The role of randomness in stochastic phase-field modelling

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Outline

Phase field modelling

Phase-field approach to phase-separation:

- evolution of a material with 2 different features, mixtures, ...
- phase variable $\varphi \in [-1, 1]$: pure phases are $\{\varphi = 1\}$ and $\{\varphi = -1\}$
- intermediate values $\varphi \in (-1, 1)$ are allowed: diffuse (narrow) interface

Advantages with respect to sharp interface models:

- no need of boundary conditions at the interface
- no need to track the interface at all times
- no difficulties in case of topological changes

Applications:

- physics and engineering: phase separation, alloys, fluid mixtures, ...
- biology: tumour growth dynamics, RNA-protein formation, ...
- ...

Deterministic approach to phase-field models

Given a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^d$, $d \in \{2,3\}$, one considers the free-energy functional

$$\mathcal{E}(\varphi) := \frac{1}{2}\int_{\mathcal{O}} |\nabla\varphi|^2 + \int_{\mathcal{O}} F(\varphi),$$

along with its variational derivative (chemical potential)

$$\mu = -\Delta\varphi + F'(\varphi).$$

Deterministic approach:
neglect other possible energy contributions (temperature, magnetic, vibrational effects)

Allen-Cahn model:

- $\partial_t\varphi + \mu = 0$
- no-flux boundary conditions for φ
- no mass-conservation

Cahn-Hilliard model:

- $\partial_t\varphi - \Delta\mu = 0$
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Classical Allen-Cahn and Cahn-Hilliard equations

Typical form of the AC equation:

$$\begin{cases} \partial_t \varphi - \Delta\varphi + F'(\varphi) = 0 & \text{in } (0,T)\times\mathcal{O}, \\ \partial_{\mathbf{n}}\varphi = 0 & \text{in } (0,T)\times\partial\mathcal{O}, \\ \varphi(0) = \varphi_0 & \text{in } \mathcal{O}. \end{cases}$$

Typical form of the CH equation:

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Physically-relevant choice of double-well potential:

$$F_{log}(r) := \frac{\theta}{2}\left[(1+r)\ln(1+r) + (1-r)\ln(1-r)\right] - \frac{\theta_0}{2}r^2, \quad r\in[-1,1], \qquad 0<\theta<\theta_0.$$

Initial datum with finite energy:

$$\varphi_0 \in H^1(\mathcal{O}), \qquad |\varphi_0| \le 1 \text{ a.e. in } \mathcal{O}.$$

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Stochastic approach to phase-field models

Phase separation may be affected by:

- unpredictable temperature oscillations, microscopic movements, magnetic effects, . . .
- measurement uncertainty, environmental disturbance, . . .

Possible solutions (Cook, 1970):

- deterministic model is too "simple" to capture such background noise
- addition of a suitable stochastic forcing in the equation: gaussian noise

General form of the stochastic Allen-Cahn ($\alpha = 0$) and Cahn-Hilliard ($\alpha = 1$) equations:

$$\begin{cases} d\varphi + (1-\alpha)\mu\, dt - \alpha\Delta\mu\, dt = G(\varphi)\, dW & \text{in } (0,T)\times\mathcal{O}\,,\\ \mu = -\Delta\varphi + F'(\varphi) & \text{in } (0,T)\times\mathcal{O}\,,\\ \partial_{\mathbf{n}}\varphi = \alpha\partial_{\mathbf{n}}\mu = 0 & \text{in } (0,T)\times\partial\mathcal{O}\,,\\ \varphi(0) = \varphi_0 & \text{in } \mathcal{O}\,. \end{cases}$$

Stochastic forcing:

- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$: filtered probability space
- W: cylindrical Wiener process on a abstract Hilbert space U i.e.

$$W = \sum_{k=0}^{\infty} \beta_k e_k\,, \qquad (e_k)_k \text{ c.o.s. of } U\,, \qquad (\beta_k)_k \text{ independent Brownian motions}\,.$$

