, t_0)$, we define

$$D_t f(\boldsymbol{x}_0, t_0) := \frac{d}{dt}\left[f(\boldsymbol{y}(t), t)\right]\bigg|_{t=t_0},$$

and $D_t f$ is called *the normal time derivative of f on $\mathcal{M}$.*

We denote the identity map on $\mathcal{M}$ by $\boldsymbol{\gamma}(\boldsymbol{x}, t) := \boldsymbol{x}$ $((\boldsymbol{x}, t) \in \mathcal{M})$. Then we have

$$D_t \boldsymbol{\gamma} = v\boldsymbol{\nu} \qquad \text{on} \quad \mathcal{M}.$$

We also remark that even if a function $f = f(\boldsymbol{x})$ does not depend on the time variable t, the normal time derivative of f may not be zero. In general, for $f(\boldsymbol{x}, t) = f(\boldsymbol{x})$, we have

$$D_t f(\boldsymbol{x}, t) = \frac{d}{dt} f(\boldsymbol{y}(t)) = v(\boldsymbol{x}, t)\,(\nabla^{\mathrm{T}} f(\boldsymbol{x}))\,\boldsymbol{\nu}(\boldsymbol{x}, t) \qquad (\boldsymbol{x} = \boldsymbol{y}(t) \in \Gamma(t)).$$

PROPOSITION 5.10. *Let $\mathcal{Q}$ be an open neighbourhood of $\mathcal{M}$ in $\mathbb{R}^{m+1}$. For $f \in C^1(\mathcal{Q})$,*

$$D_t f(\boldsymbol{x}, t) = f_t(\boldsymbol{x}, t) + v(\boldsymbol{x}, t)\frac{\partial f}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t) \qquad ((\boldsymbol{x}, t) \in \mathcal{M}),$$

holds, where $\frac{\partial f}{\partial \boldsymbol{\nu}}$ denotes the partial derivative with respect to $\boldsymbol{x}$ in the direction $\boldsymbol{\nu}(\boldsymbol{x}, t)$.

PROOF. Let $\boldsymbol{y}(t)$ be a normal trajectory on $\mathcal{M}$. Then, at $\boldsymbol{x} = \boldsymbol{y}(t) \in \Gamma(t)$, we have

$$\begin{aligned} D_t f(\boldsymbol{x}, t) &= \frac{d}{dt} f(\boldsymbol{y}(t), t) = \nabla^{\mathrm{T}} f(\boldsymbol{x}, t)\,\boldsymbol{y}'(t) + f_t(\boldsymbol{x}, t) \\ &= v(\boldsymbol{x}, t)\,\nabla^{\mathrm{T}} f(\boldsymbol{x}, t)\,\boldsymbol{\nu}(\boldsymbol{x}, t) + f_t(\boldsymbol{x}, t). \end{aligned}$$

$\square$

PROPOSITION 5.11. *For $f \in C^1(\mathcal{M})$ and a C^1-trajectory $\boldsymbol{y}(t)$ on $\mathcal{M}$, we obtain*

$$\frac{d}{dt} f(\boldsymbol{y}(t), t) = D_t f(\boldsymbol{x}, t) + \nabla^{\mathrm{T}}_{\Gamma(t)} f(\boldsymbol{x}, t)\,\boldsymbol{y}'(t) \qquad (\boldsymbol{x} = \boldsymbol{y}(t) \in \Gamma(t)).$$

PROOF. Let $\tilde{f} \in C^1(\mathcal{Q})$ be an extension of f. Then, for $\boldsymbol{x} = \boldsymbol{y}(t) \in \Gamma(t)$, we obtain

$$\begin{aligned} &\frac{d}{dt} f(\boldsymbol{y}(t), t) = \frac{d}{dt} \tilde{f}(\boldsymbol{y}(t), t) \\ &= \nabla^{\mathrm{T}} \tilde{f}(\boldsymbol{x}, t)\,\boldsymbol{y}'(t) + \tilde{f}_t(\boldsymbol{x}, t) \\ &= \left(\nabla_{\Gamma(t)} f(\boldsymbol{x}, t) + \boldsymbol{\nu}(\boldsymbol{x}, t)\frac{\partial \tilde{f}}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t)\right)^{\mathrm{T}} \boldsymbol{y}'(t) + \tilde{f}_t(\boldsymbol{x}, t) \\ &= \nabla^{\mathrm{T}}_{\Gamma(t)} f(\boldsymbol{x}, t)\,\boldsymbol{y}'(t) + \left(\frac{\partial \tilde{f}}{\partial \boldsymbol{\nu}}(\boldsymbol{x}, t)\,\boldsymbol{\nu}^{\mathrm{T}}(\boldsymbol{x}, t)\,\boldsymbol{y}'(t) + \tilde{f}_t(\boldsymbol{x}, t)\right) \\ &= \nabla^{\mathrm{T}}_{\Gamma(t)} f(\boldsymbol{x}, t)\,\boldsymbol{y}'(t) + D_t f(\boldsymbol{x}, t). \end{aligned}$$

□

2. Signed distance function for moving hypersurface

Let $\mathcal{M}$ be a $C^{2,1}$-class moving hypersurface of the form (5.1). For simplicity, we assume that there exists $\varepsilon > 0$ and $\Gamma(t)$ satisfies the condition (3.5) for each $t \in \mathcal{I}$. The signed distance function and the perpendicular foot and other quantities for $\Gamma(t)$ are denoted by $d(\boldsymbol{x}, t)$ and $\boldsymbol{\gamma}(\boldsymbol{x}, t)$, etc. We define

$$\mathcal{N}^\varepsilon(\mathcal{M}) := \{(\boldsymbol{x}, t) \in \mathbb{R}^m \times \mathbb{R};\ \boldsymbol{x} \in \mathcal{N}^\varepsilon(\Gamma(t)),\quad t \in \mathcal{I}\}.$$

Lemma 5.12. *Under the above condition, $\boldsymbol{\gamma} \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}), \mathbb{R}^m)$ and $d \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}))$ hold.*

Exercise 5.13. *Using the implicit function theorem, prove Lemma 5.12.*

Theorem 5.14. *Under the above condition, $\nabla d \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}), \mathbb{R}^m)$ and $\boldsymbol{\gamma} \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}), \mathbb{R}^m)$ hold, and, for $(\boldsymbol{x}, t) \in \mathcal{N}^\varepsilon(\mathcal{M})$, we have*

$$\begin{aligned} d_t(\boldsymbol{x}, t) &= v(\bar{\boldsymbol{x}}, t), \\ \nabla d_t(\boldsymbol{x}, t) &= (I + d(\boldsymbol{x}, t) W(\bar{\boldsymbol{x}}, t))^{-1} \nabla_{\Gamma(t)} v(\bar{\boldsymbol{x}}, t), \\ \boldsymbol{\gamma}_t(\boldsymbol{x}, t) &= v(\bar{\boldsymbol{x}}, t)\boldsymbol{\nu}(\bar{\boldsymbol{x}}, t) - d(\boldsymbol{x}, t)(I + d(\boldsymbol{x}, t) W(\bar{\boldsymbol{x}}, t))^{-1} \nabla_{\Gamma(t)} v(\bar{\boldsymbol{x}}, t), \end{aligned}$$

where $\bar{\boldsymbol{x}} = \boldsymbol{\gamma}(\boldsymbol{x}, t) \in \Gamma(t)$. Furthermore, if $W \in C^1(\mathcal{M}, \mathbb{R}^{m\times m})$, then $\nabla^2 d \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}), \mathbb{R}^{m\times m})$ holds and, for $(\boldsymbol{x}, t) \in \mathcal{M}$, we obtain

$$\begin{aligned} \nabla^2 d_t(\boldsymbol{x}, t) &= \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)} v)(\boldsymbol{x}, t) + \boldsymbol{\nu}(\boldsymbol{x}, t)\, \nabla^{\mathrm{T}}_{\Gamma(t)} v(\boldsymbol{x}, t)\, W(\boldsymbol{x}, t), \\ \Delta d_t(\boldsymbol{x}, t) &= \Delta_{\Gamma(t)} v(\boldsymbol{x}, t). \end{aligned}$$

Proof. Since

$$\nabla d(\boldsymbol{x}, t) = -\boldsymbol{\nu}(\boldsymbol{\gamma}(\boldsymbol{x}, t), t) \qquad ((\boldsymbol{x}, t) \in \mathcal{N}^\varepsilon(\mathcal{M})),$$

from Lemma 5.12, we obtain $\nabla d \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}), \mathbb{R}^m)$.

Differentiating the equality

$$\boldsymbol{x} = \boldsymbol{\gamma}(\boldsymbol{x}, t) + d(\boldsymbol{x}, t)\nabla d(\boldsymbol{x}, t) \qquad ((\boldsymbol{x}, t) \in \mathcal{N}^\varepsilon(\mathcal{M})),$$

by t, then we obtain

$$\mathbf{0} = \boldsymbol{\gamma}_t(\boldsymbol{x}, t) + d_t(\boldsymbol{x}, t)\nabla d(\boldsymbol{x}, t) + d(\boldsymbol{x}, t)\nabla d_t(\boldsymbol{x}, t). \tag{5.7}$$

Taking an inner product with $\nabla d(\boldsymbol{x}, t)$, we get

$$\begin{aligned} d_t(\boldsymbol{x}, t) &= d_t(\boldsymbol{x}, t)|\nabla d(\boldsymbol{x}, t)|^2 \\ &= -\nabla d(\boldsymbol{x}, t) \cdot (\boldsymbol{\gamma}_t(\boldsymbol{x}, t) + d(\boldsymbol{x}, t)\nabla d_t(\boldsymbol{x}, t)) \\ &= \boldsymbol{\nu}(\boldsymbol{\gamma}(\boldsymbol{x}, t), t) \cdot \boldsymbol{\gamma}_t(\boldsymbol{x}, t) - d(\boldsymbol{x}, t)\nabla d(\boldsymbol{x}, t) \cdot \nabla d_t(\boldsymbol{x}, t) \\ &= v(\boldsymbol{\gamma}(\boldsymbol{x}, t), t) - d(\boldsymbol{x}, t)\frac{1}{2}\frac{\partial}{\partial t}|\nabla d(\boldsymbol{x}, t)|^2 \\ &= v(\boldsymbol{\gamma}(\boldsymbol{x}, t), t). \end{aligned}$$

Hence, $d_t = \bar{v}$ holds and the expression of ∇d_t follows from Theorem 3.9. The time derivative of $\boldsymbol{\gamma}$ is obtained from (5.7).

In the case $W \in C^1(\mathcal{M}, \mathbb{R}^{m\times m})$, from the expression

$$\nabla^2 d(\boldsymbol{x}, t) = \left(I + d(\boldsymbol{x}, t)\bar{W}\right)^{-1} \bar{W} \qquad ((\boldsymbol{x}, t) \in \mathcal{N}^\varepsilon(\mathcal{M})),$$

where $\bar{W} = W(\boldsymbol{\gamma}(\boldsymbol{x},t),t)$, $\nabla^2 d \in C^1(\mathcal{N}^\varepsilon(\mathcal{M}), \mathbb{R}^{m\times m})$ holds. The last two equalities also follow from $d_t = \bar{v}$ and Theorem 3.9. □

3. Time derivatives of geometric quantities

The normal time derivatives of the geometric quantities of a moving hypersurface $\mathcal{M}$ are given as follows.

THEOREM 5.15. *If $\mathcal{M}$ belongs to $C^{2,1}$-class,*

$$D_t \boldsymbol{\nu} = -\nabla_{\Gamma(t)} v,$$

holds on $\mathcal{M}$. Furthermore, if $W \in C^1(\mathcal{M}, \mathbb{R}^{m\times m})$, then the following equalities hold on $\mathcal{M}$:

$$\begin{aligned} D_t W &= vW^2 + \nabla_{\Gamma(t)}(\nabla_{\Gamma(t)}^{\mathrm{T}} v) + \boldsymbol{\nu}\left(\nabla_{\Gamma(t)}^{\mathrm{T}} v\right) W, \\ D_t \kappa &= v \sum_{i=1}^{m-1} \kappa_i^2 + \Delta_{\Gamma(t)} v, \\ D_t \kappa_g &= v\kappa\kappa_g + \mathrm{adj}(W + \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}) : \nabla_{\Gamma(t)}(\nabla_{\Gamma(t)}^{\mathrm{T}} v). \end{aligned}$$

PROOF. Let $\boldsymbol{y}(t)$ be a normal trajectory on $\mathcal{M}$. Then, at $\boldsymbol{x} = \boldsymbol{y}(t) \in \Gamma(t)$, we have

$$\begin{aligned} D_t \boldsymbol{\nu}(\boldsymbol{x},t) &= \frac{d}{dt}\left(\boldsymbol{\nu}(\boldsymbol{y}(t),t)\right) = -\frac{d}{dt}\left(\nabla d(\boldsymbol{y}(t),t)\right) \\ &= -\nabla^2 d(\boldsymbol{x},t)\boldsymbol{y}'(t) - \nabla d_t(\boldsymbol{x},t) \\ &= -W(\boldsymbol{x},t)\boldsymbol{\nu}(\boldsymbol{x},t)v(\boldsymbol{x},t) - \nabla\bar{v}(\boldsymbol{x},t) \\ &= -\nabla\bar{v}(\boldsymbol{x},t) = -\nabla_{\Gamma(t)} v(\boldsymbol{x},t). \end{aligned}$$

The second equality is similarly calculated as follows.

$$D_t W(\boldsymbol{x},t) = \frac{d}{dt}\left(\nabla^2 d(\boldsymbol{y}(t),t)\right) = v(\boldsymbol{x},t)\frac{\partial}{\partial\boldsymbol{\nu}}\nabla^2 d(\boldsymbol{x},t) + \nabla^2 d_t(\boldsymbol{x},t).$$

From the equality

$$\begin{aligned} \frac{\partial}{\partial\boldsymbol{\nu}}\nabla^2 d(\boldsymbol{x},t) &= \frac{d}{dr}\left[\nabla^2 d(\boldsymbol{x} + r\boldsymbol{\nu}(\boldsymbol{x},t),t)\right]\Big|_{r=0} \\ &= \frac{d}{dr}\left[(I - rW(\boldsymbol{x},t))^{-1} W(\boldsymbol{x},t)\right]\Big|_{r=0} \\ &= W(\boldsymbol{x},t)^2, \end{aligned}$$

and Theorem 5.14, we obtain the expression of $D_t W$. The normal time derivative of the mean curvature is obtained from

$$D_t\kappa = D_t(\mathrm{tr} W) = \mathrm{tr}(D_t W).$$

For the Gauss-Kronecker curvature κ_g, from the Jacobi's formula, we have

$$D_t\kappa_g = D_t \det(W + \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}) = \mathrm{tr}\left(A\, D_t(W + \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}})\right),$$

where $A = \mathrm{adj}(W + \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}})$ denotes the adjugate matrix of $W + \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}$ (see Section 1). From Proposition A.1, it satisfies

$$AW = \kappa_g(I - \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}), \qquad A\boldsymbol{\nu} = \kappa_g\boldsymbol{\nu}, \qquad A\boldsymbol{e}_i = \left(\prod_{j\neq i}\kappa_j\right)\boldsymbol{e}_i \quad (i = 1, \cdots, m-1).$$

Since

$$\begin{aligned}
&D_t(W + \boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}) \\
&\quad = \quad D_tW + (D_t\boldsymbol{\nu})\boldsymbol{\nu}^{\mathrm{T}} + \boldsymbol{\nu}(D_t\boldsymbol{\nu})^{\mathrm{T}} \\
&\quad = \quad vW^2 + \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)}v) + \boldsymbol{\nu}\left(\nabla^{\mathrm{T}}_{\Gamma(t)}v\right)W - (\nabla_{\Gamma(t)}v)\,\boldsymbol{\nu}^{\mathrm{T}} - \boldsymbol{\nu}\,\nabla^{\mathrm{T}}_{\Gamma(t)}v \\
&\quad = \quad vW^2 + \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)}v) - (\nabla_{\Gamma(t)}v)\,\boldsymbol{\nu}^{\mathrm{T}} + \boldsymbol{\nu}\left(\nabla^{\mathrm{T}}_{\Gamma(t)}v\right)(W - I),
\end{aligned}$$

we obtain

$$\begin{aligned}
&D_t\kappa_g \\
&\quad = \quad \mathrm{tr}\left(v\kappa_gW + A\ \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)}v) - A(\nabla_{\Gamma(t)}v)\boldsymbol{\nu}^{\mathrm{T}} + \kappa_g\boldsymbol{\nu}\left(\nabla^{\mathrm{T}}_{\Gamma(t)}v\right)(W - I)\right) \\
&\quad = \quad v\kappa_g\kappa + \mathrm{tr}\left(A\ \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)}v)\right) - \left(A\,\nabla_{\Gamma(t)}v\right)\cdot\boldsymbol{\nu} + \kappa_g\boldsymbol{\nu}\cdot\left((W - I)\nabla_{\Gamma(t)}v\right) \\
&\quad = \quad v\kappa\kappa_g + \mathrm{tr}\left(A\ \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)}v)\right) \\
&\quad = \quad v\kappa\kappa_g + A : \nabla_{\Gamma(t)}(\nabla^{\mathrm{T}}_{\Gamma(t)}v)\,.
\end{aligned}$$

□

CHAPTER 6

Variational formulas

One of the most important applications of our formulation for moving hypersurfaces is to derive various variational formulas with respect to the shape deformation of hypersurfaces. In this chapter, we prove some basic transport identities, i.e., variations of volume or surface integrals in the first section. A transport identity of the symmetric polynomials of the principal curvatures is derived in the second section. This is an interesting extension of the Gauss-Bonnet theorem for two dimensional surfaces to multidimensional cases.

1. Transport identities

We suppose that $\mathcal{M}$ is an oriented $C^{2,1}$-class moving hypersurface of the form (5.1) and that $\Gamma(t)$ is compact for each $t \in \mathcal{I}$.

THEOREM 6.1. *If $f \in C^1(\mathcal{M})$, then*

$$\frac{d}{dt}\int_{\Gamma(t)} f(\boldsymbol{x},t)\, d\mathcal{H}^{m-1} = \int_{\Gamma(t)} (D_t f - f\kappa v)\, d\mathcal{H}^{m-1}.$$

In particular, we have

$$\frac{d}{dt}\,|\Gamma(t)| = -\int_{\Gamma(t)} \kappa v\, d\mathcal{H}^{m-1}. \tag{6.1}$$

PROOF. Under the condition of Proposition 5.2, we define

$$\boldsymbol{\varphi}(\boldsymbol{\xi},t) := (\boldsymbol{\xi}, u(\boldsymbol{\xi},t))^{\mathrm{T}} \in \mathbb{R}^m \quad (\boldsymbol{\xi} \in \mathcal{U},\ t \in \mathcal{I}_0).$$

We first suppose the condition:

$$\mathrm{supp}(f(\cdot,t)) \subset \boldsymbol{\varphi}(\mathcal{U},t) \qquad (t \in \mathcal{I}_0).$$

Then, from Corollary 1.12, we have

$$\int_{\Gamma(t)} f(\boldsymbol{x},t)\, d\mathcal{H}^{m-1} = \int_{\mathcal{U}} f(\boldsymbol{\varphi}(\boldsymbol{\xi},t),t)\sqrt{g(\boldsymbol{\xi},t)}\,d\boldsymbol{\xi},$$

where

$$A(\boldsymbol{\xi},t) := \left(\nabla_{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\varphi}(\boldsymbol{\xi},t),\ \boldsymbol{\nu}(\boldsymbol{\varphi}(\boldsymbol{\xi},t),t)\right),$$
$$g(\boldsymbol{\xi},t) := \left(\det A(\boldsymbol{\xi},t)\right)^2.$$

Without loss of generality, we assume that

$$\sqrt{g(\boldsymbol{\xi},t)} = \det A(\boldsymbol{\xi},t) > 0.$$

From Proposition A.3, we have

$$\frac{\partial}{\partial t}\sqrt{g(\boldsymbol{\xi},t)} = \frac{\partial}{\partial t}\det A(\boldsymbol{\xi},t) = \mathrm{tr}\left(A(\boldsymbol{\xi},t)^{-1}A_t(\boldsymbol{\xi},t)\right)\sqrt{g(\boldsymbol{\xi},t)}.$$

The last term is computed as follows. Since $\boldsymbol{x} \in \Gamma(t)$ and $\boldsymbol{\xi} \in \mathcal{U}$ are corresponding to each other by the map $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}, t)$, we can write it conversely as $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t)$, where

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}(\boldsymbol{x}, t), t) \qquad (\boldsymbol{x} \in \Gamma(t)).$$

It follows that, for $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{\xi}, t)$,

$$A(\boldsymbol{\xi}, t)^{-1} = \begin{pmatrix} \nabla^{\mathrm{T}}_{\Gamma(t)} \boldsymbol{\xi}(\boldsymbol{x}, t) \\ \boldsymbol{\nu}^{\mathrm{T}}(\boldsymbol{x}, t) \end{pmatrix},$$

and

$$\begin{aligned}
&\operatorname{tr}\left(A(\boldsymbol{\xi}, t)^{-1} A_t(\boldsymbol{\xi}, t)\right) \\
&= \operatorname{tr}\left\{ \begin{pmatrix} \nabla^{\mathrm{T}}_{\Gamma(t)} \boldsymbol{\xi} \\ \\ \boldsymbol{\nu}^{\mathrm{T}} \end{pmatrix} \left(\nabla^{\mathrm{T}}_{\boldsymbol{\xi}} \boldsymbol{\varphi}_t(\boldsymbol{\xi}, t),\ \frac{\partial}{\partial t}\left[\boldsymbol{\nu}(\boldsymbol{\varphi}(\boldsymbol{\xi}, t), t)\right] \right) \right\} \\
&= \operatorname{tr}\begin{pmatrix} (\nabla^{\mathrm{T}}_{\Gamma(t)} \boldsymbol{\xi})(\nabla^{\mathrm{T}}_{\boldsymbol{\xi}} \boldsymbol{\varphi}_t(\boldsymbol{\xi}, t)) & * \\ * & \frac{1}{2}\frac{\partial}{\partial t}|\boldsymbol{\nu}|^2 \end{pmatrix} \\
&= \operatorname{tr}\left((\nabla^{\mathrm{T}}_{\Gamma(t)} \boldsymbol{\xi})(\nabla^{\mathrm{T}}_{\boldsymbol{\xi}} \boldsymbol{\varphi}_t(\boldsymbol{\xi}, t))\right) = \operatorname{tr}\left((\nabla_{\Gamma(t)} \boldsymbol{\xi}^{\mathrm{T}})(\nabla_{\boldsymbol{\xi}} \boldsymbol{\varphi}^{\mathrm{T}}_t(\boldsymbol{\xi}, t))\right) \\
&= \operatorname{tr} \nabla_{\Gamma}\left(\boldsymbol{\varphi}^{\mathrm{T}}_t(\boldsymbol{\xi}(\boldsymbol{x}, t), t)\right) = \operatorname{div}_{\Gamma(t), \boldsymbol{x}}\left(\boldsymbol{\varphi}_t(\boldsymbol{\xi}(\boldsymbol{x}, t), t)\right).
\end{aligned}$$

Hence, applying Proposition 5.11 and the Gauss–Green formula (Theorem 2.18), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\Gamma(t)} f(\boldsymbol{x}, t)\, d\mathcal{H}^{m-1} \\
&= \frac{d}{dt}\int_{\mathcal{U}} f(\boldsymbol{\varphi}(\boldsymbol{\xi}, t), t)\sqrt{g(\boldsymbol{\xi}, t)}\, d\boldsymbol{\xi} \\
&= \int_{\mathcal{U}} \left\{ \frac{\partial}{\partial t}(f(\boldsymbol{\varphi}(\boldsymbol{\xi}, t), t))\sqrt{g} + f\frac{\partial}{\partial t}\sqrt{g(\boldsymbol{\xi}, t)} \right\} d\boldsymbol{\xi} \\
&= \int_{\mathcal{U}} \left\{ \left(D_t f + (\nabla^{\mathrm{T}}_{\Gamma(t)} f)\boldsymbol{\varphi}_t\right)\sqrt{g} + f \operatorname{div}_{\Gamma(t), \boldsymbol{x}}(\boldsymbol{\varphi}_t(\boldsymbol{\xi}(\boldsymbol{x}, t), t))\sqrt{g} \right\} d\boldsymbol{\xi} \\
&= \int_{\Gamma(t)} \left\{ D_t f + (\nabla^{\mathrm{T}}_{\Gamma(t)} f)\boldsymbol{\varphi}_t + f \operatorname{div}_{\Gamma(t), \boldsymbol{x}}(\boldsymbol{\varphi}_t(\boldsymbol{\xi}(\boldsymbol{x}, t), t)) \right\} d\mathcal{H}^{m-1}_{\boldsymbol{x}} \\
&= \int_{\Gamma(t)} (D_t f - f\kappa v)\, d\mathcal{H}^{m-1}.
\end{aligned}$$

For more general f, we can apply the above result with a partition of unity of $\mathcal{M}$. □

We additionally suppose that there exist a bounded open set $\Omega_-(t)$ and a unbounded open set $\Omega_+(t)$ of $\mathbb{R}^m$ satisfying the following conditions:

$$\mathbb{R}^m \setminus \Gamma(t) = \Omega_-(t) \cup \Omega_+(t), \quad \Omega_-(t) \cap \Omega_+(t) = \emptyset, \quad \mathcal{N}^{\varepsilon}_{\pm}(\Gamma(t)) \subset \Omega_{\pm}(t),$$

for sufficiently small $\varepsilon > 0$. We define

$$Q := \bigcup_{t\in\mathcal{I}} \Omega_-(t) \times \{t\}.$$

Then $\mathcal{M}$ is represented by the domain mapping method as follows.

PROPOSITION 6.2. *Under the above conditions, for a fixed $t_0 \in \mathcal{I}$, there exist a bounded domain $\Omega_0 \subset \mathbb{R}^m$ with a smooth boundary $\Gamma_0 = \partial\Omega_0$ and $\Phi \in C^1(\mathcal{I}_0, C^1(\overline{\Omega_0}, \mathbb{R}^m))$ with an open subinterval $\mathcal{I}_0 \subset \mathcal{I}$ including t_0, such that*

$$\Omega_-(t) = \Phi(\Omega_0, t), \quad \Gamma(t) = \Phi(\Gamma_0, t) \qquad (t \in \mathcal{I}_0),$$

and that $\Phi(\cdot, t)$ is a C^1-diffeomorphism from $\overline{\Omega_0}$ to $\overline{\Omega_-(t)}$ for each $t \in \mathcal{I}_0$.

We remark that $\Phi(\cdot, t)$ represents the map from $\boldsymbol{y} \in \overline{\Omega_0} \subset \mathbb{R}^m$ to $\boldsymbol{x} \in \overline{\Omega_-(t)} \subset \mathbb{R}^m$ as $\boldsymbol{x} = \Phi(\boldsymbol{y}, t)$, and that the condition $\Phi \in C^1(\mathcal{I}, C^1(\overline{\Omega_0}, \mathbb{R}^m))$ stands for that

$$\Phi(\boldsymbol{y}, t), \quad \nabla_{\boldsymbol{y}}\Phi^{\mathrm{T}}(\boldsymbol{y}, t), \quad \frac{\partial\Phi}{\partial t}(\boldsymbol{y}, t), \quad \nabla_{\boldsymbol{y}}\Phi_t^{\mathrm{T}}(\boldsymbol{y}, t) = \frac{\partial}{\partial t}\nabla_{\boldsymbol{y}}\Phi^{\mathrm{T}}(\boldsymbol{y}, t),$$

are all continuous with respect to $(\boldsymbol{y}, t) \in \overline{\Omega_0} \times \mathcal{I}_0$.

EXERCISE 6.3. *Prove Proposition 6.2.*

THEOREM 6.4. *Under the condition of Proposition 6.2, if $u \in C^1(\overline{Q})$, then we have*

$$\frac{d}{dt}\int_{\Omega_-(t)} u(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\Omega_-(t)} u_t(\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\Gamma(t)} u(\boldsymbol{x}, t) v(\boldsymbol{x}, t)\, d\mathcal{H}^{m-1}.$$

In particular, we have

$$\frac{d}{dt}\,|\Omega_-(t)| = -\int_{\Gamma(t)} v\, d\mathcal{H}^{m-1}.$$

PROOF. Without loss of generality, we assume that

$$\det \nabla_{\boldsymbol{y}}\Phi^{\mathrm{T}}(\boldsymbol{y}, t) > 0 \qquad (\boldsymbol{y} \in \overline{\Omega_0}).$$

We define $\Psi(\cdot, t) := \Phi(\cdot, t)^{-1} \in C^1(\overline{\Omega_-(t)}, \overline{\Omega_0})$. From the Jacobi's formula (Proposition A.3), we have

$$\begin{aligned}
\frac{\partial}{\partial t}(\det \nabla_{\boldsymbol{y}}^{\mathrm{T}}\Phi(\boldsymbol{y}, t)) &= (\det \nabla_{\boldsymbol{y}}^{\mathrm{T}}\Phi)\ \mathrm{tr}\left[(\nabla_{\boldsymbol{x}}\Psi^{\mathrm{T}})(\nabla_{\boldsymbol{y}}\Phi_t^{\mathrm{T}})\right] \\
&= (\det \nabla_{\boldsymbol{y}}^{\mathrm{T}}\Phi)\ \mathrm{tr}\,\nabla_{\boldsymbol{x}}\left[\Phi_t^{\mathrm{T}}(\Psi(\boldsymbol{x}, t), t)\right] \\
&= (\det \nabla_{\boldsymbol{y}}^{\mathrm{T}}\Phi(\boldsymbol{y}, t))\,\mathrm{div}_{\boldsymbol{x}}\left[\Phi_t(\Psi(\boldsymbol{x}, t), t)\right].
\end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_-(t)} u(\boldsymbol{x},t)\,d\boldsymbol{x}\\
&\quad= \frac{d}{dt}\int_{\Omega_0} u(\Phi(\boldsymbol{y},t),t)\ \det\nabla_{\boldsymbol{y}}\Phi^{\mathrm{T}}(\boldsymbol{y},t)\,d\boldsymbol{y}\\
&\quad= \int_{\Omega_0} \big\{(\Phi_t^{\mathrm{T}}(\boldsymbol{y},t)\nabla_{\boldsymbol{x}}u(\boldsymbol{x},t)+u_t(\boldsymbol{x},t))(\det\nabla_{\boldsymbol{y}}\Phi^{\mathrm{T}}(\boldsymbol{y},t))\\
&\qquad\quad +u(\boldsymbol{x},t)\,(\det\nabla_{\boldsymbol{y}}^{\mathrm{T}}\Phi(\boldsymbol{y},t))\,\mathrm{div}_{\boldsymbol{x}}\,[\Phi_t(\Psi(\boldsymbol{x},t),t)]\big\}\ d\boldsymbol{y}\\
&\quad= \int_{\Omega_-(t)} \big\{\Phi_t^{\mathrm{T}}(\boldsymbol{y},t)\nabla_{\boldsymbol{x}}u(\boldsymbol{x},t)+u_t(\boldsymbol{x},t)+u(\boldsymbol{x},t)\,\mathrm{div}_{\boldsymbol{x}}\,[\Phi_t(\Psi(\boldsymbol{x},t),t)]\big\}\ d\boldsymbol{x}\\
&\quad= \int_{\Omega_-(t)} u_t(\boldsymbol{x},t)\,d\boldsymbol{x}-\int_{\Gamma(t)} u(\boldsymbol{x},t)\,\Phi_t(\Psi(\boldsymbol{x},t),t)\cdot\boldsymbol{\nu}(\boldsymbol{x},t)\,d\mathcal{H}_{\boldsymbol{x}}^{m-1}.
\end{aligned}
$$

Since $v(\boldsymbol{x},t)=\Phi_t(\Psi(\boldsymbol{x},t),t)\cdot\boldsymbol{\nu}(\boldsymbol{x},t)$, the transport identity follows. □

2. Transport identities for curvatures

We again suppose that $\mathcal{M}$ is an oriented $C^{2,1}$-class moving hypersurface of the form (5.1) and that $\Gamma(t)$ is compact for each $t\in\mathcal{I}$.

THEOREM 6.5 (Transport identity for κ_g).

$$
\frac{d}{dt}\int_{\Gamma(t)} \kappa_g\,d\mathcal{H}^{m-1}=0. \tag{6.2}
$$

This theorem indicates that $\int_\Gamma \kappa_g\,d\mathcal{H}^{m-1}$ is a topological invariant. In $\mathbb{R}^2$, it is well known that

$$
\int_\Gamma \kappa\,d\mathcal{H}^{m-1}=\#(\text{connected components of }\Gamma)\times 2\pi.
$$

In the three dimensional case, Theorem 6.5 follows from the famous Gauss-Bonnet theorem for two dimensional manifolds.

If $\Gamma(t)$ is a solution of the Gaussian curvature flow $v=\kappa_g$ (Section 5), then we have

$$
\frac{d}{dt}|\Omega_-(t)|=-\int_{\Gamma(t)} v d\mathcal{H}^{m-1}=-\int_{\Gamma(t)}\kappa_g d\mathcal{H}^{m-1},
$$

where the last integral does not depend on t. This indicates that the solution of the Gaussian curvature flow possesses the constant volume speed property.

A direct approach to prove Theorem 6.5 is as follows. From Theorem 6.1 and Theorem 5.15, we have

$$
\begin{aligned}
\frac{d}{dt}\int_{\Gamma(t)}\kappa_g\,d\mathcal{H}^{m-1} &= \int_{\Gamma(t)} (D_t\kappa_g - v\kappa\kappa_g)\ d\mathcal{H}^{m-1}\\
&= \int_{\Gamma(t)} \Big(\mathrm{adj}(W+\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}) : \nabla_{\Gamma(t)}(\nabla_{\Gamma(t)}^{\mathrm{T}} v)\Big)\ d\mathcal{H}^{m-1}.
\end{aligned}
$$

Using the integration by parts on $\Gamma(t)$, it is possible to show that the last integral vanishes, but it is lengthy and complicated. So, we choose another way to prove Theorem 6.5 in a more general form.

For $r \in \mathbb{R}$, we define

$$A_r(\boldsymbol{x}, t) := I + rW(\boldsymbol{x}, t) \qquad ((\boldsymbol{x}, t) \in \mathcal{M}).$$

Then we have

$$\det A_r = \prod_{i=1}^{m-1} (1 + r\kappa_i). \tag{6.3}$$

LEMMA 6.6.

$$\frac{d}{dt} |\Gamma_r(t)| = -\int_{\Gamma(t)} \left(\frac{\partial}{\partial r} \det A_r(\boldsymbol{x}, t) \right) v(\boldsymbol{x}, t) \, d\mathcal{H}^{m-1}_{\boldsymbol{x}}.$$

PROOF. For $\boldsymbol{y} \in \Gamma_r(t)$, the perpendicular foot on $\Gamma(t)$ from $\boldsymbol{y}$ is denoted by $\boldsymbol{x} = \boldsymbol{\gamma}(\boldsymbol{y}, t) \in \Gamma(t)$. From Proposition 4.1 and (6.3), we have

$$v_r(\boldsymbol{y}, t) = \frac{\partial}{\partial t} d_r(\boldsymbol{y}, t) = \frac{\partial}{\partial t} (d(\boldsymbol{y}, t) - r) = v(\boldsymbol{x}, t).$$

and

$$\begin{aligned} \kappa_r(\boldsymbol{y}, t) &= \sum_{i=1}^{m-1} \frac{\kappa_i}{1 + r\kappa_i} = \sum_{i=1}^{m-1} \frac{\partial}{\partial r} \log(1 + r\kappa_i) \\ &= \frac{\partial}{\partial r} \log \det A_r = \frac{1}{\det A_r} \frac{\partial}{\partial r} \det A_r \,, \end{aligned}$$

where $\kappa_i = \kappa_i(\boldsymbol{x}, t)$ and $A_r = A_r(\boldsymbol{x}, t)$.

$$\begin{aligned} \frac{d}{dt} |\Gamma_r(t)| = &= -\int_{\Gamma_r(t)} \kappa_r v_r \, d\mathcal{H}^{m-1} \\ &= -\int_{\Gamma(t)} \frac{1}{\det A_r} \left(\frac{\partial}{\partial r} \det A_r \right) v \det A_r \, d\mathcal{H}^{m-1} \\ &= -\int_{\Gamma(t)} \left(\frac{\partial}{\partial r} \det A_r \right) v \, d\mathcal{H}^{m-1}. \end{aligned}$$

□

Let $S_l = S_l(\kappa_1, \cdots, \kappa_{m-1})$ be the lth elementary symmetric polynomial of $\kappa_1, \cdots, \kappa_{m-1}$, i.e.,

$$S_0 = 1, \quad S_1 = \sum_{i=1}^{m-1} \kappa_i = \kappa, \quad S_2 = \sum_{i<j} \kappa_i \kappa_j \,, \ \cdots, \quad S_{m-1} = \prod_{i=1}^{m-1} \kappa_i = \kappa_g \,.$$

Then we have

$$\det A_r = \prod_{i=1}^{m-1} (1 + r\kappa_i) = \sum_{l=0}^{m} S_l r^l,$$

where $S_m := 0$.

THEOREM 6.7.
$$\frac{d}{dt}\int_{\Gamma(t)} S_l \, d\mathcal{H}^{m-1} = -(l+1)\int_{\Gamma(t)} S_{l+1}\, v \, d\mathcal{H}^{m-1} \quad (l = 0, \cdots, m-1).$$

PROOF. Since
$$|\Gamma_r(t)| = \int_{\Gamma(t)} \det A_r \, d\mathcal{H}^{m-1} = \sum_{l=0}^{m-1} r^l \int_{\Gamma(t)} S_l \, d\mathcal{H}^{m-1},$$
we have
$$\frac{d}{dt}\,|\Gamma_r(t)| = \sum_{l=0}^{m-1} r^l \frac{d}{dt}\int_{\Gamma(t)} S_l \, d\mathcal{H}^{m-1}. \tag{6.4}$$

On the other hand, from Lemma 6.6 and the equality
$$\frac{\partial}{\partial r}\det A_r = \sum_{l=0}^{m-1}(l+1)S_{l+1}r^l,$$
we also obtain
$$\frac{d}{dt}\,|\Gamma_r(t)| = \sum_{l=0}^{m-1} r^l \left(-(l+1)\int_{\Gamma(t)} S_{l+1}\, v \, d\mathcal{H}^{m-1}\right). \tag{6.5}$$
Comparing (6.4) and (6.5), we obtain the result. □

Theorem 6.7 includes the two important formulas (6.1) with $l = 0$ and (6.2) with $l = m-1$. This theorem enables us to comprehend them in a unified approach.

CHAPTER 7

Gradient structure and moving boundary problems

Many geometric moving boundary problems have their own gradient flow structures. A gradient structure is defined by an energy functional and an inner product for each hypersurface. We consider several examples of moving boundary problems and give formal gradient structures for them. In this chapter, we suppose $\mathcal{M}$ is a sufficiently smooth moving hypersurface with compact $\Gamma(t)$.

1. General gradient flow of hypersurfaces

We illustrate a general framework of moving boundary problems with a formal gradient structure. Let $\mathcal{G}$ be a set of hypersurfaces in $\mathbb{R}^m$ with "some" regularity conditions and with a constraint $R(\Gamma) = 0$. We suppose that $\Gamma(t) \in \mathcal{G}$ for $t \in \mathcal{I}$. Then formally we can write

$$\frac{d}{dt} R(\Gamma(t)) = \langle R'(\Gamma(t)),\ v(\cdot, t)\rangle\,,$$

where the right hand side is linear with respect to the normal velocity v. This leads to the constraint for the normal velocity:

$$v \in \mathrm{T}_{\Gamma(t)}(\mathcal{G}) := \{v\ ;\ \langle R'(\Gamma(t)),\ v\rangle = 0\}.$$

An typical example of the constraint is the volume preserving condition:

$$|\Omega_-| = a_0, \quad R(\Gamma) := |\Omega_-| - a_0 = 0, \tag{7.1}$$

where $a_0 > 0$ is a fixed constant. In this case, we have

$$\frac{d}{dt} R(\Gamma(t)) = -\int_{\Gamma(t)} v \, d\mathcal{H}^{m-1},$$

$$v \in \mathrm{T}_{\Gamma}(\mathcal{G}) := \left\{ v\ ;\ \int_{\Gamma} v \, d\mathcal{H}^{m-1} = 0 \right\}. \tag{7.2}$$

We consider an energy:

$$E\ :\ \mathcal{G} \ni \Gamma \mapsto E(\Gamma) \in \mathbb{R}.$$

Then the first variation of the energy $\frac{d}{dt}E(\Gamma(t))$ becomes a linear functional of $v \in \mathrm{T}_{\Gamma}(\mathcal{G})$.

To consider a gradient flow of $E(\Gamma)$, we need to identify the first variation of the energy with a linear functional on $\mathrm{T}_{\Gamma}(\mathcal{G})$. We introduce an inner product $\langle f,\, g\rangle_{\Gamma}$

defined on a linear space $\mathrm{T}_\Gamma(\mathcal{G})$, by which we identify the variation of energy with a element of $\mathrm{T}_\Gamma(\mathcal{G})$, namely,

$$\frac{d}{dt}E(\Gamma(t)) = \langle \partial E(\Gamma(t)),\, v\rangle_{\Gamma(t)}\,.$$

The expression of the first variation $\partial E(\Gamma)$ depends on the choice of the inner product. The most standard choice is the L^2-inner product:

$$\langle f,\, g\rangle_\Gamma = \int_\Gamma fg\, d\mathcal{H}^{m-1}\,.$$

The gradient flow of $E(\Gamma)$ with respect to $\langle\cdot,\,\cdot\rangle_\Gamma$ is formally given as

$$v = -\partial E(\Gamma(t)).$$

An important property of the gradient flow is the energy decreasing property:

$$\frac{d}{dt}E(\Gamma(t)) = \langle \partial E(\Gamma(t)),\, v\rangle_{\Gamma(t)} = -\langle v,\, v\rangle_{\Gamma(t)} \le 0.$$

We will show various examples of moving boundary problems with the gradient structure in the following sections.

2. Prescribed normal velocity motion

Condition: $f \in C^\infty(\mathbb{R}^m)$ is given.
Energy: $E(\Gamma) := \displaystyle\int_{\Omega_-} f(\boldsymbol{x})d\mathcal{H}^m$
Variation of energy: $\displaystyle\frac{d}{dt}E(\Gamma(t)) = -\int_{\Gamma(t)} f(\boldsymbol{x})v\ d\mathcal{H}^{m-1}$
Inner product: $\langle f,\, g\rangle_\Gamma := \displaystyle\int_\Gamma fg\ d\mathcal{H}^{m-1}$
Gradient flow: $v = f(\boldsymbol{x})$

We remark that, if $f \equiv 1$, then $E(\Gamma) = |\Omega_-|$ and this is the constant speed motion $v = 1$, which is deeply related to the eikonal equation (see [18] etc.).

3. Mean curvature flow

The mean curvature flow is one of the most important moving boundary problems and is related to many applications. We refer [6] and [18] and reference therein.

Energy: $E(\Gamma) := \displaystyle\int_\Gamma d\mathcal{H}^{m-1}$
Variation of energy: $\displaystyle\frac{d}{dt}E(\Gamma(t)) = -\int_{\Gamma(t)} \kappa v\ d\mathcal{H}^{m-1}$
Inner product: $\langle f,\, g\rangle_\Gamma := \displaystyle\int_\Gamma fg\ d\mathcal{H}^{m-1}$
Gradient flow: $v = \kappa$

4. Anisotropic mean curvature flow

We also consider an anisotropic surface energy

$$E(\Gamma) := \int_{\Gamma} \alpha(\boldsymbol{\nu}(\boldsymbol{x}))\, d\mathcal{H}^{m-1},$$

where $\alpha \in C^2(S^{m-1}, \mathbb{R}_+)$ is a given anisotropic energy density, where $S^{m-1} := \{\boldsymbol{x} \in \mathbb{R}^m;\ |\boldsymbol{x}| = 1\}$. We extend α to the whole space as

$$\alpha(\boldsymbol{x}) := \alpha(\boldsymbol{x}/|\boldsymbol{x}|) \qquad (\boldsymbol{x} \in \mathbb{R}^m,\ \boldsymbol{x} \neq \boldsymbol{0}).$$

The variation of the energy is given as

$$\frac{d}{dt} E(\Gamma(t)) = -\int_{\Gamma(t)} \left\{ \nabla^2 \alpha(\boldsymbol{\nu}) : W + \alpha(\boldsymbol{\nu})\kappa \right\} v\ d\mathcal{H}^{m-1}. \tag{7.3}$$

Then we have the gradient flow as follows.

Inner product: $\langle f, g\rangle_\Gamma := \int_\Gamma fg\ d\mathcal{H}^{m-1}$

Gradient flow: $v = \nabla^2 \alpha(\boldsymbol{\nu}) : W + \alpha(\boldsymbol{\nu})\kappa$

In the case $m = 2$, we can write $\beta(\theta) := \alpha(\boldsymbol{\nu})$ with $(\boldsymbol{\nu} = (\cos\theta, \sin\theta)^{\mathrm{T}})$. Then the gradient flow becomes

$$v = (\beta''(\theta) + \beta(\theta))\, \kappa. \tag{7.4}$$

Exercise 7.1. *Prove (7.3) and (7.4).*

For more details about the anisotropic mean curvature flow, see [6] and the references therein.

5. Gaussian curvature flow

Let $\Gamma(t)$ be convex (i.e. $\Omega_-(t)$ is a convex domain). In the case $m = 3$, the moving boundary problem

$$v = \kappa_g,$$

is called the Gaussian curvature flow and is well studied (see [6], [19] and references therein). In particular, it possesses the constant volume speed property (see Section 2).

We consider a more general form in $\mathbb{R}^m$ using the result of Theorem 6.7.

Energy: $E(\Gamma) := \dfrac{1}{l+1} \int_\Gamma S_l(\kappa_1, \cdots, \kappa_{m-1})\, d\mathcal{H}^{m-1}$

Variation of energy: $\dfrac{d}{dt} E(\Gamma(t)) = -\int_{\Gamma(t)} S_{l+1}(\kappa_1, \cdots, \kappa_{m-1})\, v\ d\mathcal{H}^{m-1}$

Inner product: $\langle f, g\rangle_\Gamma := \int_\Gamma fg\ d\mathcal{H}^{m-1}$

Gradient flow: $v = S_{l+1}(\kappa_1, \cdots, \kappa_{m-1})$

This is an extension of the mean curvature flow ($l = 0$) and the Gaussian curvature flow ($m = 3$ and $l = 2$).

6. Willmore flow

The following gradient flow is called the Willmore flow (see [1], [17] and the references therein).

Energy: $E(\Gamma) := \dfrac{1}{2}\displaystyle\int_\Gamma \kappa^2\, d\mathcal{H}^{m-1}$

Variation of energy: $\dfrac{d}{dt}E(\Gamma(t)) = \displaystyle\int_{\Gamma(t)} \left(\Delta_\Gamma \kappa + \kappa \sum_{i=1}^{m-1} \kappa_i^2 - \frac{1}{2}\kappa^3\right) v\; d\mathcal{H}^{m-1}$

Inner product: $\langle f,\, g\rangle_\Gamma := \displaystyle\int_\Gamma fg\; d\mathcal{H}^{m-1}$

Gradient flow: $v = -\Delta_\Gamma \kappa - \kappa \displaystyle\sum_{i=1}^{m-1} \kappa_i^2 + \frac{1}{2}\kappa^3$

Gradient flow (m=3): $v = -\Delta_\Gamma \kappa - \dfrac{1}{2}\kappa^3 + 2\kappa\kappa_g$

7. Volume preserving mean curvature flow

We suppose the volume preserving condition (7.1) and the constraint on v (7.2) with the same energy and inner product as Section 3. Since we impose the constraint, the inner product is

$$\langle f,\, g\rangle_\Gamma = \int_\Gamma fg\, d\mathcal{H}^{m-1} \quad \text{for} \quad f,\ g \in \mathrm{T}_\Gamma(\mathcal{G}).$$

The variation of the energy is identified with an element of $\mathrm{T}_\Gamma(\mathcal{G})$ as follows. We define

$$\langle\kappa\rangle := \frac{1}{|\Gamma|}\int_\Gamma \kappa\, d\mathcal{H}^{m-1}.$$

Then we have

$$\begin{aligned}\frac{d}{dt}E(\Gamma(t)) &= -\int_{\Gamma(t)} \kappa v\; d\mathcal{H}^{m-1}\\ &= \int_{\Gamma(t)} (\langle\kappa\rangle - \kappa)v\; d\mathcal{H}^{m-1}\\ &= \langle \langle\kappa\rangle - \kappa,\, v(\cdot,t)\rangle_{\Gamma(t)}\,.\end{aligned}$$

Hence, we obtain

$$\partial E(\Gamma) = \langle\kappa\rangle - \kappa.$$

The gradient flow under the volume preserving condition becomes as follows.

Gradient flow: $v = \kappa - \langle\kappa\rangle$

This is called the volume preserving mean curvature flow (area preserving mean curvature flow, if $m = 2$), which keeps the volume enclosed by $\Gamma(t)$ and decreases the surface area. See [11] etc., for details.

8. Surface diffusion flow

We again suppose the volume preserving condition (7.1) and the constraint on v (7.2) with the same energy as Section 3. But we consider another inner product on $\mathrm{T}_\Gamma(\mathcal{G})$.

We consider a Sobolev space on Γ:

$$H^1(\Gamma) := \{f : \Gamma \to \mathbb{R};\ f \in L^2(\Gamma),\ |\nabla_\Gamma f| \in L^2(\Gamma)\},$$

with the norm

$$\|f\|_{H^1(\Gamma)} := \left(\int_\Gamma (|f|^2 + |\nabla_\Gamma f|^2)\, d\mathcal{H}^{m-1} \right)^{\frac{1}{2}}.$$

Its dual space is denoted by $H^{-1}(\Gamma)$. For $f \in L^2(\Gamma)$, it is identified with an element of $H^{-1}(\Gamma)$ by

$${}_{H^{-1}(\Gamma)}\langle f,\, g \rangle_{H^1(\Gamma)} = \int_\Gamma fg\, d\mathcal{H}^{m-1} \qquad ({}^\forall g \in H^1(\Gamma)).$$

We define the following Sobolev spaces with constraint.

$$\begin{aligned} \bar{H}^1(\Gamma) &:= \{f \in H^1(\Gamma);\ \int_\Gamma f\, d\mathcal{H}^{m-1} = 0\}, \\ \bar{H}^{-1}(\Gamma) &:= \{f \in H^{-1}(\Gamma);\ {}_{H^1(\Gamma)}\langle 1,\, f \rangle_{H^{-1}(\Gamma)} = 0\}. \end{aligned}$$

Then it follows that the Laplace–Beltrami operator can be extended to a topological linear isomorphism from $\bar{H}^1(\Gamma)$ onto $\bar{H}^{-1}(\Gamma)$. We denote it by $\bar{\Delta}_\Gamma \in B(\bar{H}^1(\Gamma),\ \bar{H}^{-1}(\Gamma))$. Our inner product is defined as

$$\langle f,\, g\rangle_\Gamma := -\ {}_{H^1(\Gamma)}\left\langle \bar{\Delta}_\Gamma^{-1} f,\, g \right\rangle_{H^{-1}(\Gamma)} \qquad (f,\ g \in \mathrm{T}_\Gamma(\mathcal{G}) \subset \bar{H}^{-1}(\Gamma)).$$

This can be regarded as an inner product of $\bar{H}^{-1}(\Gamma)$.

Then we can identify the variation of the energy in the following sense. Since $\int_\Gamma \Delta_\Gamma f d\mathcal{H}^{m-1} = 0$, we formally have

$$\bar{\Delta}_\Gamma^{-1}(\Delta_\Gamma f) = f - \langle f \rangle \qquad (f \in H^1(\Gamma)),$$

and

$$\begin{aligned} \frac{d}{dt} E(\Gamma(t)) &= -\int_{\Gamma(t)} (\kappa - \langle \kappa \rangle)\, v\, d\mathcal{H}^{m-1} \\ &= -\int_{\Gamma(t)} \bar{\Delta}_\Gamma^{-1}(\Delta_\Gamma \kappa)\, v\, d\mathcal{H}^{m-1} \\ &= \langle \Delta_\Gamma \kappa,\, v(\cdot, t) \rangle_{\Gamma(t)}. \end{aligned}$$

Hence, we obtain

$$\partial E(\Gamma) = \Delta_\Gamma \kappa$$

The gradient flow with respect to the inner product becomes as follows.

Gradient flow: $v = -\Delta_\Gamma \kappa$

This is called the surface diffusion flow, which keeps the volume enclosed by $\Gamma(t)$ and decreases the surface area. See [2] etc., for details.

9. Hele–Shaw moving boundary problem

Hele–Shaw flow is two-dimensional slow flow of viscous fluid between two parallel horizontal plates with a thin gap. It was studied by H. S. Hele–Shaw [9] experimentally at first, and some mathematical models have been studied in several papers [3], [4], [10], [13], etc.

A typical mathematical model for the moving boundary problem of the Hele–Shaw flow with the surface tension effect is as follows.

$$\begin{cases} \Delta p(\boldsymbol{x}, t) = 0 & (\boldsymbol{x} \in \Omega_-(t)), \\ p(\boldsymbol{x}, t) = \kappa & (\boldsymbol{x} \in \Gamma(t)), \\ v(\boldsymbol{x}, t) = -\boldsymbol{\nu}(\boldsymbol{x}, t) \cdot \nabla p(\boldsymbol{x}, t) & (\boldsymbol{x} \in \Gamma(t)), \end{cases}$$

with a given initial boundary $\Gamma(0)$, where $p(\boldsymbol{x}, t)$ is a pressure field of the fluid in $\Omega_-(t)$.

