Quantitative stochastic homogenization of convex integral functionals

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Convex integral functional with stationary and ergodic integrand:

$$\mathcal{E}_\varepsilon(u) = \int_O V(\tfrac{x}{\varepsilon}, \nabla u) - f\,u\,dx, \qquad \mathcal{E}_{\hom} = \int_O V_{\hom}(\nabla u) - f\,u\,dx$$

Qualitative homogenization: $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_{\hom}$.

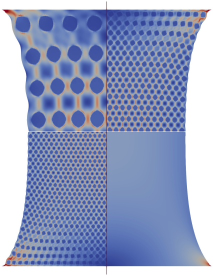
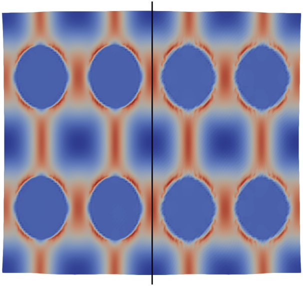
Quantitative homogenization:

1. Convergence rate for homogenization error and two-scale expansion

$$\|u_\varepsilon - u_{\hom}\| + \|\nabla u_\varepsilon - \nabla u_\varepsilon^{ts}\| \lesssim \varepsilon^\alpha$$

with optimal scaling α

2. Growth estimate of sublinear correctors

Integrand parametrized by random field

$$\omega : \mathbb{R}^d \to \mathbb{R}^N, \qquad \text{stationary random field}$$
$$V(x,\xi) = V^\omega(x,\xi) = V(\omega(x),\xi)$$

Assumptions on randomness:

- ω is **stationary** random field of centered Gaussians with joint distribution modeled over probability space $(\Omega, \mathbb{P})$
- covariance function $c(x) := \mathbb{E}[\omega(x)\omega(0)]$ is bounded and **decaying**

Assumptions on convexity: $V(\omega,\cdot) \in \mathcal{V}(\Lambda)$ for some $\Lambda > 0$, where $\mathcal{V}(\Lambda)$ denotes the class of integrands satisfying

$$V(\bullet) - \frac{1}{2\Lambda}|\bullet|^2 \text{ is convex, minimized at } 0, \quad |V(\xi)| \le \Lambda(|\xi|^2+1),$$
$$V \in C^2(\mathbb{R}^d) \text{ and } |\nabla_\xi^2 V(\xi)| \le \Lambda.$$

Assumptions on dependence on parameter:

V, $\partial_\xi V$, $\partial_\xi^2 V$ are continuously differentiable in ω, and

$$|\partial_\omega V(\omega,\xi)| \le C(|\xi|^2+1), \quad |\partial_\omega \partial_\xi V(\omega,\xi)| \le C(|\xi|+1), \quad |\partial_\omega \partial_\xi^2 V(\omega,\xi)| \le C.$$

Theorem (classic): There exists $V_{\text{hom}} \in \mathcal{V}(\Lambda)$ such that: For all $O \subset \mathbb{R}^d$ open, bounded and $f \in L^2(O)$ the functional

$$\mathcal{E}_\varepsilon : H^1_0(O) \to \mathbb{R}, \qquad \mathcal{E}_\varepsilon(u) := \int_O V(\tfrac{x}{\varepsilon}, \nabla u) - f\,u\,dx$$

$\Gamma(L^2(O))$-converges to

$$\mathcal{E}_{\text{hom}} : H^1_0(O) \to \mathbb{R}, \qquad \mathcal{E}_{\text{hom}}(u) := \int_O V_{\text{hom}}(\nabla u) - f\,u\,dx$$

- [Marcellini '78]: periodic case; [Dal Maso & Modica '86]: random case based on subadditive ergodic theorem. (C^2 and strict convexity not needed, p-growth.)
- Implies convergence of minimizers: $u_\varepsilon \rightharpoonup u_{\text{hom}}$ weakly in $H^1(O)$.
- [Geymonat, Müller, Triantifyllidis '93]: $V_{\text{hom}} \in C^2$ in the periodic case using correctors

Homogenization formula

$$V_{\text{hom}}(\xi) = \lim_{L\to\infty} \inf_{\varphi\in H_0^1(B_L)} \fint_{B_L} V(x, \nabla(\xi\cdot x + \varphi)) = \lim_{L\to\infty} \fint_{B_L} V(x, \xi + \nabla\phi_\xi)$$
$$= \mathbb{E}[V(\xi + \nabla\phi_\xi)]$$

