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Hyperbolic Approximations of Navier-Stokes-Cahn-Hilliard Systems

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TAG Workshop "Mixtures: Modeling, Analysis and Computing"

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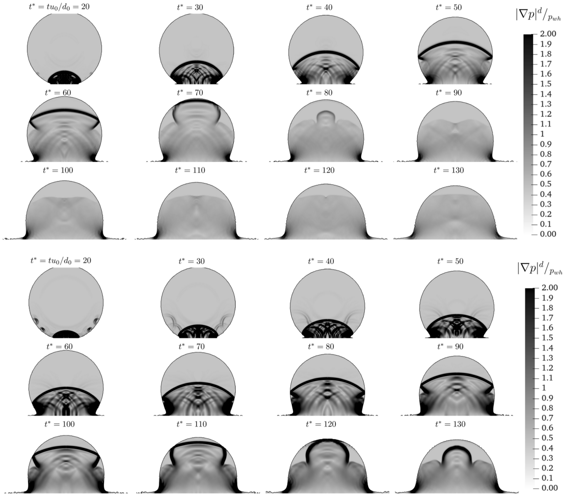
ians DROPIT SPP 2410 CoScaRa SimTech Cluster of Excellence

Detailed-Scale Liquid-Vapour Flow

Convection-dominated two-phase flow:



Droplet hitting surface (Tretola et al. ’22).

Content of the Lecture

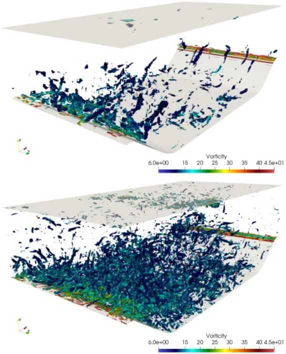
Hyperbolic Approximations of Navier-Stokes-Cahn-Hilliard Systems

1. Hyperbolic Approximations for the Incompressible Navier-Stokes System

2. Hyperbolic Approximations for the Incompressible Navier-Stokes-Cahn-Hilliard System

3. Conclusions and Outlook

1. Hyperbolic Approximations for the Incompressible Navier-Stokes System



1. Hyperbolic Approximations for the Incompressible NS System

The incompressible NS system in the domain $D \subset \mathbb{R}^d$:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} \end{aligned} \quad \text{in } D_T := D \times (0, T),$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \partial D \times (0, T) \quad + \quad \text{IC.}$$

Unknowns:

$p = p(\boldsymbol{x}, t) \in \mathbb{R}$: pressure

$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) \in \mathbb{R}^d$: velocity

$\nu > 0$ is the given viscosity.

Any classical solution of the initial boundary value problem for the NS system satisfies for all $t \in (0, T)$ the inequality

$$\frac{d}{dt} \int_D \frac{1}{2} |\mathbf{u}(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x} = - \int_D \nu |\nabla \mathbf{u}(\boldsymbol{x}, t)|^2 d\boldsymbol{x} \leq 0.$$

1. Hyperbolic Approximations for the Incompressible NS System

Artificial Compressibility (Chorin '67, Temam '69): Let $\alpha > 0$ be given.

$$\left.\begin{array}{rcl} \alpha p^\alpha_t + \nabla \cdot \mathbf{u}^\alpha &=& 0, \\ \mathbf{u}^\alpha_t + (\mathbf{u}^\alpha \cdot \nabla)\mathbf{u}^\alpha + \frac{1}{2}(\nabla \cdot \mathbf{u}^\alpha)\mathbf{u}^\alpha + \nabla p^\alpha &=& +\nu \Delta \mathbf{u}^\alpha, \end{array}\right\} \text{ in } D_T,$$

$$\boldsymbol{u}^\alpha = \mathbf{0} \quad \text{on } \partial D \times (0, T) \quad + \quad \text{IC.}$$

Unknowns:

$p^\alpha = p^\alpha(\boldsymbol{x}, t) \in \mathbb{R}$: pressure

$\boldsymbol{u}^\alpha = \boldsymbol{u}^\alpha(\boldsymbol{x}, t) \in \mathbb{R}^d$: velocity