This Hele–Shaw moving boundary problem also has a gradient structure. Let $H^{\frac{1}{2}}(\Gamma)$ be the trace space on Γ from $H^1(\Omega_-)$. Then it is well known that, for $q \in H^{\frac{1}{2}}(\Gamma)$, there exists $u = u(q) \in H^1(\Omega_-)$ with

$$\Delta u = 0 \quad \text{in} \quad \Omega_-, \quad u|_\Gamma = q.$$

We define the Dirichlet-to-Neumann map:

$$\Lambda_\Gamma q := -\frac{\partial u(q)}{\partial \boldsymbol{\nu}}, \qquad \Lambda_\Gamma \in B(H^{\frac{1}{2}}(\Gamma),\ H^{-\frac{1}{2}}(\Gamma)),$$

where $H^{-\frac{1}{2}}(\Gamma) := (H^{\frac{1}{2}}(\Gamma))'$. We define quotient spaces

$$\begin{aligned} \bar{H}^{\frac{1}{2}}(\Gamma) &:= \left\{ f \in H^{\frac{1}{2}}(\Gamma);\ \int_\Gamma f \, d\mathcal{H}^{m-1} = 0 \right\}, \\ \bar{H}^{-\frac{1}{2}}(\Gamma) &:= \left\{ f \in H^{-\frac{1}{2}}(\Gamma);\ {}_{H^{\frac{1}{2}}(\Gamma)}\langle 1,\, f\, \rangle_{H^{-\frac{1}{2}}(\Gamma)} = 0 \right\}. \end{aligned}$$

Then $\bar{\Lambda}_\Gamma := \Lambda_\Gamma|_{\bar{H}^{\frac{1}{2}}(\Gamma)}$ belongs to $B(\bar{H}^{\frac{1}{2}}(\Gamma), \bar{H}^{-\frac{1}{2}}(\Gamma))$ and is bijective. We define $U(f) := u(\Lambda_\Gamma^{-1} f)$. Their inner products are defined by

$$\begin{aligned} (f, g)_{\bar{H}^{\frac{1}{2}}(\Gamma)} &:= \int_{\Omega_-} \nabla u(f) \cdot \nabla u(g) d\boldsymbol{x}, \\ (f, g)_{\bar{H}^{-\frac{1}{2}}(\Gamma)} &:= \int_{\Omega_-} \nabla U(f) \cdot \nabla U(g) d\boldsymbol{x}, \end{aligned}$$

We remark that

$$\begin{aligned} (f, g)_{\bar{H}^{\frac{1}{2}}(\Gamma)} &= \int_\Gamma (\Lambda_\Gamma f)\, g \, d\mathcal{H}^{m-1} \quad (\text{if } \Lambda_\Gamma f \in L^2(\Gamma)), \\ (f, g)_{\bar{H}^{-\frac{1}{2}}(\Gamma)} &= \int_\Gamma (\bar{\Lambda}_\Gamma^{-1} f)\, g \, d\mathcal{H}^{m-1} \quad (\text{if } g \in L^2(\Gamma)). \end{aligned}$$

We again suppose the volume preserving condition (7.1) and the constraint on v (7.2) with the same energy as Section 3. We adopt the inner product of $\bar{H}^{-\frac{1}{2}}(\Gamma)$:

$$\langle f,\, g \rangle_\Gamma := (f, g)_{\bar{H}^{-\frac{1}{2}}(\Gamma)}.$$

Then we can identify the variation of the energy in the following sense. Since $\int_\Gamma (\Lambda_\Gamma f)\, d\mathcal{H}^{m-1} = 0$, we formally have

$$\bar{\Lambda}_\Gamma^{-1}(\Lambda_\Gamma f) = f - \langle f \rangle \qquad (f \in H^{\frac{1}{2}}(\Gamma)),$$

and

$$\begin{aligned} \frac{d}{dt} E(\Gamma(t)) &= -\int_{\Gamma(t)} (\kappa - \langle \kappa \rangle)\, v \, d\mathcal{H}^{m-1} \\ &= -\int_{\Gamma(t)} \bar{\Lambda}_\Gamma^{-1}(\Lambda_\Gamma \kappa)\, v \, d\mathcal{H}^{m-1} \\ &= -\langle \Lambda_\Gamma \kappa,\, v(\cdot, t) \rangle_{\Gamma(t)} \,. \end{aligned}$$

Hence, we obtain

$$\partial E(\Gamma) = -\Lambda_\Gamma \kappa = \frac{\partial u(\kappa)}{\partial \boldsymbol{\nu}}.$$

The gradient flow with respect to the inner product becomes as follows.

Gradient flow: $v = -\dfrac{\partial u(\kappa)}{\partial \boldsymbol{\nu}}$

This is nothing but the Hele–Shaw moving boundary problem. It also keeps the volume enclosed by $\Gamma(t)$ and decreases the surface area. Physically, these properties correspond to the volume conservation law and the surface tension effect. See the above references for details.

Bibliography

[1] K. Deckelnick and G. Dziuk, Error estimates for the Willmore flow of graphs, Interfaces and Free Boundaries Vol.8 (2006), pp.21-46.

[2] C. M. Elliott and H. Garcke, Existence results for diffusive surface motion laws, Adv. Math. Sci. Appl. Vol.7 (1997), pp.467-490.

[3] J. Escher and G. Simonett; Classical solutions of multidimensional Hele-Shaw models, SIAM J. Math. Anal. Vol.28 (1997), pp.1028-1047.

[4] J. Escher and G. Simonett; Classical solutions for Hele-Shaw models with surface tension, Adv. Diff. Eq. Vol.2 (1997), pp.619-642.

[5] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1992).

[6] Y. Giga, Surface Evolution Equations, A Level Set Approach, Monographs in Mathematics Vol.99, Birkhauser Verlag, Basel (2006).

[7] D. Gilberg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Springer-Verlag (1983).

[8] M. E. Gurtin, Thermomechanics of Evolving Phase Boundaries in the Plane, Oxford Mathematical Monographs, The Clarendon Press, Oxford (1993).

[9] H. S. Hele-Shaw, The flow of water, Nature Vol.58 (1898), pp.34-36.

[10] S. D. Howison; Fingering in Hele-Shaw cells, J. Fluid Mech. Vol.167 (1986), pp.439-453.

[11] G. Huisken, The volume preserving mean curvature flow, J. Reine Angew. Math. Vol.382 (1987), pp.35-48.

[12] M. Kimura, Numerical analysis of moving boundary problems using the boundary tracking method, Japan J. Indust. Appl. Math. Vol.14 (1997), pp.373-398.

[13] M. Kimura, Time local existence of a moving boundary of the Hele-Shaw flow with suction, Euro. J. Appl. Math. Vol.10 (2000), pp.581-605.

[14] M. Kimura, Shape derivative and linearization of moving boundary problems, GAKUTO International Series, Mathematical Sciences and Applications Vol.13 (2000), pp.167-179.

[15] M. Kimura, Numerical analysis of moving boundary problems, Sugaku Expositions, Vol.15 (2002), pp.71-88.

[16] M. Kimura and H. Notsu, A level set method using the signed distance function, Japan J. Indust. Appl. Math. Vol.19 (2002), pp.415-446.

[17] T. Oberhuber, Finite difference scheme for the Willmore flow of graphs, Kybernetika Vol.43, No. 6 (2007), pp.855-867.

[18] J. A. Sethian, Level Set Methods and Fast Marching Methods, Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science, second edition, Cambridge Monographs on Applied and

Computational Mathematics, Vol.3, Cambridge University Press, Cambridge (1999).

[19] T. K. Ushijima and H. Yagisita, Approximation of the Gauss curvature flow by a three-dimensional crystalline motion, Proceedings of Czech-Japanese Seminar in Applied Mathematics 2005, COE Lect. Note Vol.3, Kyushu Univ. (2006), pp.139-145.

[20] W. P. Ziemer, Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation, Graduate Texts in Mathematics, Vol.120, Springer-Verlag, New York (1989).

APPENDIX A

1. Adjugate matrix

For $A \in \mathbb{R}^{m\times m}$, we denote by $\mathrm{adj}A \in \mathbb{R}^{m\times m}$ the transpose of the cofactor matrix of A. It is called the adjugate of A. It is also called "adjoint", but the adjoint matrix of A often means the conjugate transpose $A^* := \overline{A}^{\mathrm{T}}$. To avoid the ambiguity, we adopt the word "adjugate" for $\mathrm{adj}A$. It is well known that

$$A\,(\mathrm{adj}A) = (\mathrm{adj}A)\,A = (\det A)\,I,$$

and, for invertible A, the inverse matrix is given by

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A\,.$$

We remark the next proposition.

Proposition A.1. *Let A be diagonalizable. Namely, there exists an invertible matrix P such that*

$$A = P^{-1}\begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_m \end{pmatrix} P,$$

where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of A. Then, the adjugate of A is represented in the form:

$$\mathrm{adj}A = P^{-1}\begin{pmatrix} \mu_1 & & O \\ & \ddots & \\ O & & \mu_m \end{pmatrix} P,$$

where

$$\mu_i := \prod_{1\le j\le m,\, j\ne i} \lambda_j \qquad (i = 1, \cdots, m).$$

Proof. The assertion is clear if $\det A = \lambda_1 \cdots \lambda_m \neq 0$. For general A, since $\det(A+\varepsilon I) = (\lambda_1+\varepsilon)\cdots(\lambda_m+\varepsilon) \neq 0$ for $0 < \varepsilon << 1$, we have

$$\mathrm{adj}(A+\varepsilon I) = P^{-1}\begin{pmatrix} \mu_1(\varepsilon) & & O \\ & \ddots & \\ O & & \mu_m(\varepsilon) \end{pmatrix} P,$$

where

$$\mu_i(\varepsilon) := \prod_{1\le j\le m,\, j\ne i} (\lambda_j+\varepsilon) \qquad (i = 1, \cdots, m).$$

Taking the limit $\varepsilon \to +0$, we obtain the assertion. □

2. Jacobi's formula

Lemma A.2. *Let* $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m)$, $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m) \in \mathbb{R}^{m\times m}$. *Then we have*

$$\mathrm{tr}\{\,(\mathrm{adj}A)\,B\,\} = \sum_{k=1}^{m} \det\,(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k\,, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m)\,.$$

Proof. It is sufficient to prove the assertion of the lemma in the case $\det A \neq 0$. We remark that $\mathrm{adj}A = (\det A)A^{-1}$. The jth column vector of the identity matrix $I \in \mathbb{R}^{m\times m}$ is denoted by $\boldsymbol{\delta}_j = (\delta_{1j}, \cdots, \delta_{mj})^{\mathrm{T}} \in \mathbb{R}^m$. We have

$$\begin{aligned}
&\sum_{k=1}^{m} \det\,(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k\,, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m) \\
&\quad = \sum_{k=1}^{m} (\det A)\, \det\big(A^{-1}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{b}_k\,, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m)\big) \\
&\quad = (\det A) \sum_{k=1}^{m} \det\big(\boldsymbol{\delta}_1, \cdots, \boldsymbol{\delta}_{k-1}, A^{-1}\boldsymbol{b}_k\,, \boldsymbol{\delta}_{k+1}, \cdots, \boldsymbol{\delta}_m\big) \\
&\quad = (\det A) \sum_{k=1}^{m} (\text{the } k\text{th component of } A^{-1}\boldsymbol{b}_k\,) \\
&\quad = (\det A)\ \mathrm{tr}\,(A^{-1}B\,).
\end{aligned}$$

For general A, we can derive the formula in the similar way to the proof of Proposition A.1. □

Proposition A.3 (Jacobi's formula). *We suppose that* $A \in C^1(\mathcal{I}, \mathbb{R}^{m\times m})$. *Then we have*

$$\frac{d}{dt} \det A(t) = \ \mathrm{tr}\{\,(\mathrm{adj}A(t))\,A'(t)\,\} \qquad (t \in \mathcal{I}).$$

In particular, if $\det A(t) \neq 0$ *then*

$$\frac{d}{dt} \det A(t) = \det A(t)\ \mathrm{tr}\big(A(t)^{-1}A'(t)\big) \qquad (t \in \mathcal{I}).$$

Proof. Let $\boldsymbol{a}_j(t) = (a_{1j}(t), \cdots, a_{mj}(t))^{\mathrm{T}} \in \mathbb{R}^m$ be the jth column vector of the matrix $A(t) \in \mathbb{R}^{m\times m}$. The t derivative is denoted by $' = \frac{d}{dt}$. We have

$$
\begin{aligned}
&\frac{d}{dt}\det A(t) \\
&\quad= \frac{d}{dt}\sum_{\sigma\in\mathcal{S}_m} \mathrm{sgn}(\sigma)\, a_{\sigma(1)1}\cdots a_{\sigma(m)m} \\
&\quad= \sum_{\sigma\in\mathcal{S}_m} \mathrm{sgn}(\sigma)\left(\sum_{k=1}^{m} a_{\sigma(1)1}\cdots a_{\sigma(k-1)k-1}\, a'_{\sigma(k)k}\, a_{\sigma(k+1)k+1}\cdots a_{\sigma(m)m}\right) \\
&\quad= \sum_{k=1}^{m}\left(\sum_{\sigma\in\mathcal{S}_m} \mathrm{sgn}(\sigma)\, a_{\sigma(1)1}\cdots a_{\sigma(k-1)k-1}\, a'_{\sigma(k)k}\, a_{\sigma(k+1)k+1}\cdots a_{\sigma(m)m}\right) \\
&\quad= \sum_{k=1}^{m}\det\left(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{k-1}, \boldsymbol{a}'_k, \boldsymbol{a}_{k+1}, \cdots, \boldsymbol{a}_m\right) \\
&\quad= \mathrm{tr}\{\,(\mathrm{adj}A(t))\, A'(t)\,\}\,.
\end{aligned}
$$

$\square$

Part 3

Large time behaviour for diffusive Hamilton–Jacobi equations

Philippe Laurençot

2000 *Mathematics Subject Classification.* Primary 35B40, 35B33, 35K15, 35K55, 49L25

Key words and phrases. diffusive Hamilton–Jacobi equation, large time behaviour, gradient estimates, extinction, convergence to self-similarity

ABSTRACT. The large time behaviour of non-negative and integrable solutions to the semilinear diffusive Hamilton–Jacobi equation $\partial_t u - \Delta u \pm |\nabla u|^q = 0$ in $(0, \infty) \times \mathbb{R}^N$ is surveyed in these lecture notes, the Hamilton–Jacobi term acting either as an absorption or a source term. Temporal decay estimates are established for the solution, its gradient and time derivative as well as one-sided estimates for its Hessian matrix. Next, according to the values of the parameters $q > 0$ and $N \geq 1$, the time evolution is shown to be either dominated by the diffusion term, the reaction term, or governed by a balance between both terms. The occurrence of finite time extinction is also investigated for $q \in (0, 1)$.

ACKNOWLEDGEMENT. These notes grew out from lectures which I gave at Fudan University, Shangai, in November 2006 and at the Nečas Center for Mathematical Modeling, Praha, in April 2008. I thank Professors Songmu Zheng and Eduard Feireisl for their kind invitations as well as both institutions for their hospitality and support. I also thank Saïd Benachour and the referee for useful comments on the manuscript.

Contents

CHAPTER 1

Introduction

The aim of these lectures is to present several qualitative properties of non-negative solutions to the Cauchy problem for a semilinear parabolic equation involving a nonlinearity depending on the gradient of the solution which might be called a diffusive (or viscous) Hamilton–Jacobi equation and reads

$$\begin{aligned} \partial_t u - \Delta u + \sigma\ |\nabla u|^q &= 0\,, \quad (t,x) \in Q_\infty := (0,\infty)\times\mathbb{R}^N\,, && (1.1)\\ u(0) &= u_0\,, \quad x\in\mathbb{R}^N\,, && (1.2)\end{aligned}$$

where $u = u(t,x)$ is a function of the time $t \geq 0$ and the space variable $x \in \mathbb{R}^N$, $N \geq 1$, and the parameters σ and q are such that

$$q \in (0,\infty) \quad \text{and} \quad \sigma \in \{-1, 1\}\,. \tag{1.3}$$

The equation (1.1) features two mechanisms acting on the space variable, the linear diffusion Δu and the nonlinearity $\sigma\ |\nabla u|^q$. Taken apart, these two mechanisms have different properties and there is thus a competition between them in (1.1). The main issue to be addressed in these notes is whether one of these two terms dominates the dynamics at large times. To start the discussion on this matter, it is obvious that the sign of σ is of utmost importance: indeed, if $\sigma = 1$, the nonlinear term $|\nabla u|^q$ is non-negative and thus acts as an absorption term which enhances the effect of the linear diffusion. While, if $\sigma = -1$, $-|\nabla u|^q$ is non-positive and is thus a source term which opposes to the diffusive effect. Observe also that, in contrast to the semilinear parabolic equation

$$\partial_t v - \Delta v \pm v^p = 0\,, \quad (t,x) \in Q_\infty\,,$$

which has been thoroughly studied, the nonlinearity in (1.1) is only effective in the spatial regions where u is steep and not flat. In particular, constants are solutions to (1.1).

In order to determine which of these mechanisms governs the large time dynamics, let us first describe briefly some salient features of their own and restrict ourselves to non-negative integrable and bounded continuous initial data: we thus assume that

$$u_0 \in L^1(\mathbb{R}^N)\cap\mathcal{BC}(\mathbb{R}^N)\,, \quad u_0 \geq 0, \quad u_0 \not\equiv 0\,. \tag{1.4}$$

On the one hand, it is well-known that the linear heat equation

$$\partial_t c - \Delta c = 0\,, \quad (t,x) \in Q_\infty\,, \tag{1.5}$$

with initial condition $c(0) = u_0$ has a unique smooth solution $c\ :t \longmapsto e^{t\Delta} u_0$ which is given by

$$c(t,x) := \left(e^{t\Delta}u_0\right)(x) = \left(g(t) * u_0\right)(x) = \frac{1}{t^{N/2}} \int_{\mathbb{R}^N} G\left(\frac{x-y}{t^{1/2}}\right)\ u_0(y)\ dy \qquad (1.6)$$

with the notation

$$g(t,x) := \frac{1}{t^{N/2}}\, G\left(\frac{x}{t^{1/2}}\right) \quad \text{and} \quad G(x) := \frac{1}{(4\pi)^{N/2}}\ \exp\left(-\frac{|x|^2}{4}\right) \qquad (1.7)$$

for $(t,x) \in Q_\infty$. An important property enjoyed by non-negative and integrable solutions to (1.5) is the invariance through time evolution of the L^1-norm, that is

$$\left\|e^{t\Delta}u_0\right\|_1 = \|u_0\|_1 \quad \text{for every} \quad t \ge 0\,. \qquad (1.8)$$

On the other hand, the Hamilton–Jacobi equation

$$\partial_t h + \sigma\ |\nabla h|^q = 0\,, \quad (t,x) \in Q_\infty\,, \qquad (1.9)$$

does not have smooth solutions and might even have several weak solutions, a drawback which is also met in the study of scalar conservation laws. Still, there is a selection principle which allows to single out one solution among the weak solutions, the so-called *viscosity solution*. We refer the reader to [4, 6, 25, 27] for the precise definition and several properties of viscosity solutions. In particular, (1.9) has a unique viscosity solution with initial condition $h(0) = u_0$. Moreover, if $q \ge 1$, it is given by the Hopf-Lax-Oleinik representation formula

$$\sigma\ h(t,x) := \inf_{y \in \mathbb{R}^N} \left\{\sigma\ u_0(y) + \frac{q-1}{q^{q/(q-1)}}\ |x-y|^{q/(q-1)}\ t^{-1/(q-1)}\right\} \qquad (1.10)$$

if $q > 1$ and

$$\sigma\ h(t,x) := \inf_{\{|y-x| \le t\}} \{\sigma\ u_0(y)\} \quad \text{if} \quad q = 1\,, \qquad (1.11)$$

for $(t,x) \in Q_\infty$. According to (1.10) and (1.11) the Hamilton–Jacobi equation (1.9) enjoys the following invariance property

$$\inf_{x \in \mathbb{R}^N} \{\sigma\ h(t,x)\} = \inf_{x \in \mathbb{R}^N} \{\sigma\ u_0(x)\} \quad \text{for every} \quad t \ge 0\,. \qquad (1.12)$$

A natural guess is then that

$$\|u(t)\|_1 \quad \text{and} \quad \inf_{x \in \mathbb{R}^N} \{\sigma\ u(t,x)\}$$

are two relevant quantities for the large time dynamics.

Let us first look at the case $\sigma = 1$. In that case, $u(t)$ turns out to be integrable for each $t \ge 0$ so that

$$\inf_{x \in \mathbb{R}^N} \{u(t,x)\} = \inf_{x \in \mathbb{R}^N} \{u_0(x)\} = 0\,,$$

and this quantity does not seem to provide any useful information on the large time behaviour. Concerning the L^1-norm of u it formally follows from (1.1) that

$$\|u(t)\|_1 = \|u_0\|_1 - \int_0^t \int_{\mathbb{R}^N} |\nabla u(s,x)|^q\ dxds\,, \quad t \ge 0\,, \qquad (1.13)$$

from which we readily deduce that $t \longmapsto \|u(t)\|_1$ is non-increasing and

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 = \|u_0\|_1 - \int_0^\infty \int_{\mathbb{R}^N} |\nabla u(s,x)|^q \ dxds \ge 0\,.$$

The question is then whether $I_1(\infty) > 0$ or $I_1(\infty) = 0$. In the latter case it means that the damping of the nonlinearity is sufficiently strong so as to drive the L^1-norm of $u(t)$ to zero as $t \to \infty$. The nonlinear term is thus expected to play a non-negligible role for large times.

In the same vein, u is a subsolution to the linear heat equation and we infer from the comparison principle that

$$\|u(t)\|_\infty \le \left\| e^{t\Delta} u_0 \right\|_\infty \le C\ t^{-N/2}\,, \quad t \in (0,\infty)\,.$$

As the nonlinear term $|\nabla u|^q$ acts as an absorption term in that case it enhances the dissipation due to the diffusion and the main issue is to figure out whether this additional dissipative mechanism speeds up the convergence to zero of the L^∞-norm of $u(t)$ as $t \to \infty$.

When $\sigma = -1$, the nonlinear term $|\nabla u|^q$ is a source term and thus slows down or even impedes the dissipation of the diffusion. Indeed, it formally follows from (1.1) by integration that

$$\|u(t)\|_1 = \|u_0\|_1 + \int_0^t \int_{\mathbb{R}^N} |\nabla u(s,x)|^q \ dxds\,, \quad t \ge 0\,.$$

Consequently $t \longmapsto \|u(t)\|_1$ and

$$t \longmapsto \inf_{x\in\mathbb{R}^N} \{-u(t,x)\} = -\|u(t)\|_\infty$$

are non-decreasing. Setting

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 \in [\|u_0\|_1, \infty] \quad \text{and} \quad I_\infty(\infty) := \lim_{t\to\infty} \|u(t)\|_\infty \in [0, \|u_0\|_\infty]\,,$$

the questions here are whether $I_1(\infty)$ is finite or not and whether $I_\infty(\infty)$ is positive or zero. While we expect the diffusion to be dominant for large times if $I_1(\infty) < \infty$ and the nonlinearity to be dominant for large times if $I_\infty(\infty) > 0$, we shall see below that the situation is more complicated than for $\sigma = 1$.

We finally collect some elementary properties of (1.1).

- ⋆ If u is a non-negative solution to (1.1), (1.2) then

$$\tilde{u}(t,x) := \frac{1-\sigma}{2}\ \|u_0\|_\infty + \sigma\ u(t,x)\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N\,, \tag{1.14}$$

  is a non-negative solution to

$$\begin{aligned} \partial_t \tilde{u} - \Delta \tilde{u} + |\nabla \tilde{u}|^q &= 0\,, \quad (t,x) \in Q_\infty\,, &(1.15)\\ \tilde{u}(0) &= (1-\sigma)\ \frac{\|u_0\|_\infty}{2} + \sigma\ u_0\,, \quad x \in \mathbb{R}^N\,. &(1.16)\end{aligned}$$

  Of course, $\tilde{u} = u$ if $\sigma = 1$.

- ⋆ The equation (1.1) is an autonomous equation, that is, if u is a solution to (1.1), (1.2), and $t_0 > 0$, then $\tilde{u}(t,x) := u(t+t_0, x)$ solves (1.1) with initial condition $u(t_0)$.

⋆ If u is a solution to (1.1), (1.2) and $\lambda > 0$, then the function u_λ defined by
$$u_\lambda(t,x) := \lambda^{2-q}\ u\left(\lambda^{2(q-1)}t, \lambda^{q-1}x\right)\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N$$
is also a solution to (1.1) with initial condition $u_\lambda(0)$.

Finally there are two cases for which (1.1) can be transformed to a linear equation:

∗ if $q = 2$, the celebrated Hopf–Cole transformation reduces (1.1) to the linear heat equation. Indeed, introducing $v = e^{-\sigma u}$, we have $v(t) = e^{t\Delta} e^{-\sigma u_0}$ for $t \ge 0$.

∗ If $q = 1$, $N = 1$ and u_0 is non-increasing on $(0,\infty)$ and non-decreasing on $(-\infty, 0)$, then so is $u(t)$ for each $t \ge 0$ and u actually solves the linear equation
$$\partial_t u - \partial_x^2 u - \sigma\ \mathrm{sign}(x)\ \partial_x u = 0\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}\,.$$

Notations. In the following, C and C_i, $i \ge 1$, denote positive constants that only depend on N and q and may vary from place to place. The dependence of these constants upon additional parameters is indicated explicitly.

For $p \in [1,\infty]$ and $w \in L^p(\mathbb{R}^N)$, $\|w\|_p$ denotes the norm of w in $L^p(\mathbb{R}^N)$. We also define the space $\mathcal{BC}(\mathbb{R}^N)$ of bounded and continuous functions in $\mathbb{R}^N$ by $\mathcal{BC}(\mathbb{R}^N) := \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ while $\mathcal{BUC}(\mathbb{R}^N)$ denotes the space of bounded and uniformly continuous functions in $\mathbb{R}^N$.

Given a real number $r \in \mathbb{R}$, we define its positive part r_+ by $r_+ := \max\{r, 0\}$.

CHAPTER 2

Well-posedness and smoothing effects

We begin with the well-posedness of (1.1), (1.2) in $\mathcal{BC}(\mathbb{R}^N)$.

THEOREM 2.1. *[32] Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$ and $\sigma \in \{-1,1\}$. Then the initial-value problem (1.1), (1.2) has a unique non-negative classical solution*

$$u \in \mathcal{BC}([0,\infty)\times\mathbb{R}^N)\cap C^{1,2}(Q_\infty)\,.$$

In addition, $t \longmapsto \|u(t)\|_\infty$ is a non-increasing function of time and

$$0 \le u(t,x) \le \|u_0\|_\infty\,, \quad (t,x)\in[0,\infty)\times\mathbb{R}^N\,. \tag{2.1}$$

We also have

$$u(t) = e^{t\Delta}u_0 - \sigma\int_0^t e^{(t-s)\Delta}|\nabla u(s)|^q\,ds\,, \quad t\ge 0\,. \tag{2.2}$$

Recall that $t \longmapsto e^{t\Delta}u_0$ denotes the solution to the linear heat equation (1.5) with initial condition u_0 and is given by (1.6), (1.7). Furthermore, if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then

$$\|\nabla u(t)\|_\infty \le \|\nabla u_0\|_\infty\,, \quad t\ge 0\,. \tag{2.3}$$

While the uniqueness stated in Theorem 2.1 is a consequence of the comparison principle [32] (see Section 5 below), the proof of the existence of a solution to (1.1), (1.2) requires more ingredients among which the following gradient estimates are crucial.

THEOREM 2.2. *[11, 32] Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$, $\sigma\in\{-1,1\}$, and denote by u the corresponding classical solution to (1.1), (1.2). Then $\nabla u(t)\in L^\infty(\mathbb{R}^N)$ and*

$$\|\nabla u(t)\|_\infty \le C_1\,\|u(s)\|_\infty\,(t-s)^{-1/2} \quad \text{if} \quad q>0\,, \tag{2.4}$$

$$\|\nabla u(t)\|_\infty \le C_2\,\|u(s)\|_\infty^{1/q}\,(t-s)^{-1/q} \quad \text{if} \quad q>0\,,\ q\ne 1\,, \tag{2.5}$$

for $t > s \ge 0$.

Furthermore, if $\sigma = 1$ and $q > 1$, then

$$\left\|\nabla u^{(q-1)/q}(t)\right\|_\infty \le C_3\,\|u(s)\|_\infty^{(q-1)/q}\,(t-s)^{-1/2}\,, \tag{2.6}$$

$$\left\|\nabla u^{(q-1)/q}(t)\right\|_\infty \le C_4\,t^{-1/q} \tag{2.7}$$

for every $t > s \ge 0$.

As a consequence of Theorem 2.2 there is a smoothing effect from $L^\infty(\mathbb{R}^N)$ to $W^{1,\infty}(\mathbb{R}^N)$ for solutions to (1.1), (1.2). But Theorem 2.2 also provides quantitative information about the temporal decay of $\|\nabla u(t)\|_\infty$ which will be used in the study of the large time behaviour. In fact, non-negative solutions to the linear heat equation (1.5) also satisfy (2.4) and (2.6) while (2.5) and (2.7) are also enjoyed by non-negative (viscosity) solutions to (1.9). Thus, in that particular case, non-negative solutions to (1.1), (1.2) inherit the smoothing effects from both terms acting on the space variable.

The proof of the gradient estimates (2.4)–(2.7) relies on a modified Bernstein technique [21]: introducing a new unknown function v defined by $v := f^{-1}(u)$ for some strictly monotone function f to be determined, one looks for time-dependent supersolutions to the nonlinear parabolic equation satisfied by $w := |\nabla v|^2$ and then apply the comparison principle.

Smoothing effects for the time derivative are also available for both (1.5) and (1.9) and it turns out that this property is still shared by (1.1).

Theorem 2.3. *[31] Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$, $\sigma \in \{-1,1\}$, and denote by u the corresponding classical solution to (1.1), (1.2). Then, for $i \in \{1,2\}$, $x \in \mathbb{R}^N$ and $t > 0$, we have*

$$-C_6\ \phi_0(t) \le \sigma\ \partial_t u(t,x) \le C_5\ \phi_i(t)\,, \quad (t,x) \in Q_\infty\,, \tag{2.8}$$

with

$$\phi_0(t) := \begin{cases} \|u_0\|_\infty\ t^{-1} & \text{if} \quad q \ne 1\,, \\ \|u_0\|_\infty\ \left(t^{-1} + t^{-1/2}\right) & \text{if} \quad q = 1\,, \end{cases}$$

$$\phi_1(t) := \begin{cases} \|u_0\|_\infty^{1/(2q-1)}\ t^{-(q+1)/(2q-1)} & \text{if} \quad q \ge 2\,, \\ \|u_0\|_\infty\ t^{-1} & \text{if} \quad q \in (0,2)\,, \end{cases}$$

$$\phi_2(t) := \begin{cases} \|u_0\|_\infty\ t^{-1} & \text{if} \quad q \ge 2\,, \\ \|u_0\|_\infty^{1/(2q-1)}\ t^{-(q+1)/(2q-1)} & \text{if} \quad q \in (1,2)\,, \\ \|u_0\|_\infty\ t^{-3/2}\ (1+\log(1+t)) & \text{if} \quad q = 1\,, \\ \|u_0\|_\infty^{(2-q)/q}\ t^{-2/q} & \text{if} \quad q \in (0,1)\,. \end{cases}$$

The proof of Theorem 2.3 also relies on the comparison principle applied to the equation satisfied by $(\delta\ \partial_t u - A)/\vartheta(\nabla u)$ for some suitable choices of $\delta \in \{-1,1\}$, $A \in \mathbb{R}$ and $\vartheta \in \mathcal{C}^2(\mathbb{R}^N;(0,\infty))$.

The situation is slightly different for the Hessian matrix: indeed, while it also enjoys a smoothing effect for (1.5), this is no longer true for (1.9). It is then not clear whether a smoothing effect is available for (1.1). Still, non-negative solutions to (1.9) become instantaneously semiconcave if $\sigma = 1$ [27, Chapter 3.3] and semiconvex if $\sigma = -1$ and non-negative solutions to (1.1) also enjoy this property if $q \in (1,2)$.

THEOREM 2.4. *[9] Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$, $\sigma \in \{-1,1\}$, and denote by u the corresponding classical solution to (1.1), (1.2). If $q \in (1,2]$, then the Hessian matrix $D^2u = (\partial_i\partial_j u)_{1\le i,j\le N}$ satisfies*

$$\sigma \ D^2u(t,x) \le C_7 \ \|u_0\|_\infty^{(2-q)/q} \ t^{-2/q} \ I_N \tag{2.9}$$

for $(t,x) \in Q_\infty$, that is,

$$\sigma \sum_{i,j=1}^N (\partial_i\partial_j u)(t,x) \ \xi_i \ \xi_j \le C_7 \ \|u_0\|_\infty^{(2-q)/q} \ |\xi|^2 \ t^{-2/q}$$

for each $\xi \in \mathbb{R}^N$. Here I_N denotes the $N \times N$ identity matrix.

Furthermore, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$,

$$\sigma \ D^2u(t,x) \le \|D^2u_0\|_\infty \ I_N \,. \tag{2.10}$$

The proof of Theorem 2.4 still relies on the comparison principle. For $q = 2$, the estimate (2.9) follows from the analysis of Hamilton [34] (since, if f is a non-negative solution to the linear heat equation $\partial_t f = \Delta f$, the function $-\log f$ solves (1.1) with $q = 2$). It is also established in [41, Lemma 5.1], still for $q = 2$.

The estimate (2.9) may also be seen as an extension to a multidimensional setting of a weak form of the Oleinik gradient estimate for scalar conservation laws. Indeed, if $N = 1$ and $U = \partial_x u$, then U is a solution to $\partial_t U - \partial_x^2 U + \sigma \ \partial_x (|U|^q) = 0$ in $(0,\infty) \times \mathbb{R}$. The estimate (2.9) then reads

$$\sigma \ \partial_x U \le C \ \|u_0\|_\infty^{(2-q)/q} \ t^{-2/q}$$

for $t > 0$, respectively, and we thus recover the results of [29, 37] in that case.

When $q = 1$ and $\sigma = -1$ it is rather $\log u$ that becomes instantaneously semi-convex, a property reminiscent from the Aronson-Bénilan one-sided inequality for the Laplacian of $\log\left(e^{t\Delta}u_0\right)$ [3].

PROPOSITION 2.5. *Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$ and denote by u the corresponding classical solution to (1.1), (1.2). If $\sigma = -1$ and $q = 1$, then the matrix $D^2 \log u(t,x) + (1/2t) \ I_N$ is a non-negative matrix for each $(t,x) \in Q_\infty$, that is, for each $\xi \in \mathbb{R}^N$,*

$$\sum_{i,j=1}^N (\partial_i\partial_j \log u)(t,x) \ \xi_i \ \xi_j + \frac{|\xi|^2}{2t} \ge 0 \,. \tag{2.11}$$

The remainder of this chapter is devoted to the proof of the above results: since the nonlinearity and the initial condition need not have the required regularity to apply the comparison principle, we give a formal proof of Theorems 2.2, 2.3, 2.4 and Proposition 2.5 in Sections 1, 2 and 3. We sketch rigorous proofs in Section 4 (using approximation arguments) and complete the proof of Theorem 2.1 in Section 5.

1. Gradient estimates

The proof of the gradient estimates listed in Theorem 2.2 relies on a modification of the Bernstein technique [21]. The cornerstone of such a technique is to find a strictly monotone function f such that the equation satisfied by $w := |\nabla v|^2$ with $v := f^{-1}(u)$ has a supersolution which is a sole function of time.

Let us thus consider a strictly monotone function f and put

$$v := f^{-1}(u) \quad \text{and} \quad w := |\nabla v|^2 \,. \tag{2.12}$$

It follows from (1.1) that v solves

$$\partial_t v - \Delta v - \left(\frac{f''}{f'}\right)(v)\ |\nabla v|^2 + \frac{1}{f'(v)}\ F\left((f'(v))^2\ |\nabla v|^2\right) = 0\,,$$

with $F(r) := \sigma\ r^{q/2}$ for $r \ge 0$. Next

$$\begin{aligned}
\partial_t w &= 2\sum_{i=1}^N \partial_i v\ \Delta\partial_i v + 2\ \left(\frac{f''}{f'}\right)(v)\ \nabla v\cdot\nabla w \\
&+ 2\ \left(\frac{f''}{f'}\right)'(v)\ w^2 + 2\ \left(\frac{f''}{f'^2}\right)(v)\ F\left((f'(v))^2\ w\right)\ w \\
&- 2\ f'(v)\ F'\left((f'(v))^2\ w\right)\ \nabla v\cdot\nabla w - 4\ f''(v)\ F'\left((f'(v))^2\ w\right)\ w^2\,.
\end{aligned}$$

Using the inequality

$$\Delta w \ge 2\sum_{i=1}^N \partial_i v\ \Delta\partial_i v\,,$$

we obtain

$$\mathcal{L}w - 2\ \left(\frac{f''}{f'}\right)'(v)\ w^2 + 2\ \left(\frac{f''}{f'^2}\right)(v)\ \Theta\left((f'(v))^2\ w\right)\ w \le 0\,, \tag{2.13}$$

where

$$\mathcal{L}z := \partial_t z - \Delta z + 2\ \left\{f'(v)\ F'\left((f'(v))^2\ w\right) - \left(\frac{f''}{f'}\right)(v)\right\}\ \nabla v\cdot\nabla z\,, \tag{2.14}$$

and

$$\Theta(r) := 2\ r\ F'(r) - F(r) = \sigma\ (q-1)\ r^{q/2}\,, \quad r\in[0,\infty)\,. \tag{2.15}$$

At this point, assuming that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and choosing $f(r) = r$ for $r \ge 0$, we find that $\mathcal{L}(\|\nabla u_0\|_\infty) = 0$ which, together with (2.13) and the comparison principle, gives (2.3).

After this preparation we are in a position to establish (2.4)–(2.7) by a suitable choice of the function f.

1.1. Gradient estimates: $\sigma = 1$ and $q > 1$. In that case, we choose $f(r) = r^{q/(q-1)}$ for $r \ge 0$ and notice that

$$-2\ \left(\frac{f''}{f'}\right)'(v) = \frac{2}{q-1}\ \frac{1}{v^2} \quad \text{and} \quad \left(\frac{f''}{f'^2}\right)(v) = \frac{1}{qf(v)}\,. \tag{2.16}$$

On the one hand, since f and Θ are non-negative and

$$-2\ \left(\frac{f''}{f'}\right)'(v) \ge \frac{2}{q-1}\ \|u_0\|_\infty^{-2(q-1)/q}$$

by (2.1) it follows from (2.13) that

$$\mathcal{L}w + \frac{2}{q-1}\ \|u_0\|_\infty^{-2(q-1)/q}\ w^2 \le 0\,. \tag{2.17}$$

The function

$$Y_1(t) := \frac{(q-1)\ \|u_0\|_\infty^{2(q-1)/q}}{2t}\,, \quad t\in(0,\infty)\,,$$

is clearly a supersolution to (2.17) and we deduce from the comparison principle that $w(t,x)\le Y_1(t)$ for $(t,x)\in Q_\infty$. We have thus proved (2.6) for $s=0$ and use the autonomous property of (1.1) to obtain (2.6) for $t>s\ge 0$.

We next turn to the proof of (2.7). For that purpose we infer from (2.13), (2.15) and (2.16) that

$$\mathcal{L}w + 2\left(\frac{q}{q-1}\right)^{q-1}\ w^{(q+2)/2} \le 0\,. \tag{2.18}$$

The function

$$Y_2(t) := \left(\frac{q-1}{q}\right)^2\ (q-1)^{-2/q}\ t^{-2/q}\,, \quad t\in(0,\infty)\,,$$

is a supersolution to (2.18) and (2.7) follows by the comparison principle. □

1.2. Gradient estimates: $q>1$. We put

$$\tilde u := \frac{1-\sigma}{2}\ \|u_0\|_\infty + \sigma\ u$$

and notice that $\tilde u$ solves (1.15), (1.16) and satisfies $0\le \tilde u(t,x)\le \|u_0\|_\infty$ for all $(t,x)\in Q_\infty$. Since

$$\nabla u(t,x) = \frac{\sigma\ q}{q-1}\ \tilde u^{1/q}\ \nabla \tilde u^{(q-1)/q}(t,x)$$

and $\tilde u$ enjoys the properties (2.6) and (2.7) as shown in the previous section 1.1, we infer from (2.6) and (2.7) that

$$\|\nabla u(t)\|_\infty \le \frac{q}{q-1}\ \|u_0\|_\infty^{1/q}\ \left\|\nabla \tilde u^{(q-1)/q}(t)\right\|_\infty \le \begin{cases} \dfrac{q\ C_3}{q-1}\ \|u_0\|_\infty\ t^{-1/2}\,, \\[2ex] \dfrac{q\ C_4}{q-1}\ \|u_0\|_\infty^{1/q}\ t^{-1/q}\,, \end{cases}$$

hence (2.4) and (2.5). □

1.3. Gradient estimates: $q=1$. In that case, the function Θ defined in (2.15) is equal to zero and we choose $f(r)=\|u_0\|_\infty - r^2$ for $r\in\left[0,\|u_0\|_\infty^{1/2}\right]$. Then

$$-2\ \left(\frac{f''}{f'}\right)'(v) = \frac{2}{v^2} \ge \frac{2}{\|u_0\|_\infty}$$

and (2.13) reads

$$\mathcal{L}w + \frac{2}{\|u_0\|_\infty}\ w^2 \le 0\,. \tag{2.19}$$

The function $t\longmapsto \|u_0\|_\infty/(2t)$ is clearly a supersolution to (2.19) and the comparison principle ensures that $|\nabla v(t,x)|\le \|u_0\|_\infty^{1/2}\ (2t)^{-1/2}$ for $(t,x)\in Q_\infty$. Finally,

$$\begin{aligned} |\nabla u(t,x)| &\le 2\ v(t,x)\ |\nabla v(t,x)| \le \sqrt{2}\ \left(\|u_0\|_\infty - u(t,x)\right)^{1/2}\ \|u_0\|_\infty^{1/2}\ t^{-1/2} \\ &\le \sqrt{2}\ \|u_0\|_\infty\ t^{-1/2}\,, \end{aligned}$$

hence (2.4) for $q=1$. □

1.4. Gradient estimates: $q \in (0,1)$. We choose

$$f(r) = \frac{1+\sigma}{2}\ \|u_0\|_\infty - \sigma\ r^2 \quad \text{for} \quad r \in \left[0, \frac{2\ \|u_0\|_\infty^{1/2}}{1+\sigma}\right) ,$$

so that

$$\begin{aligned} -2\ \left(\frac{f''}{f'}\right)'(v) &= \frac{2}{v^2} \ge \frac{2}{\|u_0\|_\infty}, \\ 2\ \left(\frac{f''}{f'^2}\right)(v)\ \Theta\left((f'(v))^2\ w\right)\ w &= 2^q\ (1-q)\ v^{q-2}\ w^{(q+2)/2} \\ &\ge 2^q\ (1-q)\ \|u_0\|_\infty^{(q-2)/2}\ w^{(q+2)/2} . \end{aligned}$$

We then deduce from (2.13) that

$$\mathcal{L}w + \frac{2}{\|u_0\|_\infty}\ w^2 \le 0 , \tag{2.20}$$

$$\mathcal{L}w + 2^q\ (1-q)\ \|u_0\|_\infty^{(q-2)/2}\ w^{(q+2)/2} \le 0 . \tag{2.21}$$

On the one hand we proceed as in the previous section to show that (2.4) follows from (2.20). On the other hand, the function

$$Y_3(t) := \left(\frac{2^{1-q}\ \|u_0\|_\infty^{(2-q)/2}}{q(1-q)}\right)^{2/q}\ t^{-2/q} , \quad t > 0 ,$$

is a supersolution to (2.21). By the comparison principle, we have $w(t,x) \le Y_3(t)$ for $(t,x) \in Q_\infty$ from which (2.5) readily follows. ☐

2. Time derivative estimates

For $\xi \in \mathbb{R}^N$, we put $F(\xi) := |\xi|^q$ and consider a positive function $\vartheta \in \mathcal{C}^2(\mathbb{R}^N)$ and two real numbers $\delta \in \{-1,1\}$ and $A \in \mathbb{R}$. Introducing

$$w := \frac{1}{\vartheta(\nabla u)}\ (\delta\ \partial_t u - A) , \tag{2.22}$$

we infer from (1.1) that

$$\begin{aligned} \partial_t w &= -\frac{(\nabla\vartheta)(\nabla u)}{\vartheta(\nabla u)} \cdot [\nabla\Delta u - \sigma\nabla\{F(\nabla u)\}]\ w \\ &+ \frac{1}{\vartheta(\nabla u)}\ [\Delta\{\vartheta(\nabla u)\ w\} - \sigma\ (\nabla F)(\nabla u)\cdot\nabla\{\vartheta(\nabla u)\ w\}] . \end{aligned}$$

Observing that

$$(\nabla\vartheta)(\nabla u)\cdot\nabla\{F(\nabla u)\} = \nabla\{\vartheta(\nabla u)\}\cdot(\nabla F)(\nabla u) ,$$

we obtain

$$\partial_t w = \frac{w}{\vartheta(\nabla u)}\ \left[\Delta\{\vartheta(\nabla u)\} - \sum_{i,j}(\partial_i\vartheta)(\nabla u)\ \partial_j^2\partial_i u\right] + b\cdot\nabla w + \Delta w ,$$

where

$$b := 2\ \frac{\nabla\{\vartheta(\nabla u)\}}{\vartheta(\nabla u)} - \sigma\ (\nabla F)(\nabla u) .$$

Since

$$\Delta\{\vartheta(\nabla u)\} = \sum_{i,j} (\partial_i \vartheta)(\nabla u)\ \partial_j^2 \partial_i u + \sum_{i,j,k} (\partial_i \partial_k \vartheta)(\nabla u)\ \partial_j \partial_k u\ \partial_j \partial_i u\,,$$

we end up with

$$\partial_t w - b \cdot \nabla w - \Delta w = \frac{w}{\vartheta(\nabla u)}\ \sum_{i,j,k} (\partial_i \partial_k)(\vartheta)(\nabla u)\ \partial_j \partial_k u\ \partial_j \partial_i u\,, \tag{2.23}$$

and we are left to find appropriate functions ϑ for which the quadratic form on the right-hand side of (2.23) will be negative.

Let $\Phi \in \mathcal{C}^2(\mathbb{R})$ be a convex and even function such that $\Phi(0) = 0$ and $\Phi'(s) \geq 0$ for $s \geq 0$. For $\varepsilon \in (0, \infty)$ and $\xi \in \mathbb{R}^N$, $|\xi| \leq \|\nabla u_0\|_\infty$, we put

$$\vartheta(\xi) := \varepsilon + \frac{N}{4}\ \left(\Phi\left(N^{1/2}\ \|\nabla u_0\|_\infty \right) - \sum_{i=1}^N \Phi(\xi_i) \right)\,, \tag{2.24}$$

and note that $\vartheta(\xi) \geq \varepsilon$ and $(\partial_i \partial_k \vartheta)(\xi) = 0$ if $i \neq k$ and $(\partial_i^2 \vartheta)(\xi) = -N\ \Phi''(\xi_i)/4$. Then (2.23) reads

$$\partial_t w - b \cdot \nabla w - \Delta w + \frac{N}{4\vartheta(\nabla u)}\ \sum_{i,j} \Phi''(\partial_i u)\ (\partial_i \partial_j u)^2\ w = 0\,. \tag{2.25}$$

Before going further, let us recall that, by the Cauchy-Schwarz inequality,

$$N\ \sum_{i,j} (\partial_i \partial_j u)^2 \geq N\ \sum_i (\partial_i^2 u)^2 \geq (\Delta u)^2\,. \tag{2.26}$$

(a) $\boxed{\Phi(s) = s^2/\mu,\ \mu > 0}$. Then $\Phi''(s) = 2/\mu$ and

$$(\Delta u)^2 = \left(\vartheta(\nabla u)\ w + A + \delta\sigma\ F(\nabla u)\right)^2\,.$$

We then infer from (2.25) that

$$\begin{aligned}
\mathscr{L} w \quad := \quad & \partial_t w - b \cdot \nabla w - \Delta w + \frac{\vartheta(\nabla u)}{2\mu}\ w^3 + \frac{A + \delta\sigma\ F(\nabla u)}{\mu}\ w^2 \\
+ \quad & \frac{1}{2\mu\vartheta(\nabla u)}\ \left[N\ \sum_{i,j} (\partial_i \partial_j u)^2 - (\Delta u)^2 + (A + \delta\sigma\ F(\nabla u))^2 \right]\ w = 0\,.
\end{aligned}$$

We first take $\varepsilon = \mu$, $\delta = -\sigma$ and $A = F(\|\nabla u_0\|_\infty) = \|\nabla u_0\|_\infty^q$. Then $\vartheta(\nabla u) \geq \mu$ and it follows from (2.3) and (2.26) that $\mathscr{L}\left(t^{-1/2}\right) \geq 0$. The comparison principle then entails that $w(t,x) \leq t^{-1/2}$ for $(t,x) \in Q_\infty$, hence

$$\sigma\ \partial_t u(t,x) \geq -\|\nabla u_0\|_\infty^q - \frac{\vartheta(\nabla u)}{t^{1/2}} \geq -\|\nabla u_0\|_\infty^q - \left(\mu + \frac{N^2\ \|\nabla u_0\|_\infty^2}{4\mu} \right)\ t^{-1/2}\,.$$

Choosing $\mu = \|\nabla u_0\|_\infty$, we conclude that

$$\sigma\ \partial_t u(t,x) \geq -\|\nabla u_0\|_\infty^q - C\ \|\nabla u_0\|_\infty\ t^{-1/2}$$

for $(t,x) \in Q_\infty$. The equation (1.1) being autonomous, we infer from the previous inequality that

$$\sigma\ \partial_t u(t,x) \geq -\|\nabla u(t/2)\|_\infty^q - \sqrt{2}\ C\ \|\nabla u(t/2)\|_\infty\ t^{-1/2}\,,$$

and we use (2.4) and (2.5) to obtain that

$$\sigma\ \partial_t u(t,x) \ge -C\ \|u_0\|_\infty\ t^{-1}\,, \quad (t,x)\in Q_\infty\,, \tag{2.27}$$

if $q\ne 1$ and

$$\sigma\ \partial_t u(t,x) \ge -C\ \|u_0\|_\infty\ \left(t^{-1/2}+t^{-1}\right)\,, \quad (t,x)\in Q_\infty\,, \tag{2.28}$$

if $q=1$.