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Choice of the noise

Noise coefficient: Hilbert-Schmidt from a abstract space U to the reference space $L^2(\mathcal{O})$, i.e.

$$G(\varphi) \in \mathscr{L}^2(U, L^2(\mathcal{O})) \quad \forall\, \varphi \in L^2(\mathcal{O})\,.$$

Possible choice of the noise:

- $G : L^2(\mathcal{O}) \to \mathscr{L}^2(U, L^2(\mathcal{O}))$ given by

$$G(\varphi)[e_k] = g_k(\varphi)\,, \quad \text{with} \quad (g_k)_k \subset C^{0,1}([-1,1])$$

- structure:

$$g_k(\pm 1) = 0\,, \quad F'' g_k^2 \in L^\infty(-1,1) \qquad \text{for all } k \in \mathbb{N}$$

- Hilbert-Schmidt condition:

$$\sum_{k\in\mathbb{N}} \left(\|g_k\|^2_{C^{0,1}([-1,1])} + \|F'' g_k^2\|_{L^\infty(-1,1)} \right) < +\infty$$

- interpretation: $G(\varphi)$ behaves as a degenerate mobility-type function
- advantage: G is of "local" type (superposition operator)
- drawback: no mass-conservation
- modelling: random forcing term giving mass proliferation (Allen-Cahn)

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Stochastic Allen-Cahn equation

Theorem (Bertacco, Orrieri, **S.** (2020-2023))

In this setting, for every $\varphi_0 \in H^1(\mathcal{O})$ with $\|\varphi_0\|_{L^\infty(\mathcal{O})} \leq 1$, there exists a unique process

$$\varphi \in L^2(\Omega; C^0([0,T]; L^2(\mathcal{O})) \cap L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2_n(\mathcal{O}))),$$
$$|\varphi| < 1 \quad \text{a.e. in } \Omega \times (0,T) \times \mathcal{O},$$

such that for every $t \in [0,T]$, $\mathbb{P}$-almost surely,

$$\varphi(t) - \int_0^t \Delta\varphi(s)\,ds + \int_0^t F'_{log}(\varphi(s))v\,ds = \varphi_0 + \int_0^t G(\varphi(s))\,dW(s) \quad \text{in } L^2(\mathcal{O}).$$

Moreover, the solution map $\varphi_0 \mapsto \varphi$ is Lipschitz-continuous as a map

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If also $\varphi_0 \in H^2_n(\mathcal{O})$ satisfies $\|\varphi_0\|_{L^\infty(\mathcal{O})} < 1$, then a random separation property holds:

$$\mathbb{P}\left(\exists\,\delta \in (0,1) : \sup_{(t,x)\in[0,T]\times\overline{\mathcal{O}}} |\varphi(t,x)| \leq 1-\delta\right) = 1.$$

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In this setting, for every $\varphi_0 \in H^1(\mathcal{O})$ *with* $\|\varphi_0\|_{L^\infty(\mathcal{O})} \leq 1$*, there exists a unique process*

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Formal energy balance

Itô formula yields the formal energy balance:

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{O}}|\nabla\varphi(t)|^2+\int_{\mathcal{O}}F(\varphi(t))+\int_0^t\int_{\mathcal{O}}|\mu(s)|^2\,ds\\&\quad=\frac{1}{2}\int_{\mathcal{O}}|\nabla\varphi_0|^2+\int_{\mathcal{O}}F(\varphi_0)+\int_0^t(\mu(s),G(\varphi(s))\,dW(s))\\&\qquad+\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}|g_k'(\varphi(s))\nabla\varphi(s)|^2\,ds+\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}F''(\varphi(s))|g_k(\varphi(s))|^2\,ds\,.\end{aligned}$$

Extra terms:

$\int_0^t(\mu(s),G(\varphi(s))\,dW(s))$: first order martingale term,

$\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}|g_k'(\varphi(s))\nabla\varphi(s)|^2\,ds$ second order proliferation term (gradient contribution),

$\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}F''(\varphi(s))|g_k(\varphi(s))|^2\,ds$ second order proliferation term (potential contribution).

