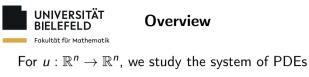


Boundary regularity for systems with symmetric gradients

Linus Behn joint work with Lars Diening

MPDE, September 2024 Prague



 $-\operatorname{div}(A(\varepsilon u)) = f$ in Ω , u = 0 on $\partial \Omega$.

The system depends on the symmetric gradient $\varepsilon u := \frac{1}{2} (\nabla u + (\nabla u)^T)$.

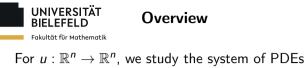
The nonlinear map A is given by, e.g.,

$$A(\varepsilon u) := (\delta + |\varepsilon u|)^{p-2} \varepsilon u,$$

where $p \in (1, \infty)$ and $\delta \geq 0$.

Application: Elasticity theory, model problem for the *p*-Stokes system

Goal: Regularity of the solution u near the boundary $\partial \Omega$.



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Application: Elasticity theory, model problem for the *p*-Stokes system

Goal: Regularity of the solution u near the boundary $\partial \Omega$.

Known results: *p*-Laplace system

$$u: \mathbb{R}^n \to \mathbb{R}^n \qquad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

If f and Ω are regular enough, then

$$u \in C^{1,\alpha}(\overline{\Omega})$$
 and $V(\nabla u) := |\nabla u|^{\frac{p-2}{2}} \nabla u \in W^{1,2}(\Omega)$

was shown by

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- Uhlenbeck; 1977: $2 \le p < \infty$,
- Acerbi, Fusco; 1989: 1 < p < 2,
- DiBenedetto, Chen; 1989: Up to the boundary,
- Diening, Stroffolini, Verde; 2009: For Orlicz-growth.

Balci, Cianchi, Diening, Maz'ya; 2022: $|\nabla u|^{p-2} \nabla u \in W^{1,2}(\Omega)$ for p > 1.1715...



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$$-\operatorname{div}(|\varepsilon u|^{p-2}\varepsilon u) = f \quad \text{in } \Omega,$$
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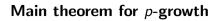
Known results:

•
$$V(\varepsilon u) := |\varepsilon u|^{(p-2)/2} \varepsilon u \in W^{1,2}(\Omega)$$
:

For 1 near a flat boundary: Seregin, Shilkin; 1997 $Extended to <math>\partial \Omega \in C^{2,1}$ and parabolic systems: Berselli, Růžička; 2017-2022

Open questions:

•
$$u \in C^{1,\alpha}(\Omega)$$
 (for $n \ge 3$)?
• $|\varepsilon u|^{p-2}\varepsilon u \in W^{1,2}(\Omega)$ (for $p \ne 2$)?.



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Theorem (Behn, Diening '24)

Let Ω be a bounded $C^{2,1}$ -domain, $1 and <math>f \in W_0^{1,p'}(\Omega)$. Then the system $-\operatorname{div}(|\varepsilon u|^{p-2}\varepsilon u) = f$ in Ω , u = 0 on $\partial\Omega$ has a unique weak solution $u \in W^{1,p}(\Omega)$ fulfilling $V(\varepsilon u) := |\varepsilon u|^{\frac{p-2}{2}}\varepsilon u \in W^{1,2}(\Omega)$ and $\|V(\varepsilon u)\|_{W^{1,2}(\Omega)}^2 \lesssim \|f\|_{W^{1,p'}(\Omega)}^{p'}$.

•
$$p \in (1,2)$$
: $u \in W^{2,\frac{np}{n+p-2}}(\Omega)$
• $p \in (2,\infty)$: $u \in W^{1+\frac{2}{p},p}(\Omega)$

Main theorem for Orlicz growth

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We replace
$$\mathcal{J}_p(u) \coloneqq \frac{1}{p} \int_{\Omega} |\varepsilon u|^p dx$$
 by energies with more general φ -growth
$$\mathcal{J}_{\varphi}(u) \coloneqq \int_{\Omega} \varphi(|\varepsilon u|) dx.$$

Theorem (Behn, Diening '24)