Any classical solution of the initial boundary value problem for the artificial-compressibility approximation satisfies for all $t \in (0, T)$ the inequality

$$\frac{d}{dt}\int_D \frac{\alpha}{2}|p^\alpha(\boldsymbol{x}, t)|^2 + \frac{1}{2}|\mathbf{u}^\alpha(\boldsymbol{x}, t)|^2 \, d\boldsymbol{x} = -\int_D \nu |\nabla \mathbf{u}^\alpha(\boldsymbol{x}, t)|^2 d\boldsymbol{x} \leq 0.$$

1. Hyperbolic Approximations for the Incompressible NS System

Artificial Compressibility for $D = \mathbb{R}$: (index α skipped)

We get for the state space $\mathcal{Q} = \{\mathbf{Q} = (p, u)^T \in \mathbb{R}^2\}$ the system

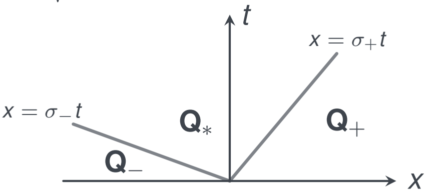
$$\begin{pmatrix} p \\ u \end{pmatrix}_t + \begin{pmatrix} \alpha^{-1} u \\ u^2 + p \end{pmatrix}_x = \begin{pmatrix} 0 \\ \nu u_{xx} \end{pmatrix} \iff \mathbf{Q}_t + \mathbf{f}(\mathbf{Q})_x = \mathbf{S}[\mathbf{Q}] \quad \text{in } \mathbb{R} \times (0, \infty).$$

This first-order system is strictly hyperbolic with two characteristic speeds

$$\lambda_{\mp}(\mathbf{Q}) = u \mp \sqrt{u^2 + 1/\alpha}.$$

Riemann problem for $\mathbf{Q}_{\pm} \in \mathcal{Q}$:

$$\mathbf{Q}(x, 0) = \begin{cases} \mathbf{Q}_-, & : x < 0 \\ \mathbf{Q}_+ & : x > 0 \end{cases}$$



Two-wave Riemann solution with intermediate state $\mathbf{Q}_* = \mathbf{Q}_*(\mathbf{Q}_-, \mathbf{Q}_+)$.

Artificial Compressibility and DG methods (Bassi et al '06):

Let a triangulation $\mathcal{T}_h$ of $D = \mathbb{R}^d$ be given with Γ_h being the set of faces. Define for $k \in \mathbb{N}_0$ the DG-spaces

$$\mathbf{U}_h = (V_h)^d,\ \mathbf{Q}_h = V_h,\ \ V_h = \left\{ v_h \in L^2(D) \,|\, v_{h|_K} \in \mathcal{P}_k(K),\ K \in \mathcal{T}_h \right\}.$$

Let $\mathbf{F}(\mathbf{Q}) = \mathbf{u} \otimes \mathbf{u} + p\mathcal{I}$.

Find $(p_h, \mathbf{u}_h) \in \mathbf{Q}_h \times \mathbf{U}_h$ with

$$\begin{aligned} -\int_D \mathbf{u}_h \cdot \nabla_h q_h \, d\mathbf{x} + \int_{\Gamma_h} [\![q_h]\!] \cdot \hat{\mathbf{u}}(\mathbf{Q}_h^-, \mathbf{Q}_h^+) \, d\xi &= 0, \\ \int_D \mathbf{v}_h \cdot \mathbf{u}_{h,t} \, d\mathbf{x} - \int_D \nabla_h \mathbf{v}_h : \mathbf{F}(\mathbf{Q}_h) \, d\mathbf{x} + \int_{\Gamma_h} [\![\mathbf{v}_h]\!] : \hat{\mathbf{F}}(\mathbf{Q}_h^-, \mathbf{Q}_h^+) \, d\xi &= a[\mathbf{v}_h, \mathbf{u}_h] \end{aligned}$$

for all $(q_h, \mathbf{v}_h) \in \mathbf{Q}_h \times \mathbf{U}_h$.