We next take $\varepsilon=\mu$, $\delta=\sigma$ and $A=0$. Then $\vartheta(\nabla u)\ge\mu$ and it follows from (2.3) and (2.26) that $\mathscr{L}\left(t^{-1/2}\right)\ge 0$. The comparison principle then entails that $w(t,x)\le t^{-1/2}$ for $(t,x)\in Q_\infty$, hence

$$\sigma\ \partial_t u(t,x) \le \frac{\vartheta(\nabla u)}{t^{1/2}} \le \left(\mu+\frac{N^2\ \|\nabla u_0\|_\infty^2}{4\mu}\right)\ t^{-1/2}\,.$$

Choosing $\mu=\|\nabla u_0\|_\infty$, we conclude that

$$\sigma\ \partial_t u(t,x)\le C\ \|\nabla u_0\|_\infty\ t^{-1/2}$$

for $(t,x)\in Q_\infty$ and we proceed as before (with the help of (2.4) and (2.5)) to conclude that

$$\sigma\ \partial_t u(t,x) \le C\ \|u_0\|_\infty\ t^{-1}\,, \quad (t,x)\in Q_\infty\,. \tag{2.29}$$

(b) $\boxed{\Phi''(s)=1/(A+|s|^q)}$**.** In that case, Φ'' is non-increasing. Consequently, $\Phi''(\partial_i u)\ge\Phi''(|\nabla u|)$ and

$$\begin{aligned}(\Delta u)^2 &= (\vartheta(\nabla u)\ w + A + \delta\sigma\ F(\nabla u))^2\\ &= 4\ \vartheta(\nabla u)\ \left(A+\delta\sigma\ F(\nabla u)\right)\ w\ + (\vartheta(\nabla u)\ w - A - \delta\sigma\ F(\nabla u))^2\,.\end{aligned}$$

We then infer from (2.25) that

$$\begin{aligned}\mathscr{L}w &:= \partial_t w - b\cdot\nabla w - \Delta w + \Phi''(|\nabla u|)\ (A+\delta\sigma\ F(\nabla u))\ w^2\\ &+ \frac{\Phi''(|\nabla u|)}{4\vartheta(\nabla u)}\ \left[N\sum_{i,j}\frac{\Phi''(\partial_i u)}{\Phi''(|\nabla u|)}\ (\partial_i\partial_j u)^2-(\Delta u)^2\right]\ w\\ &+ \frac{\Phi''(|\nabla u|)}{4\vartheta(\nabla u)}\ \left[\vartheta(\nabla u)\ w - A-\delta\sigma\ F(\nabla u)\right]^2\ w = 0\,.\end{aligned}$$

Taking $\delta=\sigma$ yields that $\Phi''(|\nabla u|)\ (A+\delta\sigma\ F(\nabla u))=1$ and thus $\mathscr{L}\left(t^{-1}\right)\ge 0$. We then deduce from the comparison principle that

$$\sigma\ \partial_t u(t,x)\le A+\left(\varepsilon+\frac{N}{4}\ \Phi\left(N^{1/2}\|\nabla u_0\|_\infty\right)\right)\ t^{-1}$$

for $(t,x)\in Q_\infty$. Letting $\varepsilon\to 0$ in the above inequality, we end up with

$$\sigma\ \partial_t u(t,x)\le A+\frac{N}{4}\ \Phi\left(N^{1/2}\|\nabla u_0\|_\infty\right)\ t^{-1}\,, \quad (t,x)\in Q_\infty\,. \tag{2.30}$$

If $q\in(0,1)$, we may take $A=0$ to obtain that $\Phi(s)=s^{2-q}/((2-q)(1-q))$ and infer from (2.30) that

$$\sigma\ \partial_t u(t,x)\le C\ \|\nabla u_0\|_\infty^{2-q}\ t^{-1}\,, \quad (t,x)\in Q_\infty\,.$$

Using once more the fact that (1.1) is an autonomous equation and (2.5) we finally obtain

$$\sigma\ \partial_t u(t,x) \le C\ \|u_0\|_\infty^{(2-q)/q}\ t^{-2/q}\,, \quad (t,x)\in Q_\infty\,. \tag{2.31}$$

If $q=1$ we have

$$\Phi(s) = \int_0^s \frac{s-z}{A+z}\ dz \le s\ \log\left(1+\frac{s}{A}\right), \quad s\ge 0\,,$$

which, together with (2.30), yields

$$\sigma\ \partial_t u(t,x) \le A + C\ \|\nabla u_0\|_\infty\ \log\left(1+\frac{\|\nabla u_0\|_\infty}{A}\right)\ t^{-1}\,, \quad (t,x)\in Q_\infty\,.$$

The choice $A = \|\nabla u_0\|_\infty\ \log(2+t)\ t^{-1}$ then gives

$$\sigma\ \partial_t u(t,x) \le C\ \|\nabla u_0\|_\infty\ \frac{1+\log(1+t)}{t}\,, \quad (t,x)\in Q_\infty\,.$$

Owing to (2.4), we may proceed as before to conclude that

$$\sigma\ \partial_t u(t,x) \le C\ \|u_0\|_\infty\ \frac{1+\log(1+t)}{t^{3/2}}\,, \quad (t,x)\in Q_\infty\,. \tag{2.32}$$

Finally, if $q>1$, we have

$$\Phi(s) = \int_0^s \frac{s-z}{A+z^q}\ dz \le A^{-(q-1)/q}\ s\ \int_0^\infty \frac{dz}{1+z^q}\,, \quad s\ge 0\,.$$

Choosing $A = \|\nabla u_0\|_\infty^{q/(2q-1)}\ t^{-q/(2q-1)}$ we infer from (2.30) that

$$\sigma\ \partial_t u(t,x) \le C\ \|\nabla u_0\|_\infty^{q/(2q-1)}\ t^{-q/(2q-1)}\,, \quad (t,x)\in Q_\infty\,.$$

Recalling (2.5) we may proceed as before to conclude that

$$\sigma\ \partial_t u(t,x) \le C\ \|u_0\|_\infty^{1/(2q-1)}\ t^{-(q+1)/(2q-1)}\,, \quad (t,x)\in Q_\infty\,, \tag{2.33}$$

and thus complete the proof of Theorem 2.3.

3. Hessian estimates

Let us first handle the case $q\in(1,2]$.

Proof of Theorem 2.4. For $1\le i,j\le N$, we put $w_{ij} := \sigma\ \partial_i\partial_j u$. It follows from (1.1) that

$$\begin{aligned}
\partial_t w_{ij} - \Delta w_{ij} &= -\sigma q\ \partial_i\left(|\nabla u|^{q-2}\ \left(\sum_{k=1}^N \partial_k u\ w_{jk}\right)\right)\\
&= -q\ |\nabla u|^{q-2}\ \sum_{k=1}^N w_{ik}\ w_{jk} - \sigma q\ |\nabla u|^{q-2}\ \sum_{k=1}^N \partial_k u\ \partial_i w_{jk}\\
&\quad - q\ (q-2)\ |\nabla u|^{q-4}\ \left(\sum_{k=1}^N \partial_k u\ w_{ik}\right)\ \left(\sum_{k=1}^N \partial_k u\ w_{jk}\right)
\end{aligned} \tag{2.34}$$

Consider now $\xi\in\mathbb{R}^N\setminus\{0\}$ and set

$$w := \sum_{i=1}^N\sum_{j=1}^N w_{ij}\ \xi_i\ \xi_j\,.$$

Multiplying (2.34) by $\xi_i\ \xi_j$ and summing up the resulting identities yield

$$\begin{aligned}\partial_t w - \Delta w \quad &= \quad -q\ |\nabla u|^{q-2}\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i\right)^2 - \sigma q\ |\nabla u|^{q-2}\ \nabla u\cdot\nabla w \\ &\qquad - q\ (q-2)\ |\nabla u|^{q-4}\ \left(\sum_{i=1}^N\sum_{j=1}^N \partial_j u\ w_{ij}\ \xi_i\right)^2 . \qquad (2.35)\end{aligned}$$

Thanks to the following inequalities

$$\begin{aligned}|\nabla u|^{q-4}\ \left(\sum_{i=1}^N\sum_{j=1}^N \partial_j u\ w_{ij}\ \xi_i\right)^2 \quad &\le \quad |\nabla u|^{q-4}\ \sum_{j=1}^N |\partial_j u|^2\ \sum_{j=1}^N \left(\sum_{i=1}^N w_{ij}\ \xi_i\right)^2 \\ &\le \quad |\nabla u|^{q-2}\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i\right)^2 ,\end{aligned}$$

and

$$w^2 \le |\xi|^2\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i\right)^2 ,$$

and since $q\le 2$, the right-hand side of (2.35) can be bounded from above. We thus obtain

$$\begin{aligned}\partial_t w - \Delta w \quad &\le \quad -q\ (q-1)\ |\nabla u|^{q-2}\ \sum_{k=1}^N \left(\sum_{i=1}^N w_{ik}\ \xi_i\right)^2 - \sigma q\ |\nabla u|^{q-2}\ \nabla u\cdot\nabla w \\ &\le \quad -\sigma q\ |\nabla u|^{q-2}\ \nabla u\cdot\nabla w - \frac{q\ (q-1)\ |\nabla u|^{q-2}}{|\xi|^2}\ w^2 .\end{aligned}$$

Consequently,

$$\mathcal{L}w \le 0 \quad \text{in} \quad \mathbb{R}^N\times(0,\infty) , \qquad (2.36)$$

where the parabolic differential operator $\mathcal{L}$ is given by

$$\mathcal{L}z := z_t - \Delta z + \sigma q\ |\nabla u|^{q-2}\ \nabla u\cdot\nabla z + \frac{q\ (q-1)\ |\nabla u|^{q-2}}{|\xi|^2}\ z^2 .$$

On the one hand, since $q\in(1,2]$ and $|\nabla u(t,x)|\le\|\nabla u_0\|_\infty$, it is straightforward to check that

$$W_1(t) := \left(\frac{1}{\|w(0)\|_\infty} + \frac{q\ (q-1)\ t}{|\xi|^2\ \|\nabla u_0\|_\infty^{2-q}}\right)^{-1} , \qquad t>0 ,$$

satisfies $\mathcal{L}W_1\ge 0$ with $W_1(0)\ge w(0,x)$ for all $x\in\mathbb{R}^N$. The comparison principle then entails that $w(t,x)\le W_1(t)$ for $(t,x)\in Q_\infty$, from which we conclude that

$$w(t,x) \le \|w(0)\|_\infty \le \|D^2u_0\|_\infty\ |\xi|^2 ,$$

whence (2.10). Observe that we also obtain that

$$w(t,x) \le \frac{|\xi|^2\ \|\nabla u_0\|_\infty^{2-q}}{q\ (q-1)\ t} \quad \text{for} \quad (t,x)\in Q_\infty . \qquad (2.37)$$

On the other hand, we infer from (2.5) that

$$W_2(t) := \frac{2\ C_2^{2-q}\ |\xi|^2}{q^2\ (q-1)}\ \frac{\|u_0\|_\infty^{(2-q)/q}}{t^{2/q}}\,, \qquad t>0\,,$$

satisfies $\mathcal{L}W_2 \ge 0$ with $W_2(0) = \infty \ge w(0,x)$ for all $x \in \mathbb{R}^N$. We then use again the comparison principle as above and obtain (2.9).

□

For further use, we report the following weaker version of Theorem 2.4.

Corollary 2.6. *Under the assumptions of Theorem 2.4 there is C_9 such that*

$$\sigma\ \Delta u(t,x) \le \frac{C_9\ \|u_0\|_\infty^{(2-q)/q}}{t^{2/q}}\,, \tag{2.38}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}^N$.

In addition, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$,

$$\sup_{x\in\mathbb{R}^N} \{\sigma\ \Delta u(t,x)\} \le \sup_{x\in\mathbb{R}^N} \{\sigma\ \Delta u_0(x)\}\ , \quad t\ge 0\,. \tag{2.39}$$

Proof. Consider $i \in \{1,\dots,N\}$ and define $\xi^i = (\xi^i_j) \in \mathbb{R}^N$ by $\xi^i_i = 1$ and $\xi^i_j = 0$ if $j \ne i$. We take $\xi = \xi^i$ in (2.36) and obtain that $\mathcal{L}(\sigma\ \partial_i^2 u) \le 0$, that is,

$$\partial_t\left(\sigma\ \partial_i^2 u\right) - \Delta\left(\sigma\ \partial_i^2 u\right) + \sigma q\ |\nabla u|^{q-2}\ \nabla u\cdot\nabla\left(\sigma\ \partial_i^2 u\right) + q\ (q-1)\ |\nabla u|^{q-2}\ \left(\sigma\ \partial_i^2 u\right)^2 \le 0$$

in Q_∞. Summing the above inequality over $i \in \{1,\dots,N\}$ and recalling that

$$|\Delta u|^2 \le N\ \sum_{i=1}^N \left(\partial_i^2 u\right)^2\,,$$

we end up with

$$\left(\sigma\ \Delta u\right)_t - \Delta\left(\sigma\ \Delta u\right) + \sigma q\ |\nabla u|^{q-2}\ \nabla u\cdot\nabla\left(\sigma\ \Delta u\right) + \frac{q\ (q-1)\ |\nabla u|^{q-2}}{N}\ |\sigma\ \Delta u|^2 \le 0$$

in Q_∞. We next proceed as in the proof of Theorem 2.4 to complete the proof of Corollary 2.6.

□

We next establish the log-semiconvexity of u in the particular case $\sigma = -1$ and $q = 1$.

Proof of Proposition 2.5. We follow the approach of [34] and introduce $v := \log u$. We deduce from (1.1) and the positivity of u [32, Corollary 4.2] that v solves

$$\partial_t v - \Delta v = |\nabla v| + |\nabla v|^2 \quad \text{in} \quad Q_\infty\,. \tag{2.40}$$

We next put $w_{ij} := \partial_i\partial_j v$ for $1 \le i,j \le N$, $P := 1/|\nabla v|$ and deduce from (2.40) that w_{ij} solves

$$\begin{aligned}\partial_t w_{ij} &= \Delta w_{ij} + 2\ \sum_{k=1}^N w_{ik}\ w_{jk} + 2\ \nabla v\cdot\nabla w_{ij} + P\ \sum_{k=1}^N w_{ik}\ w_{jk}\\ &+ P\ \nabla v\cdot\nabla w_{ij} - P^3\ \left(\sum_{k=1}^N w_{ik}\ \partial_k v\right)\ \left(\sum_{l=1}^N w_{jl}\ \partial_l v\right)\end{aligned}$$

Consider now $\xi \in \mathbb{R}^N \setminus \{0\}$ and put

$$w := \sum_{i,j=1}^{N} w_{ij}\ \xi_i\ \xi_j\,.$$

Then

$$\begin{aligned} \partial_t w &= \Delta w + 2\ \sum_{k=1}^{N} \left(\sum_{i=1}^{N} w_{ik}\ \xi_i \right)^2 + 2\ \nabla v \cdot \nabla w + P\ \sum_{k=1}^{N} \left(\sum_{i=1}^{N} w_{ik}\ \xi_i \right)^2 \\ &+ \ P\ \nabla v \cdot \nabla w - P^3\ \left(\sum_{i,k=1}^{N} w_{ik}\ \xi_i\ \partial_k v \right)^2\,. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i,k=1}^{N} w_{ik}\ \xi_i\ \partial_k v \right)^2 \le |\nabla v|^2\ \sum_{k=1}^{N} \left(\sum_{i=1}^{N} w_{ik}\ \xi_i \right)^2\,.$$

Therefore, we may estimate from below the last term of the right-hand side of the equation satisfied by w and obtain

$$\partial_t w \ge \Delta w + (2+P)\ \ \nabla v \cdot \nabla w + 2\ \sum_{k=1}^{N} \left(\sum_{i=1}^{N} w_{ik}\ \xi_i \right)^2\,.$$

Using once more the Cauchy-Schwarz inequality, we realize that

$$w^2 \le |\xi|^2\ \sum_{k=1}^{N} \left(\sum_{i=1}^{N} w_{ik}\ \xi_i \right)^2\,,$$

whence

$$\partial_t w \ge \Delta w + (2+P)\ \ \nabla v \cdot \nabla w + \frac{2\ w^2}{|\xi|^2}\,.$$

The comparison principle then ensures that

$$w(t,x) \ge -\frac{|\xi|^2}{2t}\,, \quad (t,x) \in Q_\infty\,,$$

which gives the result.

□

4. Existence

Two different approaches have been used to investigate the well-posedness of (1.1), (1.2), one relying on the comparison principle and the previous gradient estimates [11, 32] and the other one on the integral formulation (2.2) and the smoothing properties of the heat semigroup $(e^{t\Delta})_{t\ge 0}$ [20]. Both methods actually allow to handle different classes of initial data but in different ranges of the parameter q.

We sketch the first approach which is actually a compactness method and is thus based on a sequence of approximations to (1.1), (1.2). In view of the transformation (1.14), we only need to consider the case $\sigma = 1$. We thus assume that $\sigma = 1$ and consider a non-negative function $u_0 \in \mathcal{BC}(\mathbb{R}^N)$. There is a sequence of functions $(u_{0,k})_{k\geq 1}$ such that, for each integer $k \geq 1$, $u_{0,k} \in \mathcal{BC}^\infty(\mathbb{R}^N)$,

$$0 \leq u_{0,k}(x) \leq u_{0,k+1}(x) \leq u_0(x)\,, \quad x \in \mathbb{R}^N\,, \tag{2.41}$$

and $(u_{0,k})$ converges uniformly towards u_0 on compact subsets of $\mathbb{R}^N$. In addition, if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ we may assume that

$$\|\nabla u_{0,k}\|_\infty \leq \left(1 + \frac{K_1}{k}\right) \|\nabla u_0\|_\infty\,, \tag{2.42}$$

for some constant $K_1 > 0$ depending only on the approximation process. Next, since $\xi \longmapsto |\xi|^q$, $\xi \in \mathbb{R}^N$, is not regular enough for small values of q, we set

$$F_\varepsilon(r) := \left(\varepsilon^2 + r\right)^{q/2} - \varepsilon^q\,, \quad r \geq 0\,, \tag{2.43}$$

for $\varepsilon \in (0,1)$. Then the Cauchy problem

$$\partial_t u_{k,\varepsilon} - \Delta u_{k,\varepsilon} + F_\varepsilon\left(|\nabla u_{k,\varepsilon}|^2\right) = 0\,, \quad (t,x) \in Q_\infty\,, \tag{2.44}$$

$$u_{k,\varepsilon}(0) = u_{0,k} + \varepsilon\,, \quad x \in \mathbb{R}^N\,, \tag{2.45}$$

has a unique classical solution $u_{k,\varepsilon} \in \mathcal{C}^{(3+\alpha)/2,3+\alpha}([0,\infty) \times \mathbb{R}^N)$ for some $\alpha \in (0,1)$ [38]. Observing that ε and $\|u_0\|_\infty + \varepsilon$ are solutions to (2.44) with $\varepsilon \leq u_{k,\varepsilon}(0,x) \leq \|u_0\|_\infty + \varepsilon$, the comparison principle warrants that

$$\varepsilon \leq u_{k,\varepsilon}(t,x) \leq \|u_0\|_\infty + \varepsilon\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N\,.$$

Moreover, arguing as in Section 1, we obtain similar gradient estimates for $u_{k,\varepsilon}$ which does not depend on $k \geq 1$ and have a mild dependence on $\varepsilon \in (0,1)$. More precisely, introducing $v_{k,\varepsilon} := f^{-1}(u_{k,\varepsilon})$ and $w_{k,\varepsilon} := |\nabla v_{k,\varepsilon}|^2$ for some strictly monotone function f to be specified, we proceed along the lines of Section 1 to establish that:

- if $q > 1$ (with $f(r) = r^{q/(q-1)}$),

$$\left|\nabla u_{k,\varepsilon}^{(q-1)/q}(t,x)\right| \leq \left(\frac{q-1}{2}\right)^{1/2} (\|u_0\|_\infty + \varepsilon)^{(q-1)/q}\ t^{-1/2}\,, \quad t > 0\,,$$

- if $q > 2$ (with $f(r) = r^{q/(q-1)}$),

$$\left|\nabla u_{k,\varepsilon}^{(q-1)/q}(t,x)\right| \leq \left(\frac{2}{qK_2(\eta)} + \frac{K_3(\eta)}{K_2(\eta)}\ \varepsilon^{(q-1)/2}\right)^{1/q} t^{-1/q}\,, \quad t \in \left(0, \varepsilon^{(1-q)/2}\right)\,,$$

  for every $\eta \in (0, q-1)$ with

$$K_2(\eta) := \frac{2(q-1-\eta)}{q}\ \left(\frac{q}{q-1}\right)^q \quad \text{and} \quad K_3(\eta) := \frac{4}{q}\ \left(\frac{q-2}{\eta}\right)^{(q-2)/2}\,,$$

- if $q \in (1,2]$ (with $f(r) = r^{q/(q-1)}$),

$$\left|\nabla u_{k,\varepsilon}^{(q-1)/q}(t,x)\right| \leq \frac{q-1}{q}\ \left(\frac{1}{q-1} + \varepsilon^{(q-1)/2}\right)^{1/q} t^{-1/q}\,, \quad t \in \left(0, \varepsilon^{(1-q)/2}\right)\,,$$

- if $q \in (0,1]$ (with $f(r) = \|u_0\|_\infty + \varepsilon + \varepsilon^{q/2} - r^2$),

$$|\nabla u_{k,\varepsilon}(t,x)| \leq \left(2 + 2\varepsilon^{q/4}\right)^{1/2} \left(\|u_0\|_\infty + \varepsilon^{q/2}\right) t^{-1/2}\,, \quad t \in \left(0, \varepsilon^{-q/4}\right)\,,$$

- if $q \in (0,1)$ (with $f(r) = \|u_0\|_\infty + \varepsilon + \varepsilon^{q/2} - r^2$),

$$|\nabla u_{k,\varepsilon}(t,x)| \le \left(\frac{2 + q\ \varepsilon^{q/4}}{2^q\ q(1-q)} \right)^{1/q} \left(\|u_0\|_\infty + \varepsilon + \varepsilon^{q/2} \right)^{1/q}\ t^{-1/q}\,, \quad t \in \left(0, \varepsilon^{-q/4}\right)\,.$$

In addition, we have

$$\|\nabla u_{k,\varepsilon}(t)\|_\infty \le \|\nabla u_{0,k}\|_\infty\,, \quad t \ge 0\,.$$

Combining these estimates with classical parabolic regularity results we first let $\varepsilon \to 0$ and then $k \to \infty$ to obtain the existence part of Theorem 2.1, together with Theorem 2.2. The proofs of Theorems 2.3, 2.4 and Proposition 2.5 are done in a similar way.

5. Uniqueness

So far, given $u_0 \in \mathcal{BC}(\mathbb{R}^N)$, we have constructed a solution u to (1.1), (1.2) enjoying the properties stated in Theorems 2.2, 2.3, 2.4 and Proposition 2.5. To prove the uniqueness assertion of Theorem 2.1, we follow the proof of [32, Theorem 4] and show that any other solution to (1.1), (1.2) in the sense of Theorem 2.1 coincides with the solution u already obtained in the previous section. More precisely, assume that

$$v \in \mathcal{BC}([0,\infty) \times \mathbb{R}^N) \cap \mathcal{C}^{1,2}(Q_\infty)$$

is such that

$$\delta\ \{\partial_t v - \Delta v + \sigma\ |\nabla v|^q\} \ge 0\,, \quad (t,x) \in Q_\infty\,, \tag{2.46}$$

and

$$\delta\ v(0,x) \ge \delta\ u_0(x)\,, \quad x \in \mathbb{R}^N\,, \tag{2.47}$$

for some $\delta \in \{-1,+1\}$.

Then we claim that

$$\delta\ v(t,x) \ge \delta\ u(t,x)\,, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N\,. \tag{2.48}$$

Indeed, fix $T > 0$, $\varepsilon \in (0,1)$ and put $\alpha := \min\{1/q, 1\}$ and

$$A_\varepsilon := \begin{cases} 2q\ \left[N\ T^{(q-1)/q} + q\ \left(C_2\ \|u_0\|_\infty^{1/q} + \varepsilon\ T^{1/q} \right)^{q-1} \right]\ \varepsilon & \text{if} \quad q > 1\,, \\ \\ 2\ (\varepsilon^q + N\ \varepsilon) & \text{if} \quad q \in (0,1]\,. \end{cases}$$

Introducing

$$z(t,x) := \delta\ (u-v)(t,x) - A_\varepsilon\ t^\alpha - \varepsilon\ \left(1 + |x|^2\right)^{1/2}\,, \quad (t,x) \in [0,T) \times \mathbb{R}^N\,,$$

we infer from the boundedness of u and v and (2.47) that there is $R_\varepsilon > 0$ such that $z(t,x) \le -\varepsilon < 0$ for $t \in [0,T]$ and $|x| \ge R_\varepsilon$ and also for $(t,x) \in \{0\} \times \mathbb{R}^N$. Consequently, if z is positive somewhere in $[0,T] \times \mathbb{R}^N$ it must have a positive maximum at some point $(t_0,x_0) \in (0,T] \times \{|x| \le R_\varepsilon\}$. This implies that $\nabla z(t_0,x_0) = 0$, $\Delta z(t_0,x_0) \le 0$ and $\partial_t z(t_0,x_0) \ge 0$ so that

$$\partial_t z(t_0,x_0) - \Delta z(t_0,x_0) \ge 0\,. \tag{2.49}$$

Now, by (1.1) and (2.46) we have

$$\partial_t z(t_0,x_0) - \Delta z(t_0,x_0) \le \Big| |\nabla v(t_0,x_0)|^q - |\nabla u(t_0,x_0)|^q \Big| - \alpha\ A_\varepsilon\ t_0^{\alpha-1} + N\ \varepsilon$$

and we use the relation $\nabla z(t_0,x_0)=0$ to compute $\nabla v(t_0,x_0)$ and obtain

$$\begin{aligned}\partial_t z(t_0,x_0)-\Delta z(t_0,x_0) \quad &\le \quad \left| \left| \nabla u(t_0,x_0) - \delta\varepsilon\ \frac{x_0}{\left(1+|x_0|^2\right)^{1/2}} \right|^q - |\nabla u(t_0,x_0)|^q \right| \\ &\quad - \alpha\ A_\varepsilon\ t_0^{\alpha-1} + N\ \varepsilon\,.\end{aligned}$$

Either $q>1$ and it follows from the mean value theorem, (2.5) and the choice of A_ε that

$$\begin{aligned}\partial_t z(t_0,x_0)-\Delta z(t_0,x_0) \quad &\le \quad q\ \left(|\nabla u(t_0,x_0)| + \frac{\varepsilon\ |x_0|}{\left(1+|x_0|^2\right)^{1/2}} \right)^{q-1}\ \frac{\varepsilon\ |x_0|}{\left(1+|x_0|^2\right)^{1/2}} \\ &\quad -\frac{A_\varepsilon}{q}\ t_0^{-(q-1)/q} + N\ \varepsilon \\ &\le \quad q\ \varepsilon\ \left(C_2\ \|u_0\|_\infty^{1/q}\ t_0^{-1/q} + \varepsilon \right)^{q-1} - \frac{A_\varepsilon}{q}\ t_0^{-(q-1)/q} + N\ \varepsilon \\ &\le \quad -\frac{A_\varepsilon}{2q}\ t_0^{\alpha-1} < 0\,,\end{aligned}$$

which contradicts (2.49). Or $q\in(0,1]$ and we deduce from the Hölder continuity of $r\mapsto r^q$ and the choice of A_ε that

$$\partial_t z(t_0,x_0)-\Delta z(t_0,x_0) \le \left(\frac{\varepsilon\ |x_0|}{\left(1+|x_0|^2\right)^{1/2}} \right)^q - A_\varepsilon + N\ \varepsilon \le -\frac{A_\varepsilon}{2} < 0\,,$$

again contradicting (2.49). Therefore, z cannot take positive values in $[0,T]\times\mathbb{R}^N$ and thus

$$\delta\ u(t,x) \le \delta\ v(t,x) + A_\varepsilon\ t^\alpha + \varepsilon\ \left(1+|x|^2\right)^{1/2}\ , \quad (t,x)\in[0,T)\times\mathbb{R}^N\,.$$

Since $A_\varepsilon\to 0$ as $\varepsilon\to 0$ we may pass to the limit as $\varepsilon\to 0$ in the previous inequality and conclude that (2.48) holds true.

The uniqueness statement of Theorem 2.1 then readily follows.

Bibliographical notes

The gradient estimates (2.5) and (2.7) (with $\sigma=1$) are also true for non-negative viscosity solutions to the non-diffusive Hamiton-Jacobi equation (1.9): this fact is proved in [42] by a different method, still using the comparison principle. No such gradient estimate seems to be available for $q=1$ as a consequence of the lack of strict convexity (or concavity) of $|\nabla u|$.

An alternative approach to study the well-posedness of (1.1), (1.2) in Lebesgue spaces $L^p(\mathbb{R}^N)$, $p\ge 1$, is employed in [20]. It relies on the formulation (2.2) of (1.1), (1.2) which is the starting point of a fixed point procedure. This approach is restricted to $q\in[1,2)$ but provides the existence and uniqueness of a weak solution (which is classical for positive times) for $u_0\in L^r(\mathbb{R}^N)$ whenever $r\ge 1$ and

$$r > \frac{N(q-1)}{2-q} \quad \text{or} \quad r = \frac{N(q-1)}{2-q} > 1\,.$$

When $q=1$, previous existence and uniqueness results were obtained in [15, 16, 17, 18].

CHAPTER 3

Extinction in finite time

Let us first recall that, if X is a vector space, a solution $f : [0,\infty) \to X$ to an evolution problem enjoys the *extinction in finite time* property if there exists $T_\star > 0$ such that

$$\begin{cases} f(t) \neq 0 & \text{for} \quad t \in [0,T_\star)\,, \\ f(t) \equiv 0 & \text{for} \quad t \geq T_\star\,. \end{cases}$$

A typical example of an evolution problem which exhibits the previous property is the ordinary differential equation $f'(t) + f(t)^\gamma = 0$ with $f(0) = f_0 > 0$ and $\gamma \in (0,1)$. Then, for $t \geq 0$,

$$f(t) = (1-\gamma)^{1/(1-\gamma)}\ (T_\star - t)_+^{1/(1-\gamma)} \quad \text{with} \quad T_\star := \frac{f_0^{1-\gamma}}{1-\gamma}\,.$$

Extinction in finite time also shows up for non-negative solutions to the fast diffusion equation $\partial_t v = \Delta v^m$ in $\mathbb{R}^N$ if $m \in (0,(N-2)_+/N)$ and to the p-Laplacian equation $\partial_t v = \text{ div }\left(|\nabla v|^{p-2}\ \nabla v\right)$ in $\mathbb{R}^N$ if $p \in (0, 2N/(N+1))$ (see, e.g., [22, 35]). Noticing that the exponent below which extinction in finite time takes place is smaller than one in the previous examples, it is rather natural to wonder whether extinction phenomena could also occur for non-negative solutions to (1.1), (1.2) when $q \in (0,1)$. It is however obvious that extinction in finite time cannot take place if $\sigma = -1$: indeed, in that case, we have $u(t) \geq e^{t\Delta} u_0$ for each $t \geq 0$ by the comparison principle, which readily implies that $u(t,x) > 0$ for $(t,x) \in Q_\infty$ if $u_0 \not\equiv 0$.

Thus, throughout this chapter we assume that

$$\sigma = 1 \quad \text{and} \quad q \in (0,1)\,. \tag{3.1}$$

There are two different conditions on the initial condition u_0 which warrant that the corresponding classical solution u to (1.1), (1.2) vanishes after a finite time: the first one requires some integrability at infinity for u_0 while the second involves a pointwise bound.

THEOREM 3.1. *Consider a non-negative function $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ and denote by u the corresponding classical solution to (1.1), (1.2). If $u_0 \in L^1(\mathbb{R}^N; |x|^m\ dx)$ for some $m \geq 0$ and $q \in (0,(m+N)/(m+N+1))$, then there exists $T_\star \in [0,\infty)$ such that*

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in [T_\star,\infty) \times \mathbb{R}^N\,.$$

THEOREM 3.2. *[14] Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$ satisfying*

$$\limsup_{|x|\to\infty}\ |x|^{q/(1-q)}\ u_0(x) < \infty\,. \tag{3.2}$$

Denoting by u the corresponding classical solution to (1.1), (1.2), there exists $T_\star \in [0,\infty)$ such that

$$u(t,x) = 0 \quad \textit{for} \;\; (t,x) \in [T_\star,\infty) \times \mathbb{R}^N .$$

Let us first point out that the sets of initial data involved in Theorems 3.1 and 3.2 have a non-empty intersection but are different. Indeed, both results apply for compactly supported initial data but, if $m \ge 0$ and $q \in (0,(m+N)/(m+N+1))$, a function u_0 satisfying (3.2) need not be in $L^1(\mathbb{R}^N;|x|^m\,dx)$. Next, as we shall see below, the proofs of Theorems 3.1 and 3.2 rely on different arguments : indeed the latter is achieved by constructing suitable supersolutions while the former follows from a differential inequality involving the L^∞-norm of u.

The next issue to be considered is the optimality of the algebraic growth condition (3.2). In that direction, we report the following result:

Theorem 3.3. *[14] Consider $u_0 \in \mathcal{BC}(\mathbb{R}^N)$ satisfying*

$$u_0(x) > 0, \quad x \in \mathbb{R}^N , \tag{3.3}$$

$$\lim_{|x|\to\infty} |x|^{q/(1-q)}\; u_0(x) = \infty . \tag{3.4}$$

Denoting by u the corresponding classical solution to (1.1), (1.2), we have $u(t,x) > 0$ for every $(t,x) \in Q_\infty$.

It turns out that Theorem 3.3 also guarantees the optimality of the exponent $(m+N)/(m+N+1)$ in Theorem 3.1: indeed, if $m \ge 0$ and $q \in ((m+N)/(m+N+1),1)$, there is at least a non-negative function $u_0 \in L^1(\mathbb{R}^N;|x|^m\,dx)$ such that the corresponding classical solution u to (1.1), (1.2) satisfies $u(t,x) > 0$ for every $(t,x) \in Q_\infty$ (see Corollary 3.10 below).

Remark 3.4. *An explicit computation shows that, if $T > 0$ and $K \in [0,\infty)$, the function z defined by*

$$z(t,x) := (T-t)_+^{1/(1-q)} \; \left[K^{(1-q)/q} + \frac{(1-q)^{(q-1)/q}}{q}\; |x| \right]^{-q/(1-q)}$$

for $(t,x) \in Q_\infty$, is a solution to the Hamilton–Jacobi equation $\partial_t z + |\nabla z|^q = 0$ in Q_∞. Observe that $z(t,x) > 0$ for every $(t,x) \in [0,T) \times \mathbb{R}^N$ and $z(T) \equiv 0$. The supersolutions and subsolutions we will construct in the proofs of Theorems 3.2 and 3.3 are somehow related to this function.

We finally establish a lower bound for the L^∞-norm of u near the extinction time.

Proposition 3.5. *Let u_0 be a non-negative function in $\mathcal{BC}(\mathbb{R}^N)$ with compact support and u denote the corresponding classical solution to (1.1), (1.2). There is a positive real number κ depending only on N, q and $|supp\ u_0|$ such that*

$$\|u(t)\|_\infty \ge \kappa\; (T_\star - t)^{1/(1-q)} , \quad t \in [0,T_\star] , \tag{3.5}$$

where $T_\star \in (0,\infty)$ is the extinction time of u. Here, $|supp\ u_0|$ denotes the (Lebesgue) measure of $supp\ u_0$.

1. An integral condition for extinction

Consider $m \geq 0$ and a non-negative function $u_0 \in L^1(\mathbb{R}^N; |x|^m\ dx) \cap \mathcal{BC}(\mathbb{R}^N)$. We denote by u the corresponding classical solution to (1.1), (1.2) and first establish that $u(t)$ still belongs to $L^1(\mathbb{R}^N; |x|^m\ dx)$ for $t > 0$.

Lemma 3.6. *There exists $K_1 > 0$ depending only on m, N, q and u_0 such that*

$$\int_{\mathbb{R}^N} |x|^m\ u(t,x)\ dx \leq K_1\,, \quad t \geq 0\,. \tag{3.6}$$

Proof. For $x \in \mathbb{R}^N$ and $\varepsilon > 0$, we put $b_\varepsilon(x) := \left(1 + \varepsilon\ |x|^2\right)^{1/2}$. We first multiply (1.1) by $b_1(x)^m$ and integrate over $\mathbb{R}^N$: owing to the non-negativity of u and $\sigma = 1$, we obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^N} b_1(x)^m\ u(t,x)\ dx \\
\leq\ & -m \int_{\mathbb{R}^N} b_1(x)^{m-2}\ x \cdot \nabla u(t,x)\ dx \\
\leq\ & \int_{\mathbb{R}^N} \left\{ Nm\ b_1(x)^{m-2} + m(m-2)\ |x|^2\ b_1(x)^{m-4} \right\}\ u(t,x)\ dx \\
\leq\ & C \int_{\mathbb{R}^N} b_1(x)^m\ u(t,x)\ dx\,,
\end{aligned}$$

whence

$$\int_{\mathbb{R}^N} b_1(x)^m\ u(t,x)\ dx \leq C \int_{\mathbb{R}^N} b_1(x)^m\ u_0(x)\ dx\,, \quad t \in [0,1]\,. \tag{3.7}$$

Next, for $t \geq 1$, we multiply (1.1) by $b_\varepsilon(x)^m$ and integrate over $\mathbb{R}^N$ to obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^N} b_\varepsilon(x)^m\ u(t,x)\ dx + \int_{\mathbb{R}^N} b_\varepsilon(x)^m\ |\nabla u(t,x)|^q\ dx \\
\leq\ & m\varepsilon \int_{\mathbb{R}^N} b_\varepsilon(x)^{m-2}\ |x|\ |\nabla u(t,x)|\ dx \\
\leq\ & m\varepsilon^{1/2}\ \|\nabla u(t)\|_\infty^{1-q} \int_{\mathbb{R}^N} b_\varepsilon(x)^m\ |\nabla u(t,x)|^q\ dx\,.
\end{aligned}$$

Since $\|\nabla u(t)\|_\infty \leq C_2\ \|u_0\|_\infty^{1/q}$ for $t \geq 1$ by (2.5), we end up with

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^N} b_\varepsilon(x)^m\ u(t,x)\ dx \\
& + \left(1 - m\ C_2^{1-q}\ \|u_0\|_\infty^{(1-q)/q}\ \varepsilon^{1/2}\right) \int_{\mathbb{R}^N} b_\varepsilon(x)^m\ |\nabla u(t,x)|^q\ dx \leq 0
\end{aligned}$$

for $t \geq 1$. Choosing $\varepsilon = 1/\left(m^2\ C_2^{2(1-q)}\ \|u_0\|_\infty^{2(1-q)/q}\right)$, we conclude that

$$\int_{\mathbb{R}^N} b_\varepsilon(x)^m\ u(t,x)\ dx \leq \int_{\mathbb{R}^N} b_\varepsilon(x)^m\ u(1,x)\ dx\,, \quad t \geq 1\,. \tag{3.8}$$

Lemma 3.6 now readily follows from (3.7) and (3.8) as $b_\varepsilon(x) \geq \varepsilon^m\ |x|^m$.

$\square$

Another useful tool for the proof of Theorem 3.1 is the following interpolation inequality:

LEMMA 3.7. *If $w \in L^1(\mathbb{R}^N; |x|^m\ dx) \cap L^\infty(\mathbb{R}^N)$, then $w \in L^1(\mathbb{R}^N)$ and there is a constant $K_2 > 0$ depending only on N such that*

$$\|w\|_1 \le K_2\ \|w\|_\infty^{m/(m+N)}\ \left(\int_{\mathbb{R}^N} |x|^m\ w(x)\ dx\right)^{N/(m+N)}.$$

PROOF. Consider $w \in L^1(\mathbb{R}^N; |x|^m\ dx) \cap L^\infty(\mathbb{R}^N)$ and $R > 0$. Then

$$\begin{aligned} \|w\|_1 &= \int_{\{|x|\le R\}} |w(x)|\ dx + \int_{\{|x|>R\}} |w(x)|\ dx \\ &\le C\ R^N\ \|w\|_\infty + \frac{1}{R^m}\ \int_{\mathbb{R}^N} |x|^m\ w(x)\ dx\,. \end{aligned}$$

Choosing

$$R = \left(\int_{\mathbb{R}^N} |x|^m\ w(x)\ dx\right)^{1/(m+N)}\ \|w\|_\infty^{-1/(m+N)}$$

yields Lemma 3.7.

□

PROOF OF THEOREM 3.1. Owing to (2.5), (3.6), Lemma 3.7 and the Gagliardo–Nirenberg inequality

$$\|w\|_\infty \le C\ \|\nabla w\|_\infty^{N/(N+1)}\ \|w\|_1^{1/(N+1)} \quad \text{for} \quad w \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)\,, \tag{3.9}$$

we have for $t > s \ge 0$

$$\begin{aligned} \|u(t)\|_\infty^q &\le C\ \|\nabla u(t)\|_\infty^{qN/(N+1)}\ \|u(t)\|_1^{q/(N+1)} \\ &\le C\ (t-s)^{-N/(N+1)}\ \|u(s)\|_\infty^{N/(N+1)}\ \|u(t)\|_\infty^{qm/((N+1)(N+m))} \\ &\quad \times \left(\int_{\mathbb{R}^N} |x|^m\ u(t,x)\ dx\right)^{qN/((N+1)(N+m))} \\ &\le C\ (t-s)^{-N/(N+1)}\ \|u(s)\|_\infty^{N/(N+1)}\ \|u(t)\|_\infty^{qm/((N+1)(N+m))}\,, \end{aligned}$$

whence

$$\|u(t)\|_\infty^q \le C\ (t-s)^{-(m+N)/(m+N+1)}\ \|u(s)\|_\infty^{(m+N)/(m+N+1)}\,.$$

We multiply the above inequality by $1/t$ and integrate with respect to t over (s, ∞) to deduce that

$$Z(s) := \int_s^\infty \frac{\|u(t)\|_\infty^q}{t}\ dt \le C\ s^{-(m+N)/(m+N+1)}\ \|u(s)\|_\infty^{(m+N)/(m+N+1)}\,.$$

Since $dZ/ds(s) = -\|u(s)\|_\infty^q/s$, the above inequality also reads

$$s^{1-q}\ \frac{dZ}{ds}(s) + C\ Z(s)^{q(m+N+1)/(m+N)} \le 0\,, \quad s \ge 0\,.$$

Introducing $\tilde{Z}(s) := Z\left(s^{1/q}\right)$ for $s \ge 0$, we readily obtain

$$\frac{d\tilde{Z}}{ds}(s) + C\ \tilde{Z}(s)^{q(m+N+1)/(m+N)} \le 0\,, \quad s \ge 0\,,$$

whence $\tilde{Z}(s) = 0$ for s sufficiently large since $q < (m+N)/(m+N+1)$. Consequently $u(t) = 0$ for t large enough.

$\square$

2. A pointwise condition for extinction

We introduce the following positive real numbers

$$\alpha = \frac{2-q}{2(1-q)} \quad \text{and} \quad \beta = \alpha - 1 = \frac{q}{2(1-q)}. \tag{3.10}$$

A classical tool to establish extinction in finite time is to construct supersolutions to (1.1) enjoying such a property and in fact to look for self-similar supersolutions.

LEMMA 3.8. *Consider $T \in (0,\infty)$ and $B \ge 1$ such that*

$$B^{q/2} \ge \frac{\alpha + \beta + 4\ \alpha\ (\beta+1)}{\min\{\beta, (2\beta)^q\}}. \tag{3.11}$$

The function $W_{B,T}$ defined by

$$W_{B,T}(t,x) = (T-t)^\alpha\ f_B\left(|x|\ (T-t)^{-1/2}\right),\ \ (t,x) \in [0,T] \times \mathbb{R}^N,$$

$$f_B(y) = \left(A + B\ y^2\right)^{-\beta},\ \ y \in \mathbb{R},\ \ \textit{with}\ \ A = \frac{\beta}{\alpha}\ B,$$

is a supersolution to (1.1) in $[0,T) \times \mathbb{R}^N$.

PROOF. We denote by $\mathcal{L}$ the parabolic operator defined by

$$\mathcal{L}w = \partial_t w - \Delta w + |\nabla w|^q, \tag{3.12}$$

and put

$$y_0 = \left(\frac{\beta}{\alpha\ B}\right)^{1/2}.$$

Let $(t,x) \in [0,T) \times \mathbb{R}^N$ and compute $\mathcal{L}W_{B,T}(t,x)$. Putting $y = |x|\ (T-t)^{-1/2}$ we obtain

$$\mathcal{L}W_{B,T}(t,x) = (T-t)^{\alpha-1}\ \left(A + B\ y^2\right)^{-\beta-1}\ H(y),$$

where

$$\begin{aligned} H(y) &:= 2\ \beta\ N\ B - \alpha\ A + (2\ \beta\ B)^q\ y^q\ \left(A + B\ y^2\right)^{(2-q)/2} \\ &\quad -(\alpha+\beta)\ B\ y^2 - 4\ \beta\ (\beta+1)\ \frac{B^2\ y^2}{A + B\ y^2} \\ &\ge \beta\ B + (2\ \beta)^q\ B^{(2+q)/2}\ y^2 - (\alpha+\beta)\ B\ y^2 - 4\ \beta\ (\beta+1)\ \frac{B^2\ y^2}{A + B\ y^2}. \end{aligned}$$

Now, on the one hand, we have for $y \in [0, y_0]$

$$\begin{aligned} H(y) &\ge \beta\ B - (\alpha+\beta)\ B\ y_0^2 - 4\ \beta\ (\beta+1)\ \frac{B^2\ y_0^2}{A + B\ y_0^2} \\ &\ge \beta\ B - (\alpha+\beta)\ \frac{\beta}{\alpha} - 4\ \beta\ (\beta+1)\ \frac{B}{1+B}. \end{aligned}$$

As $\beta < \alpha$ and $B \ge B^{q/2} \ge 1$ we further obtain, thanks to (3.11),

$$H(y) \ge \beta\ B - (\alpha+\beta) - 4\ \beta\ (\beta+1) \ge 0.$$

On the other hand, there holds for $y \ge y_0$

$$\begin{aligned} H(y) &\ge (2\ \beta)^q\ B^{(2+q)/2}\ y^2 - (\alpha+\beta)\ B\ y^2 - 4\ \beta\ (\beta+1)\ \frac{B^2\ y^2}{A+B\ y^2} \\ &\ge \left((2\ \beta)^q\ B^{q/2} - (\alpha+\beta) - 4\ \beta\ (\beta+1)\ \frac{B}{A+B\ y_0^2} \right)\ B\ y^2 \\ &\ge \left((2\ \beta)^q\ B^{q/2} - (\alpha+\beta) - 4\ \alpha\ (\beta+1)\ \frac{B}{1+B} \right)\ B\ y^2, \end{aligned}$$

hence $H(y) \ge 0$ by (3.11). The proof of Lemma 3.8 is thus complete. □

Proof of Theorem 3.2. Owing to (3.2) and the boundedness of u_0 there is a constant $C_0 > 0$ such that

$$u_0(x) \le C_0\ (1+|x|)^{-q/(1-q)}, \quad x \in \mathbb{R}^N.$$

Since $(1+|x|)^2 \ge 1+|x|^2$ for $x \in \mathbb{R}^N$ we further obtain that u_0 satisfies

$$u_0(x) \le C_0\ \left(1+|x|^2\right)^{-\beta}, \quad x \in \mathbb{R}^N,$$

the real number β being defined in (3.10). We next fix $B \ge 1$ such that (3.11) holds true and consider $T > 0$ such that

$$T \ge 1 \quad \text{and} \quad T \ge C_0^{1/\alpha}\ B^{\beta/\alpha}. \tag{3.13}$$

With this choice of the parameters T and B the function $W_{B,T}$ defined in Lemma 3.8 is a supersolution to (1.1). In addition, $\alpha > \beta$ and we infer from (3.13) that, for $x \in \mathbb{R}^N$,

$$W_{B,T}(0,x) = T^\alpha\ B^{-\beta}\ \left(\frac{\beta}{\alpha} + \frac{|x|^2}{T}\right)^{-\beta} \ge C_0\ \left(1+|x|^2\right)^{-\beta} \ge u_0(x)\,.$$

The comparison principle [32, Theorem 4] then ensures that $u(t,x) \le W_{B,T}(t,x)$ for $(t,x) \in [0,T]\times\mathbb{R}^N$. Therefore, since u is non-negative and $W_{B,T}$ vanishes identically at time T, we conclude that $u(T) \equiv 0$, whence $u(t,x) = 0$ for $(t,x) \in [T,\infty) \times \mathbb{R}^N$. □

3. Non-extinction

The proof of Theorem 3.3 also relies on a comparison argument, but now with suitable subsolutions.

Lemma 3.9. *Let T, a and b be three positive real numbers and define*

$$w_{a,b,T}(t,x) = (T-t)^{1/(1-q)}\ \left(a+b\ |x|^2\right)^{-\beta}, \quad (t,x) \in [0,T] \times \mathbb{R}^N,$$

the parameter β being defined in (3.10). If

$$\tau := b^{q/2}\ q^q\ (1-q)^{1-q} - 1 < 0 \quad \textit{and} \quad b \le \frac{(1-q)\ |\tau|}{N\ q\ (N+2-q)}\ \frac{a}{T}, \tag{3.14}$$

then the function $w_{a,b,T}$ is a subsolution to (1.1) on $[0,T] \times \mathbb{R}^N$.

Proof. The parabolic operator $\mathcal{L}$ being defined by (3.12), we compute $\mathcal{L}w_{a,b,T}$. Let $(t,x) \in [0,T] \times \mathbb{R}^N$ and put $y = a + b\ |x|^2$. Since $b|x|^2 \le y$, we have

$$\mathcal{L}w_{a,b,T}(t,x) \le \frac{(T-t)^{q/(1-q)}}{1-q}\ y^{-2-q/(2-2q)}\ F(y),$$

where

$$F(y) := \tau\ y^2 + b\ q\ (T-t)\ \left(\frac{a\ (2-q)}{1-q} + \frac{(N-2) - q\ (N-1)}{1-q}\ y \right).$$

Now we have

$$\begin{aligned} F(y) &\le \tau\ y^2 + b\ q\ (T-t)\ \left(\frac{a\ (2-q)}{1-q} + \frac{N}{1-q}\ y \right) \\ &\le \tau\ y^2 + \frac{b\ N\ q\ T}{1-q}\ y + \frac{a\ b\ q\ (2-q)\ T}{1-q} =: G(y). \end{aligned}$$

As $G'' = 2\ \tau < 0$ by (3.14) and $y \ge a$, we infer from (3.14) that

$$G'(y) \le G'(a) = 2\ \tau\ a + \frac{b\ N\ q\ T}{1-q} \le 0\,.$$

Consequently, G is a non-increasing function on $[a,\infty)$ and (3.14) ensures that

$$G(y) \le G(a) = a\ \left(\tau\ a + \frac{b\ q\ T}{1-q}\ (N+2-q) \right) \le 0$$

for $y \in [a,\infty)$. Therefore $\mathcal{L}w_{a,b,T}(t,x) \le 0$ and the proof of Lemma 3.9 is complete. $\square$

Proof of Theorem 3.3. We fix $b \in (0,1)$ such that $b^{q/2}\ q^q\ (1-q)^{1-q} \le 1/2$ and $T > 0$. On the one hand it follows from (3.4) that there is $R_T > 0$ such that

$$u_0(x) \ge \left(b\ T^{-2/q}\ |x|^2 \right)^{-q/(2-2q)}, \quad |x| \ge R_T. \tag{3.15}$$

On the other hand the continuity of u_0 and (3.3) ensure that

$$m_T := \min_{\{|x| \le R_T\}} \{u_0(x)\} > 0\,.$$

We then define $a(T)$ by

$$a(T) = T^{2/q}\ m_T^{-(2-2q)/q} + \frac{N\ q\ (N+2-q)}{(1-q)\ |\tau|}\ b\ T, \tag{3.16}$$

where τ is defined in (3.14) and notice that

$$\left(a(T)\ T^{-2/q} \right)^{-q/(2-2q)} \le m_T\,. \tag{3.17}$$

Owing to (3.16) and Lemma 3.9 the function $w_{a(T),b,T}$ is a subsolution to (1.1) on $[0,T] \times \mathbb{R}^N$ and we infer from (3.15) and (3.17) that $w_{a(T),b,T}(0,x) \le u_0(x)$ for $x \in \mathbb{R}^N$. The comparison principle [32, Theorem 4] then entails that

$$w_{a(T),b,T}(t,x) \le u(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^N\,.$$

Therefore $u(T/2,x) > 0$ for each $x \in \mathbb{R}^N$ and, as T is arbitrary in $(0,\infty)$, Theorem 3.3 follows. $\square$

COROLLARY 3.10. *Consider $m \geq 0$ and $q \in ((m+N)/(m+N+1), 1)$. Then there is at least a non-negative function $u_0 \in \mathcal{BC}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N; |x|^m\ dx)$ such that the corresponding classical solution u to (1.1), (1.2) satisfies $u(t,x) > 0$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$.*

In other words, the assertion of Theorem 3.1 is false for $q \in ((m+N)/(m+N+1), 1)$.

PROOF. The condition on q implies that $m+N < q/(1-q)$. We may then choose $\vartheta \in (0,\infty)$ such that

$$m+N < \frac{q}{1-q} - \vartheta\,.$$

We next put

$$u_0(x) = (1+|x|)^{\vartheta - q/(1-q)}\,, \quad x \in \mathbb{R}^N.$$

Thanks to the choice of ϑ, u_0 satisfies the assumptions of Theorem 3.3 and u_0 belongs to $L^1(\mathbb{R}^N; |x|^m\ dx)$, which yields the expected result. $\square$

4. A lower bound near the extinction time

Since u_0 is compactly supported, it clearly satisfies (3.2) and Theorem 3.2 implies that $T_\star < \infty$. Introducing the positivity set

$$\mathcal{P}(t) := \left\{x \in \mathbb{R}^N \ :\ u(t,x) > 0\right\}$$

of u at time $t \geq 0$, we proceed as in [31, Theorem 9] to prove that

$$\mathcal{P}(t) \subseteq \left\{x \in \mathbb{R}^N \ :\ d(x,\mathcal{P}(0)) < \left(\frac{\|u_0\|_\infty}{A_0}\right)^{(1-q)/(2-q)}\right\} \tag{3.18}$$

where $A_0 := (1-q)^{(2-q)/(1-q)}\ (N(1-q)+q)^{-1/(1-q)}/(2-q)$. Indeed, consider $x_0 \in \mathbb{R}^N$ such that $d(x_0,\mathcal{P}(0))^{(2-q)/(1-q)} \geq \|u_0\|_\infty/A_0$. Introducing $S(x) := A_0\ |x-x_0|^{(2-q)/(1-q)}$ for $x \in \mathbb{R}^N$, we have $u_0(x) = 0 \leq S(x)$ if $x \notin \mathcal{P}(0)$ and

$$u_0(x) \leq \|u_0\|_\infty \leq A_0\ d(x_0,\mathcal{P}(0))^{(2-q)/(1-q)} \leq A_0\ |x-x_0|^{(2-q)/(1-q)} = S(x)$$

if $x \in \mathcal{P}(0)$, the last inequality following from the choice of x_0. Therefore, $u_0(x) \leq S(x)$ for $x \in \mathbb{R}^N$ and S is actually a stationary solution to (1.1). The comparison principle then entails that $u(t,x) \leq S(x)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$. In particular, $u(t,x_0) \leq S(x_0) = 0$ for $t \geq 0$ which, together with the non-negativity of u, implies that $u(t,x_0) = 0$ for $t \geq 0$ and completes the proof of (3.18).