Sublinear corrector: ϕ_ξ solution to

$$-\nabla\cdot\partial_\xi V(x,\xi+\nabla\phi_\xi) = 0 \qquad \text{in } \mathbb{R}^d,\ \mathbb{P}\text{-a.s.},$$
$$\text{subject to } \lim_{L\to\infty}\frac{1}{L}\Big(\fint_{B_L}|\phi_\xi|^2\Big)^{\frac{1}{2}} = 0 \qquad \text{(sublinear growth).}$$

Note: $\mathbf{a}(x,\xi) = \partial_\xi V(x,\xi)$ is monotone and uniformly elliptic:

$$\Lambda|\xi-\xi'|^2 \le (\mathbf{a}(x,\xi)-\mathbf{a}(x,\xi'))\cdot(\xi-\xi'), \qquad |\mathbf{a}(x,\xi)| \le \Lambda(|\xi|+1).$$

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Formal two-scale expansion: Minimizers u_ε and u_{hom} formally satisfy

$$\nabla u_\varepsilon(x) \approx \nabla u_{\text{hom}}(x) + \nabla\phi_\xi(\tfrac{x}{\varepsilon}) \text{ with } \xi = \nabla u_{\text{hom}}(x).$$

For all $\xi \in \mathbb{R}^d$ there exist unique $\phi_\xi : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $\sigma_\xi : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}_{\text{skw}}$ s.t.

$$\nabla(\phi_\xi, \sigma_\xi) \text{ is stationary, } \mathbb{E}[\nabla(\phi_\xi, \sigma_\xi)] = 0, \quad \mathbb{E}[|\nabla(\phi_\xi, \sigma_\xi)|^2]^{\frac{1}{2}} < \infty,$$

and $\mathbb{P}$-a.s.:

1. (corrector equations). $\phi_\xi, \sigma_{\xi,jk} \in H^1_{\text{loc}}(\mathbb{R}^d)$, $\fint_{B_1(0)}(\phi_\xi, \sigma_{\xi,jk}) = 0$, and

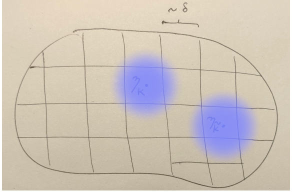
$$\left.\begin{aligned} -\nabla \cdot (\partial_\xi V(x, \nabla\phi_\xi + \xi)) &= 0 \\ \mathbf{a}_\xi &:= \partial_\xi V(x, \nabla\phi_\xi + \xi) \\ -\Delta\sigma_{\xi,jk} &= \partial_j \mathbf{a}_\xi \cdot e_k - \partial_k \mathbf{a}_\xi \cdot e_j \end{aligned}\right\} \quad \text{in } \mathbb{R}^d.$$

2. (sublinearity). $\limsup_{R\to\infty} \frac{1}{R}\left(\fint_{B_R(0)} |(\phi_\xi, \sigma_\xi)|^2 \, dx\right)^{\frac{1}{2}} = 0.$

3. (error representation).

$$-\nabla \cdot \sigma_\xi = \mathbf{a}_\xi - \mathbb{E}[\mathbf{a}_\xi] = \mathbf{a}_\xi - \mathbf{a}_{\text{hom}}(\xi).$$

- Introduce smooth partition of unity $\{\eta_k\}_{k\in K}$ of O on meso scale $\varepsilon \leq \delta \ll 1$ with $\operatorname{supp}\eta_k \subset B_{C\delta}(k)$ (+ additional properties)
- Define approximate gradients $\xi_k := \frac{1}{\int_Q \eta_k} \int_O \nabla u_{\mathrm{hom}} \eta_k$



- Define rescaled, recentered correctors $(\phi_{k,\varepsilon}, \sigma_{k,\varepsilon})(x) := \varepsilon\Big((\phi_{\xi_k}, \sigma_{\xi_k})(\frac{x}{\varepsilon}) - \fint_{B_\varepsilon(k)}(\phi_{\xi_k}, \sigma_{\xi_k})\,dy\Big)$
- Define approximate two-scale expansion $\hat{u}_\varepsilon := u_{\mathrm{hom}} + \sum_{k\in K} \eta_k \phi_{k,\varepsilon}$.