The **numerical fluxes** are defined from the Riemann problem solutions as

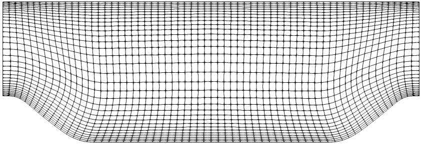
$$\hat{\mathbf{u}}(\mathbf{Q}_h^-, \mathbf{Q}_h^+) = \mathbf{u}_*(\mathbf{Q}_h^-, \mathbf{Q}_h^+), \qquad \hat{\mathbf{F}}(\mathbf{Q}_h^-, \mathbf{Q}_h^+) = \mathbf{F}(\mathbf{Q}_*(\mathbf{Q}_h^-, \mathbf{Q}_h^+)).$$

1. Hyperbolic Approximations for the Incompressible NS System

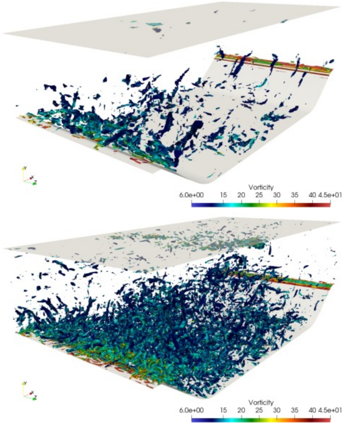
Artificial Compressibility and DG methods for convection-dominated flow:

(Bassi&Massa&Ostrowski&R., J. Comput. Phys. '22):

Implicit LES simulation for Periodic Hill at $\mathrm{Re} = 10595$ (ERCOFTAC test case)



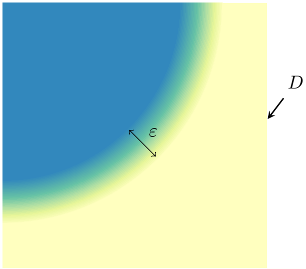
Mesh for 2D-cross section.



Isosurface of Q-15-criterion for $k = 2$ (top) and $k = 4$ (bottom), coloured with non-dimensional vorticity magnitude.

2. Hyperbolic Approximations for the Incompressible Navier-Stokes-Cahn-Hilliard System

(Joint work with Jens Keim and Hasel-Cicek Konan)



The Navier-Stokes-Cahn-Hilliard system (Hohenberg & Halperin '77):

$$\left.\begin{array}{rcl} \nabla\cdot\mathbf{u} &=& 0,\\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p &=& -c\nabla\mu(c)+\nu\Delta\mathbf{u},\\ c_t + \nabla\cdot(c\mathbf{u}) - \nabla\cdot(\nabla\mu(c)) &=& 0,\\ \mu(c) &=& W'(c)-\gamma\Delta c \end{array}\right. \quad \text{in } D_T,$$

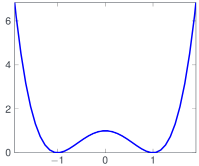
$$\boldsymbol{u}=\mathbf{0},\ \nabla c\cdot\boldsymbol{n}=0,\ \nabla\mu(c)\cdot\boldsymbol{n}=0 \quad \text{on } \partial D\times(0,T) \quad + \quad \text{IC.}$$

Unknowns:

$p = p(\boldsymbol{x},t)\in\mathbb{R}$: pressure

$\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},t)\in\mathbb{R}^d$: velocity

$c = c(\boldsymbol{x},t)\in[-1,1]$: phase field



Double-well potential $W = W(c)$.

2. Hyperbolic Approximations for the Incompressible NSCH System
Thermodynamical consistency and energy dissipation rate:

$$E[\mathbf{u}, c, \mathbf{d}] = \frac{1}{2}|\mathbf{u}|^2 + W(c) + \frac{\gamma}{2}|\mathbf{d}|^2$$

Theorem:
Any classical solution of the initial boundary value problem for the NSCH system satisfies for all $t \in (0, T)$ the inequality

$$\frac{d}{dt}\int_D E[\mathbf{u}, c, \nabla c](\boldsymbol{x}, t)\, d\boldsymbol{x} = -\int_D \left(|\nabla W(c)(\boldsymbol{x}, t)|^2 + \nu|\nabla \mathbf{u}(\boldsymbol{x}, t)|^2\right) d\boldsymbol{x} \leq 0.$$

Note:

- For the well-posedness of weak and strong solutions see e.g. Boyer' 99, Abels '09, Cao&Gal 12, Giorgini&Miranville&Temam '19, Giorgini&Temam '20...
- Extensions to account for mobility, more general double-well potentials, density variations, more complex boundary conditions...
- The NSCH model has a proper energy structure but includes nonlocalities due to the divergence constraint and the biharmonic operator in the CH part.