ISSUE: blow-up of F'' should be compensated by degeneracy of G!

Formal energy balance

Itô formula yields the formal energy balance:

$$\frac{1}{2}\int_{\mathcal{O}}|\nabla\varphi(t)|^2+\int_{\mathcal{O}}F(\varphi(t))+\int_0^t\int_{\mathcal{O}}|\mu(s)|^2\,ds$$
$$=\frac{1}{2}\int_{\mathcal{O}}|\nabla\varphi_0|^2+\int_{\mathcal{O}}F(\varphi_0)+\int_0^t(\mu(s),G(\varphi(s))\,dW(s))$$
$$+\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}|g_k'(\varphi(s))\nabla\varphi(s)|^2\,ds+\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}F''(\varphi(s))|g_k(\varphi(s))|^2\,ds\,.$$

Extra terms:

$$\int_0^t(\mu(s),G(\varphi(s))\,dW(s)):$$ first order martingale term,

$$\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}|g_k'(\varphi(s))\nabla\varphi(s)|^2\,ds$$ second order proliferation term (gradient contribution),

$$\frac{1}{2}\int_0^t\sum_{k=0}^{\infty}\int_{\mathcal{O}}F''(\varphi(s))|g_k(\varphi(s))|^2\,ds$$ second order proliferation term (potential contribution).

ISSUE: blow-up of F'' should be compensated by degeneracy of G!

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Selection of ongoing and future projects

1) Stochastic conservative Allen-Cahn equation with conservative noise

Divergence-form noise:
$$\begin{cases} d\varphi + (\mu - \overline{\mu})\,dt = -\operatorname{div}\mathbf{G}(\varphi)\,dW & \text{in } (0,T)\times\mathbb{T}, \\ \mu = -\Delta\varphi + F'(\varphi) & \text{in } (0,T)\times\mathbb{T}, \\ \varphi(0) = \varphi_0 & \text{in } \mathbb{T}, \end{cases} \tag{4.1}$$

Zero-order noise:
$$\begin{cases} d\varphi + (\mu - \overline{\mu})\,dt = (G(\varphi) - \overline{G(\varphi)})\,dW & \text{in } (0,T)\times\mathbb{T}, \\ \mu = -\Delta\varphi + F'(\varphi) & \text{in } (0,T)\times\mathbb{T}, \\ \varphi(0) = \varphi_0 & \text{in } \mathbb{T}, \end{cases} \tag{4.2}$$

- more coherent with the model derivation
- (4.1): treated in collaboration with Andrea Di Primio and Maurizio Grasselli
- (4.2): energy proliferation reads as

$$\frac{1}{2}\mathbb{E}\int_0^t \sum_{k=0}^{\infty}\int_{\mathbb{T}} |F''(\varphi(s))||g_k(\varphi(s)) - \overline{g_k(\varphi(s))}|^2\,ds$$

- problem: nonlocal term prevents to control the blow-up of F'' locally at ± 1

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- behaviour of the law of the random variable $\varphi(t) : \Omega \to L^2(\mathcal{O})$, as $t \to +\infty$
- existence uniqueness of an invariant measure ν
- convergence of the laws towards ν
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3) Coupled stochastic Allen-Cahn-Navier-Stokes systems

$$\begin{cases} d\mathbf{u} - \Delta \mathbf{u}\, dt + (\mathbf{u} \cdot \nabla)\mathbf{u}\, dt = \mu \nabla\varphi\, dt + G_1(\mathbf{u})\, dW & \text{in } (0,T) \times \mathcal{O}, \\ d\varphi + \mu\, dt + \mathbf{u} \cdot \nabla\varphi\, dt = G_2(\varphi)\, dW & \text{in } (0,T) \times \mathcal{O}, \\ \mu = -\Delta\varphi + F'(\varphi) & \text{in } (0,T) \times \mathcal{O}, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0 & \text{in } \mathcal{O}, \end{cases} \tag{4.3}$$