Let Ω be a bounded $C^{2,1}$ -domain, φ a uniformly convex N-function and $f \in W_0^{1,\varphi^*}(\Omega)$. Then the system $-\operatorname{div}(A(\varepsilon u)) = f$ in Ω , u = 0 on $\partial\Omega$

has a unique weak solution $u \in W^{1,\varphi}(\Omega)$ fulfilling $V(\varepsilon u) \in W^{1,2}(\Omega)$ and

$$\|V(\varepsilon u)\|^2_{W^{1,2}(\Omega)} \lesssim \int_{\Omega} \varphi^*(|f|) + \varphi^*(|\nabla f|) dx.$$

$$A(Q) := arphi'(|Q|)rac{Q}{|Q|}, \qquad V(Q) := \sqrt{arphi'(|Q|)|Q|}rac{Q}{|Q|}$$

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Boundary regularity for systems with symmetric gradients

Korn inequality

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Clearly, $|\varepsilon u(x)| \leq |\nabla u(x)|$. However, $|\nabla u(x)| \nleq c |\varepsilon u(x)|$.

Theorem (Korn's inequality)

Let $p \in (1,\infty)$. Then there exists c > 0 such that for all $u \in C_c^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$ we have $\|\nabla u\|_{L^p(\mathbb{R}^n)} \le c \|\varepsilon u\|_{L^p(\mathbb{R}^n)} \le c \|\nabla u\|_{L^p(\mathbb{R}^n)}.$

For p = 1 and $p = \infty$ this inequality fails to hold (Ornstein, 1962).

Thus (informally):

There are no problems coming from the function spaces involved!

Instead, the difficulty lies in the higher coupling of the system.

Korn inequality

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Coupling of the equation

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For p = 2:

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• $-\Delta u = f \iff -\Delta u_i = f_i \quad \forall \ 1 \le i \le n$ (system decouples)

• $-\Delta^{\text{sym}}u := -\operatorname{div}(\varepsilon u) = -\frac{1}{2}\Delta u - \frac{1}{2}\nabla \operatorname{div} u$ (system is highly coupled)

For $-\Delta u = f \in L^2$ we can show $u \in W^{2,2}$ near the boundary by • For tangential directions α : Testing with $\partial_{\alpha}\partial_{\alpha}u$. This gives $\partial_{\alpha}\nabla u \in L^2$. • Does not work for the normal direction $\partial_n \nabla u$!

Oblique Using the PDE:
$$\partial_n \partial_n u = f - \sum_{\alpha} \partial_{\alpha} \partial_{\alpha} u \in L^2$$
.

Main difficulty: The high coupling and nonlinear nature of the system.

To overcome this, we rely on the algebraic identity $\partial_{ij}u_k = \partial_i \varepsilon_{jk}u + \partial_j \varepsilon_{ik}u - \partial_k \varepsilon_{ij}u$.

UNIVERSITÄT Coupling of the equation BIEL FEEL D Fakultät für Mathematik For p = 2: • $-\Delta u = f \iff -\Delta u_i = f_i \quad \forall \ 1 < i < n$ (system decouples) • $-\Delta^{\text{sym}}u := -\operatorname{div}(\varepsilon u) = -\frac{1}{2}\Delta u - \frac{1}{2}\nabla\operatorname{div} u$ (system is highly coupled) For $-\Delta u = f \in L^2$ we can show $u \in W^{2,2}$ near the boundary by For tangential directions α : Testing with $\partial_{\alpha}\partial_{\alpha}u$. This gives $\partial_{\alpha}\nabla u \in L^2$. Does not work for the normal direction $\partial_n \nabla u$!

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Summary

Theorem (Behn, Diening '24)

Let Ω be a bounded $C^{2,1}$ -domain, φ a uniformly convex N-function and $f \in W_0^{1,\varphi^*}(\Omega)$. Then the system

$$-\operatorname{div}(A(\varepsilon u)) = f$$
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For p-growth:
$$p \leq 2$$
: $u \in W^{2,\frac{np}{n+p-2}}$, $p \geq 2$: $u \in W^{1+\frac{2}{p},p}$.

Thank you for your attention!

Linus Behn

Boundary regularity for systems with symmetric gradients