Long time behavior of the solution to a Stochastic Navier–Stokes–Allen–Cahn System with Singular Potential

Based on a joint work with A. Di Primio and L. Scarpa

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The deterministic 2D Allen–Cahn–Navier–Stokes equations

Let $\mathcal{O}$ be a bounded smooth domain in $\mathbb{R}^2$ and $T > 0$. Consider the Allen–Cahn–Navier–Stokes system:

$$\begin{cases} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi - \mu \nabla \varphi = 0 & \text{in } \mathcal{O} \times (0,T), \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \mathcal{O} \times (0,T), \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + \mu = 0 & \text{in } \mathcal{O} \times (0,T), \\ \mu = -\beta \Delta \varphi + F'(\varphi) & \text{in } \mathcal{O} \times (0,T), \\ \boldsymbol{u} = 0, \quad \varphi = 0 & \text{on } \partial\mathcal{O} \times (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \varphi(0) = \varphi_0 & \text{in } \mathcal{O}. \end{cases}$$

- $\boldsymbol{u}$ = velocity, π = pressure,
- φ = order parameter (difference of concentrations), $\{\varphi = \pm 1\} \to$ pure phases, $\varphi \in (-1,1) \to$ some mixing takes place.
- $\nu, \beta > 0$.
- Flory–Huggins potential ($0 < \theta < \theta_c$)
  $$F(s) = \frac{\theta}{2}[(1+s)\log(1+s) + (1-s)\log(1-s)] - \frac{\theta_c}{2}s^2.$$

The stochastic 2D Allen–Cahn–Navier–Stokes equations

Let $\mathcal{O}$ be a bounded smooth domain in $\mathbb{R}^2$, $T > 0$.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a normal filtered probability space.

$$
\begin{cases}
\mathrm{d}\boldsymbol{u} + [-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi - \mu\nabla\varphi]\,\mathrm{d}t = G_1\,\mathrm{d}W_1 & \text{in } \mathcal{O}\times(0,T), \\
\nabla\cdot\boldsymbol{u} = 0 & \text{in } \mathcal{O}\times(0,T), \\
\mathrm{d}\varphi + [\boldsymbol{u}\cdot\nabla\varphi + \mu]\,\mathrm{d}t = G_2(\varphi)\,\mathrm{d}W_2 & \text{in } \mathcal{O}\times(0,T), \\
\mu = -\beta\Delta\varphi + F'(\varphi) & \text{in } \mathcal{O}\times(0,T), \\
\boldsymbol{u} = 0, \quad \varphi = 0 & \text{on } \partial\mathcal{O}\times(0,T), \\
\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \varphi(0) = \varphi_0 & \text{in } \mathcal{O}.
\end{cases}
\tag{1}
$$

W_1 and W_2 are two independent cylindrical Wiener processes with values in the separable Hilbert spaces U_1 and U_2, respectively, defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$.

We consider an additive noise in the NS equation, a multiplicative noise in the AC equation, i.e.

$$
G_1 \in \mathscr{L}_{HS}(U_1; \boldsymbol{L}^2_{\mathsf{div}}(\mathcal{O})),
$$
$$
G_2 : \mathcal{B}_1^\infty := \{y \in L^\infty(\mathcal{O}) \ : \|y\|_{L^\infty(\mathcal{O})} \le 1\} \to \mathscr{L}_{HS}(U_2; L^2(\mathcal{O})).
$$

Assumptions: noise driving the Allen–Cahn equation

Assumption (Allen–Cahn diffusion)

Given an orthonormal basis $\{u_k^2\}_{k\in\mathbb{N}_+}$ *of* U_2*, the diffusion coefficient* $G_2 : \mathcal{B}_1^\infty \to \mathscr{L}_{HS}(U_2, H)$ *satisfies*

$$G_2(\psi)[u_k^2] = g_k(\psi) \qquad \forall\, k \in \mathbb{N}_+ \quad \forall\, \psi \in \mathcal{B}_1^\infty\,,$$

where the sequence $\{g_k\}_{k\in\mathbb{N}_+} \subset W^{1,\infty}(-1,1)$ *is such that*

$$g_k(\pm 1) = 0\,, \quad F'' g_k^2 \in L^\infty(-1,1) \qquad \forall\, k \in \mathbb{N}_+\,,$$

and

$$\sum_{k=1}^{\infty} \left(\|g_k\|^2_{W^{1,\infty}(-1,1)} + \left\|F'' g_k^2\right\|_{L^\infty(-1,1)} \right) < +\infty\,.$$

Well posedness results [1]