Consider next $t \in [0,T_\star)$ and $s \in [0,t)$. On the one hand, the gradient estimate (2.5), (3.18) and the Gagliardo–Nirenberg inequality (3.9) give

$$\begin{aligned}
\|u(t)\|_\infty &\leq C\ \|\nabla u(t)\|_\infty^{N/(N+1)}\ \|u(t)\|_1^{1/(N+1)} \\
&\leq C\ \|u(s)\|_\infty^{N/q(N+1)}\ (t-s)^{-N/q(N+1)}\ \|u(t)\|_\infty^{1/(N+1)}\ |\mathcal{P}(t)|^{1/(N+1)} \\
&\leq C(|\mathcal{P}(0)|)\ \|u(s)\|_\infty^{N/q(N+1)}\ (t-s)^{-N/q(N+1)}\ \|u(t)\|_\infty^{1/(N+1)}\,.
\end{aligned}$$

As $t < T_\star$, we have $\|u(t)\|_\infty > 0$ and we deduce from the previous inequality that

$$(t-s)\ \|u(t)\|_\infty^q \leq C(|\mathcal{P}(0)|)\ \|u(s)\|_\infty\,, \quad 0 \leq s < t < T_\star\,. \tag{3.19}$$

Now, for $T \in (0, T_\star)$, we put

$$m(T) := \inf_{s\in[0,T)} \left\{ \frac{\|u(s)\|_\infty}{(T-s)^{1/(1-q)}} \right\} > 0\,.$$

We infer from (3.19) that, for $s \in [0,T)$ and $t \in (s,T)$,

$$\begin{aligned} \frac{(t-s)(T-t)^{q/(1-q)}}{(T-s)^{1/(1-q)}} \left(\frac{\|u(t)\|_\infty}{(T-t)^{1/(1-q)}} \right)^q &= \frac{t-s}{(T-s)^{1/(1-q)}}\ \|u(t)\|_\infty^q \\ &\le C(|\mathcal{P}(0)|)\ \frac{\|u(s)\|_\infty}{(T-s)^{1/(1-q)}}\,. \end{aligned}$$

Choosing $t = (1-q)T + qs \in (s,T)$ in the previous inequality leads us to

$$C(|\mathcal{P}(0)|)\ m(T)^q \le \frac{\|u(s)\|_\infty}{(T-s)^{1/(1-q)}}\,, \quad s \in [0,T)\,.$$

Therefore, $C(|\mathcal{P}(0)|)\ m(T)^q \le m(T)$ and the positivity of $m(T)$ allows us to conclude that $m(T) \ge C(|\mathcal{P}(0)|) > 0$. We have thus proved that $\|u(s)\|_\infty \ge C(|\mathcal{P}(0)|)\ (T-s)^{1/(1-q)}$ for $s \in [0,T)$ and $T \in (0,T_\star)$ with a positive constant $C(|\mathcal{P}(0)|)$ which does not depend on T. We then let $T \to T_\star$ in the previous inequality to complete the proof of Proposition 3.5.

Bibliographical notes

Theorem 3.1 improves [13, Theorem 1] which only deals with the case $m = 0$. The proof given here is also simpler. The extinction in finite time of non-negative solutions to (1.1), (1.2) with a compactly supported initial condition u_0 is also established in [31, Corollary 9.1] by a comparison argument but with a different supersolution (travelling wave). Proposition 3.5 seems to be new and is a first step towards a better understanding of the behaviour near the extinction time.

CHAPTER 4

Temporal decay estimates for integrable initial data: $\sigma = 1$

Throughout this section we assume that

$$\sigma = 1 \quad \text{and} \quad u_0 \quad \text{is a non-negative function in} \quad L^1(\mathbb{R}^N) \cap \mathcal{BC}(\mathbb{R}^N) \tag{4.1}$$

and denote by u the corresponding classical solution to (1.1), (1.2). We then investigate the time behaviour of the L^p-norms of u for $p = 1$ and $p = \infty$. As a first step, we check that $u(t)$ actually remains in $L^1(\mathbb{R}^N)$ for $t > 0$.

LEMMA 4.1. *If $q > 0$, then $u \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N))$ and $t \longmapsto \|u(t)\|_1$ is a non-increasing function of time with*

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 \in [0, \|u_0\|_1] \,. \tag{4.2}$$

In addition, $|\nabla u|^q \in L^1(Q_\infty)$ and

$$\int_0^\infty \int_{\mathbb{R}^N} |\nabla u(t,x)|^q \; dxdt \le \|u_0\|_1 \,. \tag{4.3}$$

We now turn to the time evolution of the L^p-norms of u for $p = 1$ and $p = \infty$. The main issue here is to figure out whether the additional dissipative mechanism $|\nabla u|^q$ speeds up the convergence to zero in $L^p(\mathbb{R}^N)$ for $p = \infty$. For that purpose, we introduce the following positive real numbers:

$$q_\star := \frac{N+2}{N+1} \,, \tag{4.4}$$

$$\begin{cases} a := \dfrac{(N+1)(q_\star - q)}{2(q-1)} & \text{for} \quad q \in (1, q_\star) \\[2ex] b := \dfrac{1}{q(N+1) - N} & \text{for} \quad q > \dfrac{N}{N+1} \,, \end{cases} \tag{4.5}$$

and first study the behaviour of the L^∞-norm of u.

PROPOSITION 4.2. *[13]*

(a): *If $q = N/(N+1)$, then there is a constant C_{10} such that*

$$\|u(t)\|_\infty \le C_{10} \; \|u_0\|_\infty \; \exp\left\{ -C_{10} \; \|u_0\|_1^{-1/(N+1)} \; t \right\}, \quad t \in (0,\infty) \,. \tag{4.6}$$

(b): *If $q \in (N/(N+1), q_\star]$, then there is a constant C_{10} such that*

$$\|u(t)\|_\infty \le C_{10} \; \|u_0\|_1^{qb} \; t^{-Nb} \,, \quad t \in (0,\infty) \,. \tag{4.7}$$

(c): *If $q \geq q_\star$, then there is a constant C_{10} such that*

$$\|u(t)\|_\infty \leq C_{10}\ \|u_0\|_1\ t^{-N/2}\,, \quad t \in (0,\infty)\,. \tag{4.8}$$

Observe that, for $q \in [N/(N+1), q_\star)$ the temporal decay estimates (4.6) and (4.7) are faster for large times than the one which follows from the heat equation by comparison, thus showing the influence of the absorption term when $q < q_\star$. The decay to zero is even faster for $q \in (0, N/(N+1))$ as u vanishes identically after a finite time by Theorem 3.1 (with $m = 0$). For $q > q_\star$, the absorption term does not speed up the convergence of $\|u(t)\|_\infty$ to zero.

We next investigate the behaviour of the L^1-norm of u.

Proposition 4.3. *[11, 14, 19] We have*

$$I_1(\infty) > 0 \Longleftrightarrow q > q_\star\,, \tag{4.9}$$

where the critical exponent $q_\star$ is given by (4.4).

Recalling that the L^1-norm of non-negative solutions to the linear heat equation remains constant (and positive if $u_0 \not\equiv 0$) throughout time evolution, a consequence of Proposition 4.3 is that the absorption is sufficiently strong for $q \in (0, q_\star]$ so as to drive the L^1-norm of $u(t)$ to zero as $t \to \infty$.

When $q \in (1, q_\star)$, it is actually possible to obtain more precise information on the convergence to zero of $\|u(t)\|_1$.

Proposition 4.4. *[2] If $q \in (1, q_\star)$, there is C_{11} such that*

$$\|u(t)\|_1 \leq C_{11}\ \left(\int_{\left\{|x| \geq t^{1/2}\right\}} u_0(x)\ dx + t^{-a} \right)\,, \quad t \geq 0\,. \tag{4.10}$$

As a consequence of Proposition 4.4 we realize that, if the initial condition u_0 is such that

$$\sup_{R>0} \left\{ R^{2a}\ \int_{\{|x| \geq R\}} u_0(x)\ dx \right\} < \infty\,, \tag{4.11}$$

then $\|u(t)\|_1 \leq C\ t^{-a}$ for $t > 0$. In addition, since (1.1) is autonomous, we infer from Proposition 4.2 **(b)** that

$$\|u(t)\|_\infty \leq 2^{Nb}\ C_{10}\ \left\| u\left(\frac{t}{2}\right) \right\|_1^{qb}\ t^{-Nb} \leq C\ t^{-(2a+N)/2}$$

for $t > 0$. It is however not possible for a solution to (1.1), (1.2) to decay to zero in $L^\infty(\mathbb{R}^N)$ at a faster algebraic rate as the following result shows:

Proposition 4.5. *[13, 19] Assume that*

(a): *either $q \in (1, q_\star]$ and there is $\alpha > a$ such that*

$$\|u(t)\|_{L^\infty} \leq C\ t^{-(N/2)-\alpha} \quad \text{for } t \geq 1,$$

(b): *or $q = q_\star$ and there is $\gamma > N+1$ such that*

$$\|u(t)\|_{L^\infty} \leq C\ t^{-N/2}\ (\log t)^{-\gamma} \quad \text{for } t \geq 1,$$

then $u \equiv 0$.

Combining the outcome of Proposition 4.4 and Proposition 4.5 we conclude that, if $q \in (1, q_\star)$, there are initial data for which $\|u(t)\|_1$ decays as t^{-a} but $\|u(t)\|_1$ cannot decay at a faster algebraic rate. This is in sharp contrast with the case $q = 1$ for which exponential decay rates are possible [15, 16]. Nevertheless, $\|u(t)\|_1$ also cannot decay arbitrarily fast in that case since

$$\|u(t)\|_1 \geq C\ t^{-3/2}\ e^{-t/4}\,, \quad t \geq 1\,,$$

if $N = 1$ and

$$\|u(t)\|_1 \geq e^{-(Ct^{1/3}+t)/4}\,, \quad t \geq 1\,,$$

if $N \geq 2$.

1. Decay rates

We first state an easy consequence of the comparison principle.

Lemma 4.6. *There is a constant C such that, for $q \in (0,\infty)$ and $t > 0$,*

$$\begin{aligned} \|u(t)\|_\infty &\leq C\ \|u_0\|_1\ t^{-N/2}\,, && (4.12)\\ \|\nabla u(t)\|_\infty &\leq C\ \|u_0\|_1\ t^{-(N+1)/2}\,. && (4.13) \end{aligned}$$

Proof. Since $\sigma = 1$, u is a subsolution to the linear heat equation and the comparison principle ensures that $u(t,x) \leq \left(e^{t\Delta}u_0\right)(x)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$. Since u_0 is non-negative, we infer from the temporal decay estimates for integrable solutions to the linear heat equation that

$$\|u(t)\|_\infty \leq \left\|e^{t\Delta}u_0\right\|_\infty \leq C\ \|u_0\|_1\ t^{-N/2}$$

for $t > 0$. It next follows from (2.4) with $s = t/2$ and the previous inequality that

$$\|\nabla u(t)\|_\infty \leq C_1\ \left\|u\left(\frac{t}{2}\right)\right\|_\infty\ \left(\frac{2}{t}\right)^{1/2} \leq C\ \|u_0\|_1\ t^{-(N+1)/2}$$

for $t > 0$. □

We next use the gradient estimate (2.5) to obtain another decay rate.

Lemma 4.7. *There is a constant C such that, for $q \in (N/(N+1),\infty)$, $q \neq 1$, and $t > 0$,*

$$\begin{aligned} \|u(t)\|_\infty &\leq C\ \|u_0\|_1^{qb}\ t^{-Nb}\,, && (4.14)\\ \|\nabla u(t)\|_\infty &\leq C\ \|u_0\|_1^{b}\ t^{-(N+1)b}\,. && (4.15) \end{aligned}$$

Proof. For $t > 0$, we infer from (2.5) (with $s = t/2$), Lemma 4.1 and the Gagliardo–Nirenberg inequality (3.9) that

$$\begin{aligned} \|u(t)\|_\infty &\leq C\ \|u(t)\|_1^{1/(N+1)}\ \|\nabla u(t)\|_\infty^{N/(N+1)}\\ &\leq C\ \|u_0\|_1^{1/(N+1)}\ \left\|u\left(\frac{t}{2}\right)\right\|_\infty^{N/q(N+1)}\ \left(\frac{2}{t}\right)^{N/q(N+1)}\,. \end{aligned}$$

Multiplying both sides of the above inequality by t^{Nb} leads to

$$t^{Nb}\ \|u(t)\|_\infty \leq C\ \|u_0\|_1^{1/(N+1)}\ \left\{\left(\frac{t}{2}\right)^{Nb}\ \left\|u\left(\frac{t}{2}\right)\right\|_\infty\right\}^{N/q(N+1)}$$

for $t > 0$. Introducing

$$\omega(T) := \sup_{t\in[0,T]} \left\{ t^{Nb}\ \|u(t)\|_\infty \right\}$$

for $T > 0$, it follows from the previous inequality that, for $t \in [0,T]$,

$$t^{Nb}\ \|u(t)\|_\infty \le C\ \|u_0\|_1^{1/(N+1)}\ \omega(T)^{N/q(N+1)}\,.$$

Consequently,

$$\begin{aligned} \omega(T) &\le C\ \|u_0\|_1^{1/(N+1)}\ \omega(T)^{N/q(N+1)}\,,\\ \omega(T) &\le C\ \|u_0\|_1^{qb}\,, \end{aligned}$$

and thus

$$\|u(t)\|_\infty \le C\ \|u_0\|_1^{qb}\ t^{-Nb}$$

for $t \in [0,T]$. Since T is arbitrary, we have proved (4.14). Using again (2.5) (with $s = t/2$) and (4.14), we further obtain that

$$\|\nabla u(t)\|_\infty \le C_2\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{1/q}\ \left(\frac{2}{t}\right)^{1/q} \le C\ \|u_0\|_1^b\ t^{-(N+1)b}$$

for $t > 0$. □

The temporal decay rates stated in (b) and (c) of Proposition 4.2 for $q \in (N/(N+1), \infty)$, $q \ne 1$, follow from Lemma 4.6 for $q \ge q_\star$ and Lemma 4.7 for $q < q_\star$. We next turn to the remaining cases $q = N/(N+1)$ and $q = 1$.

1.1. Decay rates: $q = N/(N+1)$. We proceed as in the proof of Lemma 4.7 and infer from (2.5), Lemma 4.1 and the Gagliardo–Nirenberg inequality (3.9) that, for $t > s > 0$,

$$\begin{aligned} \|u(t)\|_\infty &\le C\ \|u(t)\|_1^{1/(N+1)}\ \|\nabla u(t)\|_\infty^{N/(N+1)}\\ &\le C\ \|u_0\|_1^{1/(N+1)}\ \|u(s)\|_\infty^{N/q(N+1)}\ (t-s)^{-N/q(N+1)}\\ &\le C\ \|u_0\|_1^{1/(N+1)}\ \|u(s)\|_\infty\ (t-s)^{-1}\,. \end{aligned}$$

Let B be a positive real number to be specified later and assume that $t > B$. Choosing $s = t - B > 0$ and multiplying both sides of the above inequality by $e^{t/B}$ lead to

$$e^{t/B}\ \|u(t)\|_\infty \le \frac{C\ e}{B}\ \|u_0\|_1^{1/(N+1)}\ \left\{ e^{(t-B)/B}\ \|u(t-B)\|_\infty \right\}$$

for $t > B$. Introducing

$$\omega(T) := \sup_{t\in[0,T]} \left\{ e^{t/B}\ \|u(t)\|_\infty \right\}$$

for $T > B$, it follows from the previous inequality that, for $t \in (B,T]$,

$$e^{t/B}\ \|u(t)\|_\infty \le \frac{C\ e}{B}\ \|u_0\|_1^{1/(N+1)}\ \omega(T)\,,$$

while, for $t \in [0,B]$,

$$e^{t/B}\ \|u(t)\|_\infty \le e\ \|u_0\|_\infty$$

by (2.1). Therefore,

$$\omega(T) \le \frac{C\ e}{B}\ \|u_0\|_1^{1/(N+1)}\ \omega(T) + e\ \|u_0\|_\infty\,.$$

Choosing $B = 2\ C\ e\ \|u_0\|_1^{1/(N+1)}$ we end up with

$$\omega(T) \le 2\ e\ \|u_0\|_\infty\,.$$

As T is arbitrary we conclude that

$$\|u(t)\|_\infty \le 2\ e\ \|u_0\|_\infty\ \exp\left\{ \frac{t}{2\ C\ e\ \|u_0\|_1^{1/(N+1)}} \right\}$$

for $t \ge 0$ and the proof of (4.6) is complete. □

1.2. Decay rates: $q = 1$**.** We give two proofs of Proposition 4.2 for $q = 1$, the first one relying on a Moser technique [13] and the second one on the L^1-euclidean logarithmic Sobolev inequality.

First proof of Proposition 4.2: $q = 1$. We employ a Moser technique as in [26, Section 4]. Consider $r \ge 1$, $s_2 \in (0, \infty)$ and $s_1 \in [0, s_2)$. It follows from (1.1) after multiplication by $r\ u^{r-1}$ and integration over $(s_1, s_2) \times \mathbb{R}^N$ that

$$\|u(s_2)\|_r^r + \int_{s_1}^{s_2} \int_{\mathbb{R}^N} |\nabla (u^r)(s,x)|\ dxds \le \|u(s_1)\|_r^r\,.$$

We next use the Sobolev inequality to obtain that

$$\int_{s_1}^{s_2} \|u^r(s)\|_{1^*}\ ds \le C\ \|u(s_1)\|_r^r, \tag{4.16}$$

where $1^* = \infty$ if $N = 1$ and $1^* = N/(N-1)$ otherwise.

Fix $t \in (0, \infty)$. If $N = 1$ we choose $r = 1$, $s_1 = 0$ and $s_2 = t$ in (4.16) and use the monotonicity of $s \longmapsto \|u(s)\|_\infty$ to obtain (4.7) for $q = 1$. Assume now that $N \ge 2$. As u is non-negative $s \mapsto \|u(s)\|_m$ is a non-increasing function for each $m \in [1, \infty]$ and we infer from (4.16) that

$$\|u(s_2)\|_{rN/(N-1)} \le C^{1/r}\ (s_2 - s_1)^{-1/r}\ \|u(s_1)\|_r, \quad 0 \le s_1 < s_2. \tag{4.17}$$

Introducing for $k \ge 0$ the sequences $r_k = (N/(N-1))^k$ and $t_k = t(1 - 2^{-(k+1)})$, we proceed as in [26, Section 4] and write (4.17) with $r = r_k$, $s_1 = t_k$ and $s_2 = t_{k+1}$. Arguing by induction we eventually arrive at the following inequality:

$$\|u(t)\|_{r_k} \le \|u(t_k)\|_{r_k} \le C^{\alpha_k}\ 2^{\beta_k}\ t^{-\alpha_k}\ \|u(t/2)\|_1 \tag{4.18}$$

for $k \ge 1$ with

$$\alpha_k := \sum_{n=0}^{k-1} \left(\frac{N-1}{N} \right)^n \quad \text{and} \quad \beta_k := \sum_{n=0}^{k-1} (n+2)\ \left(\frac{N-1}{N} \right)^n, \quad k \ge 0.$$

We may now let $k \to \infty$ in (4.18) to obtain (4.7) for $q = 1$.

□

We now give an alternative proof of Proposition 4.2 for $q = 1$ employing the L^1-euclidean logarithmic Sobolev inequality [8, Theorem 2] which we recall now.

THEOREM 4.8. *For each $w \in W^{1,1}(\mathbb{R}^N)$ and $\mu \in (0,\infty)$, we have*

$$\int_{\mathbb{R}^N} |w(x)| \ \log\left(\frac{|w(x)|}{\|w\|_1}\right)\ dx + N\ \log\left(\frac{\mu e}{N\mathcal{L}_1}\right)\ \|w\|_1 \le \mu\ \|\nabla w\|_1 \tag{4.19}$$

with $\mathcal{L}_1 := \Gamma((N+2)/2)^{1/N}/\left(N\pi^{1/2}\right)$.

SECOND PROOF OF PROPOSITION 4.2: $q=1$. Fix $t \in (0,\infty)$ and let $\varrho \in \mathcal{C}^1([0,t))$ be an increasing function such that $\varrho(0) = 1$ and $\varrho(s) \to \infty$ as $s \to t$. Introducing $F(s) := \|u(s)\|_{\varrho(s)}$ for $s \in [0,t)$, we infer from (1.1) that

$$\begin{aligned}
\frac{F'(s)}{F(s)} &= \frac{\varrho'(s)}{\varrho(s)^2}\int_{\mathbb{R}^N} \frac{u(s,x)^{\varrho(s)}}{F(s)^{\varrho(s)}}\ \log\left(\frac{u(s,x)^{\varrho(s)}}{F(s)^{\varrho(s)}}\right) dx\\
&+ \int_{\mathbb{R}^N} \frac{u(s,x)^{\varrho(s)-1}}{F(s)^{\varrho(s)}}\ \left(\Delta u(s,x) - |\nabla u(s,x)|\right)\ dx\\
&\le \frac{\varrho'(s)}{\varrho(s)^2}\int_{\mathbb{R}^N} \frac{u(s,x)^{\varrho(s)}}{F(s)^{\varrho(s)}}\ \log\left(\frac{u(s,x)^{\varrho(s)}}{F(s)^{\varrho(s)}}\right) dx\\
&- \frac{1}{\varrho(s)}\int_{\mathbb{R}^N} \frac{\left|\nabla u^{\varrho(s)}(s,x)\right|}{F(s)^{\varrho(s)}}\ dx\,.
\end{aligned}$$

It then follows from (4.19) with $w = u^{\varrho(s)}$ and $\mu = \varrho(s)/\varrho'(s)$ that

$$\frac{F'(s)}{F(s)} \le N\ \frac{\varrho'(s)}{\varrho(s)^2}\ \log\left(\frac{N\mathcal{L}_1}{e}\ \frac{\varrho'(s)}{\varrho(s)}\right), \quad s \in [0,t)\,.$$

With the choice $\varrho(s) = t/(t-s)$, we obtain

$$\frac{F'(s)}{F(s)} \le -\frac{N}{t}\ \log\left(\frac{e}{N\mathcal{L}_1}\ (t-s)\right), \quad s \in [0,t)\,,$$

which yields after integration

$$\begin{aligned}
\log F(s) &\le \log F(0) - \frac{N}{t}\ \left(\log\left(\frac{e}{N\mathcal{L}_1}\right) s + (t-s) - (t-s)\ \log(t-s)\right)\\
&+ N\ (1-\log t)
\end{aligned}$$

for $s \in [0,t)$. We then let $s \to t$ in the previous inequality and end up with

$$\log\|u(t)\|_\infty \le \log\|u_0\|_1 + N\ \log(N\mathcal{L}_1) - N\ \log t\,,$$

that is, (4.7) for $q=1$ with $C_{10} = (N\mathcal{L}_1)^N$.

□

2. Limit values of $\|u\|_1$

LEMMA 4.9. *[19, Lemma 3.2] If $q > 1$ and $u_0 \not\equiv 0$ then $\|u(t)\|_1 > 0$ for every $t \ge 0$.*

PROOF. Since u is a classical solution to $\partial_t u - \Delta u + A\cdot\nabla u = 0$ in Q_∞ with $A := |\nabla u|^{q-2}\ \nabla u \in \mathcal{C}(Q_\infty;\mathbb{R}^N)$, the assertion of Lemma 4.9 readily follows from the strong maximum principle.

□

PROOF OF PROPOSITION 4.3: $q > q_\star$. Combining (2.6) (with the choice $s = t/2$) and (4.12) yields

$$\left\|\nabla u^{(q-1)/q}(t)\right\|_{L^\infty} \le C\ \|u_0\|_1^{(q-1)/q}\ t^{-(q(N+1)-N)/(2q)}\,, \quad t > 0\,. \tag{4.20}$$

Consider now $s \in (0,\infty)$ and $t \in (s,\infty)$. Since $\nabla u = (q/(q-1))\ u^{1/q}\ \nabla u^{(q-1)/q}$, it follows from (1.1) and (4.20) that

$$\begin{aligned}\|u(t)\|_1 &= \|u(s)\|_1 - \left(\frac{q}{q-1}\right)^q \int_s^t \int_{\mathbb{R}^N} \left|\nabla u^{(q-1)/q}(\tau,x)\right|^q\ u(\tau,x)\ dxd\tau \\ &\ge \|u(s)\|_1 - C \int_s^t \tau^{-(q(N+1)-N)/2}\ \|u(\tau)\|_1\ d\tau\,.\end{aligned}$$

Owing to the monotonicity of $\tau \longmapsto \|u(\tau)\|_1$, we further obtain

$$\|u(t)\|_1 \ge \|u(s)\|_1\ \left(1 - C\ \int_s^t \tau^{-(q(N+1)-N)/2}\ d\tau\right)\,.$$

Since $q > q_\star$ the right-hand side of the above inequality has a finite limit as $t \to \infty$. We may then let $t \to \infty$ and use (4.2) to obtain

$$I_1(\infty) \ge \|u(s)\|_{L^1}\ \left(1 - C\ s^{-(N+1)(q-q_\star)/2}\right)\,, \quad s > 0\,.$$

Consequently, for s large enough, we have $I_1(\infty) > \|u(s)\|_1/2$ which is positive by Lemma 4.9.

□

PROOF OF PROPOSITION 4.3: $q \in [1, q_\star]$. It first follows from (4.3) that

$$\omega(t) := \int_t^\infty \|\nabla u(s)\|_q^q\ ds \underset{t\to\infty}{\longrightarrow} 0\,. \tag{4.21}$$

We next consider a $\mathcal{C}^\infty$-smooth function ϱ in $\mathbb{R}^N$ such that $0 \le \varrho \le 1$ and

$$\varrho(x) = 0 \ \text{ if } \ |x| \le 1 \ \text{ and } \ \varrho(x) = 1 \ \text{ if } \ |x| \ge 2\,.$$

For $R > 0$ and $x \in \mathbb{R}^N$ we put $\varrho_R(x) = \varrho(x/R)$. We multiply (1.1) by $\varrho_R(x)$ and integrate over $(t_1,t_2) \times \mathbb{R}^N$ to obtain

$$\int_{\mathbb{R}^N} u(t_2,x)\ \varrho_R(x)\ dx \le \int_{\mathbb{R}^N} u(t_1,x)\ \varrho_R(x)\ dx - \frac{1}{R}\int_{t_1}^{t_2} \nabla\varrho\left(\frac{x}{R}\right)\ \nabla u(s,x)\ dxds\,.$$

Using the Hölder inequality and the properties of ϱ we deduce that

$$\int_{\{|x|\ge 2R\}} u(t_2,x)\ dx \le \int_{\{|x|\ge R\}} u(t_1,x)\ dx + C\ R^{(Nq-N-q)/q}\ \omega(t_1)^{1/q}\ (t_2-t_1)^{(q-1)/q}\,.$$

Combining the above inequality with (4.7) and Lemma 4.1 yields

$$\begin{aligned} \|u(t_2)\|_1 &= \int_{\{|x|\le 2R\}} u(t_2,x)\ dx + \int_{\{|x|\ge 2R\}} u(t_2,x)\ dx \\ &\le C\ R^N\ \|u(t_2)\|_\infty + \int_{\{|x|\ge R\}} u(t_1,x)\ dx \\ &+ C\ R^{(Nq-N-q)/q}\ \omega(t_1)^{1/q}\ (t_2-t_1)^{(q-1)/q} \\ &\le \int_{\{|x|\ge R\}} u(t_1,x)\ dx + C\ R^N\ (t_2-t_1)^{-Nb} \\ &+ C\ R^{(Nq-N-q)/q}\ \omega(t_1)^{1/q}\ (t_2-t_1)^{(q-1)/q}. \end{aligned}$$

Choosing

$$R = R(t_1,t_2) := \omega(t_1)^{1/(q+N)}\ (t_2-t_1)^{(qNb+q-1)/(q+N)}$$

we are led to

$$\begin{aligned} \|u(t_2)\|_1 &\le \int_{\{|x|\ge R(t_1,t_2)\}} u(t_1,x)\ dx \\ &+ C\ \omega(t_1)^{N/(q+N)}\ (t_2-t_1)^{-qNb(N+1)(q_\star-q)/(q+N)}. \end{aligned}$$

Since $q_\star \ge q \ge 1 > N/(N+1)$ (so that $b>0$) we may let $t_2 \to \infty$ in the previous inequality to conclude that $I_1(\infty) \le 0$ if $q \in [1,q_\star)$ and $I_1(\infty) \le C\ \omega(t_1)^{N/(q_\star+N)}$ if $q = q_\star$. We have used here that $R(t_1,t_2) \to \infty$ as $t_2 \to \infty$ and that $u(t_1) \in L^1(\mathbb{R}^N)$. Owing to the non-negativity of $I_1(\infty)$, we readily obtain that $I_1(\infty)=0$ if $q\in[1,q_\star)$. When $q=q_\star$, we let $t_1\to\infty$ and use (4.21) to conclude that $I_1(\infty)=0$ also in that case.

□

Proof of Proposition 4.3: $q \in [N/(N+1),1)$. As in the previous proof we consider a $\mathcal{C}^\infty$-smooth function ϱ in $\mathbb{R}^N$ such that $0\le\varrho\le 1$ and

$$\varrho(x) = 0 \ \text{ if } \ |x|\le 1 \quad \text{and} \quad \varrho(x)=1 \quad \text{if} \quad |x|\ge 2\,.$$

For $R>0$ and $x\in\mathbb{R}^N$ we put $\varrho_R(x)=\varrho(x/R)$. We multiply (1.1) by $\varrho_R(x)$ and integrate over $(1,t)\times\mathbb{R}^N$ to obtain

$$\begin{aligned} \int_{\{|x|\ge 2R\}} u(t,x)\ dx &\le \int_{\{|x|\ge R\}} u(1,x)\ dx \\ &+ \frac{\|\nabla\varrho\|_\infty}{R} \int_1^t \|\nabla u(s)\|_\infty^{1-q} \int_{\mathbb{R}^N} |\nabla u(s)|^q\ dxds\,. \end{aligned}$$

We then infer from (2.5) and (4.3) that

$$\begin{aligned}\int_{\{|x|\ge 2R\}} u(t,x)\ dx &\le \int_{\{|x|\ge R\}} u(1,x)\ dx \\ &+ \frac{C}{R} \int_1^\infty s^{-(1-q)/q} \int_{\mathbb{R}^N} |\nabla u(s)|^q\ dxds \\ &\le \int_{\{|x|\ge R\}} u(1,x)\ dx + \frac{C}{R} \int_1^\infty \int_{\mathbb{R}^N} |\nabla u(s)|^q\ dxds \\ &\le \int_{\{|x|\ge R\}} u(1,x)\ dx + \frac{C}{R}\,.\end{aligned}$$

Therefore

$$\|u(t)\|_1 \le C\ R^N\ \left\|e^{t\Delta} u_0\right\|_\infty + \int_{\{|x|\ge R\}} u(1,x)\ dx + \frac{C}{R}\,,$$

and the choice $R = R(t) := \left\|e^{t\Delta} u_0\right\|_\infty^{-1/(N+1)}$ yields

$$\|u(t)\|_1 \le \int_{\{|x|\ge R(t)\}} u(1,x)\ dx + \left\|e^{t\Delta} u_0\right\|_\infty^{1/(N+1)}\ .$$

Letting $t \to \infty$ gives the result since $\left\|e^{t\Delta} u_0\right\|_\infty \to 0$ as $t \to \infty$. □

Proof of Proposition 4.3: $q \in (0, N/(N+1))$. As $u(t)$ vanishes identically after a finite time by Theorem 3.1 (with $m = 0$), we clearly have $I_1(\infty) = 0$. □

3. Improved decay rates: $q \in (1, q_\star)$

Proof of Proposition 4.4. We fix $T > 0$. For $R > 0$ and $t \in [0, T]$, we have

$$\|u(t)\|_1 = \int_{\{|x|\le 2R\}} u(t,x)\ dx + \int_{\{|x|>2R\}} u(t,x)\ dx\,.$$

On the one hand, it follows from (4.7) that

$$\int_{\{|x|\le 2R\}} u(t,x)\ dx \le C\ R^N\ \|u(t)\|_\infty \le C\ R^N\ \left\|u\left(\frac{t}{2}\right)\right\|_1^{qb}\ t^{-Nb}\,. \qquad (4.22)$$

On the other hand, consider a $\mathcal{C}^\infty$-smooth function ζ in $\mathbb{R}^N$ such that $0 \le \zeta \le 1$ and

$$\zeta(x) = 0 \ \text{ if } \ |x| \le 1 \quad \text{and} \quad \zeta(x) = 1 \ \text{ if } \ |x| \ge 2\,.$$

For $x \in \mathbb{R}^N$ we put $\zeta_R(x) = \zeta(x/R)$. It follows from (1.1) and the Hölder and Young inequalities that

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-1)}\ u(t,x)\ dx + \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-1)}\ |\nabla u(t,x)|^q\ dx \\
= & -\frac{q}{q-1} \int_{\mathbb{R}^N} \zeta_R(x)^{1/(q-1)}\ \nabla\zeta_R(x) \cdot \nabla u(t,x)\ dx \\
\leq & \ C\ R^{((N-1)q-N)/q} \left(\int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-1)}\ |\nabla u(t,x)|^q\ dx \right)^{1/q} \\
\leq & \ \frac{1}{2} \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-1)}\ |\nabla u(t,x)|^q\ dx + C\ R^{((N-1)q-N)/(q-1)},
\end{aligned}$$

whence, after integration with respect to time,

$$\int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-1)}\ u(t,x)\ dx \leq \int_{\mathbb{R}^N} \zeta_R(x)^{q/(q-1)}\ u_0(x)\ dx + C\ t\ R^{((N-1)q-N)/(q-1)} .$$

Using the properties of ζ and since $t \in [0,T]$, we end up with

$$\int_{\{|x|>2R\}} u(t,x)\ dx \leq \int_{\{|x|\geq R\}} u_0(x)\ dx + C\ T\ R^{((N-1)q-N)/(q-1)} . \tag{4.23}$$

Collecting (4.22) and (4.23), we obtain

$$\begin{aligned}
\|u(t)\|_1 \ \leq & \int_{\{|x|\geq R\}} u_0(x)\ dx + C\ T\ R^{((N-1)q-N)/(q-1)} \\
& + \ C\ R^N \left\| u\left(\frac{t}{2}\right) \right\|_1^{qb} t^{-Nb} .
\end{aligned} \tag{4.24}$$

Now, we put

$$\begin{aligned}
z(t) \ & := \ t^{1/(q-1)}\ R^{-1/b(q-1)}\ \|u(t)\|_1 , \quad t \in [0,T] \\
Y(T,R) \ & := \ T^{1/(q-1)}\ R^{-1/b(q-1)} \left(\int_{\{|x|\geq R\}} u_0(x)\ dx + C\ T\ R^{((N-1)q-N)/(q-1)} \right) .
\end{aligned}$$

Multiplying both sides of (4.24) by $t^{1/(q-1)}\ R^{-1/b(q-1)}$ we obtain for $t \in [0,T]$

$$\begin{aligned}
z(t) \ & \leq \ Y(T,R) \left(\frac{t}{T}\right)^{1/(q-1)} + C \left(\frac{t^b}{R}\right)^{(1-Nb(q-1)-qb)/b(q-1)} \left\{ z\left(\frac{t}{2}\right) \right\}^{qb} \\
& \leq \ Y(T,R) + C \left\{ z\left(\frac{t}{2}\right) \right\}^{qb}
\end{aligned}$$

as $Nb(q-1) - 1 + qb = 0$ by the definition (4.5) of b. Consequently,

$$\sup_{[0,T]} \{z\} \leq Y(T,R) + C \left(\sup_{[0,T]} \{z\} \right)^{qb} .$$

Since $q > 1$, we have $qb < 1$ and infer from the Young inequality that

$$\begin{aligned}
\sup_{[0,T]} \{z\} \ & \leq \ Y(T,R) + C + \frac{1}{2} \sup_{[0,T]} \{z\} \\
\sup_{[0,T]} \{z\} \ & \leq \ 2\ Y(T,R) + C .
\end{aligned}$$

In particular, $z(T) \leq 2\, Y(T,R) + C$, that is,

$$\|u(T)\|_1 \leq 2 \int_{\{|x|\geq R\}} u_0(x)\, dx + C\, T\, R^{((N-1)q-N)/(q-1)} + C\, R^{1/((q-1)b)}\, T^{-1/(q-1)}\,.$$

The choice $R = T^{1/2}$ then yields (4.10). $\square$

We finally show that, when $q \in (1, q_\star]$, the L^∞-norm of u cannot decay to zero at an arbitrary fast rate.

Proof of Proposition 4.5. It follows from (1.1) that, for $t \geq s \geq 2$,

$$\|u(s)\|_1 - \|u(t)\|_1 \leq C \int_s^t \left\|\nabla\left(u^{(q-1)/q}\right)(\tau)\right\|_\infty^q \|u(\tau)\|_1\, d\tau\,.$$

We infer from (2.6) (with $s = \tau/2$) and the time monotonicity of the L^1-norm of u that

$$\|u(s)\|_1 - \|u(t)\|_1 \leq C\, \|u(s)\|_1 \int_{s/2}^t \|u(\tau)\|_\infty^{q-1}\, \tau^{-q/2}\, d\tau\,.$$

But it is easy to check that the assumptions (a) or (b) imply that $\tau \mapsto \|u(\tau)\|_\infty^{q-1}\, \tau^{-q/2}$ belongs to $L^1(1,\infty)$. Consequently we may find s_0 large enough such that

$$\|u(s_0)\|_1 - \|u(t)\|_1 \leq \frac{1}{2}\, \|u(s_0)\|_1 \quad \text{for} \quad t \geq s_0\,.$$

We then pass to the limit as $t \to \infty$ and deduce from (4.9) that $\|u(s_0)\|_1 \leq 2\, I_1(\infty) = 0$, whence $u(s_0) \equiv 0$. Applying Lemma 4.9 we conclude that $u \equiv 0$. $\square$

Bibliographical notes

For $q \in (1,2)$, an alternative proof of Lemma 4.9 relying on L^p-estimates may be found in [2, Lemma 4.1] (and applies also if the Laplacian is replaced by a degenerate diffusion). Proposition 4.2 is proved in [13] but the argument given here is simpler. The alternative proof of Proposition 4.2 for $q = 1$ built upon the L^1-euclidean logarithmic Sobolev inequality is also new. Proposition 4.3 is proved in [15, 16] when $q = 1$ and in [1] when $q \in (1, q_\star)$ and $q = 2$. The general case $q > 1$ is performed in [11, 19] and is extended to $q \in (0,1]$ in [13, 14]. Next, if $q \in (1, q_\star)$ and u_0 fulfils (4.11), we have $\|u(t)\|_1 \leq C\, t^{-a}$ for $t > 0$ as already mentioned. This property was previously established in [12] under the stronger condition

$$R(u_0) := \inf\left\{R > 0 \ \text{ such that } \sup_{|x|\geq R}\left\{|x|^{(2-q)/(q-1)}\, u_0(x)\right\} \leq \gamma_q\right\} < \infty\,,$$

the proof combining (2.7) and the comparison with the supersolution

$$x \longmapsto \gamma_q\, |x|^{-(2-q)/(q-1)} \quad \text{with} \quad \gamma_q := \left((q-1)(2(1+a))^{1/(q-1)}\right)/(2-q)\,.$$

Proposition 4.4 is proved in [2] with a slightly different proof. It unfortunately does not cover the critical exponent $q = q_\star$. In that case, it has been shown in [30] that, for initial data decaying sufficiently rapidly as $|x| \to \infty$, then $\|u(t)\|_1 \leq C\, (\log t)^{-(N+1)}$ and $\|u(t)\|_\infty \leq C\, (\log t)^{-(N+1)}\, t^{-N/2}$ for $t \geq 1$, the proof relying on completely different arguments (invariant manifold). Note that, according to

Proposition 4.4, $t^{N/2}\|u(t)\|_\infty$ cannot decay at a faster logarithmic rate. Finally, Proposition 4.5 is actually a slight improvement of [19, Corollary 3.5].

CHAPTER 5

Temporal growth estimates for integrable initial data: $\sigma = -1$

Throughout this section we assume that

$$\sigma = -1 \quad \text{and} \quad u_0 \quad \text{is a non-negative function in} \quad L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \tag{5.1}$$

and denote by u the corresponding classical solution to (1.1), (1.2). We then investigate the time behaviour of the L^p-norms of u for $p = 1$ and $p = \infty$. As a first step, we specify conditions ensuring that $u(t)$ remains in $L^1(\mathbb{R}^N)$ for $t > 0$.

LEMMA 5.1. *If $q \geq 1$, then $u \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N))$ and $t \longmapsto \|u(t)\|_1$ is a non-decreasing function of time with*

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 \in [\|u_0\|_1, \infty] . \tag{5.2}$$

In addition,

$$\|u(t)\|_1 = \|u_0\|_1 + \int_0^t \|\nabla u(s)\|_q^q \, ds , \quad t \geq 0 . \tag{5.3}$$

REMARK 5.2. *It is not known whether Lemma 5.1 remains valid for $q \in (0,1)$ when $\sigma = -1$.*

When $\sigma = -1$, the nonlinear term $|\nabla u|^q$ is a source term and thus slows down or even impedes the dissipation of the diffusion. In that case, the counterpart of Proposition 4.3 (though less complete and more complicated) reads:

PROPOSITION 5.3. *[9, 31, 39] We have*

$$\begin{cases} I_1(\infty) \in [\|u_0\|_1, \infty) & \text{and} \quad I_\infty(\infty) = 0 \quad \text{if} \quad q \geq 2 , \\ I_1(\infty) \in [\|u_0\|_1, \infty] & \text{and} \quad I_\infty(\infty) \in [0, \|u_0\|_\infty] \quad \text{if} \quad q \in (q_\star, 2) , \\ I_1(\infty) = \infty & \text{and} \quad I_\infty(\infty) \in (0, \|u_0\|_\infty] \quad \text{if} \quad q \in [1, q_\star] , \end{cases} \tag{5.4}$$

where $q_\star$ is defined in (4.4) and

$$I_\infty(\infty) := \lim_{t\to\infty} \|u(t)\|_\infty \in [0, \|u_0\|_\infty] . \tag{5.5}$$

In addition, if $I_\infty(\infty) > 0$ then $I_1(\infty) = \infty$ and

$$\|u(t)\|_1 \geq C\, t^{N/q} , \quad t \geq 0 . \tag{5.6}$$

Observe that, if $q \in [1, q_\star]$, the strength of the source term leads to the unboundedness of the L^1-norm of $u(t)$ as $t \to \infty$. The behaviour of $\|u(t)\|_1$ is much more complex when q ranges in $(q_\star, 2)$ since it is unbounded for large times only for some initial data. Let us also emphasize at this point that the converse of the last statement of Proposition 5.3 is false, that is, there is at least one initial condition u_0 such that $I_1(\infty) = \infty$ and $I_\infty(\infty) = 0$, see Proposition 5.6 below.

Additional properties are gathered in the next proposition.

PROPOSITION 5.4. *[31]*

(a): *If $q \in (q_\star, 2)$ and $I_\infty(\infty) = 0$, then*

$$\|u(t)\|_\infty \le C\ t^{-(2-q)/(2(q-1))}\,, \quad t > 0\,. \tag{5.7}$$

(b): *If $q \in (1,2)$ and $I_\infty(\infty) > 0$, then*

$$0 \le t^{(2-q)/q}\ \left(\|u(t)\|_\infty - I_\infty(\infty)\right) \le C\ \|u_0\|_\infty^{(2-q)/q}\,, \quad t > 0\,. \tag{5.8}$$

(c): *If $q = 1$, then*

$$0 \le \frac{t^{1/2}}{\log t}\ \left(\|u(t)\|_\infty - I_\infty(\infty)\right) \le C\ \|u_0\|_\infty\,, \quad t \ge 2\,. \tag{5.9}$$

1. Limit values of $\|u\|_1$ and $\|u\|_\infty$

This section is devoted to the proof of Proposition 5.3 which requires to handle separately the cases $q \ge 2$, $q \in (q_\star, 2)$ and $q \in [1, q_\star]$.

We first check the last statement of Proposition 5.3. We thus assume that $I_\infty(\infty) > 0$ and fix $t > 0$. For $k \ge 1$, let $x_k \in \mathbb{R}^N$ (depending possibly on t) be such that $\|u(t)\|_\infty - 1/k \le u(t, x_k)$. For $R > 0$, it follows from (2.5) (with $s = 0$) and the time monotonicity of $\|u\|_\infty$ that

$$\begin{aligned}
\|u(t)\|_1 &\ge \int_{\{|x-x_k|\le R\}} u(t,x)\ dx \\
&\ge \int_{\{|x-x_k|\le R\}} \left(u(t,x_k) - |x-x_k|\ \|\nabla u(t)\|_\infty\right)\ dx \\
&\ge C\left(\frac{R^N}{N}\ \left(\|u(t)\|_\infty - \frac{1}{k}\right) - C_1\ \frac{R^{N+1}}{N+1}\ \|u_0\|_\infty^{1/q}\ t^{-1/q}\right) \\
&\ge C\ R^N\ \left(I_\infty(\infty) - \frac{1}{k} - C_1\ R\ t^{-1/q}\right)\,.
\end{aligned}$$

Letting $k \to \infty$ and choosing $R = R(t) := \left(I_\infty(\infty) t^{1/q}\right)/(2C_1)$ gives the result.

1.1. The case $q \ge 2$. We put

$$A := \|u_0\|_\infty\ \|\nabla u_0\|_\infty^{q-2} \quad \text{and} \quad B := \|\nabla u_0\|_\infty^{q-2}\,, \tag{5.10}$$

and consider $t > 0$ and an integer $p \ge 1$. We multiply (1.1) by u^p and integrate over $(0,t) \times \mathbb{R}^N$ to obtain

$$\begin{aligned}
p\int_0^t \int_{\mathbb{R}^N} u^{p-1}\ |\nabla u|^2\ dxds &= \frac{1}{p+1}\ \int_{\mathbb{R}^N} \left(u_0^{p+1} - u^{p+1}(t)\right)\ dx \\
&\quad + \int_0^t \int_{\mathbb{R}^N} u^p\ |\nabla u|^q\ dxds\,,
\end{aligned}$$

whence, thanks to (2.3),

$$\int_0^t \int_{\mathbb{R}^N} u^{p-1}\ |\nabla u|^2\ dxds \le \frac{\|u_0\|_1\ \|u_0\|_\infty^p}{p(p+1)} + \frac{B}{p}\ \int_0^t \int_{\mathbb{R}^N} u^p\ |\nabla u|^2\ dxds\,. \qquad (5.11)$$

We next claim that

$$\int_0^t \int_{\mathbb{R}^N} |\nabla u|^q\ dxds \le \|u_0\|_1\ \sum_{i=1}^k \frac{A^i}{(i+1)!} + \frac{B^{k+1}}{k!}\ \int_0^t \int_{\mathbb{R}^N} u^k\ |\nabla u|^2\ dxds \qquad (5.12)$$

for every integer $k \ge 1$. Indeed, it follows from (2.3) and (5.11) with $p = 1$ that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q\ dxds &\le B\ \int_0^t \int_{\mathbb{R}^N} |\nabla u|^2\ dxds \\ &\le \frac{A}{2}\ \|u_0\|_1 + B^2\ \int_0^t \int_{\mathbb{R}^N} u\ |\nabla u|^2\ dxds \end{aligned}$$

whence (5.12) for $k = 1$. We next proceed by induction and assume that (5.12) holds true for some $k \ge 1$. Inserting (5.11) with $p = k+1$ in (5.12) yields

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q\ dxds &\le \|u_0\|_1\ \sum_{i=1}^k \frac{A^i}{(i+1)!} + \frac{B^{k+1}}{k!}\ \frac{\|u_0\|_1\ \|u_0\|_\infty^{k+1}}{(k+1)(k+2)} \\ &+ \frac{B^{k+2}}{(k+1)!}\ \int_0^t \int_{\mathbb{R}^N} u^{k+1}\ |\nabla u|^2\ dxds \\ &\le \|u_0\|_1\ \sum_{i=1}^{k+1} \frac{A^i}{(i+1)!} + \frac{B^{k+2}}{(k+1)!}\ \int_0^t \int_{\mathbb{R}^N} u^{k+1}\ |\nabla u|^2\ dxds\,, \end{aligned}$$

whence (5.12) for $k+1$ and the proof of (5.12) is complete.

Finally, let $p_0 \ge 1$ be the smallest integer satisfying $p_0 > A$. Using (5.11) with $p = p_0 + 1$ and (2.3), we obtain that

$$(p_0+1)\ \int_0^t \int_{\mathbb{R}^N} u^{p_0}\ |\nabla u|^2\ dxds \le \frac{\|u_0\|_1\ \|u_0\|_\infty^{p_0+1}}{p_0+2} + A\ \int_0^t \int_{\mathbb{R}^N} u^{p_0}\ |\nabla u|^2\ dxds\,,$$

from which we deduce, since $p_0 + 1 - A \ge 1$,

$$\int_0^t \int_{\mathbb{R}^N} u^{p_0}\ |\nabla u|^2\ dxds \le \frac{\|u_0\|_1\ \|u_0\|_\infty^{p_0+1}}{p_0+2}\,.$$

We insert the above estimate in (5.12) with $k = p_0$ and end up with

$$\int_0^t \int_{\mathbb{R}^N} |\nabla u|^q\ dxds \le \|u_0\|_1\ \sum_{i=1}^{p_0} \frac{A^i}{(i+1)!} + \frac{B^{p_0+1}}{p_0!}\ \frac{\|u_0\|_1\ \|u_0\|_\infty^{p_0+1}}{p_0+2} \le \|u_0\|_1\ \sum_{i=1}^{p_0+1} \frac{A^i}{i!}\,.$$

Consequently, $|\nabla u|^q \in L^1(Q_\infty)$ and we may integrate (1.1) over Q_∞ to conclude that

$$I_1(\infty) \le \|u_0\|_1\ \sum_{i=0}^{p_0+1} \frac{A^i}{i!} < \infty\,,$$

and the proof of (5.4) for $q \ge 2$ is complete. □

1.2. The case $q \in (q_\star, 2)$. In that case, the statement of Proposition 5.3 is an obvious consequence of Theorem 2.1 and Lemma 5.1. □

Nevertheless, some additional information on $I_1(\infty)$ and $I_\infty(\infty)$ may be obtained when $q \in (q_\star, 2)$. The first result in that direction is the following:

PROPOSITION 5.5. *Assume that $q \in (q_\star, 2)$. There is $\gamma = \gamma(N, q) > 0$ such that $I_1(\infty) < \infty$ (and thus $I_\infty(\infty) = 0$ by Proposition 5.3) whenever*

$$\|u_0\|_1 \ \|\nabla u_0\|_\infty^{(N+1)(q-q_\star)} \le \gamma\,. \tag{5.13}$$

PROOF. Owing to (5.3), it is clear that $I_1(\infty) < \infty$ whenever $t \longmapsto \|\nabla u(t)\|_q^q \in L^1(0,\infty)$. Such a property is certainly true if we know that

$$t \longmapsto \|\nabla u(t)\|_k^k \in L^1(0,\infty) \quad \text{for some} \quad k \in [1,q] \tag{5.14}$$

since $\|\nabla u\|_q^q \le \|\nabla u_0\|_\infty^{q-k}\ \|\nabla u\|_k^k$ by (2.3). We proceed in two steps. Under a smallness assumption on u_0, we obtain a first temporal decay estimate for $\|\nabla u(t)\|_k$. In a second step, we improve this estimate so as to obtain the required time integrability of $t \longmapsto \|\nabla u(t)\|_k^k$.

Step 1. Fix $k \in (q_\star, q]$ (close to $q_\star$) such that

$$\left\{\begin{array}{l} q_\star < k < \min\left\{\dfrac{N}{N-1}, 2\right\}, \quad p := \dfrac{N(k-1)}{2-k} \in (1,k) \quad \text{and} \\ \dfrac{1}{2(k-1)} - \dfrac{N}{2k} > \dfrac{1}{q}\,. \end{array}\right. \tag{5.15}$$

Such a choice is always possible since the function $k \longmapsto N(k-1)/(2-k)$ is equal to 1 for $k = q_\star$ and the function $k \longmapsto (1/(2(k-1))) - (N/(2k))$ is equal to $1/q_\star$ for $k = q_\star$. Putting $A(s,x) := |\nabla u(s,x)|^{q-k}$, the integral identity (2.2) reads

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} A(s) |\nabla u(s)|^k \ ds\,, \quad t \ge 0\,. \tag{5.16}$$

Recall that $\|A(s)\|_\infty \le A_\infty := \|\nabla u_0\|_\infty^{q-k}$ in view of (2.3).

We next introduce

$$\alpha := \frac{N}{2}\ \left(\frac{1}{p} - \frac{1}{k}\right) + \frac{1}{2} = \frac{1}{2(k-1)} - \frac{N}{2k} > \frac{1}{2}$$

and

$$M(t) := \sup_{s \in (0,t)} \{s^\alpha \|\nabla u(s)\|_k\}$$

for $t > 0$. Using a fixed point argument in a suitable space and the uniqueness assertion of Theorem 2.1, we proceed as in [20, 39] to establish that $M(t)$ is finite for each $t > 0$ and converges to zero as $t \to 0$. It follows from (5.16) and the decay properties of the heat semigroup that

$$\begin{aligned} \|\nabla u(t)\|_k &\le C\ t^{-\alpha}\ \|u_0\|_p + C\ \int_0^t (t-s)^{-(k+N(k-1))/(2k)}\ A_\infty\ \left\||\nabla u(s)|^k\right\|_1\ ds \\ &\le C\ t^{-\alpha}\ \|u_0\|_p + C\ A_\infty\ M(t)^k\ \int_0^t (t-s)^{-(k+N(k-1))/(2k)}\ s^{-k\alpha}\ ds\,. \end{aligned}$$

Consequently,

$$t^\alpha \ \|\nabla u(t)\|_k \le C \ \|u_0\|_p + C \ A_\infty \ M(t)^k \ \int_0^1 (1-s)^{-(k+N(k-1))/(2k)} \ s^{-k\alpha} \ ds \,,$$

and the last integral in the right-hand side of the previous inequality is finite thanks to the choice of $k \in (q_\star, N/(N-1))$. Thus

$$M(t) \le K_1 \ \left(\|u_0\|_p + A_\infty \ M(t)^k\right) \quad \text{for} \quad t > 0 \,. \tag{5.17}$$

Now, assume that

$$\|u_0\|_p^{k-1} \ A_\infty \le 2^{-(k+1)} \ K_1^{-k} \,. \tag{5.18}$$

Since $M(t) \to 0$ as $t \to 0$ and M is non-decreasing, there is a maximal $t_0 \in (0, \infty]$ such that $M(t) \le 2K_1 \ \|u_0\|_p$ for $t \in [0, t_0)$. Assuming for contradiction that $t_0 < \infty$, we have $M(t_0) = 2K_1 \ \|u_0\|_p$ while (5.17) implies that

$$M(t_0) \le K_1 \ \left(\|u_0\|_p + A_\infty \ (2K_1)^k \ \|u_0\|_p^k\right) \le \frac{3K_1}{2} \ \|u_0\|_p \,,$$

hence a contradiction. Consequently, $t_0 = \infty$ and we deduce that

$$M(t) \le 2K_1 \ \|u_0\|_p \quad \text{for all} \quad t > 0 \tag{5.19}$$

whenever u_0 fulfils (5.18). Observe that, since $k\alpha < 1$, the estimate (5.19) does not guarantee (5.14). A better estimate will be obtained in the next step.