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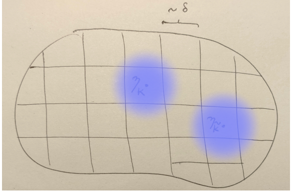

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- Define approximate two-scale expansion $\hat u_\varepsilon := u_{\mathrm{hom}} + \sum_{k\in K}\eta_k\phi_{k,\varepsilon}$.
- Error representation for residuum [Fischer & N. 21]

$$-\nabla\cdot(\mathbf{a}(\tfrac{x}{\varepsilon},\nabla\hat u_\varepsilon)) = -\nabla\cdot(\mathbf{a}_{\mathrm{hom}}(\nabla u_{\mathrm{hom}})) + \nabla\cdot R_\varepsilon$$

where the residuum R_ε satisfies for all $\ell\in K$:

$$\int_O \eta_\ell |R_\varepsilon|^2\,dx \le C\Bigg(\int_{B_{2\delta}(\ell)\cap O}\delta^2|\nabla^2 u_{\mathrm{hom}}|^2\,dx + \frac{1}{\delta^2}\sum_{\substack{k\in K\\ |\ell-k|\le 4\delta}}\int_{B_{6\delta}(\ell)}|\phi_{\ell,\varepsilon}-\phi_{k,\varepsilon}|^2 + |\sigma_{\ell,\varepsilon}-\sigma_{k,\varepsilon}|^2\,dx\Bigg).$$

Lemma [Fischer & N. 21] Assume that there exists $\mu : [1,\infty) \to [0,\infty]$ (non-decreasing) s.t. for all $r \geq 1$,

$$\left(\fint_{B_{6r}(x)} |(\phi_\xi, \sigma_\xi)|^2 \right)^{\frac{1}{2}} \leq \mathscr{C}_x (|\xi| + 1)\mu(r + |x|),$$

for all $x \in \mathbb{R}^d$ where $\mathscr{C}_x$ is a r.v. with integrability $\mathbb{E}[|\mathscr{C}_x|^p]^{\frac{1}{p}} \leq C(p)$.

Consider three scales $0 < \varepsilon \ll \delta \ll 1, \quad 0 < \varepsilon \leq \tau \leq \delta$. Then

$$\begin{aligned} &\|u_\varepsilon - u_{\mathrm{hom}}\|_{L^2(O)} + \|\nabla \hat{u}_\varepsilon - \nabla u_\varepsilon\|_{L^2(O)} \\ &\qquad \leq \mathscr{C}_\varepsilon \|\nabla u_{\mathrm{hom}}\|_{H^1(O)} \big(\delta^2 + (\tfrac{\varepsilon}{\delta})^2 \mu^2(\tfrac{\delta}{\varepsilon}) + \frac{\varepsilon^2}{\tau}\mu^2(\tfrac{\delta}{\varepsilon}) + \tau\big)^{\frac{1}{2}} \end{aligned}$$

for a random constant $\mathscr{C}_\varepsilon$ satisfying $\mathbb{E}[|\mathscr{C}_\varepsilon|^p]^{\frac{1}{p}} \leq C(\Lambda, d, O)C(p)$.

Lemma [Fischer & N. 21] Assume additionally that there exists $\widetilde{\mu} : [1,\infty) \to [0,\infty]$ (non-decreasing) s.t. for all $r \geq 1$,

$$\left(\fint_{B_{6r}(x)} |(\phi_\xi, \sigma_\xi) - (\phi_{\xi'}, \sigma_{\xi'})|^2 \right)^{\frac{1}{2}} \leq \mathscr{C}_x (1 + |\xi| \vee |\xi'|) |\xi - \xi'| \tilde{\mu}(r + |x|),$$

for all $x \in \mathbb{R}^d$ where $\mathscr{C}_x$ is a r.v. with integrability $\mathbb{E}[|\mathscr{C}_x|^p]^{\frac{1}{p}} \leq C(p)$.