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- m: mobility function, degenerate ($m(x) = 1 - x^2$) or non-degenerate ($m(x) = 1$)
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A long-time future project

5) Uniqueness by noise for phase-field models

Ill-posed deterministic evolution equation: $\partial_t u + Au + B(u) = 0.$

$$\Downarrow$$

Well-posed stochastic evolution equation: $du + Au\,dt + B(u)\,dt = G\,dW.$

- uniqueness by noise: well-known in finite dimensions
- infinite dimensions with perturbation B of order zero: Da Prato, Flandoli, Priola, Röckner (2013)
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- infinite dimensions for doubly-nonlinear equations: Orrieri, **S.**, Stefanelli (ongoing)

- idea: restore uniqueness by noise for ill-posed phase-field models: **local Cahn-Hilliard with degenerate mobility and logarithmic potential!**
- modelling: validation of the original Cahn-Hilliard separation model
- maths: regularising properties of degenerate elliptic operators in infinite dimensions
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Another long-time future project

6) Joint hydrodynamic and nonlocal-to-local convergence for Cahn-Hilliard

Microscopic derivation of the nonlocal Cahn-Hilliard equation:

- Giacomin, Lebowitz (1997): probabilistic limit
- from lattice gas model to nonlocal Cahn-Hilliard equation ($n \to \infty$)

Convergence of the nonlocal Cahn-Hilliard equation to the local one:

- Davoli, **S.**, Trussardi (2020-2021): constant mobility and singular potential
- Abels, Terasawa (2022): coupled Cahn-Hilliard-Navier-Stokes
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- interaction kernels suitably scaled around a Dirac delta ($\varepsilon \to 0$)

Possible idea for microscopic derivation of the local Cahn-Hilliard equation:

- lattice gas (n particles) with scaled interaction potential (ε)
- joint limiting procedure as $n \to \infty$ and $\varepsilon \to 0$.

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Another long-time future project

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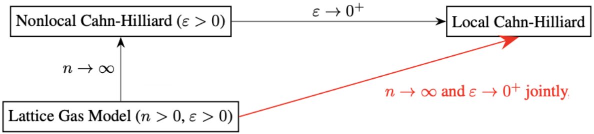

References:

- L. Scarpa. **The stochastic Cahn-Hilliard equation with degenerate mobility and logarithmic potential.** *Nonlinearity* 34 (2021), no. 6, 3813–3857.
- F. Bertacco. **Stochastic Allen–Cahn equation with logarithmic potential.** *Nonlinear Anal.* 202 (2021), 112122, 1–22.
- F. Bertacco, C. Orrieri, L. Scarpa. **Random separation property for stochastic Allen-Cahn-type equations.** *Electron. J. Probab.* 27 (2022), 1–32.
- E. Davoli, L. Scarpa, L. Trussardi. **Nonlocal-to-local convergence of Cahn-Hilliard equations: Neumann boundary conditions and viscosity terms**. *Arch. Ration. Mech. Anal.* 239 (2021), no. 1, 117–149.
- A. Di Primio, M. Grasselli, L. Scarpa. **A stochastic Allen-Cahn-Navier-Stokes system with singular potential.** *J. Differential Equations* 387 (2024), 378–431.
- L. Scarpa, M. Zanella. **Degenerate Kolmogorov equations and ergodicity for the stochastic Allen-Cahn equation with logarithmic potential.** *Stoch. Partial Differ. Equ. Anal. Comput.* 12 (2024), 281–325.
- A. Di Primio, M. Grasselli, L. Scarpa. **Stochastic Cahn-Hilliard and conserved Allen-Cahn equations with logarithmic potential and conservative noise.** *Nonlinearity* 37 (2024), no. 12, Paper No. 125005, 57 pp.
- F. Bertacco, C. Orrieri, L. Scarpa. **Weak uniqueness by noise for singular stochastic PDEs.** Submitted (2023), arXiv:2308.01642

THANK YOU FOR YOUR ATTENTION!