Step 2. We combine (2.5) and (5.19) to obtain

$$\|\nabla u(t)\|_q^q \le \|\nabla u(t)\|_\infty^{q-k} \ \|\nabla u(t)\|_k^k \le C \ \|u_0\|_\infty^{(q-k)/q} \ t^{-1-k(\alpha-(1/q))}$$

for $t \ge 1$. Recalling that $\alpha > 1/q$ by (5.15), we conclude that $t \longmapsto \|\nabla u(t)\|_q^q$ belongs to $L^1(0,\infty)$, whence $I_1(\infty) < \infty$. Finally, owing to the Gagliardo–Nirenberg inequality

$$\|w\|_p \le C \ \|\nabla w\|_\infty^{(N(p-1))/(p(N+1))} \ \|w\|_1^{(p+N)/(p(N+1))} \,, \quad w \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \,,$$

we realize that

$$\|u_0\|_p^{k-1} \ A_\infty = \|u_0\|_p^{k-1} \ \|\nabla u_0\|_\infty^{q-k} \le C \ \|u_0\|_1^{1/(N+1)} \ \|\nabla u_0\|_\infty^{q-q_\star} \,,$$

so that choosing γ sufficiently small in (5.13) implies (5.18).

□

Before identifying a class of initial data for which $I_\infty(\infty) > 0$ (and thus $I_1(\infty) = \infty$ by Proposition 5.3) we report the following result.

PROPOSITION 5.6. *[20] Assume that $q \in (q_\star, 2)$. There exists $u_0 \in L^1(\mathbb{R}^N) \cap \mathcal{BC}^2(\mathbb{R}^N)$ such that $I_1(\infty) = \infty$ and $I_\infty(\infty) = 0$. More precisely, this particular solution to (1.1), (1.2) enjoys the following properties: for $t \ge 0$,*

$$\|u(t)\|_1 = (1+t)^{((N+1)(q-q_\star))/(2(q-1))} \ \|u_0\|_1 \quad \textit{and} \quad \|u(t)\|_\infty = \frac{\|u_0\|_\infty}{(1+t)^{(2-q)/(2(q-1))}} \,.$$

Proposition 5.6 is established in [20, Theorem 3.5] by constructing a self-similar solution to (1.1) of the form

$$(t,x) \longmapsto t^{-(2-q)/(2(q-1))} \ W_s \left(|x| t^{-1/2}\right) \,.$$

The profile W_s then satisfies a nonlinear ordinary differential equation, which turns out to have an integrable and smooth solution W_s if $q \in (q_\star, 2)$. Taking $u_0 = W_s$ gives the result.

We end up this section with the following result:

PROPOSITION 5.7. *[9] There is a constant $K = K(q) > 0$ such that, if $u_0 \in W^{2,\infty}(\mathbb{R}^N)$ satisfies*

$$\|u_0\|_\infty \ \left| \inf_{y\in\mathbb{R}^N} \{\Delta u_0\} \right|^{-(2-q)/q} > K\,, \tag{5.20}$$

then $I_\infty(\infty) > 0$.

PROOF. Since u is a classical solution to (1.1), (1.2) with $\sigma = -1$, it follows from (1.1) that

$$u(t,x) \ge u_0(x) + \int_0^t \Delta u(s,x)\ ds \ge u_0(x) + \int_0^t \inf_{y\in\mathbb{R}^N} \{\Delta u(s,y)\}\ ds$$

for every $x \in \mathbb{R}^N$ and $t \ge 0$. Therefore,

$$\|u(t)\|_\infty \ge \|u_0\|_\infty + \int_0^t \inf_{y\in\mathbb{R}^N} \{\Delta u(s,y)\}\ ds\,,$$

and we infer from (2.38) and (2.39) that

$$\|u(t)\|_\infty \ge \|u_0\|_\infty + T \inf_{y\in\mathbb{R}^N} \{\Delta u_0\} - C\ \|u_0\|_\infty^{(2-q)/q} \int_T^t s^{-2/q}\ ds$$

for $T > 0$ and $t > T$. Since $q < 2$, we may let $t \to \infty$ in the above inequality and obtain with the choice $T = \|u_0\|_\infty^{(2-q)/2} \ \left|\inf_{y\in\mathbb{R}^N} \{\Delta u_0\}\right|^{-q/2}$ that there is a constant K depending only on q such that

$$I_\infty(\infty) \ge \|u_0\|_\infty - K^{q/2} \left| \inf_{y\in\mathbb{R}^N} \{\Delta u_0\} \right|^{(2-q)/2} \|u_0\|_\infty^{(2-q)/2}\,. \tag{5.21}$$

Therefore, if $\|u_0\|_\infty > K \ \left|\inf_{y\in\mathbb{R}^N} \{\Delta u_0\}\right|^{(2-q)/q}$, we readily conclude from (5.21) that $I_\infty(\infty) > 0$, whence Proposition 5.7.

□

Unfortunately, the conditions (5.13) and (5.20) do not involve the same quantities. Still, we can prove that if u_0 fulfils

$$\|u_0\|_\infty \ \|D^2 u_0\|_\infty^{-(2-q)/q} > \Gamma$$

(which clearly implies (5.20) since $q < 2$), the quantity $\|u_0\|_1 \ \|\nabla u_0\|_\infty^{(N+1)(q-q_\star)}$ cannot be small. Indeed, there is a constant C depending only on q and N such that

$$\left(\|u_0\|_\infty \ \|D^2 u_0\|_\infty^{-(2-q)/q} \right)^{q(N+1)/2} \le C \|u_0\|_1 \ \|\nabla u_0\|_\infty^{(N+1)(q-q_\star)}\,. \tag{5.22}$$

To prove (5.22), we put $B = \|u_0\|_\infty \ \|D^2 u_0\|_\infty^{-(2-q)/q}$ and note that the Gagliardo–Nirenberg inequalities

$$\begin{aligned} \|u_0\|_\infty &\le C \ \|\nabla u_0\|_\infty^{N/(N+1)} \ \|u_0\|_1^{1/(N+1)}\,, \\ \|\nabla u_0\|_\infty &\le C \ \|D^2 u_0\|_\infty^{(N+1)/(N+2)} \ \|u_0\|_1^{1/(N+2)}\,, \end{aligned}$$

imply that

$$\begin{aligned}
\|\nabla u_0\|_\infty^{(2-q)(N+2)} & \le C\ \|D^2 u_0\|_\infty^{(2-q)(N+1)}\ \|u_0\|_1^{2-q} \\
& = C\ B^{-q(N+1)}\ \|u_0\|_\infty^{q(N+1)}\ \|u_0\|_1^{2-q} \\
& \le C\ B^{-q(N+1)}\ \|\nabla u_0\|_\infty^{qN}\ \ \|u_0\|_1^2\,,
\end{aligned}$$

whence the above claim.

1.3. The case $q \in [1, q_\star]$. We first show that $I_1(\infty) = \infty$.

LEMMA 5.8. *Assume that $q \in [1, q_\star]$. Then $I_1(\infty) = \infty$.*

PROOF. For $T > 0$ and $t > T$, we infer from (1.1) that

$$\|u(t)\|_1 = \|u(T)\|_1 + \int_T^t \|\nabla u(s)\|_q^q\ ds \ge \int_T^t \|\nabla u(s)\|_q^q\ ds\,. \tag{5.23}$$

Case 1: $N \ge 2$. Since $1 \le q \le q_\star \le 2$, it follows from (5.23), the pointwise inequality $u(s,x) \ge \left(e^{s\Delta} u_0\right)(x)$ and the Sobolev inequality that

$$\|u(t)\|_1 \ge C\ \int_T^t \|u(s)\|_Q^q\ ds \ge C\ \int_T^t \left\|e^{s\Delta} u_0\right\|_Q^q\ ds$$

with $Q := Nq/(N-q)$. Since

$$\begin{aligned}
\left\|e^{s\Delta} u_0\right\|_Q & \ge \|u_0\|_1\ \|g(s)\|_Q - \left\|e^{s\Delta} u_0 - \|u_0\|_1\ g(s)\right\|_Q \\
& = s^{-N(Q-1)/(2Q)}\ \|u_0\|_1\ \|G\|_Q - \left\|e^{s\Delta} u_0 - \|u_0\|_1\ g(s)\right\|_Q \\
& \ge \frac{\|u_0\|_1\ \|G\|_Q}{2}\ s^{(N-q(N+1))/(2q)}
\end{aligned}$$

for $s \ge T$ and T large enough by Proposition A.1 (the functions g and G being defined in (1.7)), we conclude that

$$\|u(t)\|_1 \ge C\ \left(t^{((N+1)(q_\star-q))/2} - T^{((N+1)(q_\star-q))/2}\right) \quad \text{if} \quad q \in [1, q_\star)$$

and

$$\|u(t)\|_1 \ge C\ (\log t - \log T) \quad \text{if} \quad q = q_\star\,,$$

which gives the expected result for $N \ge 2$.

Case 2: $N = 1$. It follows from (1.1), the time monotonicity of $\|u\|_1$ and the Gagliardo–Nirenberg inequality

$$\|w\|_\infty^{2q-1} \le C\ \|\partial_x w\|_q^q\ \|w\|_1^{q-1}\,, \quad w \in L^1(\mathbb{R}) \cap W^{1,q}(\mathbb{R})\,,$$

that, for $t > T > 0$,

$$\|u(t)\|_1 \ge C\ \int_T^t \|u(s)\|_\infty^{2q-1}\ \|u(s)\|_1^{-(q-1)}\ ds \ge C\ \|u(t)\|_1^{-(q-1)}\ \int_T^t \|u(s)\|_\infty^{2q-1}\ ds\,.$$

Consequently,

$$\|u(t)\|_1^q \ge C\ \int_T^t \|u(s)\|_\infty^{2q-1}\ ds\,,$$

and we proceed as in the previous case to complete the proof.

□

Having excluded the finiteness of $I_1(\infty)$, we turn to $I_\infty(\infty)$ and establish the following lemma:

LEMMA 5.9. *[31] Assume that $q \in (1,2)$ and $I_\infty(\infty) = 0$. Then $\|u(t)\|_\infty \leq C\ t^{-(2-q)/(2(q-1))}$ for all $t \geq 0$.*

PROOF. infer from (1.1) that, for each $T > t$,

$$u(T,x) = u(t,x) + \int_t^T \partial_t u(s,x)\ ds \geq u(t,x) + \int_t^T \Delta u(s,x)\ ds\,.$$

Since (1.1) is autonomous, we infer from (2.9) that

$$-\Delta u(s,x) \leq N\ C_7\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(2-q)/q}\ \left(s - \frac{t}{2}\right)^{-2/q} \quad \text{for} \quad s \in (t,T)\,,$$

so that

$$\begin{aligned} u(T,x) &\geq u(t,x) - N\ C_7\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(2-q)/q}\ \int_t^T \left(s - \frac{t}{2}\right)^{-2/q}\ ds \\ &\geq u(t,x) - C\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(2-q)/q}\ t^{-(2-q)/q}\,. \end{aligned}$$

Letting $T \to \infty$ we obtain

$$\begin{aligned} u(t,x) &\leq C\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(2-q)/q}\ t^{-(2-q)/q} + I_\infty(\infty) \\ &= C\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(2-q)/q}\ t^{-(2-q)/q}\,, \end{aligned}$$

whence

$$t^{(2-q)/(2q-2)}\ \|u(t)\|_\infty \leq C\ \left(\left(\frac{t}{2}\right)^{(2-q)/(2q-2)}\ \left\| u\left(\frac{t}{2}\right)\right\|_\infty\right)^{(2-q)/q}\,.$$

Introducing

$$A(t) := \sup_{s\in[0,t]} \left\{ s^{(2-q)/(2q-2)}\ \|u(s)\|_\infty \right\},$$

we deduce from the previous inequality that $A(t) \leq C\ A(t)^{(2-q)/q}$, hence $A(t) \leq C^{q/(2q-2)}$. This bound being valid for each $t \geq 0$, the proof of Lemma 5.9 is complete.

□

PROOF OF PROPOSITION 5.3: $q \in [1, q_\star]$. Since $u(t) \geq e^{(t-T)\Delta} u(T)$ for $T \geq 0$ and $t > T$ by the comparison principle, we have

$$\begin{aligned} (t-T)^{N/2}\ \|u(t)\|_\infty &\geq \|u(T)\|_1\ \|G\|_\infty \\ &- (t-T)^{N/2}\ \left\| e^{(t-T)\Delta} u(T) - \|u(T)\|_1\ g(t-T) \right\|_\infty\,, \end{aligned}$$

and we infer from Proposition A.1 that

$$\liminf_{t\to\infty} \left\{ t^{N/2}\ \|u(t)\|_\infty \right\} \geq \liminf_{t\to\infty} \left\{ (t-T)^{N/2}\ \|u(t)\|_\infty \right\} \geq \|u(T)\|_1\ \|G\|_\infty \quad (5.24)$$

for every $T > 0$, the functions g and G being still defined by (1.7).

We first consider the case $q > 1$ and assume for contradiction that $I_\infty(\infty) = 0$. Then Lemma 5.9 entails that

$$t^{N/2}\, \|u(t)\|_\infty \le C\, t^{((N+1)(q-q_\star))/(2(q-1))}\,,$$

from which we deduce that

$$\limsup_{t\to\infty} \left\{ t^{N/2}\, \|u(t)\|_\infty \right\} \le C \tag{5.25}$$

as $q \in [1, q_\star]$. Combining (5.24) and (5.25) yields that $\|u(T)\|_1\, \|G\|_\infty \le C$ for every $T > 0$ and thus contradicts Lemma 5.8. Therefore $I_\infty(\infty) > 0$.

Next, if $q = 1$, we observe that (2.3) ensures that

$$\partial_t u - \Delta u \ge \frac{|\nabla u|^{q_\star}}{\|\nabla u_0\|_\infty^{q_\star - 1}}$$

in Q_∞. Consequently, $\tilde{u} := u/\|\nabla u_0\|_\infty$ satisfies $\partial_t \tilde{u} - \Delta \tilde{u} \ge |\nabla \tilde{u}|^{q_\star}$ in Q_∞. We then infer from the comparison principle that $\tilde{u}(t,x) \ge v(t,x)$ for $(t,x) \in [0,\infty) \times \mathbb{R}^N$, where v denotes the unique classical solution to $\partial_t v - \Delta v = |\nabla v|^{q_\star}$ in Q_∞ with initial condition $v(0) = u_0/\|\nabla u_0\|_\infty$. Therefore,

$$I_\infty(\infty) = \|\nabla u_0\|_\infty \lim_{t\to\infty} \|\tilde{u}(t)\|_\infty \ge \|\nabla u_0\|_\infty \lim_{t\to\infty} \|v(t)\|_\infty > 0\,,$$

which completes the proof of Proposition 5.3.

□

2. Growth rates

First, the assertion **(a)** of Proposition 5.4 is nothing but Lemma 5.9. We then turn to the proof of the assertions **(b)** and **(c)**.

Proof of Proposition 5.4 (b). For $(t,x) \in Q_\infty$, we infer from (1.1) and (2.38) that, for each $T > t$,

$$\begin{aligned}
u(T,x) &= u(t,x) + \int_t^T \partial_t u(s,x)\, ds \ge u(t,x) + \int_t^T \Delta u(s,x)\, ds \\
&\ge u(t,x) - C_9\, \|u_0\|_\infty^{(2-q)/q} \int_t^T s^{-2/q}\, ds \\
&\ge u(t,x) - C\, \|u_0\|_\infty^{(2-q)/q}\, t^{-(2-q)/q}\,, \\
\|u(T)\|_\infty &\ge \|u(t)\|_\infty - C\, \|u_0\|_\infty^{(2-q)/q}\, t^{-(2-q)/q}\,.
\end{aligned}$$

Letting $T \to \infty$ gives (5.8).

□

PROOF OF PROPOSITION 5.4 (C). We proceed as in the previous proof but use (2.8) instead of (2.38) to obtain

$$\begin{aligned}
u(T,x) &= u(t,x) + \int_t^T \partial_t u(s,x)\ ds \ge u(t,x) - C_5\ \int_t^T \phi_2(s)\ ds \\
&\ge u(t,x) - C\ \|u_0\|_\infty\ \int_t^T s^{-3/2}\ \left(1+\log\left(1+s\right)\right) ds \\
&\ge u(t,x) - C\ \|u_0\|_\infty\ t^{-1/2}\ \left(2+\log\left(1+t\right)\right)\ , \\
\|u(T)\|_\infty &\ge \|u(t)\|_\infty - C\ \|u_0\|_\infty\ t^{-1/2}\ \left(2+\log\left(1+t\right)\right)\ .
\end{aligned}$$

Letting $T \to \infty$ gives (5.9). □

Bibliographical notes

Lemma 5.8 was first proved in [39]. The proof given here is slightly different and is inspired by the proof of [31, Theorem 3]. The fact that $I_\infty(\infty) > 0$ for $q \in [1, q_\star]$ was first establish in [9] for sufficiently large initial data (fulfilling a condition similar to (5.20)). The proof for general initial data is due to Gilding [31, Theorem 3]. When $q \in (1,2)$ and $I_\infty(\infty) > 0$ the growth rate (5.6) seems to be optimal, see Section 2. When $q = 1$ the growth rate (5.6) is also optimal in the sense that there is a large class of initial data (including compactly supported ones) such that $\|u(t)\|_1 \le C\ t^N$ for $t \ge 1$ [40]. Concerning self-similar solutions to (1.1) for $\sigma = -1$, their existence is investigated in [20, 33].

CHAPTER 6

Convergence to self-similarity

1. The diffusion-dominated case: $\sigma = 1$

In this section, we shall prove that, when $\sigma = 1$, the diffusion governs the large time dynamics as soon as $I_1(\infty)$ is non-zero. More precisely, we have the following result which is similar to the one known for the linear heat equation (1.5) and recalled in Section 1 below.

THEOREM 6.1. *Assume that $\sigma = 1$ and $q > q_\star$. Let $u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be a non-negative function and denote by u the corresponding classical solution to (1.1), (1.2). Then*

$$I_1(\infty) := \lim_{t\to\infty} \|u(t)\|_1 \in (0,\infty)\,, \tag{6.1}$$

and, for every $p \in [1,\infty]$,

$$\lim_{t\to\infty} t^{N(p-1)/2p}\|u(t) - I_1(\infty)\ g(t)\|_p = 0\,, \tag{6.2}$$

$$\lim_{t\to\infty} t^{N(p-1)/2p+(1/2)}\|\nabla u(t) - I_1(\infty)\ \nabla g(t)\|_p = 0\,, \tag{6.3}$$

where the functions g and G are defined in (1.7).

The proof of Theorem 6.1 relies on the following properties of the inhomogeneous heat equation.

LEMMA 6.2. *Assume that v is the solution to the Cauchy problem for the linear inhomogeneous heat equation*

$$\begin{aligned} \partial_t v - \Delta v &= f\,, \qquad (t,x) \in Q_\infty\,,\\ v(0) &= v_0\,, \qquad x \in \mathbb{R}^N\,, \end{aligned}$$

with $v_0 \in L^1(\mathbb{R}^N)$ and $f \in L^1(Q_\infty)$. Then

$$\lim_{t\to\infty} \|v(t) - K_\infty\ g(t)\|_1 = 0\,, \tag{6.4}$$

where

$$K_\infty := \lim_{t\to\infty} \int_{\mathbb{R}^N} v(t,x)\ dx = \int_{\mathbb{R}^N} v_0(x)\ dx + \int_0^\infty \int_{\mathbb{R}^N} f(t,x)\ dx\ dt\,.$$

Assume further that there is $p \in [1,\infty]$ such that $f(t) \in L^p(\mathbb{R}^N)$ for every $t > 0$ and

$$\lim_{t\to\infty} t^{1+(N/2)(1-1/p)}\ \|f(t)\|_p = 0\,. \tag{6.5}$$

Then

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)}\ \|v(t) - K_\infty\ g(t)\|_p = 0\,, \tag{6.6}$$

and

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2}\ \|\nabla v(t) - K_\infty\ \nabla g(t)\|_p = 0\,. \tag{6.7}$$

PROOF. We first observe that the assumptions on v_0 and f warrant that K_∞ is finite and put

$$K(T) := \int_{\mathbb{R}^N} v(T,x)\ dx = \int_{\mathbb{R}^N} v_0(x)\ dx + \int_0^T \int_{\mathbb{R}^N} f(t,x)\ dx\ dt \quad \text{for} \quad T>0\,.$$

Moreover, by the Duhamel formula,

$$\|v(t)\|_1 \le \|g(t)*v_0\|_1 + \int_0^t \|g(t-\tau)*f(\tau)\|_1\ d\tau \le \|v_0\|_1 + \int_0^\infty \|f(\tau)\|_1\ d\tau$$

for $t\ge 0$ and thus $v \in L^\infty(0,\infty;L^1(\mathbb{R}^N))$.

We now prove (6.4). Let $T>0$ and $t\in(T,\infty)$. By the Duhamel formula,

$$v(t) = g(t-T)*v(T) + \int_T^t g(t-\tau)*f(\tau)\ d\tau\,,$$

so that, by the Young inequality,

$$\|v(t) - g(t-T)*v(T)\|_1 \le \int_T^t \|g(t-\tau)\|_1\ \|f(\tau)\|_1\ d\tau \le \int_T^\infty \|f(\tau)\|_1\ d\tau\,.$$

Then

$$\begin{aligned}
\|v(t) - K_\infty\ g(t)\|_1 &\le \|v(t) - g(t-T)*v(T)\|_1 + \|(g(t-T)-g(t))*v(T)\|_1 \\
&+ \|g(t)*v(T) - K(T)\ g(t)\|_1 + |K(T)-K_\infty|\ \|g(t)\|_1 \\
&\le 2\int_T^\infty \|f(\tau)\|_1\ d\tau + \sup_{\tau\ge 0}\{\|v(\tau)\|_1\}\ \|(g(t-T)-g(t))\|_1 \\
&+ \|g(t)*v(T) - K(T)\ g(t)\|_1\,.
\end{aligned}$$

Owing to Proposition A.1 and the properties of g, we may let $t\to\infty$ in the above inequality and deduce that

$$\limsup_{t\to\infty} \|v(t) - K_\infty\ g(t)\|_1 \le 2\int_T^\infty \|f(\tau)\|_1\ d\tau\,.$$

We next use the time integrability of f to pass to the limit as $T\to\infty$ and complete the proof of (6.4).

We next assume (6.5) and prove (6.7). Let $T>0$ and $t\in(T,\infty)$. By the Duhamel formula,

$$\nabla v(t) = \nabla g(t-T)*v(T) + \int_T^t \nabla g(t-\tau)*f(\tau)\ d\tau\,.$$

It follows from the Young inequality that

$$\begin{aligned}
& t^{(N/2)(1-1/p)+1/2}\ \|\nabla v(t) - \nabla g(t-T) * v(T)\|_p \\
\leq\ & C\ t^{(N/2)(1-1/p)+1/2} \int_T^{(T+t)/2} (t-\tau)^{-(N/2)(1-1/p)-1/2}\ \|f(\tau)\|_1\ d\tau \\
+\ & C\ t^{(N/2)(1-1/p)+1/2} \int_{(T+t)/2}^{t} (t-\tau)^{-1/2}\ \|f(\tau)\|_p\ d\tau \\
\leq\ & C\ \left(\frac{t}{t-T}\right)^{(N/2)(1-1/p)+1/2} \int_T^\infty \|f(\tau)\|_1\ d\tau \\
+\ & C\ \sup_{\tau\geq T} \left\{\tau^{(N/2)(1-1/p)+1}\ \|f(\tau)\|_p\right\} \int_{(T+t)/2}^{t} (t-\tau)^{-1/2}\ \tau^{-1/2}\ d\tau \\
\leq\ & C\ \left(\frac{t}{t-T}\right)^{(N/2)(1-1/p)+1/2} \int_T^\infty \|f(\tau)\|_1\ d\tau \\
+\ & C\ \sup_{\tau\geq T} \left\{\tau^{(N/2)(1-1/p)+1}\ \|f(\tau)\|_p\right\}\ .
\end{aligned}$$

It also follows from Proposition A.1 and the properties of g that

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2}\, \|\nabla g(t-T) * v(T) - K(T)\ \nabla g(t-T)\|_p = 0\,,$$

and

$$\lim_{t\to\infty} t^{(N/2)(1-1/p)+1/2}\, \|\nabla g(t-T) - \nabla g(t)\|_p = 0$$

for every $p \in [1,\infty]$. Since

$$\begin{aligned}
\|\nabla v(t) - K_\infty\ \nabla g(t)\|_p \quad \leq\ & \|\nabla v(t) - \nabla g(t-T) * v(T)\|_p \\
+\ & \|\nabla g(t-T) * v(T) - K(T)\ \nabla g(t-T)\|_p \\
+\ & |K(T) - K_\infty|\ \|\nabla g(t-T)\|_p \\
+\ & |K_\infty|\ \|\nabla g(t-T) - \nabla g(t)\|_p\,,
\end{aligned}$$

the previous relations imply that

$$\begin{aligned}
& \limsup_{t\to\infty}\ t^{(N/2)(1-1/p)+1/2}\ \|\nabla v(t) - K_\infty\ \nabla g(t)\|_p \\
\leq\ & C\ \left(\int_T^\infty \|f(\tau)\|_1\ d\tau + \sup_{\tau\geq T}\left\{\tau^{(N/2)(1-1/p)+1}\ \|f(\tau)\|_p\right\}\right)\ .
\end{aligned}$$

The above inequality being valid for any $T > 0$, we may let $T \to \infty$ and conclude that (6.7) holds true. The assertion (6.6) then follows from (6.4) and (6.7) by the Gagliardo–Nirenberg inequality.

□

Proof of Theorem 6.1. Consider $p \in [1,\infty]$ and $t > 0$. Since $\sigma = 1$, u is a subsolution to the linear heat equation and therefore satisfies

$$\|u(t)\|_p \leq \|e^{t\Delta} u_0\|_p \leq C\ t^{-(N/2)(1-1/p)}\ \|u_0\|_1$$

as a consequence of the comparison principle. Combining this estimate with (2.6) (with $s = t/2$) and (4.12) gives

$$\begin{aligned} \| |\nabla u(t)|^q \|_p &= \left(\frac{q}{q-1} \right)^q \left\| u(t) \left| \nabla u^{(q-1)/q}(t) \right|^q \right\|_p \\ &\le \left(\frac{q}{q-1} \right)^q \left\| \nabla u^{(q-1)/q}(t) \right\|_\infty^q \|u(t)\|_p \\ &\le C \|u_0\|_1 \left\| u\left(\frac{t}{2}\right) \right\|_\infty^{q-1} t^{-q/2} t^{-N(p-1)/2p} \\ &\le C \|u_0\|_1^q t^{-(q+N(q-1))/2} t^{-N(p-1)/2p}, \end{aligned}$$

whence

$$t^{(N/2)(1-1/p)+1} \| |\nabla u(t)|^q \|_p \le C \|u_0\|_1^q t^{(N+1)(q_\star - q)/2} \underset{t\to\infty}{\longrightarrow} 0$$

because $q > q_\star$. Theorem 6.1 then readily follows from Lemma 6.2 with $f(t,x) = -|\nabla u(t,x)|^q$. $\square$

Remark 6.3. *The convergence (6.2) in Theorem 6.1 is also true when $\sigma = -1$ and either $q \ge 2$ or $q \in (q_\star, 2)$ and u_0 is suitably small. Indeed, in both cases, $I_1(\infty)$ is positive and finite by Proposition 5.3 and Proposition 5.5, respectively. The proof again relies on Lemma 6.2 but establishing that $f = |\nabla u|^q$ fulfils the condition (6.5) for all $p \in [1,\infty]$ is more delicate [9].*

2. The reaction-dominated case: $\sigma = -1$

The aim of this section is to show that, when $\sigma = -1$ and $I_\infty(\infty) > 0$, the large time behaviour of the solution u to (1.1), (1.2) is dominated by the nonlinear Hamilton–Jacobi term in the sense that u behaves as a self-similar solution to (1.9) for large times (recall that $I_\infty(\infty)$ is the limit of $\|u(t)\|_\infty$ as $t \to \infty$, see (5.5)). Such a result however holds for initial data decaying to zero as $|x| \to \infty$ and we define the space $\mathcal{C}_0(\mathbb{R}^N)$ of such functions by

$$\mathcal{C}_0(\mathbb{R}^N) := \left\{ z \in \mathcal{BC}(\mathbb{R}^N) \ : \ \lim_{R\to\infty} \sup_{\{|x|\ge R\}} \{|z(x)|\} = 0 \right\}.$$

Theorem 6.4. *Assume that $\sigma = -1$ and $q \in (1,2)$. Let $u_0 \in \mathcal{C}_0(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be a non-negative function and denote by u the corresponding classical solution to (1.1), (1.2). Assume further that*

$$I_\infty(\infty) := \lim_{t\to\infty} \|u(t)\|_\infty > 0. \tag{6.8}$$

Then

$$\lim_{t\to\infty} \|u(t) - h_\infty(t)\|_\infty = 0, \tag{6.9}$$

where h_∞ is given by

$$h_\infty(t,x) := H_\infty\left(\frac{x}{t^{1/q}}\right) \quad \text{and} \quad H_\infty(x) := \left(I_\infty(\infty) - \gamma_q |x|^{q/(q-1)} \right)_+$$

for $(t,x) \in Q_\infty$ and $\gamma_q := (q-1)\, q^{-q/(q-1)}$.

A similar result is valid for viscosity solutions to (1.9), see Section 2 below. Thus Theorem 6.4 clearly asserts the sole domination of the Hamilton–Jacobi term for large times. In fact, h_∞ is the unique viscosity solution in $\mathcal{BUC}(Q_\infty)$ to (1.9) with the bounded and upper semicontinuous initial condition $h_\infty(0,x) = 0$ if $x \neq 0$ and $h_\infty(0,0) = I_\infty(\infty)$.

We next point out that there is no loss of generality in assuming that $q < 2$ in Theorem 6.4. Indeed, according to Proposition 5.3, we always have $I_\infty(\infty) = 0$ if $q \geq 2$.

Proof of Theorem 6.4. We change the variables and the unknown function so that the convergence (6.9) reduces to a convergence to a steady state. More precisely, we introduce the self-similar (or scaling) variables

$$\tau = \frac{1}{q}\ \log(1+t)\,, \qquad y = \frac{x}{(1+t)^{1/q}}\,,$$

and the new unknown function v defined by

$$u(t,x) = v\left(\frac{\log(1+t)}{q}, \frac{x}{(1+t)^{1/q}}\right)\,, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^N\,.$$

Then $v(\tau,y) = u\,(e^{q\tau} - 1, ye^{\tau})$ and it follows from (1.1), (1.2) that v solves

$$\begin{aligned} \partial_\tau v - y \cdot \nabla v - q\ |\nabla v|^q - q\ e^{-(2-q)\tau}\ \Delta v &= 0\,, \quad (\tau,y) \in Q_\infty\,, && (6.10)\\ v(0) &= u_0\,, \quad y \in \mathbb{R}^N\,. && (6.11) \end{aligned}$$

We also infer from (2.1), (2.3), (2.5) and (2.9) that there is a positive constant K_1 depending only on N, q and u_0 such that

$$\begin{aligned} \|v(\tau)\|_\infty + \|\nabla v(\tau)\|_\infty &\leq K_1\,, \qquad \tau \geq 0\,, && (6.12)\\ \Delta v(\tau,y) &\geq -K_1\,, \qquad (\tau,y) \in [1,\infty) \times \mathbb{R}^N\,. && (6.13) \end{aligned}$$

At this point, since $q < 2$, it is clear that the diffusion in (6.10) becomes weaker and weaker as time increases to infinity and the behaviour of v for large times is expected to look like that of the solution to the Hamilton–Jacobi equation $\partial_\tau z - y \cdot \nabla z - q\ |\nabla z|^q = 0$. To investigate the large time behaviour of first-order Hamilton–Jacobi equations, a useful technique has been developed in [43, 45] which relies on the relaxed half-limits method introduced in [7]. More precisely, for $(\tau,y) \in Q_\infty$ we define the relaxed half-limits v_* and v^* by

$$v_*(y) := \liminf_{(\sigma,z,s)\to(\tau,y,\infty)} v(\sigma+s,z) \quad \text{and} \quad v^*(y) := \limsup_{(\sigma,z,s)\to(\tau,y,\infty)} v(\sigma+s,z) \quad (6.14)$$

and first note that the right-hand sides of the above definitions indeed do not depend on $\tau > 0$. In addition,

$$0 \leq v_*(x) \leq v^*(x) \leq I_\infty(\infty) \quad \text{for} \quad x \in \mathbb{R}^N \qquad (6.15)$$

by (5.5) and (6.14), while (6.12) and the Rademacher theorem clearly ensure that v_* and v^* both belong to $W^{1,\infty}(\mathbb{R}^N)$. Finally, by [6, Théorème 4.1] applied to equation (6.10), v^* and v_* are viscosity subsolution and supersolution, respectively, to the Hamilton–Jacobi equation

$$H(y,\nabla z) := -y \cdot \nabla z - q\ |\nabla z|^q = 0 \quad \text{in} \quad \mathbb{R}^N\,. \qquad (6.16)$$

The next step is to show that v^* and v_* actually coincide. However, the equation (6.16) has several solutions as $y \longmapsto \left(c - \gamma_q\ |y|^{q/(q-1)}\right)_+$ solves (6.16) for any $c > 0$. The information obtained so far on v_* and v^* are thus not sufficient. The next two results provide the lacking information.

LEMMA 6.5. *Given $\varepsilon \in (0,1)$, there is $R_\varepsilon > 1/\varepsilon$ such that*

$$v(\tau, y) \le \varepsilon \quad \textit{for} \quad \tau \ge 0 \quad \textit{and} \quad y \in \mathbb{R}^N \setminus B(0, R_\varepsilon)\,. \tag{6.17}$$

PROOF. We first construct a supersolution to (6.10) in $(0,\infty) \times \mathbb{R}^N \setminus B(0,R)$ for R large enough. To this end, consider $R \ge \left(4 + q\ 2^{q-1}\ \|u_0\|_\infty^{q-1}\right)^{1/q}$ and define $\Sigma_R(y) = \|u_0\|_\infty\ R^2\ |y|^{-2}$ for $y \in \mathbb{R}^N$. Let $\mathcal{L}$ be the parabolic operator defined by

$$\mathcal{L}w(\tau, y) := \partial_\tau w(\tau, y) - y \cdot \nabla w(\tau, y) - q\ |\nabla w(\tau, y)|^q - q\ e^{-(2-q)\tau}\ \Delta w(\tau, y)$$

for $(\tau, y) \in Q_\infty$ (so that $\mathcal{L}v = 0$). Then, if $y \in \mathbb{R}^N \setminus B(0,R)$, we have

$$\begin{aligned}
\mathcal{L}\Sigma_R(y) &= 2\ \|u_0\|_\infty\ \frac{R^2}{|y|^2} - q\ 2^q\ \|u_0\|_\infty^q\ \frac{R^{2q}}{|y|^{3q}} - 2\ (4-N)\ \|u_0\|_\infty\ \frac{R^2}{|y|^4}\ e^{-(2-q)\tau} \\
&= 2\ \|u_0\|_\infty\ \frac{R^2}{|y|^2}\ \left(1 + \frac{N-4}{|y|^2}\ e^{-(2-q)\tau} - q\ \frac{(2\ \|u_0\|_\infty)^{q-1}}{|y|^q}\ \left(\frac{R^2}{|y|^2}\right)^{q-1}\right) \\
&\ge 2\ \|u_0\|_\infty\ \frac{R^2}{|y|^2}\ \left(1 - \frac{4}{|y|^2}\ e^{-(2-q)\tau} - q\ \frac{(2\ \|u_0\|_\infty)^{q-1}}{R^q}\right) \\
&\ge 2\ \|u_0\|_\infty\ \frac{R^2}{|y|^2}\ \left(1 - \left(4 + q\ (2\ \|u_0\|_\infty)^{q-1}\right)\ R^{-q}\right) \\
&\ge 0\,.
\end{aligned}$$

Consequently, for $R \ge \left(4 + q\ 2^{q-1}\ \|u_0\|_\infty^{q-1}\right)^{1/q}$, Σ_R is a supersolution to (6.10) in $(0,\infty) \times \mathbb{R}^N \setminus B(0,R)$.

Consider next $\varepsilon \in (0,1)$. Since $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$, there is $r_\varepsilon \ge \left(4 + q\ 2^{q-1}\ \|u_0\|_\infty^{q-1}\right)^{1/q}$, $r_\varepsilon \ge 1/\varepsilon$ such that $u_0(y) \le \varepsilon/2$ if $|y| \ge r_\varepsilon$. Then, if $y \in \mathbb{R}^N \setminus B(0, r_\varepsilon)$, we have $u_0(y) - \varepsilon/2 \le 0 \le \Sigma_{r_\varepsilon}(y)$ while $v(\tau,y) - \varepsilon/2 \le \|u_0\|_\infty = \Sigma_{r_\varepsilon}(y)$ if $\tau \ge 0$ and $|y| = r_\varepsilon$. Since Σ_{r_ε} is a supersolution to (6.10) as previously established and $v - \varepsilon/2$ is a solution to (6.10), the comparison principle entails that $v(\tau,y) - \varepsilon/2 \le \Sigma_{r_\varepsilon}(y)$ for $\tau \ge 0$ and $y \in \mathbb{R}^N \setminus B(0, r_\varepsilon)$. Since $\Sigma_{r_\varepsilon}(y) \to 0$ as $|y| \to \infty$, there is $R_\varepsilon \ge r_\varepsilon$ such that $\Sigma_{r_\varepsilon}(y) \le \varepsilon/2$ for $|y| \ge R_\varepsilon$ which completes the proof of (6.17). □

□

It actually follows from Lemma 6.5 that $v(\tau)$ belongs to $\mathcal{C}_0(\mathbb{R}^N)$ for every $\tau \ge 0$ in a way which is "uniform" with respect to $\tau \ge 0$.

LEMMA 6.6. *For $y \in \mathbb{R}^N$, we have*

$$H_\infty(y) \le v_*(y) \le v^*(y)\,. \tag{6.18}$$

PROOF. For $\tau \ge 0$, $y \in \mathbb{R}^N$, $\xi_0 \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, we set $\mathcal{H}(\tau, y, \xi_0, \xi) := \xi_0 - y \cdot \xi - q\ |\xi|^q + q\ K_1\ e^{-(2-q)\tau}$, the constant K_1 being defined in (6.12) and (6.13). Since $\Delta v(\tau, y) \ge -K_1$ for $(\tau, y) \in (1,\infty) \times \mathbb{R}^N$ by (6.13), we deduce from (6.10) that $\mathcal{H}(\tau,y,\partial_\tau v(\tau,y),\nabla v(\tau,y)) = q\ e^{-(2-q)\tau}\ (\Delta v(\tau,y)+K_1)\ge 0$ for $(\tau,y)\in(1,\infty)\times\mathbb{R}^N$. Consequently,

$$v \quad \text{is a supersolution to} \quad \mathcal{H}(\tau,y,\partial_\tau z,\nabla z)=0 \quad \text{in} \quad (1,\infty)\times\mathbb{R}^N\,. \tag{6.19}$$

Next fix $\tau_0>1$ and denote by w the (viscosity) solution to

$$\begin{aligned} \partial_\tau w - y\cdot\nabla w - q\ |\nabla w|^q &= 0\,, \qquad (\tau,y)\in(\tau_0,\infty)\times\mathbb{R}^N\,,\\ w(\tau_0) &= v(\tau_0)\,, \qquad y\in\mathbb{R}^N\,. \end{aligned}$$

On the one hand, the function $\tilde{w}$ defined by

$$\tilde{w}(\tau,y) := w(\tau,y) - q\ K_1 \int_{\tau_0}^{\tau} e^{-(2-q)s}\ ds\,, \qquad (\tau,y)\in(\tau_0,\infty)\times\mathbb{R}^N\,,$$

is the (viscosity) solution to $\mathcal{H}(\tau,y,\partial_\tau\tilde{w},\nabla\tilde{w})=0$ in $(\tau_0,\infty)\times\mathbb{R}^N$ with initial condition $\tilde{w}(\tau_0)=v(\tau_0)$. Recalling (6.19), we infer from the comparison principle that

$$\tilde{w}(\tau,y)\le v(\tau,y) \quad \text{for} \quad (\tau,y)\in(\tau_0,\infty)\times\mathbb{R}^N\,. \tag{6.20}$$

On the other hand, it follows from Corollary A.4 that

$$\lim_{\tau\to\infty}\ \sup_{y\in\mathbb{R}^N} \left| w(\tau) - \left(\|v(\tau_0)\|_\infty - \gamma_q\ |y|^{q/(q-1)}\right)_+ \right| = 0\,.$$

We may then pass to the limit as $\tau\to\infty$ in (6.20) to conclude that

$$\left(\|v(\tau_0)\|_\infty - \gamma_q\ |y|^{q/(q-1)}\right)_+ - q\ K_1 \int_{\tau_0}^{\infty} e^{-(2-q)s}\ ds \le v_*(y)\le v^*(y)$$

for $y\in\mathbb{R}^N$. Letting $\tau_0\to\infty$ in the above inequality gives (6.18).

□

We are now in a position to complete the proof of Theorem 6.4. Fix $\varepsilon\in(0,1)$. On the one hand, it readily follows from Lemma 6.5 that $v^*(y)\le\varepsilon$ for $|y|\ge R_\varepsilon\ge 1/\varepsilon$. On the other hand, $H_\infty(0)=I_\infty(\infty)$ and the continuity of H_∞ ensures that there is $r_\varepsilon\in(0,\varepsilon)$ such that $H_\infty(y)\ge I_\infty(\infty)-\varepsilon$ for $y\in B(0,r_\varepsilon)$. Recalling (6.15), we conclude that

$$\begin{cases} v^*(y)-\varepsilon\le 0\le H_\infty(y) & \text{if} \quad |y|=R_\varepsilon\,,\\ \\ v^*(y)-\varepsilon\le I_\infty(\infty)-\varepsilon\le H_\infty(y) & \text{if} \quad |y|=r_\varepsilon\,. \end{cases} \tag{6.21}$$

Now, introducing $\phi(y)=-\gamma_q\ |y|^{q/(q-1)}/2$ and recalling that H is defined in (6.16), we have

$$H(y,\nabla\phi(y)) = \frac{q\gamma_q}{2(q-1)}\ |y|^{q/(q-1)}\ \left(1-\frac{1}{2^{q-1}}\right)>0 \quad \text{if} \quad r_\varepsilon<|y|<R_\varepsilon\,. \tag{6.22}$$

Summarizing, putting $\Omega_\varepsilon := \left\{y\in\mathbb{R}^N\ :\ r_\varepsilon<|y|<R_\varepsilon\right\}$, we have shown that H_∞ and $v^*-\varepsilon$ are supersolution and subsolution to (6.16) in Ω_ε, respectively, with $v^*-\varepsilon\le H_\infty$ on $\partial\Omega_\varepsilon$ by (6.21). Owing to (6.22) and the concavity of H with respect to its second variable, we may apply [36, Theorem 1] to conclude that $v^*-\varepsilon\le H_\infty$ in Ω_ε. Letting $\varepsilon\to 0$, we end up with $v^*\le H_\infty$ in $\mathbb{R}^N$. Recalling (6.18), we have thus established that $v^*=v_*=H_\infty$ in $\mathbb{R}^N$. In addition, owing to [4, Lemma V.1.9] or [6, Lemme 4.1], the equality $v_*=v^*$ and (6.14) provide the

convergence of $(v(\tau))_{\tau>0}$ towards H_∞ uniformly on every compact subset of $\mathbb{R}^N$ as $\tau \to \infty$. Combining this last property with Lemma 6.5 implies that

$$\lim_{\tau\to\infty} \|v(\tau) - H_\infty\|_\infty = 0\,. \tag{6.23}$$

Theorem 6.4 then readily follows after writing the convergence (6.23) in the original variables (t,x) and noticing that $\|h_\infty(1+t) - h_\infty(t)\|_\infty \longrightarrow 0$ as $t \to \infty$.

□

Bibliographical notes

Theorem 6.1 was first proved for $p = 1$ in [23] when $\sigma = 1$ and [39] when $\sigma = -1$. The extension to the other L^p-norms and the gradient was done in [9]. Theorem 6.4 is proved in [9, Theorem 2.6] with a different method which uses rather the Hopf-Lax-Oleinik representation formula than the relaxed half-limits technique.

When $\sigma = 1$ and $q \in (1, q_\star)$, the large time behaviour of non-negative solutions to (1.1), (1.2) is also self-similar (at least for initial data vanishing sufficiently rapidly for large values of x) but neither the diffusion nor the absorption dominate for large times. It is actually a self-similar solution W to (1.1) which characterises the large time behaviour of non-negative solutions u to (1.1), (1.2) with an initial condition satisfying (1.4) and the decay condition

$$\operatorname*{ess\,lim}_{|x|\to\infty} |x|^{(2-q)/(q-1)}\ u_0(x) = 0\,.$$

More precisely, it has been shown in [9] that

$$\lim_{t\to\infty} \left\{ t^{(N+1)(q_\star-q)/(2(q-1))}\ \|u(t) - W(t)\|_1 + t^{(2-q)/(2(q-1))}\ \|u(t) - W(t)\|_\infty \right\} = 0$$

and W satisfies $W(t,x) = t^{-(2-q)/(2(q-1))}\ W\left(1, xt^{-1/2}\right)$ for $(t,x) \in Q_\infty$. In fact, W is a *very singular solution* to (1.1) in the sense of [24], that is, W is a classical solution to (1.1) in Q_∞, $W(t) \in L^1(\mathbb{R}^N)$ for each $t > 0$,

$$\lim_{t\to 0} \int_{B(0,r)} W(t,x)\ dx = \infty \quad \text{and} \quad \lim_{t\to 0} \int_{\{|x|\ge r\}} W(t,x)\ dx = 0$$

for every $r > 0$. The existence and uniqueness of such a solution have been studied in [10, 12, 28, 44].

Finally, convergence to self-similarity also takes place for the critical exponent $q = q_\star$ when $\sigma = 1$, the scaling profile being the heat kernel g as for $q > q_\star$ (see Theorem 6.1) but with an additional logarithmic scaling factor. More precisely, it is shown in [30] that, if $q = q_\star$ and u_0 satisfies (1.4) together with $u_0 \in L^2(\mathbb{R}^N; (1 + |x|^{2m})\ dx)$ for some $m > N/2$, the corresponding solution u to (1.1), (1.2) satisfies

$$\lim_{t\to\infty} t^{(N(p-1))/2p}\ (\log t)^{N+1} \left\| u(t) - \frac{M_\star}{(\log t)^{N+1}}\ g(t) \right\|_p = 0$$

with $M_\star := (N+1)^{N+1}\ \|\nabla G\|_{q_\star}^{-(N+2)}$ (recall that g and G are defined in (1.7)).

Bibliography

[1] Laurent Amour and Matania Ben-Artzi, *Global existence and decay for viscous Hamilton–Jacobi equations*, Nonlinear Anal. **31** (1998), 621–628.

[2] Daniele Andreucci, Anatoli F. Tedeev and Maura Ughi, *The Cauchy problem for degenerate parabolic equations with source and damping*, Ukrainian Math. Bull. **1** (2004), 1–23.

[3] Donald G. Aronson and Philippe Bénilan, *Régularité des solutions de l'équation des milieux poreux dans* $\mathbb{R}^N$, C. R. Acad. Sci. Paris Sér. I Math. **288** (1979), 103–105.

[4] Martino Bardi and Italo Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton–Jacobi-Bellman Equations*, Systems Control Found. Appl., Birkhäuser, Boston, 1997.

[5] Guy Barles, *Asymptotic behavior of viscosity solutions of first order Hamilton–Jacobi equations*, Ricerche Mat. **34** (1985), 227–260.

[6] Guy Barles, *Solutions de Viscosité des Equations d'Hamilton–Jacobi*, Mathématiques & Applications **17**, Springer-Verlag, Berlin, 1994.

[7] Guy Barles and Benoît Perthame, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Optim. **26** (1988), 1133–1148.

[8] William Beckner, *Geometric asymptotics and the logarithmic Sobolev inequality*, Forum Math. **11** (1999), 105–137.

[9] Saïd Benachour, Grzegorz Karch and Philippe Laurençot, *Asymptotic profiles of solutions to viscous Hamilton–Jacobi equations*, J. Math. Pures Appl. (9) **83** (2004), 1275–1308.

[10] Saïd Benachour, Herbert Koch and Philippe Laurençot, *Very singular solutions to a nonlinear parabolic equation with absorption. II – Uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 39–54.

[11] Saïd Benachour and Philippe Laurençot, *Global solutions to viscous Hamilton–Jacobi equations with irregular initial data*, Comm. Partial Differential Equations **24** (1999), 1999–2021.

[12] Saïd Benachour and Philippe Laurençot, *Very singular solutions to a nonlinear parabolic equation with absorption. I. Existence*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 27–44.

[13] Saïd Benachour, Philippe Laurençot and Didier Schmitt, *Extinction and decay estimates for viscous Hamilton–Jacobi equations in* $\mathbb{R}^N$, Proc. Amer. Math. Soc. **130** (2002), 1103–1111.

[14] Saïd Benachour, Philippe Laurençot, Didier Schmitt and Philippe Souplet, *Extinction and non-extinction for viscous Hamilton–Jacobi equations in* $\mathbb{R}^N$, Asymptot. Anal. **31** (2002), 229–246.

[15] Saïd Benachour, Bernard Roynette and Pierre Vallois, *Solutions fondamentales de* $u_t - \frac{1}{2}\, u_{xx} = \pm|u_x|$, Astérisque **236** (1996), 41–71.

[16] Saïd Benachour, Bernard Roynette and Pierre Vallois, *Asymptotic estimates of solutions of* $u_t - \frac{1}{2}\, \Delta u = -|\nabla u|$ *in* $\mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 2$, J. Funct. Anal. **144** (1997), 301–324.

[17] Matania Ben-Artzi, *Global existence and decay for a nonlinear parabolic equation*, Nonlinear Anal. **19** (1992), 763–768.

[18] Matania Ben-Artzi, Jonathan Goodman and Arnon Levy, *Remarks on a nonlinear parabolic equation*, Trans. Amer. Math. Soc. **352** (1999), 731–751.

[19] Matania Ben-Artzi and Herbert Koch, *Decay of mass for a semilinear parabolic equation*, Comm. Partial Differential Equations **24** (1999), 869–881.

[20] Matania Ben-Artzi, Philippe Souplet and Fred B. Weissler, *The local theory for viscous Hamilton–Jacobi equations in Lebesgue spaces*, J. Math. Pures Appl. (9) **81** (2002), 343–378.

[21] Philippe Bénilan, *Evolution equations and accretive operators*, Lecture notes taken by S. Lenhardt, Univ. of Kentucky, Spring 1981.

[22] Philippe Bénilan and Michael G. Crandall, *The continuous dependence on* φ *of solutions of* $u_t - \Delta\varphi(u) = 0$, Indiana Univ. Math. J. **30** (1981), 161–177.

[23] Piotr Biler, Mohammed Guedda and Grzegorz Karch, *Asymptotic properties of solutions of the viscous Hamilton–Jacobi equation*, J. Evolution Equations **4** (2004), 75–97.

[24] Haïm Brezis, Lambertus A. Peletier and David Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. **95** (1986), 185–209.

[25] Michael G. Crandall, Hitoshi Ishii and Pierre-Louis Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[26] Lawrence C. Evans, *Regularity properties for the heat equation subject to nonlinear boundary constraints*, Nonlinear Anal. **1** (1977), 593–602.

[27] Lawrence C. Evans, *Partial Differential Equations*, Grad. Stud. Math. **19**, Amer. Math. Soc., Providence, RI, 1998.

[28] Zhong Bo Fang and Minkyu Kwak, *Complete classification of shape functions of self-similar solutions*, J. Math. Anal. Appl. **330** (2007), 1447–1464.

[29] Eduard Feireisl and Philippe Laurençot, *The* L^1*-stability of constant states of degenerate convection-diffusion equations*, Asymptot. Anal. **19** (1999), 267–288.

[30] Thierry Gallay and Philippe Laurençot, *Asymptotic behavior for a viscous Hamilton–Jacobi equation with critical exponent*, Indiana Univ. Math. J. **56** (2007), 459–479.

[31] Brian H. Gilding, *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$, large-time behaviour*, J. Math. Pures Appl. (9) **84** (2005), 753–785.