Consider three scales $0 < \varepsilon \ll \delta \ll 1, \quad 0 < \varepsilon \leq \tau \leq \delta$. Then

$$\begin{aligned} &\|u_\varepsilon - u_{\text{hom}}\|_{L^2(O)} + \|\nabla \hat{u}_\varepsilon - \nabla u_\varepsilon\|_{L^2(O)} \\ &\quad \leq \mathscr{C}_\varepsilon \|\nabla u_{\text{hom}}\|_{W^{1,\infty}(O)} \left(\delta^2 + \varepsilon^2 \tilde{\mu}^2(\tfrac{\delta}{\varepsilon}) + \frac{\varepsilon^2}{\tau} \mu^2(\tfrac{\delta}{\varepsilon}) + \tau\right)^{\frac{1}{2}} \end{aligned}$$

for a random constant $\mathscr{C}_\varepsilon$ satisfying $\mathbb{E}[|\mathscr{C}_\varepsilon|^p]^{\frac{1}{p}} \leq C(\Lambda, d, O) C(p)$.

$$(\star_1) \qquad \left(\fint_{B_{6r}(x)} |(\phi_\xi, \sigma_\xi)|^2 \right)^{\frac{1}{2}} \leq \mathscr{C}_x(|\xi| + 1)\mu(r + |x|),$$

$$(\star_2) \qquad \left(\fint_{B_{6r}(x)} |(\phi_\xi, \sigma_\xi) - (\phi_{\xi'}, \sigma_{\xi'})|^2 \right)^{\frac{1}{2}} \leq \mathscr{C}_x(1+|\xi| \vee |\xi'|)|\xi - \xi'|\tilde{\mu}(r + |x|),$$

Assume: $c(x) = \mathbb{E}[\omega(x)\omega(0)], \qquad |c(x)| \lesssim (|x| + 1)^{-\beta}.$

Theorem [Fischer & N. ARMA 21] Assume $\beta > d$. Then $(\star_1)$ holds with

$$\mu(s) = \begin{cases} s^{\frac{1}{2}} & \text{if } d = 1, \\ \log(s+2)^{\frac{1}{2}} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$$

If $\partial_\xi V$ satisfies a small-scale regularity assumptions, then $(\star_2)$ holds with $\tilde{\mu} = \mu$. In both cases the random constant has stretched exponential moments $C(p) \leq Cp^\nu$ for some $\nu \leq 1$.

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- [Gloria, N., Otto APDE '21]: $\mathbf{a}(x, \xi) = A(x)\xi$ (linear case):

$$\mu(s) = \begin{cases} (s + 1)^{1-\frac{\beta}{2}} & \text{for } \beta < 2 \\ \log(s + 2) & \text{for } \min\{d, \beta\} = 2 \\ 1 & \text{for } \beta, d > 2 \end{cases}$$

[Fischer, N. ARMA' 21]

$$\mu(s), \tilde{\mu}(s) = \begin{cases} s^{\frac{1}{2}} & \text{if } d = 1, \\ |\log(2+s)|^{\frac{1}{2}} & \text{if } d = 2, \\ 1 & \text{if } d \geq 3. \end{cases}$$

Combine with error representation and optimize choice of $\varepsilon \leq \tau \leq \delta$:

$$\left(\delta^2 + (\tfrac{\varepsilon}{\delta})^2 \mu^2(\tfrac{\delta}{\varepsilon}) + \frac{\varepsilon^2}{\tau}\mu^2(\tfrac{\delta}{\varepsilon}) + \tau\right)^{\frac{1}{2}} \quad \longrightarrow \quad \begin{cases} \varepsilon^{\frac{1}{3}} & \text{if } d = 1, \\ \varepsilon^{\frac{1}{2}} |\log \varepsilon|^{\frac{1}{2}} & \text{if } d = 2, \\ \varepsilon^{\frac{1}{2}} & \text{if } d \geq 3. \end{cases}$$

$$\left(\delta^2 + \varepsilon^2 \tilde{\mu}^2(\tfrac{\delta}{\varepsilon}) + \frac{\varepsilon^2}{\tau}\mu^2(\tfrac{\delta}{\varepsilon}) + \tau\right)^{\frac{1}{2}} \quad \longrightarrow \quad \sqrt{\varepsilon}$$