For every initial datum $(\boldsymbol{u}_0, \varphi_0)$ satisfying

(i) $\boldsymbol{u}_0 \in L^p(\Omega; \boldsymbol{L}^2_{\mathsf{div}}(\mathcal{O}))$;

(ii) $\varphi_0 \in L^p(\Omega; H^1_0(\mathcal{O}))$;

(iii) $F(\varphi_0) \in L^{\frac{p}{2}}(\Omega; L^1(\mathcal{O}))$,

with $p > 2$, there exists a unique (analytically-weak) probabilistically-strong solution $(\boldsymbol{u}, \varphi)$ for problem (1) such that,

$$\boldsymbol{u} \in L^p(\Omega; L^\infty(0,T; \boldsymbol{L}^2_{\mathsf{div}}(\mathcal{O}))) \cap L^p(\Omega; L^2(0,T; \boldsymbol{H}^1_{\mathsf{div}}(\mathcal{O}))),$$

$$\varphi \in L^p(\Omega; C^0([0,T]; L^2(\mathcal{O}))) \cap L^p(\Omega; L^\infty(0,T; H^1_0(\mathcal{O}))) \cap L^p(\Omega; L^2(0,T; H^2(\mathcal{O}))),$$

$$|\varphi| < 1 \quad \text{a.e. in } \Omega \times \mathcal{O} \times [0,T],$$

$$w := -\beta\Delta\varphi + F'(\varphi) \in L^p(\Omega; L^2(0,T; H)),$$

$$(\boldsymbol{u}(0), \varphi(0)) = (\boldsymbol{u}_0, \varphi_0).$$

[1] A. Di Primio, M. Grasselli, and L. Scarpa. *A stochastic Allen-Cahn-Navier-Stokes system with singular potential.* J. Differential Equations 387 (2024), pp. 378–431.

Invariant measures. An intuitive idea: energy balance

The energy of the system is given by

$$\mathcal{E}(\boldsymbol{u},\varphi) := \frac{1}{2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O})} + \frac{\beta}{2}\|\nabla\varphi\|^2_{L^2(\mathcal{O})} + \int_{\mathcal{O}} F(\varphi).$$

For every $t \geq 0$, we obtain the (mean) energy equality

$$\begin{aligned}
&\frac{1}{2}\,\mathbb{E}\,\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O})} + \frac{\beta}{2}\,\mathbb{E}\,\|\nabla\varphi(t)\|^2_{L^2(\mathcal{O})} + \mathbb{E}\,\|F(\varphi(t))\|_{L^1(\mathcal{O})} \\
&+ \mathbb{E}\int_0^t \Big[\nu\|\nabla\boldsymbol{u}(s)\|^2_{\boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O})} + \beta^2\,\mathbb{E}\,\|\Delta\varphi(s)\|^2_{L^2(\mathcal{O})} + \mathbb{E}\,\|F'(\varphi(s))\|^2_{L^2(\mathcal{O})}\Big]\,\mathrm{d}s \\
= &\frac{1}{2}\,\mathbb{E}\,\|\boldsymbol{u}_0\|^2_{\boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O})} + \frac{\beta}{2}\,\mathbb{E}\,\|\nabla\varphi_0\|^2_{L^2(\mathcal{O})} + \mathbb{E}\,\|F(\varphi_0)\|_{L^1(\mathcal{O})} + \frac{t}{2}\|G_1\|^2_{\mathscr{L}_{HS}(U_1,\boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O}))} \\
&\qquad \frac{1}{2}\,\mathbb{E}\int_0^t \sum_{k\in\mathbb{N}}\int_{\mathcal{O}} \Big[\beta|g_k'(\varphi(s))\nabla\varphi(s)|^2 + F''(\varphi(s))|g_k(\varphi(s))|^2\Big]\,\mathrm{d}s.
\end{aligned}$$

- We expect that the energy injected by the noise is dissipated by the (dissipative) deterministic part of the system.
- If there is some balance it is meaningful to look for invariant measures, that is a statistical equilibrium of the system.
- Questions: existence, uniqueness, asymptotic stability of invariant measures.

Invariant measures

Set $\boldsymbol{X} := \boldsymbol{L}^2_{\mathrm{div}}(\mathcal{O})$, $\qquad Y := H^1_0(\mathcal{O}) \cap \{y \in L^\infty(\mathcal{O}) \ : \|y\|_{L^\infty} \le 1\}$.