[32] Brian H. Gilding, Mohammed Guedda and Robert Kersner, *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$*, J. Math. Anal. Appl. **284** (2003), 733–755.

[33] Mohammed Guedda and Robert Kersner, *Self-similar solutions to the generalized deterministic KPZ equation*, NoDEA Nonlinear Differential Equations Appl. **10** (2003), 1–13.

[34] Richard S. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. **1** (1993), 113–126.

[35] Miguel A. Herrero and Juan Luis Vázquez, *Asymptotic behaviour of the solutions of a strongly nonlinear parabolic problem*, Ann. Fac. Sci. Toulouse Math. (5) **3** (1981), 113–127.

[36] Hitoshi Ishii, *A simple, direct proof of uniqueness for solutions of the Hamilton–Jacobi equations of eikonal type*, Proc. Amer. Math. Soc. **100** (1987), 247–251.

[37] Yong Jung Kim, *An Oleinik type estimate for a convection-diffusion equation and convergence to N-waves*, J. Differential Equations **199** (2004), 269–289.

[38] Olga A. Ladyženskaja, Vsevolod A. Solonnikov and Nina N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. **23**, Amer. Math. Soc., Providence, RI, 1988.

[39] Philippe Laurençot and Philippe Souplet, *On the growth of mass for a viscous Hamilton–Jacobi equation*, J. Anal. Math. **89** (2003), 367–383.

[40] Philippe Laurençot and Philippe Souplet, *Optimal growth rates for a viscous Hamilton–Jacobi equation*, J. Evolution Equations **5** (2005), 123–135.

[41] Chi-Tien Lin and Eitan Tadmor, *L^1-stability and error estimates for approximate Hamilton–Jacobi solutions*, Numer. Math. **87** (2001), 701–735.

[42] Pierre-Louis Lions, *Regularizing effects for first-order Hamilton–Jacobi equations*, Appl. Anal. **20** (1985), 283–307.

[43] Gawtum Namah and Jean-Michel Roquejoffre, *Remarks on the long time behaviour of the solutions of Hamilton–Jacobi equations*, Comm. Partial Differential Equations **24** (1999), 883–893.

[44] Yuanwei Qi and Mingxin Wang, *The self-similar profiles of generalized KPZ equation*, Pacific J. Math. **201** (2001), 223–240.

[45] Jean-Michel Roquejoffre, *Convergence to steady states or periodic solutions in a class of Hamilton–Jacobi equations*, J. Math. Pures Appl. (9) **80** (2001), 85–104.

APPENDIX A

Self-similar large time behaviour

1. Convergence to self-similarity for the heat equation

We recall the following well-known result.

PROPOSITION A.1. *Consider* $u_0 \in L^1(\mathbb{R}^N)$ *and put*

$$M_0 := \int_{\mathbb{R}^N} u_0(x)\ dx\,.$$

Then, for $p \in [1,\infty]$, *we have*

$$\lim_{t\to\infty} t^{N(p-1)/(2p)}\ \left\|e^{t\Delta}u_0 - M_0\ g(t)\right\|_p \quad = \quad 0\,, \tag{A.1}$$

$$\lim_{t\to\infty} t^{N(p-1)/(2p)+(1/2)}\ \left\|\nabla e^{t\Delta}u_0 - M_0\ \nabla g(t)\right\|_p \quad = \quad 0\,, \tag{A.2}$$

where the functions g *and* G *are given by (1.7).*

PROOF. For $(t,x) \in Q_\infty$, we put

$$c(t,x) := \left(e^{t\Delta}u_0\right)(x) = \frac{1}{t^{N/2}}\ \int_{\mathbb{R}^N} G\left(\frac{x-y}{t^{1/2}}\right)\ u_0(y)\ dy$$

and $z(t,x) := c(t,x) - M_0\ g(t,x)$. An alternative formula for z reads

$$z(t,x) = \frac{1}{t^{N/2}}\ \int_{\mathbb{R}^N} \left[G\left(\frac{x-y}{t^{1/2}}\right) - G\left(\frac{x}{t^{1/2}}\right)\right]\ u_0(y)\ dy\,.$$

On the one hand,

$$\|z(t)\|_1 \le \int_{\mathbb{R}^N} |u_0(y)|\ \int_{\mathbb{R}^N} \left|G(x) - G\left(x - yt^{-1/2}\right)\right|\ dxdy\,.$$

Noting that

$$\int_{\mathbb{R}^N} \left|G(x) - G\left(x - yt^{-1/2}\right)\right|\ dx \le 2\ \|G\|_1 < \infty$$

and

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} \left|G(x) - G\left(x - yt^{-1/2}\right)\right|\ dx = 0$$

for each $y \in \mathbb{R}^N$, we infer from the Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty} \|z(t)\|_1 = 0\,. \tag{A.3}$$

On the other hand, since $c(t) = e^{(t/2)\Delta}c(t/2)$ and $g(t) = e^{(t/2)\Delta}g(t/2)$, we have

$$|z(t,x)| \le \left|\frac{1}{(2\pi t)^{N/2}}\ \int_{\mathbb{R}^N} z\left(\frac{t}{2},y\right)\ \exp\left\{-\frac{|x-y|^2}{2t}\right\}\ dy\right| \le \frac{1}{(2\pi t)^{N/2}}\ \left\|z\left(\frac{t}{2}\right)\right\|_1\,,$$

hence

$$t^{N/2}\ \|z(t)\|_\infty \le \frac{1}{(2\pi)^{N/2}}\ \left\| z\left(\frac{t}{2}\right)\right\|_1 . \tag{A.4}$$

Combining (A.3) and (A.4) yields (A.1) for $p=\infty$. The result for $p\in(1,\infty)$ then follows by interpolation with (A.3).

Similarly,

$$\nabla z(t,x) = \frac{2^{(N+1)/2}}{t^{(N+1)/2}}\ \int_{\mathbb{R}^N} \nabla G\left(\frac{2^{1/2}(x-y)}{t^{1/2}}\right)\ z\left(\frac{t}{2},y\right)\ dy$$

and we infer from the Young inequality that

$$\|\nabla z(t)\|_p \le \frac{2^{1/2}}{t^{1/2}}\ \|\nabla G\|_1\ \left\| z\left(\frac{t}{2}\right)\right\|_p$$

for $p\in[1,\infty]$. We now use (A.1) to estimate the right-hand side of the previous inequality and obtain (A.2).

□

2. Convergence to self-similarity for Hamilton–Jacobi equations

This section is devoted to the large time behaviour of non-negative (viscosity) solutions to the Hamilton–Jacobi equation (1.9). We begin with the absorption case $\sigma=1$ and establish the following result for compactly supported initial data.

THEOREM A.2. *Assume that $q>1$ and $\sigma=1$. Consider a non-negative function $u_0\in\mathcal{C}(\mathbb{R}^N)$ with compact support and denote by h the corresponding viscosity solution to the Hamilton–Jacobi equation (1.9). Then*

$$\lim_{t\to\infty} \left\| t^{1/(q-1)}\ h(t) - h_\infty \right\|_\infty = 0 \tag{A.5}$$

with

$$h_\infty(x) = \gamma_q\ d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)^{q/(q-1)}, \qquad x\in\mathbb{R}^N,$$

$\gamma_q := (q-1)\ q^{-q/(q-1)}$ *and* $\mathcal{P}_0 := \left\{x\in\mathbb{R}^N\ :\ u_0(x)>0\right\}$.

PROOF. We first recall that h is given by the Hopf–Lax–Oleinik formula (1.10)

$$h(t,x) := \inf_{y\in\mathbb{R}^N} \left\{ u_0(y) + \gamma_q\ |x-y|^{q/(q-1)}\ t^{-1/(q-1)} \right\} \quad \text{for} \quad (t,x)\in Q_\infty . \tag{A.6}$$

We next introduce the positivity set $\mathcal{P}(t)$ of h at time $t>0$ and claim that

$$\mathcal{P}(t) := \left\{x\in\mathbb{R}^N\ :\ h(t,x)>0\right\} = \mathcal{P}_0 . \tag{A.7}$$

Indeed, if $t>0$ and $x\notin\mathcal{P}_0$, we have $u_0(x)=0$ and the choice $y=x$ in the right-hand side of (A.6) ensures that $0\le h(t,x)\le u_0(x)=0$, whence $x\notin\mathcal{P}(t)$. Next, if $t>0$ and $x\in\mathcal{P}_0$, we assume for contradiction that $h(t,x)=0$. Since $u_0(y)+\gamma_q\ |x-y|^{q/(q-1)}\ t^{-1/(q-1)} \longrightarrow \infty$ as $|y|\to\infty$, the infimum in (A.6) is attained and there is $y\in\mathbb{R}^N$ such that

$$0 = h(t,x) = u_0(y) + \gamma_q\ |x-y|^{q/(q-1)}\ t^{-1/(q-1)} .$$

This readily implies that $y=x$ and $u_0(y)=u_0(x)=0$, and a contradiction. Therefore, $x\in\mathcal{P}(t)$ and the proof of (A.7) is complete.

We next introduce $\ell(t,x) := t^{1/(q-1)}\ h(t,x)$ so that

$$\ell(t,x) := \inf_{y\in\mathbb{R}^N} \left\{ t^{1/(q-1)}\ u_0(y) + \gamma_q\ |x-y|^{q/(q-1)} \right\} \tag{A.8}$$

for $(t,x) \in [0,\infty)\times\mathbb{R}^N$. We first note that

$$0 \le \ell(t_1,x) \le \ell(t_2,x) \le \gamma_q\ d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)^{q/(q-1)} = h_\infty(x) \tag{A.9}$$

for $x\in\mathbb{R}^N$ and $0<t_1<t_2$, the last inequality being deduced from (A.8) by taking the infimum only on $\mathbb{R}^N\setminus\mathcal{P}_0$ where u_0 vanishes. We next consider $x\in\mathcal{P}_0$ and $\varepsilon\in\left(0, d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)/2\right)$. Setting

$$m_\varepsilon := \inf\left\{u_0(y)\ \ :\ \ d\left(y,\mathbb{R}^N\setminus\mathcal{P}_0\right)\ge\varepsilon\right\} > 0\,, \qquad t_\varepsilon := \left(\frac{\gamma_q}{m_\varepsilon}\right)^{q-1}\ d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)^q$$

and choosing $t\ge t_\varepsilon$, we have the following alternative for $y\in\mathbb{R}^N$: either $d\left(y,\mathbb{R}^N\setminus\mathcal{P}_0\right)\ge\varepsilon$ and

$$t^{1/(q-1)}\ u_0(y) + \gamma_q\ |x-y|^{q/(q-1)} \ge m_\varepsilon\ t_\varepsilon^{1/(q-1)} \ge \gamma_q\ d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)^{q/(q-1)}$$

or $d\left(y,\mathbb{R}^N\setminus\mathcal{P}_0\right)<\varepsilon$ and

$$\begin{aligned} t^{1/(q-1)}\ u_0(y) + \gamma_q\ |x-y|^{q/(q-1)} &\ge \gamma_q\ \left(d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right) - d\left(y,\mathbb{R}^N\setminus\mathcal{P}_0\right)\right)^{q/(q-1)} \\ &\ge \gamma_q\ \left(d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)-\varepsilon\right)^{q/(q-1)}\ . \end{aligned}$$

Consequently,

$$\ell(t,x) \ge \gamma_q\ \left(d\left(x,\mathbb{R}^N\setminus\mathcal{P}_0\right)-\varepsilon\right)^{q/(q-1)} \quad \text{for} \quad t\in[t_\varepsilon,\infty)\times\mathbb{R}^N\,. \tag{A.10}$$

The convergence (A.5) then readily follows from (A.9) and (A.10) after passing to the limit $\varepsilon\to 0$.

$\square$

We next turn to the case $\sigma=-1$ and prove the following result.

THEOREM A.3. *Assume that $q>1$ and $\sigma=-1$. Consider a non-negative function $u_0\in\mathcal{C}_0(\mathbb{R}^N)$ and denote by h the corresponding viscosity solution to the Hamilton–Jacobi equation (1.9). Then*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^N}\left|h(t,x) - H_\infty\left(\frac{x}{t^{1/q}}\right)\right| = 0 \tag{A.11}$$

with

$$H_\infty(x) = \left(\|u_0\|_\infty - \gamma_q\ |x|^{q/(q-1)}\right)_+\,, \qquad x\in\mathbb{R}^N\,.$$

PROOF. As before, h is given by the Hopf–Lax–Oleinik formula (1.10)

$$h(t,x) := \sup_{y\in\mathbb{R}^N}\left\{u_0(y) - \gamma_q\ |x-y|^{q/(q-1)}\ t^{-1/(q-1)}\right\} \quad \text{for} \quad (t,x)\in Q_\infty\,. \tag{A.12}$$

We first show that

$$\|h(t)\|_\infty = \|u_0\|_\infty\,, \qquad t\ge 0\,. \tag{A.13}$$

Indeed, recall that $h(t,x)\ge u_0(x)$ for $(t,x)\in[0,\infty)\times\mathbb{R}^N$ which ensures that $\|h(t)\|_\infty\ge\|u_0\|_\infty$ while the reverse inequality readily follows from the representation formula (A.12).

Next, it is rather easy to deduce from (A.12) that $h(t,x) \longrightarrow \|u_0\|_\infty$ as $t \to \infty$ for each $x \in \mathbb{R}^N$. The goal is now to study more precisely this convergence. To this end we introduce the self-similar (or scaling) variables

$$s = \frac{1}{q}\ \log(1+t)\,, \qquad \xi = \frac{x}{(1+t)^{1/q}}\,,$$

and the new unknown function ℓ defined by

$$h(t,x) = \ell\left(\frac{\log(1+t)}{q}, \frac{x}{(1+t)^{1/q}}\right)\,, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^N\,.$$

Then $\ell(s,\xi) = h\left(e^{qs}-1, \xi e^s\right)$ and it follows from (A.12) that

$$\ell(s,\xi) = \sup_{y\in\mathbb{R}^N} \left\{ u_0(y) - \gamma_q\ \left|\xi - y\ e^{-s}\right|^{q/(q-1)}\ \left(1-e^{-qs}\right)^{-1/(q-1)} \right\}$$

for $(s,\xi) \in [0,\infty) \times \mathbb{R}^N$. Since $\ell(s,\xi) \ge u_0(\xi e^s) \ge 0$, we have in fact

$$\ell(s,\xi) = \sup_{y\in\mathbb{R}^N} \left\{ \left(u_0(y) - \gamma_q\ \left|\xi - y\ e^{-s}\right|^{q/(q-1)}\ \left(1-e^{-qs}\right)^{-1/(q-1)} \right)_+ \right\}$$

for $(s,\xi) \in [0,\infty) \times \mathbb{R}^N$. Let us point out here that, as a consequence of (1.9), ℓ is actually the viscosity solution to

$$\begin{aligned} \partial_s \ell &= \xi \cdot \nabla \ell + q\, |\nabla \ell|^q\,, \quad (s,\xi) \in Q_\infty\,, && \text{(A.14)} \\ \ell(0) &= u_0\,, \quad \xi \in \mathbb{R}^N\,. && \text{(A.15)} \end{aligned}$$

Consider now $\varepsilon \in (0,1)$. As $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$, there is $R_\varepsilon > 0$ such that

$$u_0(y) \le \varepsilon \quad \text{ for } \quad |y| \ge R_\varepsilon\,. \tag{A.16}$$

On the one hand, if $(s,\xi) \in [\log R_\varepsilon, \infty) \times \mathbb{R}^N$ and $y \in \mathbb{R}^N$, we have either $|y| \ge R_\varepsilon$ and

$$\begin{aligned} & \left| \left(u_0(y) - \gamma_q\ \frac{|\xi - y\ e^{-s}|^{q/(q-1)}}{\left(1-e^{-qs}\right)^{1/(q-1)}} \right)_+ - \left(u_0(y) - \gamma_q\ |\xi|^{q/(q-1)}\ \right)_+ \right| \\ \le\ & \left(u_0(y) - \gamma_q\ \frac{|\xi - y\ e^{-s}|^{q/(q-1)}}{\left(1-e^{-qs}\right)^{1/(q-1)}} \right)_+ + \left(u_0(y) - \gamma_q\ |\xi|^{q/(q-1)} \right)_+ \\ \le\ & 2\varepsilon \end{aligned}$$

by (A.16) or $y \in B(0, R_\varepsilon)$ and

$$\begin{aligned} & \left| \left(u_0(y) - \gamma_q \ \frac{|\xi - y\ e^{-s}|^{q/(q-1)}}{(1-e^{-qs})^{1/(q-1)}} \right)_+ - \left(u_0(y) - \gamma_q \ |\xi|^{q/(q-1)} \ \right)_+ \right| \\ \leq\ & \gamma_q \ \left|\xi - y\ e^{-s}\right|^{q/(q-1)} \ \left\{ \left(1-e^{-qs}\right)^{-1/(q-1)} - 1 \right\} \\ & + \gamma_q \ \left| \left|\xi - y\ e^{-s}\right|^{q/(q-1)} - |\xi|^{q/(q-1)} \right| \\ \leq\ & \gamma_q \ \left(|\xi| + R_\varepsilon\ e^{-s}\right)^{q/(q-1)} \ \left\{ \left(1-e^{-qs}\right)^{-1/(q-1)} - 1 \right\} \\ & + \frac{q\ \gamma_q}{q-1} \ \left(|\xi| + |y|\ e^{-s}\right)^{1/(q-1)} \ |y|\ e^{-s} \\ \leq\ & \gamma_q \ (1+|\xi|)^{1/(q-1)} \ \left\{ \frac{q}{q-1} + 1 + |\xi| \right\} \ \left\{ \left(1-e^{-qs}\right)^{-1/(q-1)} - 1 + R_\varepsilon\ e^{-s} \right\} \end{aligned}$$

as $s \geq \log R_\varepsilon$. Combining the above two estimates gives

$$\begin{aligned} & \left| \ell(s,\xi) - \sup_{y \in \mathbb{R}^N} \left\{ \left(u_0(y) - \gamma_q \ |\xi|^{q/(q-1)} \ \right)_+ \right\} \right| \\ \leq\ & C \ (|\xi|+1)^{q/(q-1)} \ \left\{ \left(1-e^{-qs}\right)^{-1/(q-1)} - 1 + R_\varepsilon\ e^{-s} \right\} + 2\varepsilon\,, \end{aligned}$$

whence

$$\begin{aligned} & |\ell(s,\xi) - H_\infty(\xi)| \\ \leq\ & C \ (|\xi|+1)^{q/(q-1)} \ \left\{ \left(1-e^{-qs}\right)^{-1/(q-1)} - 1 + R_\varepsilon\ e^{-s} \right\} + 2\varepsilon \end{aligned} \tag{A.17}$$

for $(s,\xi) \in [\log R_\varepsilon, \infty) \times \mathbb{R}^N$. On the other hand, if $s \geq \log(R_\varepsilon)$, $|\xi| \geq \Xi := 1 + (\|u_0\|_\infty/\gamma_q)^{(q-1)/q}$ and $y \in \mathbb{R}^N$, we have either $|\xi - y\ e^{-s}| \geq \Xi - 1$ and

$$\begin{aligned} & u_0(y) - \gamma_q \ \left|\xi - y\ e^{-s}\right|^{q/(q-1)} \ \left(1-e^{-qs}\right)^{-1/(q-1)} \\ \leq\ & \left(1-e^{-qs}\right)^{-1/(q-1)} \ \left\{ \|u_0\|_\infty \ \left(1-e^{-qs}\right)^{1/(q-1)} - \gamma_q \ \left|\xi - y\ e^{-s}\right|^{q/(q-1)} \right\} \\ \leq\ & \left(1-e^{-qs}\right)^{-1/(q-1)} \ \left\{ \|u_0\|_\infty - \gamma_q \ (\Xi-1)^{q/(q-1)} \right\} \\ \leq\ & 0\,, \end{aligned}$$

or $|\xi - y\ e^{-s}| < \Xi - 1$ and

$$|y| \geq |\xi\ e^s| - |y - \xi\ e^s| \geq \Xi\ e^s - (\Xi - 1)\ e^s = e^s \geq R_\varepsilon\,,$$

so that

$$u_0(y) - \gamma_q \ \left|\xi - y\ e^{-s}\right|^{q/(q-1)} \ \left(1-e^{-qs}\right)^{-1/(q-1)} \leq \varepsilon$$

by (A.16). Therefore,

$$\ell(s,\xi) \leq \varepsilon \quad \text{for} \quad (s,\xi) \in [\log R_\varepsilon, \infty) \times \mathbb{R}^N \setminus B(0,\Xi)\,. \tag{A.18}$$

It then follows easily from (A.17) and (A.18) that $\|\ell(s) - H_\infty\|_\infty \longrightarrow 0$ as $s \to \infty$. Returning to the original variables (t,x), we conclude that

$$\lim_{t\to\infty} \sup_{x \in \mathbb{R}^N} \left| h(t,x) - H_\infty\left(\frac{x}{(1+t)^{1/q}} \right) \right| = 0\,.$$

Since

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| H_\infty\left(\frac{x}{(1+t)^{1/q}}\right) - H_\infty\left(\frac{x}{t^{1/q}}\right) \right| = 0\,,$$

the convergence (A.11) follows. □

We have actually shown the following result.

COROLLARY A.4. *Let $q > 1$ and consider a non-negative function $u_0 \in \mathcal{C}_0(\mathbb{R}^N)$. If ℓ denotes the corresponding viscosity solution to (A.14), (A.15), then*

$$\lim_{s\to\infty} \|\ell(s) - H_\infty\|_\infty = 0\,,$$

the function H_∞ being defined in Theorem A.3.

REMARK A.5. *Related results may be found in [5, Section II] for the Hamilton–Jacobi equation $\partial_t w + H(\nabla w) = 0$ in Q_∞, the function H being non-negative and continuous with suitable properties.*

Part 4

An area-preserving crystalline curvature flow equation

Shigetoshi Yazaki

2000 *Mathematics Subject Classification.* 34A34, 34K25, 39A12, 52A10, 53A04, 74N05, 34K12, 34K28, 65L12, 65M06, 65M12, 65D99, 41A25, 82D25

Key words and phrases. evolving interfaces, polygonal curves, curvature, curve-shortening, crystalline curvature, anisotropy, interfacial energy, the Frank diagram, the Wulff shape, admissibility, area-preserving, single ice crystal, internal melting, Tyndall figure, negative crystal, vapor figure, crystal growth

Abstract. In this note, we will show how to derive crystalline curvature flow equation and its area-preserving version. This flow is regarded as a gradient flow of total interfacial energy with the singular anisotropy, then the usual differential calculation is not allowed in the classical sense. Because of this, a difference calculation is used instead of differential calculation and polygonal curves are used instead of smooth curves. This consideration will be focused in the former and main part of this note, which includes the following three topics:

- plane curves, geometric quantities on them, and evolution of them;
- anisotropy, the Frank diagram, and the Wulff shape, etc.; and
- crystalline curvature flow equation and its area-preserving version.

In the latter part, we will see the following two applications:

- a numerical scheme of the area-preserving crystalline curvature flow; and
- a modeling perspective on producing negative ice crystals.

Contents

Preface

The so-called crystalline curvature flow equation appeared at the end of 1980's in the field of mathematical sciences. This equation is regarded as a weak form of an equation of interfacial motion with a singular interfacial energy. The equation is described as a system of ODEs and governs motion of polygonal curves in two dimensional case. If the number of vertices of curves is large, the motion can approximate the one described by PDE with a smooth interfacial energy. On the other hand, if the number is small, the motion is sometimes strikingly different from the one in the smooth case. This phenomena suggest that research of this field will continue to a wide range of applications.

The lecture note will target beginners of the research field. Therefore, only two dimensional closed curves will be focused and I will put emphasis on a fundamental part of mathematical introduction to crystalline curvature flow equations. I hope all the readers will follow several calculations and challenge to solve given exercises. In addition, we will touch on topics of a numerical scheme and a modeling of negative crystal, and they may give directions of applications.

The note is based on the lectures which were held at Czech Technical University in Prague, September 4 and 6, 2006, and December 4, 6, 11 and 13, 2007, while the author was visiting Czech Republic within the period from August to September, 2006, and from March, 2007 to March, 2008; this visit was sponsored by Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, within the Jindřich Nečas Center for Mathematical Modeling (Project of the Czech Ministry of Education, Youth and Sports LC 06052). The author would like to thank Professor Michal Beneš for giving me this opportunity not only to have the series of lectures but also to visit Czech Republic as a young researcher.

Shigetoshi Yazaki
Miyazaki, June 2008

CHAPTER 1

An area-preserving crystalline curvature flow equation: introduction to mathematical aspects, numerical computations, and a modeling perspective

1. Introduction

Motion by curvature is generally referred as a motion of curves in the plane or surfaces in space which change its shape in time and depend on its bend, especially on its curvature. They are also called curvature flows, since one tracks flows of the family of curves and surfaces parametrized in time. The curvature flow equation is a general term which describes such flows, and has been investigated by many scientists and mathematicians since the 1950's. At the end of 1980's, J. E. Taylor, and S. Angenent and M. E. Gurtin focused on motion of polygonal curves by crystalline curvature in the plane, and since then crystalline curvature flow equation has been studied under various kinds of evolution law by several authors. We refer the reader to the pioneer works Taylor [Tay91a, Tay93] and Angenent and Gurtin [AG89], and the surveys by Taylor, Cahn and Handwerker [TCH92] and the books by Gurtin [Gur93] for a geometric and physical background. Also one can find essentially the same method of crystalline as a numerical scheme for curvature flow equation in Roberts [Rob93]. We refer the reader Almgren and Taylor [AT95] for detailed history. Besides this crystalline strategy, other strategies by subdifferential and level-set method have been extensively studied. See Giga [Gig00] and references therein.

This note will focus mainly on motion of smooth and piecewise linear curves in the plane, and touch on a numerical scheme of crystalline curvature flow equation which is a kind of so-called direct approach. On motion of surfaces, indirect approach (ex. level set methods), other curvature flows, and physical background, the reader is referred to the books by Gurtin [Gur93] and Sethian [Set96], and the surveys by [TCH92] and Giga [Gig95, Gig98, Gig00], and references therein, respectively. In the last section, a modeling perspective will be reported.

2. Plane curve

In this section we will explain plane curve and several geometric quantities on it.

A **plane curve** is defined as the following map $\boldsymbol{x}$:

$$\Gamma = \{\boldsymbol{x}(\sigma) \in \mathbb{R}^2;\ \sigma \in Q\},$$

where $Q \subset \mathbb{R}$ is an interval[1], and the map $\boldsymbol{x} : Q \mapsto \Gamma \subset \mathbb{R}^2$ is continuous and surjective[2]. The plane curve $\boldsymbol{x}$ is said to be **closed** if $Q = [a, b]$ and $\boldsymbol{x}(a) = \boldsymbol{x}(b)$. If $\boldsymbol{x}$ is not closed, then it is called **open**. The plane curve $\boldsymbol{x}$ is said to be **simple** if the map[3] $\boldsymbol{x}$ is injective[4]. A simple and closed plane curve is called **Jordan curve**. If $\boldsymbol{x}$ is a Jordan curve, then Γ is also called Jordan curve.

REMARK 1.1 (Jordan curve theorem). *If Γ is a Jordan curve, then $\mathbb{R}^2 \backslash \Gamma$ has two components*[5]*, where Γ is the boundary of each.*

In what follows, curves mean plane curves, and we will omit the term "plane". A curve $\boldsymbol{x}$ is said to be $\boldsymbol{C^k}$**-curve** $(k = 1, 2, \ldots)$ if the map $\boldsymbol{x}$ is C^k-class function. We put

$$g(\sigma) = |\partial_\sigma \boldsymbol{x}(\sigma)|.$$

Here and hereafter, we denote $\partial_\xi \mathsf{F} = \partial \mathsf{F}/\partial \xi$. A curve $\boldsymbol{x}$ is said to be **immersed** if $\boldsymbol{x}$ is C^1-curve and $g(\sigma) \neq 0$ holds for all σ.

For closed curves, we take $\Sigma = \mathbb{R}/\mathbb{Z}$ with parameter σ (modulo 1), and use the interval $\Sigma = [0, 1]$. Then a closed plane curve $\boldsymbol{x}$ can be described as

$$\Gamma = \{\boldsymbol{x}(\sigma) \in \mathbb{R}^2;\ \sigma \in \Sigma\}.$$

REMARK 1.2. *If $\boldsymbol{x}$ is a closed immersed curve, then*

$$s(\sigma) = \int_0^\sigma g(\sigma)\, d\sigma$$

is increasing C^1-function for $\sigma \in \Sigma$, since g is continuous function and $\partial_\sigma s(\sigma) = g(\sigma) > 0$ holds. Therefore, by virtue of an implicit function theorem, σ is a function of s, and the closed immersed curve $\boldsymbol{x}$ can be parametrized by the arc-length parameter s:

$$\Gamma = \{\boldsymbol{x}(s) \in \mathbb{R}^2;\ s \in [0, L]\}, \quad L = \int_\Gamma ds = \int_\Sigma g(\sigma)\, d\sigma,$$

where L is the total length of Γ.

A curve $\boldsymbol{x}$ is said to be **embedded** if $\boldsymbol{x}$ is immersed and simple. Hence, if Γ has self-intersection or self-tangency, then $\boldsymbol{x}$ is not embedded curve. Note that all the above terminologies are defined for the map $\boldsymbol{x}$, not for the set Γ. This distinction is important. See section 1.

Let $\boldsymbol{x}$ be an immersed curve. The **unit tangent vector** is defined as $\boldsymbol{t} = \partial_\sigma \boldsymbol{x}/|\partial_\sigma \boldsymbol{x}| = \partial_s \boldsymbol{x}$, and the unit normal vector $\boldsymbol{n}$ is defined such as $\det(\boldsymbol{n}, \boldsymbol{t}) = 1$, or $\boldsymbol{t} = \boldsymbol{n}^\perp$, where $(a, b)^\perp = (-b, a)$. Note that the derivative with respect to the arc-length ∂_s along the curve is uniquely defined, while the arc-length s is unique only up to a constant. The **positive direction** of $\boldsymbol{x}$ is the direction which σ is increasing. If the curve is a Jordan curve, we take the positive direction as

---

[1]Non-empty and connected subset in $\mathbb{R}$.

[2]Onto map, i.e., for any $\overline{\boldsymbol{x}} \in \Gamma$ there exists a $\sigma \in Q$ for which $\boldsymbol{x}(\sigma) = \overline{\boldsymbol{x}}$.

[3]If $\boldsymbol{x}$ is closed, we take $Q = [a, b)$.

[4]One-to-one map, i.e., for any $\sigma_1, \sigma_2 \in Q$ if $\sigma_1 \neq \sigma_2$, then $\boldsymbol{x}(\sigma_1) \neq \boldsymbol{x}(\sigma_2)$ holds.

[5]So-called "inside" D and "outside" E. Both D and E are connected, respectively, but its direct sum $D \oplus E$ is not connected.

anticlockwise, i.e., the direction which one can always see enclosed region on the left side when traveling along the curve in the direction of σ increasing. See Figure 1.

Let $\boldsymbol{x}$ be an immersed C^2-curve. The **curvature** K of $\boldsymbol{x}$ is defined as

$$\partial_s \boldsymbol{t} = -K\boldsymbol{n} \quad \text{and} \quad \partial_s \boldsymbol{n} = K\boldsymbol{t}, \tag{1.1}$$

and it is often called the curvature in the direction of $-\boldsymbol{n}$. The signed convention of the curvature K is that $K = 1$ if and only if $\boldsymbol{x}$ is the unit circle. See Figure 1.

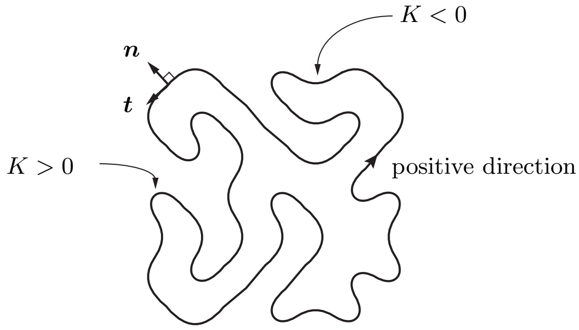


FIGURE 1. The normal vector $\boldsymbol{n}$, the tangent vector $\boldsymbol{t}$, and the curvature K on a closed embedded curve.

REMARK 1.3. *Equations (1.1) are called* **Frenet–Serret formulae** *for plane curves. From $\boldsymbol{n}\cdot\boldsymbol{n} = 1$, we obtain $\partial_s(\boldsymbol{n}\cdot\boldsymbol{n}) = 2(\partial_s\boldsymbol{n})\cdot\boldsymbol{n} = 0$. Here and hereafter, $\boldsymbol{a}\cdot\boldsymbol{b}$ is the Euclidean inner product between $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence there exists a function $c(s)$ such as $\partial_s\boldsymbol{n} = c\boldsymbol{t}$. Similarly, from $\boldsymbol{t}\cdot\boldsymbol{t} = 1$ and $\boldsymbol{t}\cdot\boldsymbol{n} = 0$, we obtain $\partial_s(\boldsymbol{t}\cdot\boldsymbol{t}) = 2(\partial_s\boldsymbol{t})\cdot\boldsymbol{t} = 0$ and $\partial_s(\boldsymbol{t}\cdot\boldsymbol{n}) = (\partial_s\boldsymbol{t})\cdot\boldsymbol{n} + \boldsymbol{t}\cdot(\partial_s\boldsymbol{n}) = (\partial_s\boldsymbol{t})\cdot\boldsymbol{n} + c = 0$. Therefore $\partial_s\boldsymbol{t} = -c\boldsymbol{n}$ holds. We defined c as the curvature K.*

Let Γ be a Jordan curve and let Ω be a region enclosed by Γ[6]. The Jordan curve Γ is said to be **convex** or **oval** if for any $\sigma_1, \sigma_2 \in \Sigma$ ($\sigma_1 \neq \sigma_2$) the line segment[7] $S = [\boldsymbol{x}(\sigma_1), \boldsymbol{x}(\sigma_2)]$ is not contained outside of Γ, i.e., $S \in \overline{\Omega}$ holds. In particular, if $S \cap \Gamma = \{\boldsymbol{x}(\sigma_1), \boldsymbol{x}(\sigma_2)\}$ holds for any $\sigma_1, \sigma_2 \in \Sigma$, then Γ is called **strictly convex**. The curve is called **concave** or **nonconvex**, if it is not convex.

LEMMA 1.4. *A closed embedded C^2-curve is convex if and only if $K(\sigma) \geq 0$ holds for $\sigma \in \Sigma$. Equivalently, the curve is concave if and only if K takes negative value at some point.*

LEMMA 1.5. *A closed embedded C^2-curve is strictly convex, if $K(\sigma) > 0$ holds for $\sigma \in \Sigma$.*

EXERCISE 1.6. *Prove the above two lemmas.*

REMARK 1.7. *Convex curves admit including a line segment ($K = 0$ on some portion of Σ), while strictly convex curve does not admit it. This does not mean that the converse of Lemma 1.5 is true. See section 2.*

[6] Ω is called inside of Γ, and outside of Γ is $\mathbb{R}^2\backslash\overline{\Omega}$.
[7] In detail, $S = \{(1-\lambda)\boldsymbol{x}(\sigma_1) + \lambda\boldsymbol{x}(\sigma_2);\ \lambda \in [0,1]\}$.

3. Moving plane curve

We consider a moving curve with the time t in some interval, for instance $t \in [0,T)$ with some $T > 0$. Let a smooth-closed-immersed moving curve be

$$\Gamma(t) = \{\boldsymbol{x}(\sigma,t) \in \mathbb{R}^2;\ \sigma \in \Sigma\}.$$

By V we denote the growth speed at each point of $\Gamma(t)$ in the direction of $\boldsymbol{n}$:

$$V = \partial_t \boldsymbol{x} \cdot \boldsymbol{n}.$$

It is well known that information of V determines the shape of $\Gamma(t)$, while the tangential component v does not effect on the shape [EG87]:

$$\partial_t \boldsymbol{x} = V\boldsymbol{n} + v\boldsymbol{t}.$$

Here and hereafter $\partial_t \mathsf{F}$ means $\partial_t \mathsf{F}(\sigma,t)$, and $\partial_s \mathsf{F}$ is given in terms of σ:

$$\partial_s \mathsf{F} = g^{-1}\partial_\sigma \mathsf{F}, \quad g = \sqrt{(\partial_\sigma x)^2 + (\partial_\sigma y)^2} = |\partial_\sigma \boldsymbol{x}|, \quad \boldsymbol{x}(\sigma,t) = (x(\sigma,t), y(\sigma,t)).$$

That is, the arc-length element is $ds = g\,d\sigma$.

In what follows, we will see that fundamental properties of evolution of geometric quantities.

Since ∂_t and ∂_σ can commute: $\partial_t\partial_\sigma = \partial_\sigma\partial_t$, using the Frenet–Serret formulae we obtain:

Lemma 1.8. $\partial_t g = (KV + \partial_s v)g$.

This lemma and $\boldsymbol{t} = g^{-1}\partial_\sigma \boldsymbol{x}$ yield:

Lemma 1.9. *$\partial_t \boldsymbol{t} = (\partial_s V - Kv)\boldsymbol{n}$ and $\partial_t \boldsymbol{n} = -(\partial_s V - Kv)\boldsymbol{t}$.*

Let $\theta = \theta(\sigma,t)$ be the normal angle such as $\boldsymbol{n} = (\cos\theta, \sin\theta)$. Then we have:

Lemma 1.10. *$\partial_s \theta = K$ and $\partial_t \theta = -\partial_s V + Kv$.*

By using the above lemmas, we have the evolution of K:

Lemma 1.11. $\partial_t K = -(\partial_s^2 V + K^2 V) + v\partial_s K$.

Since s depends on t, ∂_t and ∂_s can not commute:

Lemma 1.12. $\partial_t\partial_s \mathsf{F} = \partial_s\partial_t \mathsf{F} - (KV + \partial_s v)\partial_s \mathsf{F}$.

We denote by $L(t)$ the total length of $\Gamma(t)$: $L = \int_\Gamma ds = \int_\Sigma g\,d\sigma$. The evolution of $L(t)$ is given by:

Lemma 1.13. $\partial_t L = \int_\Gamma KV\,ds$.

We denote by $\Omega(t)$ the enclosed region of $\Gamma(t)$ and by $A(t)$ the area of $\Omega(t)$. Using the relation

$$A = \frac{1}{2}\int_\Omega \operatorname{div}\begin{pmatrix} x \\ y \end{pmatrix} dxdy = \frac{1}{2}\int_\Gamma \boldsymbol{x}\cdot\boldsymbol{n}\,ds = \frac{1}{2}\int_\Sigma (\boldsymbol{x}\cdot\boldsymbol{n})g\,d\sigma,$$

we have the evolution of $A(t)$:

Lemma 1.14. $\partial_t A = \int_\Gamma V\,ds$.

Exercise 1.15. *Show above lemmas.*

4. Curvature flow equations

To catch mathematical characteristics of curvature flows, we shall formulate the typical problem. We mainly focus on smooth, embedded and closed curve, and sometimes mention the case where curves are immersed.

Classical curvature flow. The classical and the typical equation which motivates investigation of curvature flow equations is

$$V = -K. \tag{1.2}$$

This is called the **classical curvature flow equation**. Figure 2 indicates evolution of curve by (1.2). Convex part ($K > 0$) of nonconvex curve $\Gamma(t)$ moves towards the direction of $-\boldsymbol{n}$ (inward), and concave part ($K < 0$) moves towards the direction of $\boldsymbol{n}$ (outward). Any embedded curve becomes convex in finite time [Gra87], and any convex curve shrinks to a single point in finite time and its asymptotic shape is a circle [GH86]. In the case where the solution curve is immersed, for example,

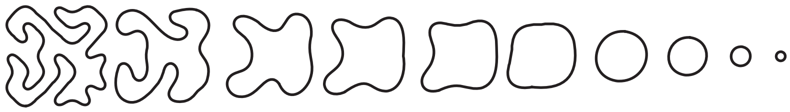


FIGURE 2. Evolution of a simple closed curve by (1.2) (from left to right).

the curve has a single small loop as in Figure 3, we can observe that the single loop shrinks and the curve may have cusp in finite time. A detailed analysis of the singularity near the extinction time was done by Angenent and Velázquez [AV95].

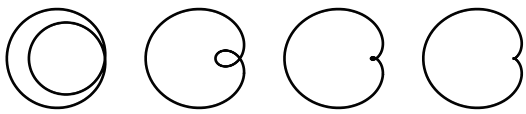


FIGURE 3. Evolution of a immersed closed curve by (1.2) (from left to right). The small loop disappears in finite time.

Example of graph. In the case where $\Gamma(t)$ is described by a graph $y = u(x, t)$, we have the partial differential equation $\partial_t u = \frac{\partial_x^2 u}{1+(\partial_x u)^2}$ equivalent to (1.2) with $V = (0, \partial_t u)^T \cdot \boldsymbol{n}$, $\boldsymbol{n} = \frac{(-\partial_x u, 1)^T}{\sqrt{1+(\partial_x u)^2}}$ and $K = \frac{-\partial_x^2 u}{(1+(\partial_x u)^2)^{3/2}}$, where $(a, b)^T = \begin{pmatrix} a \\ b \end{pmatrix}$. Figure 4 indicates its numerical example.

Gradient flow. As Figure 2 and Figure 4 suggested, the circumference $L(t)$ of the solution curve $\Gamma(t)$ is decreasing in time. This is the reason why (1.2) is often called the **curve-shortening equation**. Actually, (1.2) is characterized by the gradient flow of L as follows. By Lemma 1.13, the rate of change of the total length $L = L(\Gamma(t))$ of $\Gamma(t)$ is

$$\partial_t L(\Gamma(t)) = \int_\Gamma KV\, ds.$$

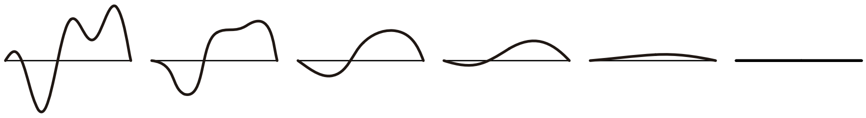


FIGURE 4. Evolution of a graph $y = u(x,t)$ by the curvature flow equation $\partial_t u = \frac{\partial_x^2 u}{1+(\partial_x u)^2}$ $(0 < x < 1)$ (from left to right). The boundary condition is $u(0) = u(1) = 0$, and the initial curve is given by $u(x,0) = 0.1\sin(\pi x) - 0.3\sin(2\pi x) + 0.2\sin(5\pi x)$.

Let the inner product on Γ be

$$\langle \mathsf{F}, \mathsf{G} \rangle_\Gamma = \int_\Gamma \mathsf{F}\mathsf{G}\, ds.$$

Then we have $\partial_t L(\Gamma(t)) = \langle K, V \rangle_\Gamma$. If the left-hand-side is regarded as a linear functional of V in a suitable space with the metric $\langle \cdot, \cdot \rangle_\Gamma$, then the gradient flow is given by $V = -K$, and $\partial_t L(\Gamma(t)) = -\langle K, K \rangle_\Gamma \le 0$. Hence, (1.2) is the equation which requires motion of curve in the most decreasing direction of L in the sense of the above metric.

EXERCISE 1.16. *Show* $\partial_t A(t) = -2\pi$.

Area-preserving curvature flow. Besides the curve-shortening flow, there has also been interest in area-preserving flows. The typical flow is the gradient flow of L along curves which enclose a fixed area as follows:

$$V = \overline{K} - K, \tag{1.3}$$

where $\overline{K}$ is the average of K: $\overline{K} = \int_\Gamma K\, ds / \int_\Gamma ds = 2\pi/L$ if Γ is simple and closed, $\overline{K} = 2\eta\pi/L$ if Γ is immersed with the rotation number being $\eta = 2, 3, \ldots$ We have (1.3) as the gradient flow of L in the metric $\langle \cdot, \cdot \rangle_\Gamma$ under the constraint for V: $\langle 1, V \rangle_\Gamma = 0$.

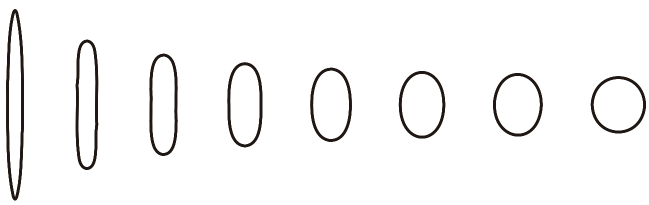


FIGURE 5. Evolution of a convex curve by (1.3) (from left to right).

EXERCISE 1.17. *Show that* $\partial_t L(t) \le 0$ *and* $\partial_t A(t) = 0$.

Gage [Gag86] proved that any convex curve converges to a circle as time tends to infinity (see Figure 5), and conjectured that a nonconvex curve may intersect; this conjecture was proved rigorously by Mayer and Simonett [MS00].

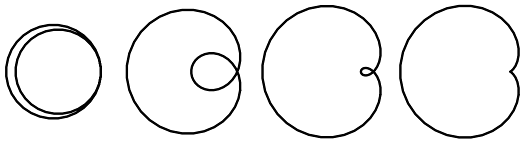


FIGURE 6. Evolution of a immersed closed curve by (1.3) (from left to right). The small loop disappears in finite time (at least numerically).

Open problems. The following two problems are still open: One is the problem whether the small loop may shrink or not. Figure 6 is a numerical example, and it suggests that extinction occurs in finite time. The other one is the problem whether any embedded curve becomes eventually convex or not, even if self-intersection occurred. In the curve-shortening case, these problems have been already analyzed, as it was mentioned above.

5. Anisotropy

In the field of material sciences and crystallography, we need to explain the anisotropy—phenomenon of interface motion which depends on the normal direction, i.e., model equations have to contain effects of anisotropy of materials or crystals.

Note. There are two kinds of anisotropy: one is kinetic anisotropy which appears in the problem of growth form, and the other is equilibrium anisotropy which appears in the problem of equilibrium form.

Interfacial energy. For example, if the crystal growth of snow flakes is regarded as motion of closed plane curve, then some points on the interface Γ are easy to growth in the case where their normal directions are in the six special directions. To explain these phenomena, it is convenient to define an interfacial energy on the curve Γ which has line density $\gamma(\boldsymbol{n}) > 0$. Integration of γ over Γ is the total interfacial energy:

$$E_\gamma = \int_\Gamma \gamma(\boldsymbol{n})\, ds.$$

In the case where $\gamma \equiv 1$, E_γ is nothing but the total length L, and its gradient flow was (1.2). For a general γ, what is the gradient flow of E_γ?

Extension of $\boldsymbol{\gamma}$. The function $\gamma(\boldsymbol{n})$ can be extended to the function $\boldsymbol{x} \in \mathbb{R}^2$ by putting

$$\gamma(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}|\, \gamma\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), & \boldsymbol{x} \neq \boldsymbol{0}, \\ 0, & \boldsymbol{x} = \boldsymbol{0}. \end{cases}$$

This extension is called the extension of positively homogeneous of degree 1 (in short, **PHD1**) , since

$$\gamma(\lambda \boldsymbol{x}) = \lambda \gamma(\boldsymbol{x})$$

holds for $\lambda \geq 0$ and $\boldsymbol{x} \in \mathbb{R}^2$. We will use the same notation γ for the extended function. Hereafter we assume that γ is positive for $\boldsymbol{x} \neq \boldsymbol{0}$ and positively homogeneous of degree 1 function. Following Kobayashi and Giga [KG01], we call such γ a **PPHD1** function.

REMARK 1.18. *The isotropic case $\gamma(\boldsymbol{n}) \equiv 1$ holds if and only if $\gamma(\boldsymbol{x}) = |\boldsymbol{x}|$.*

We assume that $\gamma \in C^2(\mathbb{R}^2 \backslash \{\mathbf{0}\})$. Then the following properties hold:

- $\nabla\gamma(\lambda \boldsymbol{x}) = \nabla\gamma(\boldsymbol{x})$ ($\nabla\gamma$ is homogeneous of degree 0),
- $\gamma(\boldsymbol{x}) = \boldsymbol{x} \cdot \nabla\gamma(\boldsymbol{x})$ (In particular, $\gamma(\boldsymbol{n}) = \boldsymbol{n} \cdot \nabla\gamma(\boldsymbol{n})$ holds),
- $\mathrm{Hess}\,\gamma(\lambda \boldsymbol{x}) = \frac{1}{\lambda} \mathrm{Hess}\,\gamma(\boldsymbol{x})$ ($\mathrm{Hess}\,\gamma$ is homogeneous of degree -1),

where $\mathrm{Hess}\,\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$ is the Hessian of γ ($\gamma_{ij} = \partial^2\gamma/\partial x_i \partial x_j$ for $i, j \in \{1, 2\}$ and $\boldsymbol{x} = (x_1, x_2)$).

EXERCISE 1.19. *Show these properties. (The answer is in [KG01, Appendix 1].)*

Weighted curvature flow. The gradient flow of the total interfacial energy

$$E_\gamma(\Gamma(t)) = \int_\Gamma \gamma(\boldsymbol{n})\, ds = \int_\Sigma \gamma(\boldsymbol{n}) g\, d\sigma$$

is given by

$$V = -\Lambda_\gamma(\boldsymbol{n}), \quad \Lambda_\gamma(\boldsymbol{n}) = (\mathrm{Hess}\,\gamma(\boldsymbol{n})\boldsymbol{n}^\perp) \cdot \boldsymbol{n}^\perp K. \tag{1.4}$$

This $\Lambda_\gamma(\boldsymbol{n})$ is called **weighted curvature** or **anisotropic curvature** , and (1.4) is called the **weighted curvature flow equation** or **anisotropic curvature flow equation** .

EXERCISE 1.20. *Show $\partial_t E_\gamma(\Gamma(t)) = \int_\Gamma \Lambda_\gamma(\boldsymbol{n}) V\, ds$.*

EXERCISE 1.21. *Try to calculate the gradient flow of $F_\gamma = \int_\Gamma \gamma(\boldsymbol{x})\, ds$.*

EXERCISE 1.22. *Try to calculate the gradient flow of $G_\gamma = \int_\Gamma \Lambda_\gamma(\boldsymbol{n})^\alpha\, ds$, $\alpha = 1, 2$.*

Weighted curvature. Meaning of the weighted curvature $\Lambda_\gamma(\boldsymbol{n})$ will be clearer as follows: Let θ be the exterior normal angle such as $\boldsymbol{n} = \boldsymbol{n}(\theta) = (\cos\theta, \sin\theta)$ and $\boldsymbol{t} = \boldsymbol{t}(\theta) = (-\sin\theta, \cos\theta)$. Put $\widehat{\gamma}(\theta) = \gamma(\boldsymbol{n}(\theta))$. Then we obtain

$$\Lambda_\gamma(\boldsymbol{n}(\theta)) = (\widehat{\gamma}(\theta) + \partial_\theta^2 \widehat{\gamma}(\theta))K.$$

EXERCISE 1.23. *Prove this equation.*

Example of $\boldsymbol{\gamma}$. Let us construct an example of characteristic γ. For a natural number M, we extend the line segment $y = x$ ($x \in [0, 2\pi/M]$) to $2\pi/M$-periodic function $p(x)$ such as

$$p(x) = \frac{2}{M}\left(\arctan\left(\tan\left(\frac{M}{2}x - \frac{\pi}{2}\right)\right) + \frac{\pi}{2}\right),$$

and by $\eta(x)$ we denote a concave and symmetric function with respect to the central line $x = \pi/M$ defined on $[0, 2\pi/M]$. For example,

$$\eta(x) = \frac{\sin x + \sin(2\pi/M - x)}{\sin(2\pi/M)}.$$

The following function γ_p is a $2\pi/M$-periodic function with the number of peak being M on $[0, 2\pi/M]$ (Figure 8 (left)).

$$\gamma_p(\theta) = \eta(p(\theta)), \quad \theta \in \mathbb{R}.$$

Note that γ_p is not differentiable at $2\pi k/M$ ($k \in \mathbb{Z}$). Then by using $\rho_\mu(x) = \sqrt{x^2 + \mu^{-2}}$, we construct a smooth function γ_μ (Figure 8 (left)):

$$\gamma_\mu(\theta) = \rho_\mu(\gamma_p(\theta) - 1) + 1, \quad \theta \in \mathbb{R}.$$

As μ tends to infinity, $\rho_\mu(x)$ converges to $|x|$, and then γ_μ converges to γ_p formally.

In general, if the curve $\Gamma(t)$ is strictly convex, then the solution K of (1.4) satisfies the partial differential equation $\partial_t K = K^2(\partial_\theta^2(-V) + (-V))$. See the book of Gurtin [Gur93] or Exercise 1.24 below. Hence if $\widehat{\gamma} + \partial_\theta^2\widehat{\gamma} > 0$ holds, then the PDE is quasi-linear strictly parabolic type and its Cauchy problem is solvable. We note that γ_μ satisfies $\gamma_\mu + \partial_\theta^2\gamma_\mu > 0$.

Exercise 1.24. *Check the following story: Using the height function $h = \boldsymbol{x}\cdot\boldsymbol{n}$, the curve $\Gamma = \{\boldsymbol{x}\}$ is described as $\boldsymbol{x} = h\boldsymbol{n} + (\partial_\theta h)\boldsymbol{t}$, if Γ is strictly convex. Hence $\partial_\theta^2 h = K^{-1} - h$ holds, and then we have $\partial_t(K^{-1}) = \partial_\theta^2 V + V$ by $V = \partial_t h$.*

Figure 7 indicates evolution of solution curves of (1.4) by using the interfacial energy density $\widehat{\gamma} = \gamma_\mu$ in the case where $M = 6$ and $\mu = 15$. The asymptotic shape seems to be a round hexagon which is strikingly different from the one in Figure 2.