Ignoring error due to boundary layer (the case $O = \mathbb{R}^d$)

$$\left(\delta^2 + \varepsilon^2 \mu^2(\tfrac{\delta}{\varepsilon})\right)^{\frac{1}{2}} \quad \longrightarrow \quad \begin{cases} \varepsilon^{\frac{1}{2}} & \text{if } d = 1, \\ \varepsilon |\log \varepsilon|^{\frac{1}{2}} & \text{if } d = 2, \\ \varepsilon & \text{if } d \geq 3. \end{cases}$$

Spectral gap inequality — (SG) [Gloria, Otto, N., Duerinkx, ...] Assume $\beta > d$. Then for all random variables F

$$\mathbb{E}\Big[\big|F-\mathbb{E}[F]\big|^2\Big]^{1/2} \le C\mathbb{E}\bigg[\int_{\mathbb{R}^d}\bigg(\fint_{B_1(x)}\Big|\frac{\partial F}{\partial\omega}\Big|\,dy\bigg)^2 dx\bigg]^{1/2}$$

where the "L^1-norm of functional derivative" is defined by

$$\int_{B_1(x)}\Big|\frac{\partial F(\omega,y)}{\partial\omega}\Big|\,dy := \sup_{\substack{\operatorname{supp}\delta\omega\subset B_1(x)\\ |\delta\omega|\le 1}}\bigg\{\limsup_{h\to 0}\Big|\frac{F(\omega_\varepsilon+h\delta\omega)-F(\omega_\varepsilon)}{h}\Big|\bigg\}$$

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(SG) implies decay of fluctuations: Let $f\in C^1$ with $|Df|\leq C$. Consider

$$F_R(\omega) := \fint_{B_R(0)} f(\omega(x))\,dx$$

Then

$$\int_{B_1(x)}\Big|\frac{\partial F_R(\omega,y)}{\partial\omega}\Big|\,dy = \begin{cases} 0 & \text{if } B_1(x)\cap B_R(0)=\emptyset\\ \leq CR^{-d} & \text{else.}\end{cases}$$

Conclude for $R\geq 1$,

$$\mathbb{E}[|F_R-\mathbb{E}[F_R]|^2] \leq C^2\Big(\frac{R+1}{R}\Big)^d R^{-d}$$

1. Reduction to $r = 1$: Assume estimate for $r = 1$, then for $r \geq 1$,

$$\fint_{B_r(\bar{x})} |\phi_\xi|^2 \leq \underbrace{\sum_{\substack{k\in\mathbb{Z}\\ |k-\bar{x}|\leq r+1}} \frac{|B_1(k)|}{|B_r(\bar{x})|}}_{\lesssim 1} \underbrace{\fint_{B_1(k)} |\phi_\xi|^2}_{\lesssim(|\xi|^2+1)(|k|+1)} \leq C(|\xi|^2+1)(|\bar{x}|+r+1)$$

2. Uniform Lipschitz estimate for $d = 1$.

$$|\nabla\phi_\xi| \leq C(|\xi|+1) \qquad \text{a.e. in } \Omega. \tag{LIP}$$

Argument: Monotone corrector equation in $d = 1$ implies that $\mathbf{a}(\omega, \xi + \nabla\phi_\xi(\omega)) = \mathbf{a}_{\hom}(\xi)$.

3. (Argument for $r = 1$)

$$\left(\fint_{B_1(\bar{x})} |\phi_\xi|^2\right)^{\frac{1}{2}} \leq \left(\fint_{B_1(\bar{x})} \Big|\phi_\xi - \fint_{B_1(\bar{x})} \phi_\xi\Big|^2\right)^{\frac{1}{2}} + \Big|\fint_{B_1(\bar{x})} \phi_\xi - \fint_{B_1(0)} \phi_\xi\Big|.$$

By Poincaré and (LIP), $\bullet \leq C(|\xi|+1)$.

$$\bullet = \int \nabla\phi_\xi g_{\bar{x}} =: F(\omega),$$

with weight $g_{\bar{x}} \in W^{1,\infty}(\mathbb{R})$, $\operatorname{supp} g_{\bar{x}} \subset B_{|\bar{x}|+1}(0)$ and $|g_{\bar{x}}| + |\nabla g_{\bar{x}}| \leq 2$.