The solution of (1) is a stochastic process, in particular, for $t \ge 0$,

$$(\boldsymbol{u}^{\boldsymbol{u}_0}(t), \varphi^{\varphi_0}(t)) : (\Omega, \mathscr{F}, \mathbb{P}) \to (\boldsymbol{X} \times Y, \mathcal{B}(\boldsymbol{X} \times Y)) \quad \text{is a random variable.}$$

We can consider the pushforward measure

$$\mathrm{Law}_{\mathbb{P}}(\boldsymbol{u}^{\boldsymbol{u}_0}(t), \varphi^{\varphi_0}(t))(A) = \mathbb{P}((\boldsymbol{u}^{\boldsymbol{u}_0}(t), \varphi^{\varphi_0}(t)) \in A), \qquad A \in \mathcal{B}(\boldsymbol{X} \times Y)$$

and study the evolution in time of this probability measure.
Formally, we introduce the semigroup of operators $P^* := (P^*_t)_{t \ge 0}$ as

$$P^*_t : \mathscr{P}(\boldsymbol{X} \times Y) \to \mathscr{P}(\boldsymbol{X} \times Y), \quad P^*_t \mu(A) := \int_{\boldsymbol{X} \times Y} \mathrm{Law}_{\mathbb{P}}(\boldsymbol{u}^{\boldsymbol{x}}(t), \varphi^{y}(t))(A) \, \mu(\mathrm{d}\boldsymbol{x}, \mathrm{d}y).$$

In particular, for $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{X} \times Y$, $P^*_t \delta_{(\boldsymbol{u}_0, \varphi_0)}(A) = \mathrm{Law}_{\mathbb{P}}(\boldsymbol{u}^{\boldsymbol{u}_0}(t), \varphi^{\varphi_0}(t))(A)$.

Definition (Invariant measure)

An invariant measure for (1) *is a probability measure* ϑ *on* $\boldsymbol{X} \times Y$ *s.t.*

$$P^*_t \vartheta(A) = \vartheta(A), \quad \forall\, t \ge 0, \quad \forall\, A \in \mathcal{B}(\boldsymbol{X} \times Y).$$

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Existence and uniqueness of the invariant measure: the stochastic 2D Navier-Stokes equations (additive noise)

To get existence of an invariant measure usually one works under the same assumptions that ensure the existence and uniqueness of solutions.

To get uniqueness and asymptotic stability of the invariant measure one can work under different assumptions on the noise:

- **elliptic case:** the noise acts on an infinite number of modes,
- **effectively elliptic case:** the noise acts on a (finite) sufficiently large number of modes. Then noise forces the unstable direction of the system in the spirit of Foias and Prodi [2],
- **hypoelliptic case:** the noise acts on two modes and the nonlinear term spread the randomness into the system [3].

[2] C. Foias and G. Prodi. *Sur le comportement global des solutions non-stationnaires des equations de Navier-Stokes en dimension 2.* Rend. Sem. Mat. Univ. Padova, 39:1–34, 1967.

[3] M. Hairer and J. C. Mattingly. *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing.* Ann. of Math. (2), 164(3):993–1032, 2006.

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Main results [4]

Theorem (Existence, uniqueness and asymptotic stability of invariant measures)

There exists at least one invariant measure for system (1)*. Moreover, for every* $\nu > 0$*, there exist a positive integer* $\bar{N}$ *and a positive real number* $\bar{\beta}$*, sufficiently large, such that, if* $\beta \geq \bar{\beta}$ *and* $Rg\ G_1 \supseteq P_N[\boldsymbol{L}^2_{div}(\mathcal{O})]$*, for some* $N \geq \bar{N}$*, then there exists at most one invariant measure* ϑ *for system* (1) *and it is asympotically stable, that is*

$$P_t^* \delta_{(\boldsymbol{u}_0, \varphi_0)} \rightharpoonup \vartheta, \quad \textit{for} \quad \forall\, (\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{X} \times Y, \quad \textit{as} \quad t \to \infty.$$

- **Effectively elliptic setting** for the NS equations: the noise acts in a non-degenerate way on the (finite) number of unstable directions of the system,

$$G_1 \mathrm{d}W(t) = \sum_{k=1}^{N} \sigma_k e_k \, \mathrm{d}\beta_k(t), \qquad \sigma_k > 0.$$

- **Large viscosity regime** for the AC equation (here the noise is degenerate!)

[4] A. Di Primio, L. Scarpa, M. Z., *Existence, uniqueness and asymptotic stability of invariant measures for the stochastic Allen-Cahn-Navier-Stokes system with singular potential*, arXiv:2501.06174, 2025

Some open problems

- Uniqueness and asymptotic stability of the invariant measure in the hypoelliptic setting,
- speed of convergence to the invariant measure,
- what noise assumptions in the Allen-Cahn equation to remove the large dissipation condition (β sufficiently large)?

Thank you for your attention!