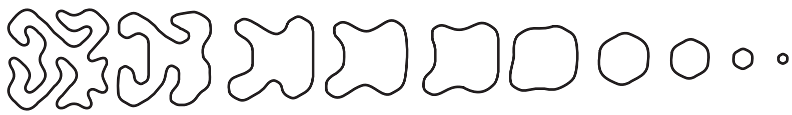


Figure 7. Evolution of a simple closed curve by (1.4) (from left to right). The initial curve is the same as in Figure 2.

6. The Frank diagram and the Wulff shape

We will consider γ_p and γ_μ by graphical constructions.

The Frank diagram. Figure 8 (left) indicates the graph of $\gamma = \gamma_p(\theta)$ and $\gamma = \gamma_p(\theta)$, respectively; and Figure 8 (middle) indicates the polar coordinate of each γ:

$$\mathcal{C}_\gamma = \left\{\gamma(\boldsymbol{n}(\theta))\boldsymbol{n}(\theta);\ \theta \in S^1\right\}.$$

Incidentally, to observe the characteristic of γ, the following **Frank diagram** (Figure 8 (right)) is more useful than a graph of γ and $\mathcal{C}_\gamma$:

$$\mathcal{F}_\gamma = \left\{\frac{\boldsymbol{n}(\theta)}{\gamma(\boldsymbol{n}(\theta))};\ \theta \in S^1\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^2;\ \gamma(\boldsymbol{x}) = 1\right\}.$$

From this definition $\mathcal{F}_\gamma = \mathcal{C}_{1/\gamma}$ holds.

Exercise 1.25. *Show the following three proposition are equivalent, if γ is a PPHD1 function. (The answer is in [KG01, Appendix 2].)*

(i) We denote the set enclosed by $\mathcal{F}_\gamma$ by $\widetilde{\mathcal{F}}_\gamma = \left\{\boldsymbol{x} \in \mathbb{R}^2;\ \gamma(\boldsymbol{x}) \le 1\right\}$. Then $\widetilde{\mathcal{F}}_\gamma$ is convex.

(ii) The function γ satisfies $\gamma((1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y}) \le (1-\lambda)\gamma(\boldsymbol{x}) + \lambda\gamma(\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ and $\lambda \in [0,1]$. (That is, γ is convex.)

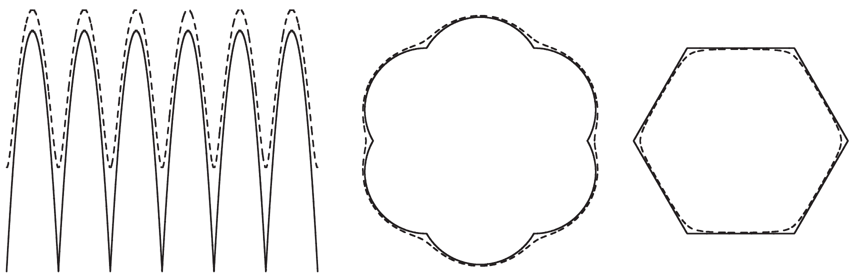


FIGURE 8. In each figure, the solid line is $\gamma_p(\theta)$, and the dotted line is $\gamma_\mu(\theta)$ ($\mu = 15$), respectively. $M = 6$ in any case. (Left): graphs of $\gamma_p(\theta)$ and $\gamma_\mu(\theta)$. The horizontal axis is $0 \leq \theta \leq 2\pi$, and the vertical axis is magnified for easy to observe. At each point $\theta = 2\pi k/M$ ($k \in \mathbb{Z}$), the function $\gamma_p(\theta)$ (resp. $\gamma_\mu(\theta)$) achieves the minimum value 1 (resp. $1+1/\mu$). (Middle): $\mathcal{C}_{\gamma_p}$ and $\mathcal{C}_{\gamma_\mu}$. (Right): the Frank diagram $\mathcal{F}_{\gamma_p}$ and $\mathcal{F}_{\gamma_\mu}$. $\mathcal{F}_{\gamma_p}$ is a regular M-polygon.

(iii) The function γ satisfies $\gamma(\boldsymbol{x}+\boldsymbol{y}) \leq \gamma(\boldsymbol{x}) + \gamma(\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. (That is, γ is subadditive.)

The interfacial energy γ can be classified by the Frank diagram $\mathcal{F}_\gamma$. Since the sign of the curvature of the curve $\mathcal{F}_\gamma$ agrees with the sign of $\widehat{\gamma} + \partial_\theta^2\widehat{\gamma}$, then as mentioned above, the initial value problem of (1.4) can be caught in the framework of parabolic partial differential equations. The function $\gamma = \gamma_\mu$ is such an example. However, in the case where $\gamma = \gamma_p$, its Frank diagram $\mathcal{F}_{\gamma_p}$ is a regular M-polygon, and on the each edge $\widehat{\gamma}_p + \partial_\theta^2\widehat{\gamma}_p = 0$, and $\widehat{\gamma}_p$ may not be differentiable at each vertex. To treat such interfacial energy, crystalline curvature flow in the title will enter the stage. It will be mentioned later. Before it, let us define another shape of characterizing γ.

The Wulff shape. The following shape is called the **Wulff shape**:

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^1} \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot \boldsymbol{n}(\theta) \leq \gamma(\boldsymbol{n}(\theta)) \right\}.$$

Figure 9 indicates the Wulff shape of γ_p and γ_μ, respectively, in the case where $M = 4$ and $M = 6$.

The Wulff shape is a convex set by means of constructions. In particular, if γ is smooth and $\mathcal{F}_\gamma$ is strictly convex, then $\mathcal{W}_\gamma$ is also a strictly convex set with a smooth boundary. In this case, the distance from the origin (which is inside of $\mathcal{W}_\gamma$) to L_θ equals to $\gamma(\boldsymbol{n}(\theta)) = \widehat{\gamma}(\theta)$, where L_θ is the tangent line which is passing through the point on $\partial\mathcal{W}_\gamma$ with its outward normal vector being $\boldsymbol{n}(\theta)$. From this fact, we note that the curvature of $\partial\mathcal{W}_\gamma$ is given by $(\widehat{\gamma} + \partial_\theta^2\widehat{\gamma})^{-1}$.

EXERCISE 1.26. *Show that the curvature of $\partial\mathcal{W}_\gamma$ is $(\widehat{\gamma} + \partial_\theta^2\widehat{\gamma})^{-1}$ if $\mathcal{W}_\gamma$ is a strictly convex set with a smooth boundary.*

In the physical context, $\mathcal{W}_\gamma$ describes the equilibrium of crystal, i.e., in the plane case, $\mathcal{W}_\gamma$ is the answer of the following ≪Wulff problem≫ on the equilibrium shape of crystals (by Gibbs (1878), Curie (1885), and Wulff (1901)):

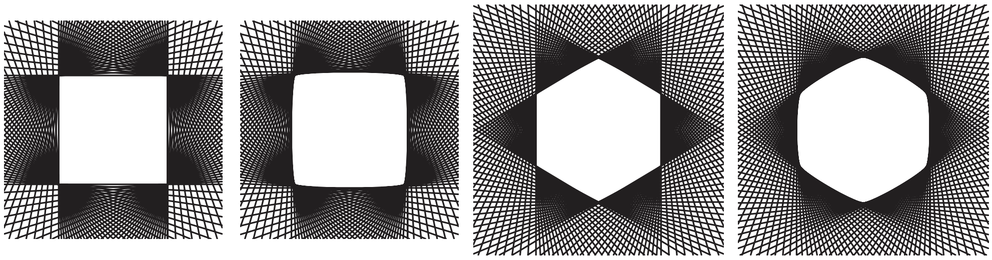


FIGURE 9. Comparison of $\gamma_p(\theta)$ and $\gamma_\mu(\theta)$ ($\mu = 15$) by the Wulff shape. (Left, middle left): The Wulff shapes $\mathcal{W}_{\gamma_p}$ and $\mathcal{W}_{\gamma_\mu}$ with $M = 4$. (Middle right, right): The Wulff shapes $\mathcal{W}_{\gamma_p}$ and $\mathcal{W}_{\gamma_\mu}$ with $M = 6$. Two $\mathcal{W}_{\gamma_p}$'s are regular M-polygons, and two $\mathcal{W}_{\gamma_\mu}$'s are round regular M-polygons, respectively.

What is the shape which has the least total interfacial energy E_γ of the curve for the fixed enclosed area?

This suggests that the asymptotic shape of a solution curve of the gradient flow of E_γ (1.4) is $\partial\mathcal{W}_{\gamma_\mu}$ in the case $\gamma = \gamma_\mu$ (Figure 7). However, in the mathematical context, there are not so many known results.

Open problems. Convexified phenomena or formation of convexity is one of the interesting problems. In the isotropic case $\widehat{\gamma} \equiv 1$, Grayson [Gra87] proved that convexity is formed (see Figure 2). However in the anisotropic case, even if γ is smooth, the formation of convexity has not been completed, except in the symmetric case $\widehat{\gamma}(\theta + \pi) = \widehat{\gamma}(\theta)$ by Chou and Zhu [CZ99].

Incidentally, the Frank diagram $\mathcal{F}_\gamma$ has been considered after the Wulff shape $\mathcal{W}_\gamma$ has appeared (Frank (1963) and Meijering (1963)). See the book on crystal growth in detail. We also refer the reader to Kobayashi and Giga [KG01] for a lot of examples of $\mathcal{C}_\gamma$, $\mathcal{F}_\gamma$ and $\mathcal{W}_\gamma$, and for discussion of anisotropy and the effect of curvature.

Numerical examples. Now, going back the story, we observe the behavior of a solution curve $y = u(x,t)$ by (1.4) with $\gamma = \gamma_\mu$ ($\mu = 1000$) from a view point of an approximation of $\gamma = \gamma_p$.

Since the total interfacial energy is given by

$$E_\gamma = \int_0^1 W(\partial_x u)\,dx, \quad W(\xi) = \widehat{\gamma}(\theta(\xi))\sqrt{1+\xi^2}, \quad \theta(\xi) = -\arctan\frac{1}{\xi},$$

we obtain the partial differential equation

$$\partial_t u = \sqrt{1+(\partial_x u)^2}\,\partial_x(\partial_\xi W(\partial_x u)) = \frac{\widehat{\gamma} + \partial_\theta^2\widehat{\gamma}}{1+(\partial_x u)^2}\partial_x^2 u,$$

which is equivalent to (1.4).

Figure 10 indicates a numerical example of the case where $M = 4$ and $\mu = 1000$. As time goes by, one can observe that line segment parts satisfying $\partial_x u = 0$ expands, and the other parts (which are put between two parallel line segments satisfying $\partial_x u = 0$) do not move, and will disappear in finite time. Actually, since $\widehat{\gamma}_\mu + \partial_\theta^2\widehat{\gamma}_\mu$

takes a huge value only at $\theta = \pi k/2$ ($k \in \mathbb{Z}$) and almost zero at other points, the movement of points satisfying $\partial_x u \sim 0$ can be observed and other points do not seem to move.

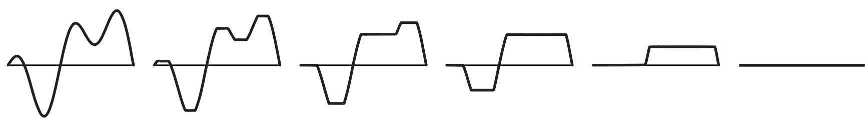


FIGURE 10. Evolution of a graph $y = u(x,t)$ by the curvature flow equation $\partial_t u = \sqrt{1+(\partial_x u)^2}\,\partial_x(\partial_\xi W(\partial_x u))$ $(0 < x < 1)$ (from left to right). Here $W(\xi) = \widehat{\gamma}_\mu(\theta(\xi))\sqrt{1+\xi^2}$ $(\mu = 1000, M = 4)$, $\theta(\xi) = \pi/2 + \arctan\xi$. The boundary conditions are $u(0) = u(1) = 0$, and the initial curve is the same as the one in Figure 4.

Figure 11 indicates a numerical example of the case where $M = 6$ and $\mu = 1000$. In this case, $\mathcal{W}_\gamma$ is almost a regular hexagon (it almost agrees with $\mathcal{W}_{\gamma_p}$ in Figure 9 (middle right), and in the set of normal angles of $\mathcal{W}_{\gamma_\mu}$ one can observe the movement of points on the graph satisfying that their normal angles almost agree with $\theta = \pm\pi k/3$ $(k = 1, 2)$, and other points do not seem to move. Since the end points are fixed, Figure 11 (far right) is the final shape.

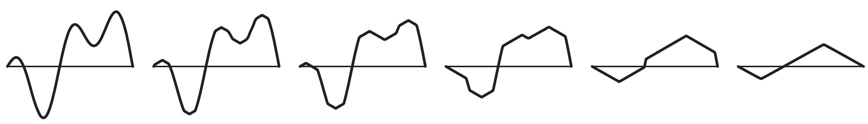


FIGURE 11. Evolution of a graph $y = u(x,t)$ by the curvature flow equation $\partial_t u = \sqrt{1+(\partial_x u)^2}\,\partial_x(\partial_\xi W(\partial_x u))$ $(0 < x < 1)$ (from left to right). Here $W(\xi) = f_\mu(\theta(\xi))\sqrt{1+\xi^2}$ $(\mu = 1000, M = 6)$, $\theta(\xi) = \pi/2 + \arctan\xi$. The boundary conditions are $u(0) = u(1) = 0$, and the initial curve is the same as the one in Figure 4.

When μ tends to infinity, how does the equation

$$\partial_t u = \sqrt{1+(\partial_x u)^2}\partial_x(\partial_\xi W(\partial_x u))$$

change? In the case where $M = 4$, $\widehat{\gamma}_p(\theta) = |\cos\theta| + |\sin\theta|$ holds, and then we have $W(\xi) = |\xi| + 1$. From this we obtain formally $\partial_\xi^2 W(\xi) = 2\delta(\xi)$, i.e., the partial differential equation becomes $\partial_t u = 2\sqrt{1+(\partial_x u)^2}\,\delta(\partial_x u)\partial_x^2 u$, where δ is Dirac delta function. As Figure 10 suggested, the right-hand-side is considerable as a nonlocal quantity [Gig97].

By the observation of Figure 10 and Figure 11, it seems that there is no problem if curves are restricted to the following special class of polygonal curves: the set of normal vectors of the curves agrees with the one of the Wulff shape.

7. Crystalline energy

If the Frank diagram $\mathcal{F}_\gamma$ is a convex polygon, γ is called **crystalline energy**. When $\mathcal{F}_\gamma$ is a J-sided convex polygon, there exists a set of angles $\{\phi_i \mid \phi_1 < \phi_2 < \cdots < \phi_J < \phi_1 + 2\pi\}$ such that the position vectors of vertices are labeled $\boldsymbol{n}(\phi_i)/\gamma(\boldsymbol{n}(\phi_i))$ in an anticlockwise order ($\phi_{J+1} = \phi_1,\ \phi_0 = \phi_J$) and $\phi_{i+1} - \phi_i \in (0, \pi)$ holds for $i = 1, 2, \ldots, J$:

$$\mathcal{F}_\gamma = \bigcup_{i=1}^{J} \left[\frac{\boldsymbol{\nu}_i}{\gamma(\boldsymbol{\nu}_i)}, \frac{\boldsymbol{\nu}_{i+1}}{\gamma(\boldsymbol{\nu}_{i+1})} \right].$$

Here and hereafter, we denote $\boldsymbol{\nu}_i = \boldsymbol{n}(\phi_i)$ ($\forall i$). See Figure 12 (left). In this case, the Wulff shape is also a J-sided convex polygon with the outward normal vector of the i-th edge being $\boldsymbol{\nu}_i$:

$$\mathcal{W}_\gamma = \bigcap_{i=1}^{J} \left\{ \boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot \boldsymbol{\nu}_i \le \gamma(\boldsymbol{\nu}_i) \right\}.$$

See Figure 12 (right). We define a set of normal vectors of $\mathcal{W}_\gamma$ by $\mathcal{N}_\gamma = \{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_J\}$.

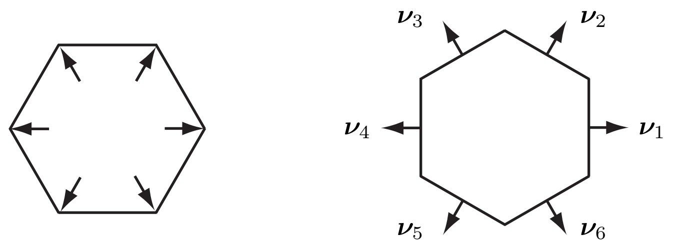


FIGURE 12. Example of the Frank diagram (left) and the Wulff shape (right) in the case where γ is a crystalline energy and $J = 6$. This Frank diagram is the same regular hexagon $\mathcal{F}_{\gamma_p}$ as in Figure 8 (right), and this Wulff shape is the same regular hexagon $\mathcal{W}_{\gamma_p}$ as in Figure 9 (middle right).

REMARK 1.27. *A point on the $(i-1)$-th edge of J-sided polygon $\mathcal{F}_\gamma$ can be described as*

$$\frac{\boldsymbol{\nu}}{\gamma(\boldsymbol{\nu})} = (1-\mu)\frac{\boldsymbol{\nu}_{i-1}}{\gamma(\boldsymbol{\nu}_{i-1})} + \mu\frac{\boldsymbol{\nu}_i}{\gamma(\boldsymbol{\nu}_i)}$$

for some $\mu \in (0, 1)$, where $\boldsymbol{\nu} = \boldsymbol{n}(\phi)$ with some $\phi \in (\phi_{i-1}, \phi_i)$; or one can describe $\boldsymbol{\nu}$ without using ϕ:

$$\boldsymbol{\nu} = \frac{(1-\lambda)\boldsymbol{\nu}_{i-1} + \lambda\boldsymbol{\nu}_i}{|(1-\lambda)\boldsymbol{\nu}_{i-1} + \lambda\boldsymbol{\nu}_i|}, \quad \lambda = \frac{\mu\gamma(\boldsymbol{\nu}_{i-1})}{(1-\mu)\gamma(\boldsymbol{\nu}_i) + \mu\gamma(\boldsymbol{\nu}_{i-1})}.$$

EXERCISE 1.28. *Show this expression of $\boldsymbol{\nu}$.*

Let L_i be the straight line passing through $\gamma(\boldsymbol{\nu}_i)\boldsymbol{\nu}_i$ in the direction $\boldsymbol{\nu}_i^\perp$ for all i, and let $\boldsymbol{y}_i$ be the intersection point of L_{i-1} and L_i:

$$\boldsymbol{y}_i = \gamma(\boldsymbol{\nu}_i)\boldsymbol{\nu}_i - l_\gamma^- \boldsymbol{\nu}_i^\perp, \quad l_\gamma^- = \frac{\gamma(\boldsymbol{\nu}_{i-1}) - (\boldsymbol{\nu}_{i-1}\cdot\boldsymbol{\nu}_i)\gamma(\boldsymbol{\nu}_i)}{-\boldsymbol{\nu}_{i-1}\cdot\boldsymbol{\nu}_i^\perp}.$$

It is easy to check that the following claim.

PROPOSITION 1.29. *$\boldsymbol{y}_i \cdot \boldsymbol{\nu} = \gamma(\boldsymbol{\nu})$ holds for each i.*

EXERCISE 1.30. *Prove Proposition 1.29.*

Proposition 1.29 asserts that the straight line passing through $\gamma(\boldsymbol{\nu})\boldsymbol{\nu}$ in the direction $\boldsymbol{\nu}^\perp$ passes through $\boldsymbol{y}_i$.

We have

$$l_\gamma^- = \frac{1-a^-}{b^-}, \quad a^- = (\boldsymbol{\nu}_{i-1}\cdot\boldsymbol{\nu}_i)\frac{\gamma(\boldsymbol{\nu}_i)}{\gamma(\boldsymbol{\nu}_{i-1})}, \quad b^- = -\frac{\boldsymbol{\nu}_{i-1}\cdot\boldsymbol{\nu}_i^\perp}{\gamma(\boldsymbol{\nu}_{i-1})}.$$

In the same way, we have

$$\boldsymbol{y}_{i+1} = \gamma(\boldsymbol{\nu}_i)\boldsymbol{\nu}_i + l_\gamma^+\boldsymbol{\nu}_i^\perp, \quad l_\gamma^+ = \frac{\gamma(\boldsymbol{\nu}_{i+1}) - (\boldsymbol{\nu}_{i+1}\cdot\boldsymbol{\nu}_i)\gamma(\boldsymbol{\nu}_i)}{\boldsymbol{\nu}_{i+1}\cdot\boldsymbol{\nu}_i^\perp} = \frac{1-a^+}{b^+},$$

where

$$a^+ = (\boldsymbol{\nu}_{i+1}\cdot\boldsymbol{\nu}_i)\frac{\gamma(\boldsymbol{\nu}_i)}{\gamma(\boldsymbol{\nu}_{i+1})}, \quad b^+ = \frac{\boldsymbol{\nu}_{i+1}\cdot\boldsymbol{\nu}_i^\perp}{\gamma(\boldsymbol{\nu}_{i+1})}.$$

Since the angle between $\boldsymbol{\nu}_{i-1}$ and $\boldsymbol{\nu}_i$ satisfies $\phi_{i-1} - \phi_i \in (0, \pi)$, $b^- > 0$ holds, and also $b^+ > 0$ holds by the similar reason.

PROPOSITION 1.31. *$(\boldsymbol{y}_{i+1} - \boldsymbol{y}_i)\cdot\boldsymbol{\nu}_i^\perp = l_\gamma^- + l_\gamma^+ > 0$ holds.*

Since $\mathcal{F}_\gamma$ is a J-sided convex polygon, there exists $\mu \in (0,1)$ such that for each i

$$\boldsymbol{\eta} = (1-\mu)\frac{\boldsymbol{\nu}_{i-1}}{\gamma(\boldsymbol{\nu}_{i-1})} + \mu\frac{\boldsymbol{\nu}_{i+1}}{\gamma(\boldsymbol{\nu}_{i+1})}$$

holds, and that

$$\boldsymbol{\eta}\cdot\boldsymbol{\nu}_i < \frac{1}{\gamma(\boldsymbol{\nu}_i)}, \ \boldsymbol{\eta}\cdot\boldsymbol{\nu}_i^\perp = 0$$

hold. From the inequality and equation, and from the facts $b^- > 0$ and $b^+ > 0$, we have

$$\mu = \frac{b^-}{b^- + b^+}, \quad a^- - \mu(a^- - a^+) = \frac{a^-b^+ + a^+b^-}{b^- + b^+} < 1.$$

Hence it holds that

$$l_\gamma^- + l_\gamma^+ = \frac{1}{b^-b^+}(b^- + b^+ - (a^-b^+ + a^+b^-)) > 0.$$

Therefore $\mathcal{W}_\gamma$ is J-sided convex polygon with the i-th vertex being $\boldsymbol{y}_i$ and the the length of the i-th edge being $l_\gamma^- + l_\gamma^+ > 0$ for each i.

EXERCISE 1.32. *Show that if $\mathcal{F}_\gamma$ is the regular J-sided polygon, $\mathcal{W}_\gamma$ is also the regular J-sided polygon.*

REMARK 1.33. *There exist two crystalline energies γ_1 and γ_2 such that $\mathcal{W}_{\gamma_1}$ agrees with $\mathcal{W}_{\gamma_2}$ except the position of the center. See Figure 13.*

REMARK 1.34. *As we saw the above, if the Frank diagram is a J-sided convex polygon, then the Wulff shape is also the J-sided convex polygon. But the converse is not true. See Figure 14.*

REMARK 1.35. *Remark 1.33 and Remark 1.34 suggest that $\mathcal{W}_\gamma$ does not have all the information of γ. Actually, $\mathcal{W}_{\bar\gamma} = \mathcal{W}_\gamma$ holds, where $\bar\gamma$ is convexification of γ. See [KG01].*

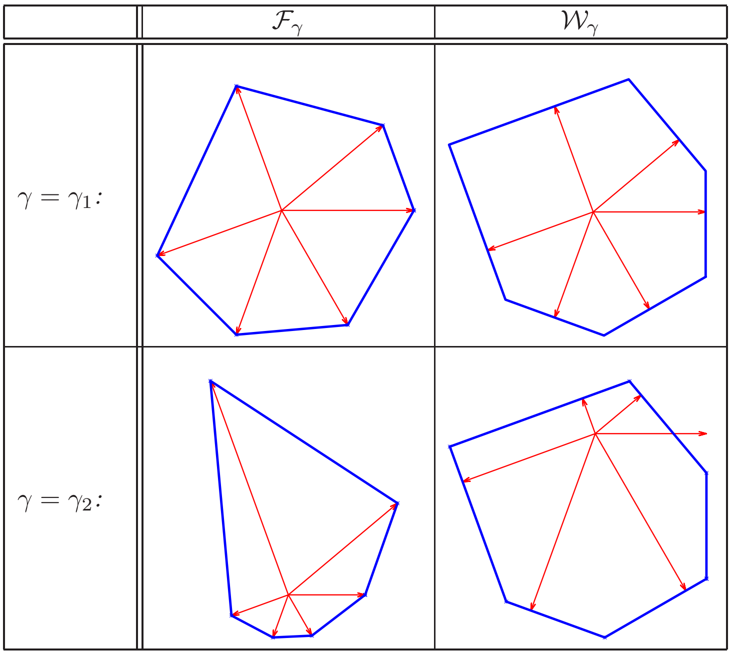


FIGURE 13. Example of the same shape of the Wulff shape with different γ: $\gamma_1 \neq \gamma_2$.

8. Crystalline curvature flow equations

We will define crystalline curvature flow equations.

Polygonal curves. Let $\mathcal{P}$ be a simple closed N-sided polygonal curve in the plane $\mathbb{R}^2$, and label the position vector of vertices $\boldsymbol{p}_i$ $(i = 1, 2, \ldots, N)$ in an anti-clockwise order:

$$\mathcal{P} = \bigcup_{i=1}^{N} \mathcal{S}_i,$$

where $\mathcal{S}_i = [\boldsymbol{p}_i, \boldsymbol{p}_{i+1}]$ is the i-th edge ($\boldsymbol{p}_{N+1} = \boldsymbol{p}_1$, $\boldsymbol{p}_0 = \boldsymbol{p}_N$). The length of $\mathcal{S}_i$ is $d_i = |\boldsymbol{p}_{i+1} - \boldsymbol{p}_i|$, and then the i-th unit tangent vector is $\boldsymbol{t}_i = (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)/d_i$ and the i-th unit outward normal vector is $\boldsymbol{n}_i = -\boldsymbol{t}_i^{\perp}$. We define a set of normal vectors of $\mathcal{P}$ by $\mathcal{N} = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_N\}$. Let θ_i be the exterior normal angle of $\mathcal{S}_i$. Then $\boldsymbol{n}_i = \boldsymbol{n}(\theta_i)$ and $\boldsymbol{t}_i = \boldsymbol{t}(\theta_i)$ hold. We define the i-th hight function $h_i = \boldsymbol{p}_i \cdot \boldsymbol{n}_i = \boldsymbol{p}_{i+1} \cdot \boldsymbol{n}_i$. See Figure 15. By using N-tuple $\underline{h} = (h_1, h_2, \ldots, h_N)$, d_i is described as $d_i = D_i(\underline{h})$, where

$$D_i(\underline{h}) = \frac{\chi_{i-1,i}}{\sqrt{1 - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i)^2}}(h_{i-1} - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i)h_i) + \frac{\chi_{i,i+1}}{\sqrt{1 - (\boldsymbol{n}_i \cdot \boldsymbol{n}_{i+1})^2}}(h_{i+1} - (\boldsymbol{n}_i \cdot \boldsymbol{n}_{i+1})h_i),$$

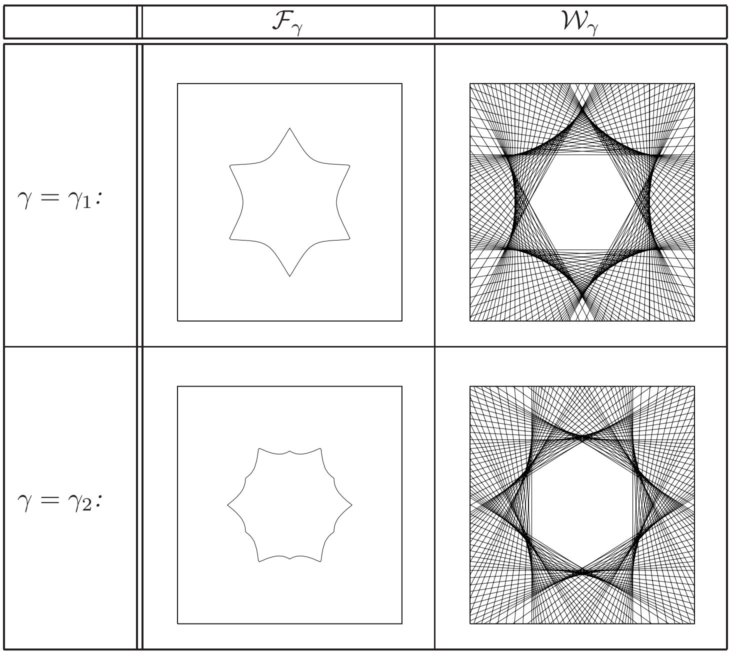


FIGURE 14. The Wulff shapes are both regular hexagons, while the corresponding Frank diagrams are not polygons with different γ: $\gamma_1 \neq \gamma_2$.

where $\chi_{i,j} = \mathrm{sgn}(\det(\boldsymbol{n}_i, \boldsymbol{n}_j))$ for $i = 1, 2, \ldots, N$ $(h_{N+1} = h_1, h_0 = h_N)$. Since $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \cos(\theta_i - \theta_j)$, we have another expression:

$$D_i(\underline{h}) = -(\cot \vartheta_i + \cot \vartheta_{i+1})h_i + h_{i-1} \operatorname{cosec} \vartheta_i + h_{i+1} \operatorname{cosec} \vartheta_{i+1}, \tag{1.5}$$

where $\vartheta_i = \theta_i - \theta_{i-1}$ for $i = 1, 2, \ldots, N$. Note that $0 < |\vartheta_i| < \pi$ holds for all i. Furthermore, the i-th vertex $\boldsymbol{p}_i$ $(i = 1, 2, \ldots, N)$ is described as follows:

$$\boldsymbol{p}_i = h_i \boldsymbol{n}_i + \frac{h_{i-1} - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i) h_i}{\boldsymbol{n}_{i-1} \cdot \boldsymbol{t}_i} \boldsymbol{t}_i. \tag{1.6}$$

EXERCISE 1.36. *From the relation* $\boldsymbol{n}_{i-1}^{\mathrm{T}} \boldsymbol{p}_i = h_{i-1}$ *and* $\boldsymbol{n}_i^{\mathrm{T}} \boldsymbol{p}_i = h_i$*, we obtain*

$$\boldsymbol{p}_i = \begin{pmatrix} \boldsymbol{n}_{i-1}^{\mathrm{T}} \\ \boldsymbol{n}_i^{\mathrm{T}} \end{pmatrix}^{-1} \begin{pmatrix} h_{i-1} \\ h_i \end{pmatrix}.$$

From this, show (1.6). This idea can be found in [BKY08].

EXERCISE 1.37. *From (1.6) and the relation* $\boldsymbol{t}_i \cdot \boldsymbol{n}_{i+1} = \sin \vartheta_{i+1}$*, show (1.5).*

EXERCISE 1.38. *From (1.6), check that* $d_i = |\boldsymbol{p}_{i+1} - \boldsymbol{p}_i|$ *and* $d_i = (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot \boldsymbol{t}_i$ *hold.*

EXERCISE 1.39. *Fix any* $\boldsymbol{a} \in \mathbb{R}^2$*. By* $k_i = \boldsymbol{a} \cdot \boldsymbol{n}_i$ *we denote distance from the origin to* $\boldsymbol{a}$ *in the direction* $\boldsymbol{n}_i$*, and by* $\underline{k} = (k_1, k_2, \ldots, k_N)$ *we denote their* N*-tuple* $(k_{N+1} = k_1, k_0 = k_N)$*. Show that* $D_i(\underline{k}) = 0$ *holds for* $i = 1, 2, \ldots, N$*. From this*

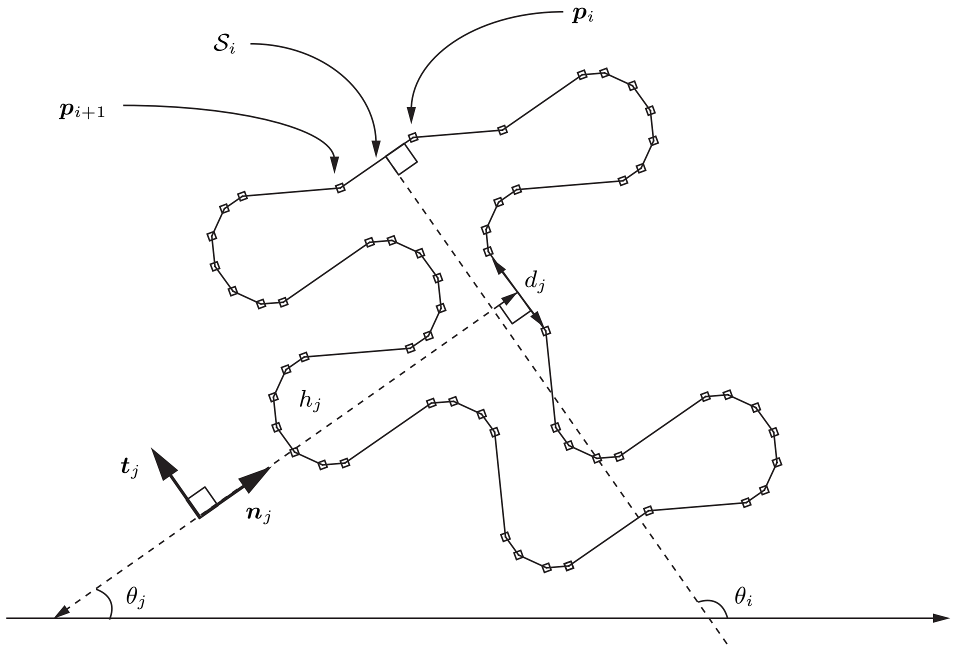


FIGURE 15. Example of polygonal curves ($N = 56$).

fact, $D_i(\underline{h} + \underline{k}) = D_i(\underline{h}) + D_i(\underline{k}) = d_i$ *holds for all* i*. This means that the shape of curve does not change for a shift of the origin.*

Admissible curves. Following [HGGD05], we call curve $\mathcal{P}$ **essentially admissible** if and only if the consecutive outward unit normal vectors $\boldsymbol{n}_i, \boldsymbol{n}_{i+1} \in \mathcal{N}$ ($\boldsymbol{n}_{N+1} = \boldsymbol{n}_1$, $\boldsymbol{n}_0 = \boldsymbol{n}_N$) satisfy

$$\boldsymbol{n}_{i+\lambda} = \frac{(1-\lambda)\boldsymbol{n}_i + \lambda \boldsymbol{n}_{i+1}}{|(1-\lambda)\boldsymbol{n}_i + \lambda \boldsymbol{n}_{i+1}|} \notin \mathcal{N}_\gamma$$

for $\lambda \in (0, 1)$ and $i = 1, 2, \ldots, N$. See Figure 16 (middle).

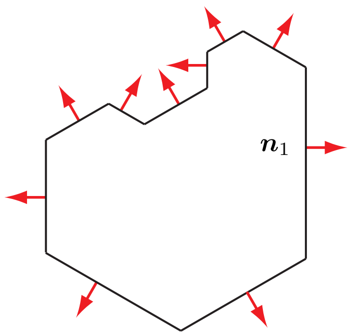
EXERCISE 1.40. *If* $\mathcal{P}$ *is an essentially admissible curve, then* $\mathcal{N} \supseteq \mathcal{N}_\gamma$ *holds. But the converse is not true. Make a counterexample.*

REMARK 1.41. *In the case where* $\mathcal{P}$ *is a convex polygon,* $\mathcal{P}$ *is an essentially admissible if and only if* $\mathcal{N} \supseteq \mathcal{N}_\gamma$ *holds.*

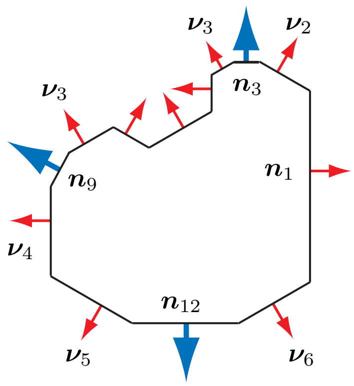
EXERCISE 1.42. *Check this remark.*

We call curve $\mathcal{P}$ **admissible** if and only if $\mathcal{P}$ is an essentially admissible curve and $\mathcal{N} = \mathcal{N}_\gamma$ (especially, $\mathcal{N} \subseteq \mathcal{N}_\gamma$) holds. In other words, $\mathcal{P}$ is admissible if and only if $\mathcal{N} = \mathcal{N}_\gamma$ holds and any adjacent two normal vectors in the set $\mathcal{N}_\gamma$, for example $\boldsymbol{\nu}_i$ and $\boldsymbol{\nu}_{i+1}$ are also adjacent in the set $\mathcal{N}$, i.e., $\{\boldsymbol{\nu}_i, \boldsymbol{\nu}_{i+1}\} = \{\boldsymbol{n}_j, \boldsymbol{n}_{j+1}\} \subset \mathcal{N}$ holds for some i and j ($\boldsymbol{\nu}_{J+1} = \boldsymbol{\nu}_1$, $\boldsymbol{n}_{N+1} = \boldsymbol{n}_1$). See Figure 16 (left).

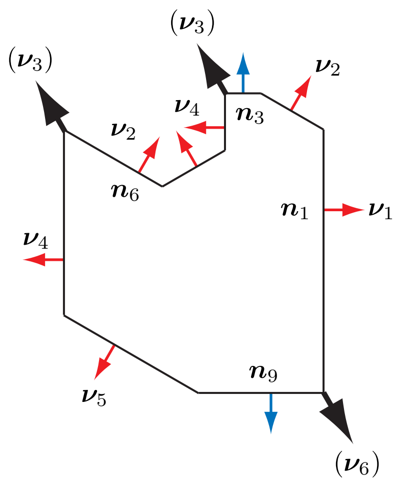
Gradient flow. We consider a moving essentially admissible and an N-sided curve $\mathcal{P}(t)$ with the time t in some interval and with the N-tuple of hight functions $\underline{h}(t) = (h_1(t), h_2(t), \ldots, h_N(t))$. Then we have $d_i(t) = D_i(\underline{h}(t))$. The total



(a) Admissible polygonal curve ($N = 10$).



(b) Essentially admissible polygonal curve ($N = 13$). Note that $\mathcal{N}_\gamma \oplus \{\boldsymbol{n}_3, \boldsymbol{n}_9, \boldsymbol{n}_{12}\} = \mathcal{N}$ holds.



(c) This polygonal curve ($N = 9$) is not admissible, nor essentially admissible, since $\mathcal{N}_\gamma \not\subseteq \mathcal{N}$ ($\boldsymbol{\nu}_6 \notin \mathcal{N}$) holds, and in addition there exist $\lambda, \lambda', \lambda'' \in (0,1)$ such that $\boldsymbol{n}_{3+\lambda} = \boldsymbol{n}_{6+\lambda'} = \boldsymbol{\nu}_3, \boldsymbol{n}_{9+\lambda''} = \boldsymbol{\nu}_6 \in \mathcal{N}_\gamma$ holds.

FIGURE 16. Three typical examples of polygonal curves for the Wulff shape in Figure 12 (right).

crystalline energy on $\mathcal{P}(t)$ is defined as $E_\gamma(\mathcal{P}(t)) = \sum_{i=1}^N \gamma(\boldsymbol{n}_i) d_i(t)$, and the rate of change of it is given as

$$\partial_t E_\gamma(\mathcal{P}(t)) = \sum_{i=1}^N \gamma(\boldsymbol{n}_i)\partial_t D_i(\underline{h}) = \sum_{i=1}^N \gamma(\boldsymbol{n}_i) D_i(\underline{V}),$$

where $\underline{V} = (V_1, V_2, \dots, V_N)$ and $V_i = \partial_t h_i(t)$ is the velocity on $\mathcal{S}_i$ in the direction of $\boldsymbol{n}_i$ for $i = 1, 2, \dots, N$. Here we have used the relation $\partial_t d_i(t) = D_i(\underline{V})$ from (1.5), which is equivalent to the time derivative of (1.6):

$$\partial_t \boldsymbol{p}_i = V_i \boldsymbol{n}_i + v_i \boldsymbol{t}_i, \quad v_i = \frac{V_{i-1} - (\boldsymbol{n}_{i-1} \cdot \boldsymbol{n}_i) V_i}{\boldsymbol{n}_{i-1} \cdot \boldsymbol{t}_i}, \quad i = 1, 2, \dots, N. \tag{1.7}$$

We denote $\underline{\gamma} = (\gamma(\boldsymbol{n}_1), \gamma(\boldsymbol{n}_2), \dots, \gamma(\boldsymbol{n}_N))$, and then we have

$$\sum_{i=1}^{N} \gamma(\boldsymbol{n}_i) D_i(\underline{V}) = \sum_{i=1}^{N} V_i D_i(\underline{\gamma}) = \sum_{i=1}^{N} \frac{D_i(\underline{\gamma})}{d_i} V_i d_i.$$

Hence we have

$$\partial_t E_\gamma(\mathcal{P}(t)) = \sum_{i=1}^{N} \Lambda_\gamma(\boldsymbol{n}_i) V_i d_i, \quad \Lambda_\gamma(\boldsymbol{n}_i) = \frac{D_i(\underline{\gamma})}{d_i}, \quad i = 1, 2, \dots, N.$$

Here $\Lambda_\gamma(\boldsymbol{n}_i)$ is called **crystalline curvature** on the i-th edge $\mathcal{S}_i$, which is the crystalline version of weighted curvature derived from $\partial_t E_\gamma(\Gamma(t)) = \int_\Gamma \Lambda_\gamma(\boldsymbol{n}) V \, ds$. For two N-tuples $\underline{\mathsf{F}} = (\mathsf{F}_1, \mathsf{F}_2, \dots, \mathsf{F}_N)$, $\underline{\mathsf{G}} = (\mathsf{G}_1, \mathsf{G}_2, \dots, \mathsf{G}_N) \in \mathbb{R}^N$, let us define the inner product on $\mathcal{P}$ as follows:

$$\langle \underline{\mathsf{F}}, \underline{\mathsf{G}} \rangle_{\mathcal{P}} = \sum_{i=1}^{N} \mathsf{F}_i \mathsf{G}_i d_i.$$

Therefore by analogue of gradient flow of $E_\gamma(\Gamma(t))$, we have the gradient flow of $E_\gamma(\mathcal{P}(t))$ such as

$$V_i = -\Lambda_\gamma(\boldsymbol{n}_i), \quad i = 1, 2, \dots, N.$$

This is called **crystalline curvature flow equation**.

The numerator of crystalline curvature $\Lambda_\gamma(\boldsymbol{n}_i)$ is described as

$$D_i(\underline{\gamma}) = \chi_i l_\gamma(\boldsymbol{n}_i), \tag{1.8}$$

where $\chi_i = (\chi_{i-1,i} + \chi_{i,i+1})/2$ takes $+1$ (resp. -1) if $\mathcal{P}$ is convex (resp. concave) around $\mathcal{S}_i$ in the direction of $-\boldsymbol{n}_i$, otherwise $\chi_i = 0$; and $l_\gamma(\boldsymbol{n})$ is the length of the j-th edge of $\mathcal{W}_\gamma$ if $\boldsymbol{n} = \boldsymbol{\nu}_j$ for some j, otherwise $l_\gamma(\boldsymbol{n}) = 0$.

REMARK 1.43. *Equation (1.8) can be derived as follows. We define*

$$l_\gamma^-(\phi', \phi) = \frac{\gamma(\boldsymbol{n}(\phi')) - (\boldsymbol{n}(\phi) \cdot \boldsymbol{n}(\phi'))\gamma(\boldsymbol{n}(\phi))}{-\boldsymbol{n}(\phi') \cdot \boldsymbol{n}(\phi)^\perp} = \frac{\gamma(\boldsymbol{n}(\phi')) - \gamma(\boldsymbol{n}(\phi)) \cos(\phi - \phi')}{\sin(\phi - \phi')},$$

and $l_\gamma^+(\phi, \phi') = -l_\gamma^-(\phi', \phi)$. *Then we have*

$$l_\gamma(\boldsymbol{\nu}_j) = l_\gamma^-(\phi_{j-1}, \phi_j) + l_\gamma^+(\phi_j, \phi_{j+1}), \quad D_i(\underline{\gamma}) = l_\gamma^-(\theta_{i-1}, \theta_i) + l_\gamma^+(\theta_i, \theta_{i+1}).$$

By geometric consideration, we have

$$l_\gamma^-(\theta, \phi_j) = \begin{cases} l_\gamma^-(\phi_{j-1}, \phi_j), & \theta \in [\phi_{j-1}, \phi_j), \\ -l_\gamma^+(\phi_j, \theta) = -l_\gamma^+(\phi_j, \phi_{j+1}), & \theta \in (\phi_j, \phi_{j+1}], \end{cases}$$

and

$$l_\gamma^+(\phi_j, \theta) = \begin{cases} l_\gamma^+(\phi_j, \phi_{j+1}), & \theta \in (\phi_j, \phi_{j+1}], \\ -l_\gamma^-(\theta, \phi_j) = -l_\gamma^-(\phi_{j-1}, \phi_j), & \theta \in [\phi_{j-1}, \phi_j). \end{cases}$$

The two cases are possible. The first case is $\boldsymbol{n}_i \in \mathcal{N}_\gamma$*, i.e., there exists* j *such that* $\boldsymbol{n}_i = \boldsymbol{\nu}_j$ *and* $\theta_i = \phi_j$*; and we have four subcases as follows:*

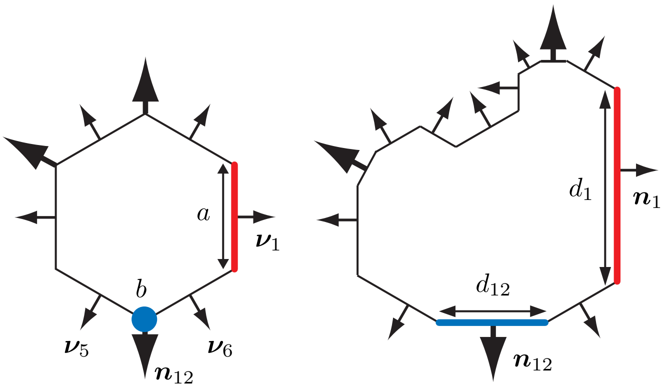
(i) $\theta_{i-1} \in [\phi_{j-1}, \phi_j)$ *and* $\theta_{i+1} \in (\phi_j, \phi_{j+1}]$,
(ii) $\theta_{i-1} \in (\phi_j, \phi_{j+1}]$ *and* $\theta_{i+1} \in [\phi_{j-1}, \phi_j)$,
(iii) $\theta_{i-1}, \theta_{i+1} \in (\phi_j, \phi_{j+1}]$, *and*
(iv) $\theta_{i-1}, \theta_{i+1} \in [\phi_{j-1}, \phi_j)$.

Hence, $D_i(\underline{\gamma})$ *is equal to*

(i) $l_\gamma(\boldsymbol{\nu}_j)$,
(ii) $-l_\gamma(\boldsymbol{\nu}_j)$,
(iii) 0, *and*
(iv) 0, *respectively.*

The second case is $\boldsymbol{n}_i \notin \mathcal{N}_\gamma$, *i.e., there exists* j *such that* $\theta_i \in (\phi_{j-1}, \phi_j)$; *and we have four subcases similarly as above. In this case, we regard that there is a zero-length edge of* $\mathcal{W}_\gamma$ *between the* $(j-1)$*-th and the* j*-th edges, and for all subcases* $D_i(\underline{\gamma}) = 0$ *holds. Thus we obtain (1.8).*

EXERCISE 1.44. *Follow this remark.*



(a) a is the length of edge $l_\gamma(\boldsymbol{n}_1) > 0$ and b is the length of degenerate edge $l_\gamma(\boldsymbol{n}_{12}) = 0$.

(b) d_1 (resp. d_{12}) is the length of the edge whose normal vector is $\boldsymbol{n}_1$ (resp. $\boldsymbol{n}_{12}$). Hence the crystalline curvature is $\Lambda_\gamma(\boldsymbol{n}_1) = l_\gamma(\boldsymbol{n}_1)/d_1$ (resp. $\Lambda_\gamma(\boldsymbol{n}_{12}) = 0$).

FIGURE 17. The Wulff shape (left) is the same hexagon as in Figure 12 (right). Essentially admissible polygonal curve (right) is the same curve as in Figure 16(b).

REMARK 1.45. *If* $\mathcal{P}$ *is an admissible and convex polygon, then* $\boldsymbol{n}_i = \boldsymbol{\nu}_i$ *and* $\chi_i = 1$ *for all* $i = 1, 2, \ldots, N = J$; *and moreover, if* $\mathcal{P} = \partial\mathcal{W}_\gamma$, *then the crystalline curvature is* 1.

Note. To handle the gradient flow of E_γ (1.4) in the case where γ is crystalline, we have restricted smooth curve to admissible curve and have introduced the crystalline curvature defined on each edge. This strategy was proposed by Taylor [Tay90, Tay91a, Tay91b, Tay93] and independently by Angenent and Gurtin [AG89]. Also one can find essentially the same method as a numerical scheme for curvature flow equation in Roberts [Rob93]. We refer the reader Almgren and Taylor [AT95] for detailed history. In the physical context, the region enclosed by $\mathcal{P}$ represents the crystal. See also Taylor, Cahn and Handwerker [TCH92] and Gurtin [Gur93] for physical background. Besides this crystalline strategy, other strategies by subdifferential and level-set method have been extensively studied. See Giga [Gig97, Gig98, Gig00] and references therein.

9. An area-preserving motion by crystalline curvature

The enclosed area $A(t)$ of $\mathcal{P}(t)$ is given by

$$A = \frac{1}{2}\sum_{i=1}^{N} h_i d_i,$$

and its rate of change is

$$\partial_t A(t) = \frac{1}{2}\sum_{i=1}^{N} V_i d_i + \frac{1}{2}\sum_{i=1}^{N} h_i D_i(\underline{V}) = \sum_{i=1}^{N} V_i d_i.$$

Then the gradient flow of E_γ along $\mathcal{P}$ which encloses a fixed area is

$$V_i = \overline{\Lambda}_\gamma - \Lambda_\gamma(\boldsymbol{n}_i), \quad i = 1, 2, \dots, N, \tag{1.9}$$

where

$$\overline{\Lambda}_\gamma = \frac{\sum_{i=1}^N \Lambda_\gamma(\boldsymbol{n}_i) d_i}{\sum_{k=1}^N d_k} = \frac{\sum_{i=1}^N D_i(\underline{\gamma})}{L} = \frac{\sum_{i=1}^N \chi_i l_\gamma(\boldsymbol{n}_i)}{L}$$

is the average of the crystalline curvature. The gradient flow (1.9) is in the metric $\langle \cdot, \cdot \rangle_{\mathcal{P}}$ under the constraint for $\underline{V}$: $\langle \underline{1}, \underline{V} \rangle_{\mathcal{P}} = 0$, $\underline{1} = (1, 1, \dots, 1)$.

Exercise 1.46. *Check the following two basic properties: $\partial_t E_\gamma(\mathcal{P}(t)) \le 0$ and $\partial_t A(t) = 0$.*

Problem 1. For a given N-sided essentially admissible closed curve $\mathcal{P}_0$, find a family of N-sided essentially admissible curves $\bigcup_{0 \le t < T} \mathcal{P}(t)$ satisfying

$$\begin{cases} V_i(t) = \overline{\Lambda}_\gamma - \Lambda_\gamma(\boldsymbol{n}_i), \quad 0 \le t < T, \quad i = 1, 2, \dots, N, \\ \mathcal{P}(0) = \mathcal{P}_0. \end{cases}$$

Remark 1.47. *Problem 1 is equivalent to $\partial_t d_i(t) = D_i(\underline{V})$, or equivalent to (1.7); and since it is described as the N-system of ordinary differential equations, the maximal existence time is positive: $T > 0$.*

Remark 1.48. *Problem 1 is closely related to general area-preserving motion of polygonal curves. See [BKY07, BKY08].*

Known results. What might happen to $\mathcal{P}(t)$ as t tends to $T \le \infty$? For this question, we have the following three results. The first result is the case where motion is isotropic and polygon is admissible.

Theorem 1.49. *Let the interfacial energy be isotropic $\gamma \equiv 1$. Assume the initial polygon $\mathcal{P}_0$ is an N-sided admissible convex polygon. Then a solution admissible polygon $\mathcal{P}(t)$ exists globally in time keeping the area enclosed by the polygon constant A, and $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the*

Hausdorff metric as t tends to infinity, where $\gamma_* = \sqrt{2A/\sum_{k=1}^{N} l_1(\boldsymbol{n}_k)}$ is constant. In particular, if $\mathcal{P}_0$ is centrally symmetric with respect to the origin, then we have an exponential rate of convergence.

This theorem is proved by Yazaki [Yaz02] by using the isoperimetric inequality and the theory of dynamical systems. We note that $\partial\mathcal{W}_{\gamma_*}$ is the circumscribed polygon of a circle with radius γ_*, and then this result is a semidiscrete version of Gage [Gag86].

The second result is the case where motion is anisotropic and polygon is admissible.