- Consider variations of ω by $\delta\omega$,

$$\operatorname{supp}\delta\omega \subset B_1(s),\ |\delta\omega| \le 1, \qquad \delta F = \partial_{\delta\omega} F, \qquad \delta\phi_\xi = \partial_{\delta\omega}\phi_\xi$$

- $\delta\phi_\xi$ solves

$$-\nabla\cdot(A_\xi \nabla\delta\phi_\xi) = \nabla\cdot(\partial_\omega \mathbf{a}(\xi+\nabla\phi_\xi)\delta\omega), \qquad A_\xi := \partial_\xi^2 V(\xi+\nabla\phi_\xi)$$

- Introduce h as solution to

$$-\nabla\cdot(A_\xi^\top \nabla h) = \nabla\cdot g_{\bar{x}}.$$

  Since $d=1$, have $\nabla h = -A_\xi^{-\top} g_{\bar{x}}$ and thus $|\nabla h| \le C 1_{B_{|\bar{x}|+1}(0)}$.

- Represent δF with h:

$$\delta F = \int \nabla\delta\phi_\xi g_{\bar{x}} = -\int A_\xi^\top \nabla h\cdot\nabla\delta\phi_\xi = \int \nabla h\cdot\partial_\omega \mathbf{a}(\xi+\nabla\phi_\xi)\delta\omega$$

$$\le |B_{|\bar{x}|+1}(0)\cap B_1(s)|C(|\xi|+1)$$

$$\Rightarrow \qquad \int_{B_1(s)} \left|\frac{\partial F}{\partial\omega}\right| \le \begin{cases} C(|\xi|+1) & \text{if } |s|\le|\bar{x}|+2 \\ 0 & \text{else.}\end{cases}$$

$$\Rightarrow \int_{\mathbb{R}} \left(\int_{B_1(s)} \left|\frac{\partial F}{\partial\omega}\right|\right)^2 ds \le C(|\xi|^2+1)(|\bar{x}|+1).$$

- For $d \geq 2$ no deterministic Lipschitz estimate available. Replacement: Large Scale Lipschitz estimate

$$r \geq r_{*,\xi}(\omega) \, : \, \fint_{B_r} |\xi + \nabla\phi_\xi|^2 \leq C(|\xi|^2 + 1).$$

- Representation with $g_{\bar{x}}$ with decay at infinity (instead of compact support)

$$\fint_{B_1(\bar{x})} \phi_\xi - \fint_{B_1(0)} \phi_\xi = \int \nabla\phi_\xi \cdot g_{\bar{x}}$$

- Only weighted Meyers estimate (instead of Lipschitz estimate) for linearized equation available

$$-\nabla \cdot A_\xi^\top \nabla h = \nabla \cdot g_{\bar{x}}.$$

- Moment bounds for $r_{*,\xi}$

$$\text{bound for } r_{*,\xi} \Rightarrow \text{smallness of } \phi_\xi \Rightarrow \text{bound for } r_{*,\xi}.$$

  Buckling argument uses sensitivity estimate, Spectral Gap, deterministic gain of regularity from hole-filling

$$\fint_{B_r} |\nabla\phi_\xi + \xi|^2 \leq C\left(\frac{R}{r}\right)^{d-\kappa} \fint_{B_R} |\nabla\phi_\xi + \xi|^2$$

Assumption: Nonlinear elastic, random laminate

- $W(x,F) = W(\omega(x_1),F)$ with $\omega : \mathbb{R} \to \mathbb{R}^N$ Gaussian random field with $\beta > 1$.
- $W(x,F) = W(x,RF)$ $\forall F \in \mathbb{R}^{d\times d}$, $R \in \mathrm{SO}(d)$ (**frame indifferent**)
- $W(x,\mathrm{Id}) = \min W(x,\cdot) = 0$ (**reference configuration = natural state**)
- $W(x,F) \geq \alpha\, \mathrm{dist}^2(F,\mathrm{SO}(d))$ $\forall F \in \mathbb{R}^{d\times d}$ (**non-degeneracy**)
- $W(x,\cdot) \in C^3$ in neighbourhood of $\mathrm{SO}(d)$.