THEOREM 1.50. *Let the crystalline energy be $\gamma > 0$. Assume the initial polygon $\mathcal{P}_0$ is an N-sided admissible convex polygon. Then a solution admissible polygon $\mathcal{P}(t)$ exists globally in time keeping the area enclosed by the polygon constant A, and $\mathcal{P}(t)$ converges to the shape of the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ in the Hausdorff metric as t tends to infinity, where $\gamma_*(\boldsymbol{n}_i) = \gamma(\boldsymbol{n}_i)/w_*$, $w_* = \sqrt{|\mathcal{W}_\gamma|/A}$ for $i = 1, 2, \ldots, N$ and $|\mathcal{W}_\gamma| = \sum_{k=1}^{N} \gamma(\boldsymbol{n}_k) l_\gamma(\boldsymbol{n}_k)/2$ is enclosed area of $\mathcal{W}_\gamma$.*

This theorem is proved in Yazaki [Yaz04, Part I] by using the anisoperimetric inequality or Brünn and Minkowski's inequality and the theory of dynamical systems. For reader's convenience, the proof will be shown in the next section.

The last result is the case where motion is anisotropic and polygon is essentially admissible.

THEOREM 1.51. *Let the crystalline energy be $\gamma > 0$. Assume the initial polygon $\mathcal{P}_0$ is an N-sided essentially admissible convex polygon. If the maximal existence time of a solution essentially admissible polygon $\mathcal{P}(t)$ is finite $T < \infty$, then there exists the i-th edge $\mathcal{S}_i$ such that $\lim_{t\to T} d_i(t) = 0$ and $l_\gamma(\boldsymbol{n}_i) = 0$ hold. That is, the normal vector of vanishing edge does not belong to $\mathcal{N}_\gamma$, and $\inf_{0<t<T} d_k(t) > 0$ holds for all $\boldsymbol{n}_k \in \mathcal{N}_\gamma$.*

This theorem is proved in [Yaz07b].

Open problems. For any essentially admissible convex polygon $\mathcal{P}_0$, is T a finite value? This is still open. If the answer of this question is yes, then we have the finite time sequence $T_1 < T_2 < \cdots < T_M$ such that $\mathcal{P}(T_i)$ is essentially admissible for $i = 1, 2, \ldots, M-1$ and $\mathcal{P}(T_M)$ is admissible. In the general case where $V_i = F(\boldsymbol{n}_i, \Lambda_\gamma(\boldsymbol{n}_i))$ for all i under certain conditions of F, the answer of the above question is yes. See [Yaz07a]. However, F does not include $\overline{\Lambda}_\gamma$.

Open problems in the case where $\boldsymbol{\mathcal{P}_0}$ is nonconvex. Even if $\mathcal{P}_0$ is admissible, we have no theoretical results at this stage. See section 11 in detail.

10. Scenario of the proof of Theorem 1.50

Put the crystalline curvature such as

$$w_i(t) = \Lambda_\gamma(\boldsymbol{n}_i) = \frac{l_\gamma(\boldsymbol{n}_i)}{d_i}, \quad i = 1, 2, \ldots, N.$$

Then we have

$$\partial_t w_i(t) = -l_i^{-1} w_i(t)^2 D_i(\underline{V}), \quad D_i(\underline{V}) = D_i(\underline{1})\overline{\Lambda}_\gamma - D_i(\underline{w}),$$

where $l_i = l_\gamma(\boldsymbol{n}_i)$, $\underline{1} = (1, 1, \ldots, 1)$ and $\underline{w} = (w_1, w_2, \ldots, w_N)$. Note that

$$D_i(\underline{1}) = \tan\frac{\vartheta_i}{2} + \tan\frac{\vartheta_{i+1}}{2}$$

holds for $i = 1, 2, \ldots, N$. Hence from the average $\overline{\Lambda}_\gamma = \sum_{k=1}^N l_k/L$ and the length $L = \sum_{j=1}^N l_j w_j(t)^{-1}$, we can restate Problem 1 as the following Problem 2.

Problem 2. Find a function $\underline{w}(t) = (w_1(t), w_2(t), \ldots, w_N(t)) \in (C[0,T) \cap C^1(0,T))^N$ satisfying

$$\partial_t w_i(t) = l_i^{-1} D_i(\underline{w}(t)) w_i(t)^2 - \frac{l_i^{-1} D_i(\underline{1}) \sum_{k=1}^N l_k}{\sum_{j=1}^N l_j w_j(t)^{-1}} w_i(t)^2, \qquad i = 1, 2, \ldots, N, \quad 0 < t < T$$

and

$$w_i(0) = w_i^0, \quad i = 1, 2, \ldots, N,$$
$$w_{N+1}(t) = w_1(t), \quad w_0(t) = w_N(t), \quad 0 < t < T,$$

where w_i^0 is the i-th initial crystalline curvature of $\mathcal{P}_0$.

Problem 1 and Problem 2 are equivalent except the indefiniteness of position of a solution polygon. See [Yaz02, Remark 2.1]. Since Problem 2 is the initial value problem of ordinary differential equations, there exists a unique time local solution. Moreover, by using a similar argument as in Taylor [Tay93, Proposition 3.1], Ishii and Soner [IS99, Lemma 3.4], Yazaki [Yaz02, Lemma 3.1], we obtain the time global solvability.

LEMMA 1.52 (time global existence). *A solution $\underline{w}$ of Problem 2 and a solution polygon of Problem 1 exist globally in time, i.e., a solution polygon does not develop singularities in a finite time.*

In the following, we will show three lemmas, which play an important role in a scenario of the proof of Theorem 1.50.

Let the anisoperimetric ratio be

$$\mathcal{J}_\gamma(\mathcal{P}) = \frac{E_\gamma(\mathcal{P})^2}{4|\mathcal{W}_\gamma| A}$$

for an N-sided admissible convex polygon $\mathcal{P}$ associated with $\mathcal{W}_\gamma$. Here A (resp. $|\mathcal{W}_\gamma|$) is the enclosed area of $\mathcal{P}$ (resp. the area of $\mathcal{W}_\gamma$). The first key lemma is the anisotropic version of the isoperimetric inequality.

LEMMA 1.53 (anisoperimetric inequality). *For a polygon $\mathcal{P}$ associated with $\mathcal{W}_\gamma$, the anisoperimetric inequality*

$$\mathcal{J}_\gamma(\mathcal{P}) \geq 1$$

holds. The equality $\mathcal{J}_\gamma(\mathcal{P}) \equiv 1$ holds if and only if $w_i = \Lambda_\gamma(\boldsymbol{n}_i) \equiv$ const. for all i, i.e., Ω (the enclosed region of $\mathcal{P}$) satisfies $\Omega = k\mathcal{W}_\gamma$ for some constant $k > 0$.

We will prove this lemma in Appendix 3 using the mixed area and the Brünn and Minkowski's inequality.

From this lemma and lemma 1.52, for a solution polygon $\mathcal{P}(t)$,

$$\mathcal{J}_\gamma(\mathcal{P}(t)) = \frac{E_\gamma(\mathcal{P}(t))^2}{4|\mathcal{W}_\gamma|A} \geq 1$$

holds for $t \geq 0$. Moreover, we have the following second key lemma.

LEMMA 1.54. $\lim_{t\to\infty} E_\gamma(\mathcal{P}(t)) = 2\sqrt{|\mathcal{W}_\gamma|A}$ *and* $\lim_{t\to\infty} \mathcal{J}_\gamma(\mathcal{P}(t)) = 1$ *hold.*

Proof of this lemma closely follows [Yaz02, Lemma 5.6].

From this lemma, if a solution polygon $\mathcal{P}(t)$ is N-sided polygon at the time infinity, then $\Omega(t)$ approaches to $k\mathcal{W}_\gamma$ ($k > 0$) as t tends to infinity. Therefore, to complete the proof of Theorem 1.50, it is required to show the estimate $\inf_{0<t<\infty} \min_{1\leq i\leq N} d_i(t) > 0$. As a matter of fact, the following the third key lemma holds. This lemma is a strong assertion compared with the above estimate.

LEMMA 1.55. *Let w_* be the equilibrium point of the first evolution equation in Problem 2. Then $w_* = \sqrt{|\mathcal{W}_\gamma|/A}$ holds. Moreover, the equilibrium point w_* is asymptotically stable and*

$$\lim_{t\to\infty} w_i(t) = w_*$$

holds for $i = 1, 2, \ldots, N$.

One can prove this lemma by the general theory of dynamical systems or the Lyapunov theorem (see [Yaz02, Lemma 5.8]).

PROOF OF THEOREM 1.50. From Lemma 1.55, we have

$$\lim_{t\to\infty} d_i(t) = \frac{l_i}{w_*}, \quad i = 1, 2, \ldots, N.$$

From this limit and the theory of the generalized eigenvalue space, there exists a vector $\boldsymbol{c}(t) \in \mathbb{R}^2$ such that $h_i(t) - \boldsymbol{c}(t) \cdot \boldsymbol{n}_i$ converges to $\gamma_*(\boldsymbol{n}_i) = \gamma(\boldsymbol{n}_i)/w_*$ for all i as t tends to infinity. Hence for any $\varepsilon > 0$ there exists $t' > 0$ such that $\mathcal{P}(t) \subset \mathcal{W}_{(1+\varepsilon)\gamma_*} \backslash \mathcal{W}_{(1-\varepsilon)\gamma_*}$ holds for $t \geq t'$. Then the assertion holds. □

11. Numerical scheme

The aims are to construct a numerical scheme which enjoys two basic properties $\partial_t E_\gamma \leq 0$ and $\partial_t A = 0$ (see Exercise 1.46), and to investigate what might happen to solution $\mathcal{P}(t)$ of the evolution equation (1.9) as t tends to $T \leq \infty$. We discretize the system of ordinary equations $\partial_t d_i(t) = D_i(\underline{V})$ or (1.7) with the initial essentially admissible closed curve $\mathcal{P}^0 = \mathcal{P}_0$ at the time $t_0 = 0$. Let $m = 0, 1, 2, \ldots$ be a step number. By a^m, we denote the approximation of $a(t_m)$ at the time $t_m = \sum_{i=0}^{m-1} \tau_i$ (the time step τ_m will be defined in the following procedure).

The following procedure is an extension of [UY04].

Procedure 1. Fix parameters $\mu \in [0, 1]$ and $\lambda, \varepsilon \in (0, 1)$. For a given essentially admissible N-sided closed curve $\mathcal{P}^m = \bigcup_{i=1}^N [\boldsymbol{p}_i^m, \boldsymbol{p}_{i+1}^m]$, we define $\mathcal{P}^{m+1} = \bigcup_{i=1}^N [\boldsymbol{p}_i^{m+1}, \boldsymbol{p}_{i+1}^{m+1}]$ as follows:

(i) the i-th length: $d_i^m = |\boldsymbol{p}_{i+1}^m - \boldsymbol{p}_i^m|$ ($\forall i$);

(ii) the m-th variable time step: $\tau_m = \rho(d_{\min}^m)^2/\Delta$, where $\rho = \varepsilon(1-\mu\lambda)\min\{\lambda, 1-\mu\lambda\}$, $\Delta = 2|\chi l_\gamma(\boldsymbol{n})|_{\max}(2/|\sin\vartheta|_{\min} + |\tan(\vartheta/2)|_{\max})$;

(iii) the i-th length d_i^{m+1}:
$(D_\tau d)_i^m = -(\cot\vartheta_i + \cot\vartheta_{i+1})V_i^{m+\mu} + V_{i-1}^{m+\mu}\operatorname{cosec}\vartheta_i + V_{i+1}^{m+\mu}\operatorname{cosec}\vartheta_{i+1}$ $(\forall i)$,

$$V_i^{m+\mu} = \frac{\sum_{j=1}^N \chi_j l_\gamma(\boldsymbol{n}_j)}{\sum_{k=1}^N d_k^{m+\mu}} - \frac{\chi_i l_\gamma(\boldsymbol{n}_i)}{d_i^{m+\mu}}, \quad d_i^{m+\mu} = (1-\mu)d_i^m + \mu d_i^{m+1};$$

(iv) the i-th hight h_i^{m+1}: $(D_\tau h)_i^m = V_i^{m+\mu}$ $(\forall i)$;

(v) the i-th vertex: $\boldsymbol{p}_i^{m+1} = h_i^{m+1}\boldsymbol{n}_i + \frac{h_{i-1}^{m+1} - (\boldsymbol{n}_{i-1}\cdot\boldsymbol{n}_i)h_i^{m+1}}{\boldsymbol{n}_{i-1}\cdot\boldsymbol{t}_i}\boldsymbol{t}_i$ $(\forall i)$;

(vi) the new time: $t_{m+1} = t_m + \tau_m$.

Here we have used the notation: $a_{\min} = \min_{1\le i\le N} a_i$, $|a|_{\min} = \min_{1\le i\le N}|a_i|$, $|a|_{\max} = \max_{1\le i\le N}|a_i|$, $(D_\tau a)_i^m = (a_i^{m+1} - a_i^m)/\tau_m$, and the periodicity $\mathsf{F}_{N+1} = \mathsf{F}_1$, $\mathsf{F}_0 = \mathsf{F}_N$.

Two basic properties. One is that the total energy E_γ^m is decreasing in steps: $(D_\tau E_\gamma)^m \le 0$ for any $\mu \in [0,1]$. The other is that the enclosed area A^m of $\mathcal{P}^m$ is preserved: $(D_\tau A)^m = 0$ if $\mu = 1/2$.

Iteration. In (3), if $\mu \in (0,1]$, we solve the following iteration starting from $z_i^0 = d_i^m$:

$$\frac{z_i^{k+1} - z_i^0}{\tau_m} = -(\cot\vartheta_i + \cot\vartheta_{i+1})\widetilde{V}_i^{m+\mu} + \widetilde{V}_{i-1}^{m+\mu}\operatorname{cosec}\vartheta_i + \widetilde{V}_{i+1}^{m+\mu}\operatorname{cosec}\vartheta_{i+1},$$

$$\widetilde{V}_i^{m+\mu} = \frac{\sum_{j=1}^N \chi_j l_\gamma(\boldsymbol{n}_j)}{\sum_{k=1}^N \widetilde{d}_k^{m+\mu}} - \frac{\chi_i l_\gamma(\boldsymbol{n}_i)}{\widetilde{d}_i^{m+\mu}}, \quad \widetilde{d}_i^{m+\mu} = (1-\mu)z_i^0 + \mu z_i^k, \quad k = 0, 1, \ldots$$

Convergence $\lim_{k\to\infty} z_i^k = d_i^{m+1}$ and positivity $d_i^{m+1} \ge (1-\lambda)d_i^m > 0$ hold for all i ([UY04]).

The maximal existence time. Since the above positivity $d_i^m > 0$ holds, we can keep iterating Procedure 1 in finitely many steps (even if $\mathcal{P}^m$ self-intersects at a step m, we can continue). Then the maximal existence time is $t_\infty = \lim_{m\to\infty}\sum_{k=0}^m \tau_k$. Here we have two questions: one is whether t_∞ is finite or not, and the other is what might happen to $\mathcal{P}^m$ as m tends to infinity. It is known that $t_\infty = \infty$ holds if W_γ is an N-sided regular polygon and $\mathcal{P}^0$ is an admissible convex polygon ([UY04]).

Extension. At the maximal existence time t_∞, it is possible that at least one edge, for instance the i-th edge, may disappear. If $\chi_i = 0$ or $l_\gamma(\boldsymbol{n}_i) = 0$, then $\mathcal{P}^\infty$ is still essentially admissible. Hence we can continue Procedure 1 starting from the initial curve $\mathcal{P}^\infty$. In practice, if some edges are small enough, we eliminate them artificially as the following procedure.

Procedure 2. Put a positive parameter $\delta \ll 1$. For every step m, do the followings:

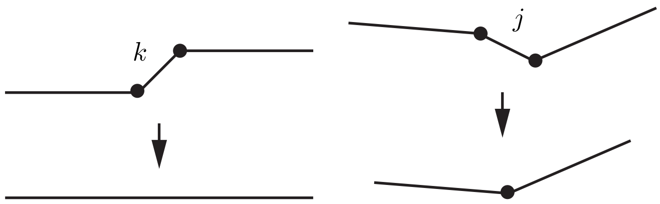
(i) define $D = \min\{d_i^m;\ \chi_i \ne 0,\ l_\gamma(\boldsymbol{n}_i) > 0\}$ (this is well-defined);

(ii) find the value k such that $d_k^m = \min\{d_i^m;\ \chi_i = 0\}$ (if it exists);
find the value j such that $d_j^m = \min\{d_i^m;\ \chi_i \ne 0,\ l_\gamma(\boldsymbol{n}_i) = 0\}$ (if it exists);

(iii) if k or j exists, check the followings:
  (a) if $d_k^m/D < \delta$ and if $d_k^m \le d_j^m$ or the value j does not exist, then eliminate the k-th edge (see Figure 18 (left));
  (b) if $d_j^m/D < \delta$ and if $d_j^m < d_k^m$ or the value k does not exist, then eliminate the j-th edge (see Figure 18 (right));
  (c) otherwise, exit from Procedure 2;

(iv) in (3), if (a) occurred, then do renumbering within the new number $N := N - 2$;
else if (b) occurred, then do renumbering within the new number $N := N - 1$.

Numerical computation will be continued repeating Procedure 1 and 2.



(a) In the type of (3) (a), the number of particles decreases from N to $N - 2$.

(b) In the type of (3) (b), the number of particles decreases from N to $N - 1$.

FIGURE 18. Two types of elimination of some edges in Procedure 2 (3).

Numerical simulations. In the following 8 figures on each line, from left to right, they indicate $\mathcal{W}_\gamma$, $\mathcal{P}^0, \mathcal{P}^{m_1}, \ldots, \mathcal{P}^{m_6}$ $(0 < m_1 < \cdots < m_6)$.

The case where $\mathcal{P}_0$ is convex and admissible. Figure 19 and Figure 20 indicate that asymptotic behavior of solution polygon which does not break the results of Theorem 1.49 and Theorem 1.50. See also Gage [Gag86] for the smooth case. On the convergence between $\Gamma(t_m)$ and $\mathcal{P}(t_m)$, see Ushijima and Yazaki [UY04].

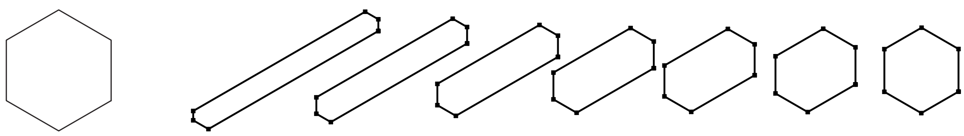


FIGURE 19. A numerical example of Theorem 1.49: Convergence of a hexagon to the regular hexagon $\mathcal{W}_\gamma$.

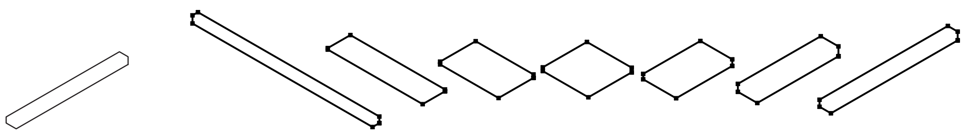


FIGURE 20. A numerical example of Theorem 1.50: Even when the initial polygon is quite different shape from $\mathcal{W}_\gamma$, it converges to $\mathcal{W}_\gamma$.

The case where $\mathcal{P}_0$ is convex and essentially admissible. Known result is only Theorem 1.51. Figure 21 suggests possibility of positive answer of the question in open problem just after Theorem 1.51.

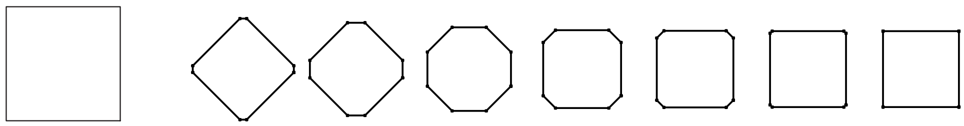


FIGURE 21. A numerical example of Theorem 1.51: An essentially admissible convex polygon converges to $\mathcal{W}_\gamma$ in a finite time.

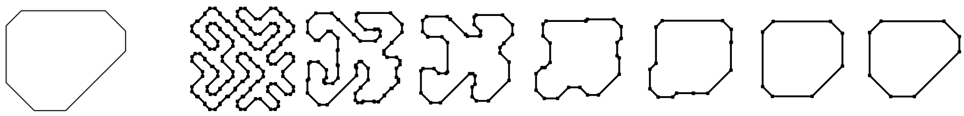


FIGURE 22. A numerical example of convexified phenomena.

The case where $\boldsymbol{\mathcal{P}_0}$ is nonconvex and admissible. Figure 22 suggests that convexified phenomena holds.

An example of self-intersection. In smooth case, a self-intersection is conjectured in Gage [Gag86], and is proved in Mayer and Simonett [MS00]. Figure 23 suggests that self-intersection is possible to occur.

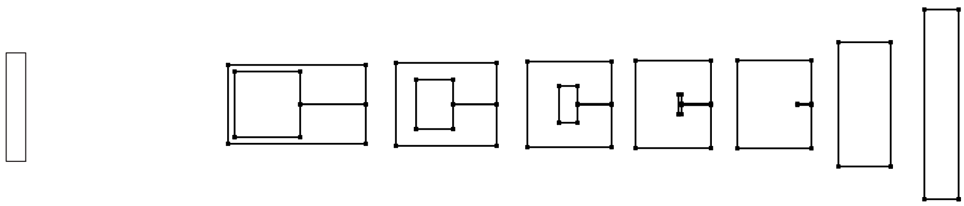


FIGURE 23. A numerical example of self-intersection. The motion is delicate. See the symbolic motion in Figure 24.

Open problems. At this stage, we have the following three open problems:

(i) Does $\mathcal{P}(t)$ become convex in finite time?
(ii) Will the admissibility be preserved?
(iii) Does $\mathcal{P}(t)$ self-intersect?

We have information related to these three questions. (1) In the case where $V_i = -\gamma(\boldsymbol{n}_i)|\Lambda_\gamma(\boldsymbol{n}_i)|^{\alpha-1}\Lambda_\gamma(\boldsymbol{n}_i)$, there exist $\alpha \in (0,1)$, γ and $\mathcal{P}_0$ such that nonconvex solution curve $\mathcal{P}(t)$ shrinks homothetically, i.e., there exists a nonconvex self-similar solution polygonal curve. See Ishiwata, Ushijima, Yagisita and Yazaki [IUYY04]. (2) In the case where $V_i = -a(\boldsymbol{n}_i)|\Lambda_\gamma(\boldsymbol{n}_i)|^{\alpha-1}\Lambda_\gamma(\boldsymbol{n}_i)$, for any $\alpha \geq 1$, $a(\cdot)$ and $\mathcal{P}_0$ if γ is symmetric, then the solution curve keeps admissibility. See Giga and Giga [GGig00]. However, there exist $\alpha \in (0,1)$, $a(\cdot)$ and $\mathcal{P}_0$ such that the admissibility collapses in finite time, i.e., we have the examples that admissible nonconvex polygonal curve becomes nonadmissible in finite time. See Hirota, Ishiwata and Yazaki [HIY06, HIY07]. (3) Figure 23 suggests the possibility of self-intersection.

REMARK 1.56. *Thanks to the tangential velocity v_i in (1.7), numerical computation of crystalline curvature flow equations is quite stable. The v_i corresponds to*

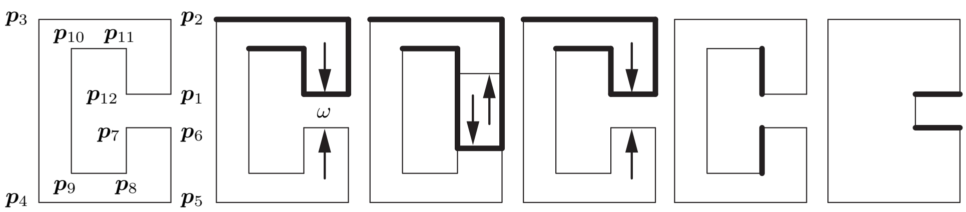


FIGURE 24. Symbolic motion of a numerical example in Figure 23. The initial 12-sided nonconvex polygonal curve $\mathcal{P}^0$ in Figure 23 is (far left). Let ω be the distance between the 6-th and the 12-th edges (left), which is small enough. Then ω disappears immediately, and moreover, becomes negative (middle left), and becomes positive again (middle right). After that, the vertical two edges, the thick line segments in (right), disappear. Finally, the horizontal two edges, the thick line segments in (far right), disappear, and becomes rectangle such as Figure 23 (far right).

$v = \partial_s V/K$ (this is equivalent to $\partial_t \theta = 0$ by Lemma 1.10.) in the continuous case (see [Yaz07c]), and it is utilized as redistribution of grid points on evolving curve. See [ŠY07, ŠY, BKPŠTY07].

12. Towards modeling the formation of negative ice crystals or vapor figures produced by freezing of internal melt figures

This section is based on the work with Tetsuya Ishiwata [IY07, IY08].

When a block of ice is exposed to solar beams or other radiation, internal melting of ice occurs. That is, internal melting starts from some interior points of ice without melting the exterior portions, and each water region forms a flower of six petals, which is called "Tyndall figure" (see Figure 25). The figure is filled

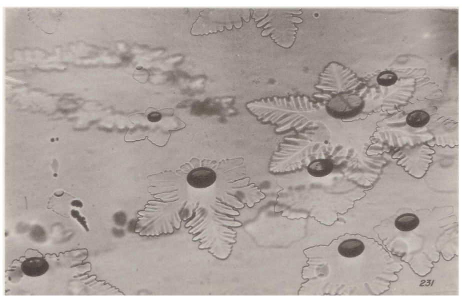


FIGURE 25. Tyndall figures seen from the direction of 45° to c-axis [Nak56, No. 17].

with water except for a vapor bubble (black spot in Figure 25). This phenomenon was first observed by Tyndall (1858). When Tyndall figure is refrozen, the vapor bubble remains in the ice as a hexagonal disk (see Figure 26). This hexagonal disk is filled with water vapor saturated at that temperature and surrounded by

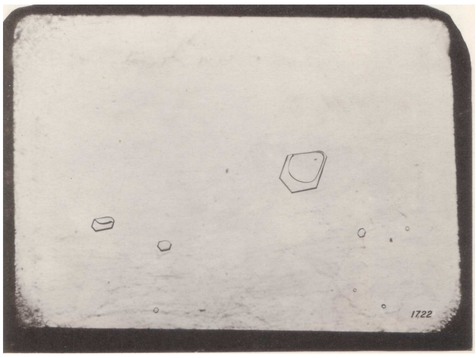


Figure 26. Natural negative crystals in an ice single crystal [Nak56, No. 1].

ice. McConnel found these disks in the ice of Davos lake [McC89]. Nakaya called this hexagonal disk "vapor figure" and investigated its properties precisely [Nak56]. Adams and Lewis (1934) called it "negative crystal." (Although Nakaya said "this term does not seem adequate" with a certain reason, hereafter we use the term negative crystal for avoiding confusion.)

Negative crystal is useful to determine the structure and orientation of ice or solids. Because, within a single ice crystal, all negative crystals are similarly oriented, that is, corresponding edges of hexagon are parallel each other (see Figure 27). Furukawa and Kohata made hexagonal prisms experimentally in a single

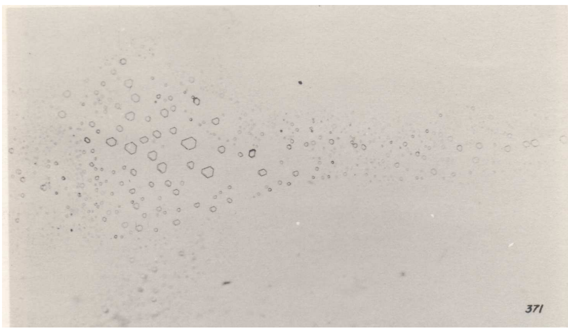


Figure 27. A cluster of minute negative crystals [Nak56, No. 53].

ice crystal, and investigated the habit change of negative crystals with respect to the temperatures and the evaporation mechanisms of ice surfaces [FK93].

To the best of author's knowledge, after the Furukawa and Kohata's experimental research, there have been no published results on negative crystals, and there are no dynamical model equations describing the process of formation of negative crystals. In the present talk, we will focus on the process of formation of negative crystals after Tyndall figures are refrozen, and try to propose a model equation of interfacial motion which tracks the deformation of negative crystals in time.

Formation of negative crystals. Figure 28 indicates aftereffect of freezing of Tyndall figures from the initial stage of refrozen process to the final stage of the formation of negative crystals. The aim of this talk is to propose a model equation,

FIGURE 28. From left to right, upper to lower: (a) Start of freezing, $t = 0$min. (b) Freezing proceeds, $t = 3$min. (c) Freezing proceeds further, $t = 11$min. (d) The bubble is separated, $t = 17$min. (e) The separated liquid film migrates, $t = 28$min. (f) After freezing, cloudy layers and a vapor figure are left, $t = 1$hr 21min. [Nak56, No. 52a–52f].

revealed the process in Figure 28 from (e) to (f). This process may be described as the following:

Negative crystal changes the shape from oval to hexagon.

Thus, our model will be assumed that

(i) water vapor region is simply connected and bounded region in the plane $\mathbb{R}^2$ (we denote it by Ω);
(ii) Ω is surrounded by a single ice crystal (i.e., ice region is $\mathbb{R}^2 \setminus \Omega$);

(iii) moving boundary $\partial\Omega$ is interface of the water vapor region and the single ice crystal; and
(iv) c-axis (main axis) of the single ice crystal is perpendicular to the plane.

The evolution law of moving interface is similar to the growth of snow crystal, since deformation of negative crystal is regarded as crystal growth in the air. As a model of snow crystal growth, we refer the Yokoyama-Kuroda model [YK90], which is based on the diffusion process and the surface kinetic process by Burton-Cabrera-Frank (BCF) theory [BCF51]. Meanwhile, we assume the existence of interfacial energy (density) on the boundary $\partial\Omega$. The equilibrium shape of negative crystal is a regular hexagon, and if the region Ω is very close to a regular hexagon, then the evolution process may be described as a gradient flow of total interfacial energy subject to a fixed enclosed area. Therefore, we can divide the process of formation of negative crystals into two stages as follows:

(i) in the former stage by the diffusion and the surface kinetic, oval Ω changes to a hexagon; and
(ii) in the latter stage by a gradient flow of interfacial energy, the hexagon converges to a regular hexagon.

A simple modeling of these stages is proposed as an area-preserving crystalline curvature flow for "negative" polygonal curves. See [IY07, IY08].

Bibliography

[AT95] F. Almgren and J. E. Taylor, *Flat flow is motion by crystalline curvature for curves with crystalline energies*, J. Diff. Geom. **42** (1995) 1–22.

[AG89] S. Angenent and M. E. Gurtin, *Multiphase thermomechanics with interfacial structure,* 2. *Evolution of an isothermal interface*, Arch. Rational Mech. Anal. **108** (1989) 323–391.

[AV95] S. Angenent and J. J. L Velázquez, *Asymptotic shape of cusp singularities in curve shortening*, Duke Math. J. **77** (1995) 71–110.

[BKPŠTY07] M. Beneš, M. Kimura, P. Pauš, D. Ševčovič, T. Tsujikawa and S. Yazaki, *Application of a curvature adjusted method in image segmentation*, submitted to Bulletin of Inst. of Mathematics, Academia Sinica at Taipei (arXiv:0712.2334v1).

[BKY07] M. Beneš, M. Kimura and S. Yazaki, *Analytical and numerical aspects on motion of polygonal curves with constant area speed*, Proceedings of Slovak-Austrian Mathematical Congress (Podbanske, 2007), within MAGIA 2007, Dept. of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, ISBN 978-80-227-2796-9 (2007) 127–141. (arXiv:0802.2204v3 is a new version including a minor change.)

[BKY08] M. Beneš, M. Kimura and S. Yazaki, *Second order numerical scheme for motion of polygonal curves with constant area speed*, preprint (arXiv:0805.4469v1).

[BCF51] W. K. Burton, N. Cabrera and F. C. Frank, *The growth of crystals and the equilibrium structure of their surfaces*, Philos. Trans. R. Soc. Landon, Ser. A **243** (1951) 299–358.

[CZ99] K.-S. Chou and X.-P. Zhu, *A convexity theorem for a class of anisotropic flows of plane curves*, Indiana Univ. Math. J. **48** (1999) 139–154.

[EG87] C. L. Epstein and M. Gage, *The curve shortening flow*, Wave motion: theory, modelling, and computation (Berkeley, Calif., 1986), Math. Sci. Res. Inst. Publ., **7**, Springer, New York (1987) 15–59.

[FK93] Y. Furukawa and S. Kohata, *Temperature dependence of the growth form of negative crystal in an ice single crystal and evaporation kinetics for its surfaces*, J. Crystal Growth **129** (1993) 571–581.

[Gag86] M. Gage, *On an area-preserving evolution equations for plane curves*, Contemporary Math. **51** (1986) 51–62.

[GH86] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Diff. Geom. **23** (1986) 69–96.

[Gig95] Y. Giga, *A level set method for surface evolution equations*, Sūgaku **47** (1995) 321–340; English transl., Sugaku Expositions **10** (1997) 217–241.

[Gig97] Y. Giga, *Interesting diffusion equations*, Sūgaku **49** (1997) 193–196. (Japanese)

[Gig98] Y. Giga, *Interface dynamics — effects of curvature*, Hokkaido University Technical Report Series **56** (1998). (Japanese)

[Gig00] Y. Giga, *Anisotropic curvature effects in interface dynamics*, Sūgaku **52** (2000) 113–127; English transl., Sugaku Expositions **16** (2003) 135–152.

[GGig00] M.-H. Giga and Y. Giga, *Crystalline and level set flow – convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane*, Free boundary problems: theory and applications, I (Chiba, 1999), GAKUTO Internat. Ser. Math. Sci. Appl., Gakkōtosho, Tokyo **13** (2000) 64–79.

[Gra87] M. A. Grayson, *The heat equation shrinks emmbedded plane curves to round points*, J. Diff. Geom. **26** (1987) 285–314.

[Gur93] M. E. Gurtin, *Thermomechanics of evolving phase boundaries in the plane*, Oxford, Clarendon Press (1993).

[HIY06] C. Hirota, T. Ishiwata and S. Yazaki, *Note on the asymptotic behavior of solutions to an anisotropic crystalline curvature flow*, Recent Advances on Elliptic and Parabolic Issues (Editors: Michel Chipot and Hirokazu Ninomiya), Proceedings of the 2004 Swiss-Japan Seminar, World Scientific (2006) 129–143.

[HIY07] C. Hirota, T. Ishiwata and S. Yazaki, *Numerical study and examples on singularities of solutions to anisotropic crystalline curvature flows of nonconvex polygonal curves*, Proceedings of MSJ-IRI 2005 "Asymptotic Analysis and Singularities" (Sendai, 2005), Advanced Studies in Pure Mathematics (ASPM) **47** (2007) 543–564.

[HGGD05] H. Hontani, M.-H. Giga, Y. Giga and K. Deguchi, *Expanding selfsimilar solutions of a crystalline flow with applications to contour figure analysis*, Discrete Applied Mathematics **147** (2005) 265–285.

[IS99] K. Ishii and H. M. Soner , *Regularity and convergence of crystalline motion*, SIAM J. Math. Anal. **30** (1999) 19–37.

[IUYY04] T. Ishiwata, T. K. Ushijima, H. Yagisita and S. Yazaki, *Two examples of nonconvex self-similar solution curves for a crystalline curvature flow*, Proc. Japan Academy **80**, Ser. A, No. 8 (2004) 151–154.

[IY07] T. Ishiwata and S. Yazaki, *Towards modelling the formation of negative ice crystals or vapor figures produced by freezing of internal melt figures.* Symposium on Mathematical Models of Phenomena and Evolution Equations (Kyoto 2006). RIMS Kôkyûroku **1542** (2007) 1–11.

[IY08] T. Ishiwata and S. Yazaki, *Interface motion of a negative crystal and its analysis.* Nonlinear Evolution Equations and Mathematical Modeling (Kyoto 2007). RIMS Kôkyûroku **1588** (2008) 23–29.

[KG01] R. Kobayashi and Y. Giga, *On anisotropy and curvature effects for growing crystals*, Japan J. Indust. Appl. Math. **18** (2001) 207–230.

[MS00] U. F. Mayer and G. Simonett, *Self-intersections for the surface diffusion and the volume-preserving mean curvature flow*, Differential Integral Equations **13** (2000) 1189–1199.

[McC89] J. C. McConnel, *The crystallization of lake ice*, Nature **39** (1889) 367.

[Nak56] U. Nayaka, *Properties of single crystals of ice, revealed by internal melting*, SIPRE (Snow, Ice and Permafrost Reseach Establishment) Research Paper

13 (1956).

[Rob93] S. Roberts, *A line element algorithm for curve flow problems in the plane*, CMA Research Report **58** (1989); J. Austral. Math. Soc. Ser. B **35** (1993) 244–261.

[Sch93] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press (1993).

[Set96] J. A. Sethian, *Level set methods: evolving interfaces in geometry, fluid mechanics, computer vision, and materials science*, Cambridge Monographs on Applied and Computational Mathematics **3**, Cambridge University Press, Cambridge (1996).

[ŠY07] D. Ševčovič and S. Yazaki, *On a motion of plane curves with a curvature adjusted tangential velocity*, submitted to Proceedings of Equadiff 2007 Conference (arXiv:0711.2568v1).

[ŠY] D. Ševčovič and S. Yazaki, *Evolution of plane curves with curvature adjusted tangential redistribution velocity*, in preparation.

[Tay90] J. E. Taylor, *Crystals, in equilibrium and otherwise*, videotape of 1989 AMS-MAA lecture, Selected Lectures in Math., Amer. Math. Soc. (1990).

[Tay91a] J. E. Taylor, *Constructions and conjectures in crystalline nondifferential geometry*, Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs Surveys Pure Appl. Math. **52** (1991) 321–336, Pitman London.

[Tay91b] J. E. Taylor, *Motion by crystalline curvature*, Computing Optimal Geometries, Selected Lectures in Math., Amer. Math. Soc. (1991) 63–65 plus video.

[Tay93] J. E. Taylor, *Motion of curves by crystalline curvature, including triple junctions and boundary points*, Diff. Geom.: partial diff. eqs. on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., **54** (1993), Part I, 417–438, AMS, Providencd, RI.

[TCH92] J. E. Taylor, J. Cahn and C. Handwerker, *Geometric models of crystal growth*, Acta Metall., **40** (1992) 1443–1474.

[Tot72] L. F. Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer-Verlag, Heidelberg (1972); Japanese transl., Misuzu Shobo (1983).

[UY04] T. K. Ushijima and S. Yazaki, *Convergence of a crystalline approximation for an area-preserving motion*, Journal of Computational and Applied Mathematics **166** (2004) 427–452.

[Yaz02] S. Yazaki, *On an area-preserving crystalline motion*, Calc. Var. **14** (2002) 85–105.

[Yaz04] S. Yazaki, *On an anisotropic area-preserving crystalline motion and motion of nonadmissible polygons by crystalline curvature*, RIMS Kôkyûroku **1356** (2004) 44–58.

[Yaz07a] S. Yazaki, *Motion of nonadmissible convex polygons by crystalline curvature*, Publications of Research Institute for Mathematical Sciences **43** (2007), 155–170.

[Yaz07b] S. Yazaki, *Asymptotic behavior of solutions to an area-preserving motion by crystalline curvature*, Kybernetika **43** (2007) 903–912.

[Yaz07c] S. Yazaki, *On the tangential velocity arising in a crystalline approximation of evolving plane curves*, Kybernetika **43** (2007) 913–918.

[YK90] E. Yokoyama and T. Kuroda, *Pattern formation in growth of snow crystals occurring in the surface kinetic process and the diffusion process*, Physical Review A **41**(4) (1990) 2038–2049.

Index

APPENDIX A

1. Strange examples

The following set Γ describes the unit circle:

$$\Gamma = \{\boldsymbol{x}(\sigma) = (\cos f(\sigma), \sin f(\sigma));\ \sigma \in [0,1]\},$$

if we take the range of f is greater than or equals 2π. However, the following circles are strange in the sense of contrary to intuition.

Non-closed circle: If $f(\sigma) = (2\pi+1)\sigma$, then $\boldsymbol{x}$ is not closed, since $\boldsymbol{x}(0) \neq \boldsymbol{x}(1)$.

Non-simple circle: If $f(\sigma) = \pi(2\sigma-1)(4(2\sigma-1)^2-1)/3$, then the map $f : [0,1] \to [0,2\pi]$ is surjective but not injective. Hence $\boldsymbol{x}$ is not simple.

Non-immersed circle: If $f(\sigma) = \pi((2\sigma-1)^3+1)$, then the map $f : [0,1] \to [0,2\pi]$ is bijective. Hence $\boldsymbol{x}$ is simple and C^1-closed curve but not immersed, since $g(\sigma) = |\partial_\sigma f(\sigma)| = 0$ holds at $\sigma = 1/2$.

Non-embedded circle: If $f(\sigma) = 2n\pi\sigma$ ($n = 2,3,\ldots$), then $\boldsymbol{x}$ is the n-multiply covered unit circle. Since the map $\boldsymbol{x} : \Sigma \to \Gamma$ is not injective, $\boldsymbol{x}$ is immersed but not embedded.

2. A non-concave curve

We need some attention concerning the converse of Lemma 1.5. Strictly convex curves admit finite number of zeros of K (such points are considered as degenerate line segment). Figure 1 indicates such an example. Thus we can say that

> A closed embedded C^2-curve is strictly convex, if and only if $K > 0$ holds in Σ except finite number of zero points.

3. Anisotropic inequality—proof of Lemma 1.53—

The result of Lemma 1.53 follows from a classical convex geometry by using a concept of mixed area and the Brünn and Minkowski's inequality.

Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be N-sided admissible convex polygons associated with $\mathcal{W}_\gamma$. For $i = 0,1$, by $h_j^{(i)}$ and $d_j^{(i)}$, we denote the j-th hight function and the length of the j-th edge, respectively. Note that $d_j^{(i)} = D_j(\underline{h}^{(i)})$ holds for $i = 1,2,\ldots,N$, $\underline{h}^{(i)} = (h_1^{(i)}, h_2^{(i)}, \ldots, h_N^{(i)})$. Let the enclosed region of $\mathcal{P}_i$ be $\Omega_i = \bigcap_{j=1}^N \{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} \cdot \boldsymbol{n}_j \leq h_j^{(i)}\}$ for $i = 0,1$. Define the linear interpolant of Ω_0 and Ω_1 by $\Omega_\mu = (1-\mu)\Omega_0 + \mu\Omega_1 = \{(1-\mu)\boldsymbol{x} + \mu\boldsymbol{y};\ \boldsymbol{x} \in \Omega_0,\ \boldsymbol{y} \in \Omega_1\}$, and $\mathcal{P}_\mu = (1-\mu)\mathcal{P}_0 + \mu\mathcal{P}_1$ for $0 \leq \mu \leq 1$. Then the j-th height function and the length of the j-th edge of $\mathcal{P}_\mu$

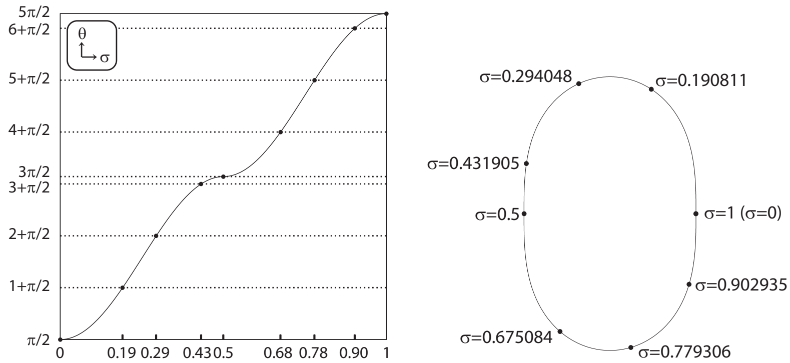


FIGURE 1. Graph of θ versus σ (left), and the C^2-class strictly convex curve $\Gamma = \{\boldsymbol{x}(\sigma);\ \sigma \in [0,1]\}$ (right). Here $\theta(\sigma) = \eta(\sigma) = \pi(2-\cos(2\pi\sigma))/2$ if $\sigma \in [0, 0.5)$, $\theta(\sigma) = \eta(\sigma - 0.5) + \pi$ if $\sigma \in [0.5, 1]$, and $\boldsymbol{x}(\sigma) = \int_0^\sigma (\cos\theta(\sigma), \sin\theta(\sigma))\, d\sigma$. In the right figure, the set $\{\sigma;\ \theta(\sigma) \in \Theta\}$ is indicated, where $\Theta = \{i + \pi/2\ (i = 1, 2, \ldots, 6), 3\pi/2, 5\pi/2\}$. Note that $\theta(0.5) = 3\pi/2$ and $\theta(1) = 5\pi/2$. Since $g(\sigma) \equiv 1$, the curvature is given by $K(\sigma) = \partial_\sigma\theta = \pi^2|\sin(2\pi\sigma)|$. Hence $K(\sigma)$ is positive except two points $\sigma = 0.5$ and $\sigma = 1$ ($\sigma = 0$).

are given as $h_j^{(\mu)} = (1-\mu)h_j^{(0)} + \mu h_j^{(1)}$ and $d_j^{(\mu)} = (1-\mu)d_j^{(0)} + \mu d_j^{(1)}$, respectively. Let $A(\Omega_\mu)$ be the area of Ω_μ. Then we have

$$A(\Omega_\mu) = \frac{1}{2}\sum_{j=1}^N d_j^{(s)} h_j^{(s)} = (1-\mu)^2 A(\Omega_0) + \mu^2 A(\Omega_1) + \frac{\mu(1-\mu)}{2}\sum_{j=1}^N \left(d_j^{(1)} h_j^{(0)} + d_j^{(0)} h_j^{(1)}\right).$$

By summation by parts:

$$\sum_{j=1}^N d_j^{(1)} h_j^{(0)} = \sum_{j=1}^N D_j(\underline{h}^{(1)}) h_j^{(0)} = \sum_{j=1}^N D_j(\underline{h}^{(0)}) h_j^{(1)} = \sum_{j=1}^N d_j^{(0)} h_j^{(1)}, \tag{A.1}$$

we have

$$A(\Omega_\mu) = \frac{1}{2}\sum_{j=1}^N d_j^{(\mu)} h_j^{(\mu)} = (1-\mu)^2 A(\Omega_0) + \mu^2 A(\Omega_1) + 2\mu(1-\mu)\frac{1}{2}\sum_{j=1}^N d_j^{(0)} h_j^{(1)}.$$

The coefficient of $2\mu(1-\mu)$ is called the **mixed area** of Ω_0 and Ω_1, and denoted by

$$A(\Omega_0, \Omega_1) = \frac{1}{2}\sum_{j=1}^N d_j^{(0)} h_j^{(1)}.$$

Note that $A(\Omega_0, \Omega_1) = A(\Omega_1, \Omega_0)$ holds by (A.1).

H. Brünn and H. Minkowski proved the following inequality:

$$\sqrt{A(\Omega_\mu)} \geq (1-\mu)\sqrt{A(\Omega_0)} + \mu\sqrt{A(\Omega_1)}, \quad 0 \leq \mu \leq 1. \tag{A.2}$$

Equality holds if and only if $\Omega_0 = k\Omega_1$ ($k > 0$). The Brünn and Minkowski's inequality (A.2) is equivalent to the following inequality:

$$A(\Omega_0, \Omega_1) \geq \sqrt{A(\Omega_0)A(\Omega_1)}. \tag{A.3}$$

From this inequality, we obtain the anisoperimetric inequality as follows. Let $\Omega = \Omega_0$ be the enclosed region of a polygon $\mathcal{P}$ and let the enclosed area be $A(\Omega) = A$. For the crystalline energy $\gamma > 0$, let Ω_1 be the Wulff shape $\mathcal{W}_\gamma$. Then the area of $\mathcal{W}_\gamma$ is $A(\mathcal{W}_\gamma) = |\mathcal{W}_\gamma| = \sum_{k=1}^N \gamma(\boldsymbol{n}_k) l_\gamma(\boldsymbol{n}_k)/2$, and the mixed area is $A(\Omega, \mathcal{W}_\gamma) = \sum_{j=1}^N \gamma(\boldsymbol{n}_j) d_j/2 = E_\gamma(\mathcal{P})/2$, which is a half of the total interfacial energy on $\mathcal{P}$. Hence by (A.3), $E_\gamma(\mathcal{P})/2 \geq \sqrt{A|\mathcal{W}_\gamma|}$, namely,

$$\mathcal{J}_\gamma(\mathcal{P}) = \frac{E_\gamma(\mathcal{P})^2}{4|\mathcal{W}_\gamma|A} \geq 1. \tag{A.4}$$

The equality $\mathcal{J}_\gamma(\mathcal{P}) \equiv 1$ holds if and only if $\Omega = k\mathcal{W}_\gamma$ for some constant $k > 0$, i.e., $\Lambda_\gamma(\boldsymbol{n}_j) \equiv const.$ for $j = 1, 2, \ldots, N$.

In particular, if $\gamma \equiv 1$, then $E_\gamma = L$ and $|\mathcal{W}_\gamma| = \sum_{j=1}^N l_\gamma(\boldsymbol{n}_j)/2$, so we have

$$\mathcal{I} = \frac{L^2}{2\sum_{j=1}^N l_1(\boldsymbol{n}_j)A} \geq 1, \tag{A.5}$$

which is the isoperimetric inequality of polygons. Note that L. Fejes Tóth writes in his book [Tot72] such that this inequality was done by S. Lhuilier. We refer R. Schneider [Sch93] for general theory of convex bodies. See also [Yaz02] for the proof of (A.5) by using a solution of crystalline curvature flow equation.

The isoperimetric inequality (A.5) represents the variational problem: what is the shape which has the least total length of a polygon for the fixed enclosed area? The answer (it corresponds to the case where $\mathcal{I} = 1$) is the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$, which is the circumscribed polygon of the circle with radius $\gamma_* = \sqrt{2A/\sum_{i=1}^N l_1(\boldsymbol{n}_i)}$ (cf. Theorem 1.49). Similarly, the anisoperimetric inequality (A.4) represents the variational problem: what is the shape which has the least total interfacial energy of a polygon for the fixed enclosed area? The answer (it corresponds to the case where $\mathcal{J}_\gamma = 1$) is the boundary of the Wulff shape $\partial\mathcal{W}_{\gamma_*}$ (cf. Theorem 1.50).

(Eds.) M. Beneš and E. Feireisl

Topics in mathematical modeling

Published by
MATFYZPRESS
Publishing House of the Faculty of Mathematics and Physics
Charles University, Prague
Sokolovská 83, CZ – 186 75 Praha 8
as the 254. publication

The volume was typeset by the authors using LaTeX

Printed by
Reproduction center UK MFF
Sokolovská 83, CZ – 186 75 Praha 8

First edition

Praha 2008

ISBN 978-80-7378-060-9

Jindřich Nečas

Jindřich Nečas was born in Prague on December 14$^{\text{th}}$, 1929. He studied mathematics at the Faculty of Natural Sciences at the Charles University from 1948 to 1952. After a brief stint as a member of the Faculty of Civil Engineering at the Czech Technical University, he joined the Czechoslovak Academy of Sciences where he served as the Head of the Department of Partial Differential Equations. He held joint appointments at the Czechoslovak Academy of Sciences and the Charles University from 1967 and became a full time member of the Faculty of Mathematics and Physics at the Charles University in 1977. He spent the rest of his life there, a significant portion of it as the Head of the Department of Mathematical Analysis and the Department of Mathematical Modeling.

His initial interest in continuum mechanics led naturally to his abiding passion to various aspects of the applications of mathematics. He can be rightfully considered as the father of modern methods in partial differential equations in the Czech Republic, both through his contributions and through those of his numerous students. He has made significant contributions to both linear and non-linear theories of partial differential equations. That which immediately strikes a person conversant with his contributions is their breadth without the depth being compromised in the least bit. He made seminal contributions to the study of Rellich identities and inequalities, proved an infinite dimensional version of Sard's Theorem for analytic functionals, established important results of the type of Fredholm alternative, and most importantly established a significant body of work concerning the regularity of partial differential equations that had a bearing on both elliptic and parabolic equations. At the same time, Nečas also made important contributions to rigorous studies in mechanics. Notice must be made of his work, with his collaborators, on the linearized elastic and inelastic response of solids, the challenging field of contact mechanics, a variety of aspects of the Navier-Stokes theory that includes regularity issues as well as important results concerning transonic flows, and finally non-linear fluid theories that include fluids with shear-rate dependent viscosities, multi-polar fluids, and finally incompressible fluids with pressure dependent viscosities.

Nečas was a prolific writer. He authored or co-authored eight books. Special mention must be made of his book "Les méthodes directes en théorie des équations elliptiques" which has already had tremendous impact on the progress of the subject and will have a lasting influence in the field. He has written a hundred and forty seven papers in archival journals as well as numerous papers in the proceedings of conferences all of which have had a significant impact in various areas of applications of mathematics and mechanics.

Jindřich Nečas passed away on December 5$^{\text{th}}$, 2002. However, the legacy that Nečas has left behind will be cherished by generations of mathematicians in the Czech Republic in particular, and the world of mathematical analysts in general.

Jindřich Nečas Center for Mathematical Modeling

The Nečas Center for Mathematical Modeling is a collaborative effort between the Faculty of Mathematics and Physics of the Charles University, the Institute of Mathematics of the Academy of Sciences of the Czech Republic and the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University.

The goal of the Center is to provide a place for interaction between mathematicians, physicists, and engineers with a view towards achieving a better understanding of, and to develop a better mathematical representation of the world that we live in. The Center provides a forum for experts from different parts of the world to interact and exchange ideas with Czech scientists.

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