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Lemma (Existence and bounds on correctors for small strains) [N.,Schäffner, Varga AIHP 24]. There exists $\rho > 0$ s.t. for all $F \in U_\rho := \{F \,:\, \mathrm{dist}(F,\mathrm{SO}(d)) \leq \rho\}$ there exists a sublinear corrector ϕ_F such that

$$W_{\mathrm{hom}}(F) = \lim_{L\to\infty} \min_{\varphi\in W_0^{1,p}(B_L)} \fint_{B_L} W(y,F+\nabla\varphi) = \mathbb{E}[W(F+\nabla\phi_F)].$$

Moreover, we have the sublinear growth estimate

$$\mu(s), \tilde{\mu}(s) \leq s^{\frac{1}{2}}.$$

- [Fischer, Neukamm ARMA 21]: We proved optimal order estimate for bounded and unbounded domains

$$\|\nabla \hat{u}_\varepsilon - \nabla u_\varepsilon\|_{L^2(O)} \leq \mathscr{C}_\varepsilon \begin{cases} O \text{ bdd} & O = \mathbb{R}^d & \\ \varepsilon^{\frac{1}{3}} & \varepsilon^{\frac{1}{2}} & d = 1, \\ \varepsilon^{\frac{1}{2}} |\log(\varepsilon)|^{\frac{1}{2}} & \varepsilon |\log(\varepsilon)|^{\frac{1}{2}} & d = 2, \\ \varepsilon^{\frac{1}{2}} & \varepsilon & d \geq 3 \end{cases}$$

  with $\mathscr{C}_\varepsilon$ having stretched exponential moments independent of ε.
- [Armstrong, Ferguson, Kuusi ARMA 20, CPAM 21]: Corrector bounds with suboptimal order, but optimal stochastic integrability.
- [Clozeau, Gloria ARMA 23]: Extensions to monotone operators of the form $-\nabla \cdot A(1 + |\nabla u|^{p-2})\nabla u$ with $2 \leq p$
- Open: Strong correlations. (Linear case: [Gloria, Neukamm, Otto APDE 23])
- Open: Boundary correctors. (Linear case: [Josien,Raithel, Schäffner SIMA 24])
- Open: Nonlinear elasticity. (Random laminates: [Neukamm, Schäffner, Varga AIHP 24])
- Open: Gradient flow, fluctuations, ...

Lemma. Let $O = \mathbb{R}^d$ or a bounded C^1-domain or a bounded, convex Lipschitz domain. For all $0 < \delta \le c_O$ there exists

$$\{\eta_k\}_{k\in K} \subset C_c^\infty(\mathbb{R}^d) \qquad \text{with} \qquad K \subset \overline{O} \text{ at most countable,}$$

such that for all $k \in K$,

$$\sum_{k\in K} \eta_k = 1 \quad \text{in } O, \qquad 0 \le \eta_k \le 1, \qquad \delta|\nabla \eta_k|_\infty \le c_O, \tag{1a}$$

$$\operatorname{supp} \eta_k \subset B_{2\sqrt{d}\delta}(k), \qquad \frac{1}{c_O}\delta^d \le \int_O \eta_k \le c_O\delta^d, \tag{1b}$$

and $\#(K \cap B_{4\sqrt{d}\delta}(k)) \le c_O$. For all $g \in H^1(O)$ the local averages $\xi_k := \frac{1}{\int_O \eta_k\,dx}\int_O g(x)\eta_k(x)\,dx$ satisfies

$$|\xi_k| \le C\left(\delta^{-d}\int_{B_{2\sqrt{d}\delta}(k)\cap O} |g|^2\,dx\right)^{\frac{1}{2}}, \tag{2a}$$

$$\left(\int_{B_{2\delta}(k)\cap O} |\xi_k - g|^2\,dx\right)^{\frac{1}{2}} \le C\delta\left(\int_{B_{2\sqrt{d}\delta}(k)\cap O} |\nabla g|^2\,dx\right)^{\frac{1}{2}}, \tag{2b}$$

Moreover, if $\operatorname{supp}(\eta_k) \cap \operatorname{supp}(\eta_{\tilde{k}}) \ne \emptyset$, then

$$|\xi_k - \xi_{\tilde{k}}| \le C\delta\left(\delta^{-d}\int_{B_{6\delta}(\tilde{k})} |\nabla g|^2\,dx\right)^{\frac{1}{2}}. \tag{2c}$$