2. Hyperbolic Approximations for the Incompressible NSCH System

Artificial Compressibility and hyperbolic Cahn-Hilliard:

Define $\varepsilon = (\alpha, \delta)$.

$$\begin{aligned}
\alpha p_t^\varepsilon \quad + \quad & \nabla \cdot \mathbf{u}^\varepsilon && = 0, \\
\mathbf{u}_t^\varepsilon \quad + \quad & (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon + \nabla p^\varepsilon && = -c^\varepsilon(\nabla(W'(c^\varepsilon) - \gamma \Delta c^\varepsilon)) \\
& && \quad + \nu \Delta \mathbf{u}^\varepsilon, \quad \text{in } D_T, \\
\delta c_{tt}^\varepsilon \quad + \quad & \nabla \cdot (c^\varepsilon \mathbf{u}^\varepsilon) - \nabla \cdot (\nabla \mu(c^\varepsilon)) && = -c_t^\varepsilon, \\
& \mu(c^\varepsilon) && = W'(c^\varepsilon) - \gamma^2 \Delta c^\varepsilon
\end{aligned}$$

$$\boldsymbol{u}^\varepsilon = \mathbf{0}, \ \nabla c^\varepsilon \cdot \boldsymbol{n} = 0, \ \nabla \mu c^\varepsilon \cdot \boldsymbol{n} = 0 \quad \text{on } \partial D \times (0, T) \quad + \quad \text{IC.}$$

Unknowns:

$p^\varepsilon = p^\varepsilon(\boldsymbol{x}, t) \in \mathbb{R}$ **: pressure**

$c^\varepsilon = c^\varepsilon(\boldsymbol{x}, t) \in [-1, 1]$ **: phase field**

$\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\varepsilon(\boldsymbol{x}, t) \in \mathbb{R}^d$ **: velocity**

2. Hyperbolic Approximations for the Incompressible NSCH System

Artificial Compressibility and friction-type approximation:

Define $\varepsilon = (\alpha, \delta)$.

$$
\begin{aligned}
\alpha p_t^\varepsilon \; + \; & \nabla \cdot \mathbf{u}^\varepsilon && = \; 0, \\
\mathbf{u}_t^\varepsilon \; + \; & (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon + \nabla p^\varepsilon && = \; -c^\varepsilon(\nabla(W'(c^\varepsilon) - \gamma \Delta c^\varepsilon)) \\
& && \quad + \nu \Delta \mathbf{u}^\varepsilon, \qquad \text{in } D_T, \\
c_t^\varepsilon \; + \; & \nabla \cdot (c^\varepsilon \mathbf{u}^\varepsilon) + \nabla \cdot \mathbf{v}^\varepsilon && = \; 0, \\
\delta \mathbf{v}_t^\varepsilon \; + \; & \nabla(W'(c^\varepsilon) - \gamma \Delta c^\varepsilon) && = \; -\mathbf{v}^\varepsilon
\end{aligned}
$$

$$\boldsymbol{u}^\varepsilon = \mathbf{0}, \; \nabla c^\varepsilon \cdot \boldsymbol{n} = 0, \; \boldsymbol{v}^\varepsilon \cdot \boldsymbol{n} = 0 \quad \text{on } \partial D \times (0, T) \quad + \quad \text{IC.}$$

Unknowns:

- $p^\varepsilon = p^\varepsilon(\boldsymbol{x}, t) \in \mathbb{R}$ **: pressure**
- $c^\varepsilon = c^\varepsilon(\boldsymbol{x}, t) \in [-1, 1]$ **: phase field**
- $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\varepsilon(\boldsymbol{x}, t) \in \mathbb{R}^d$ **: velocity**
- $\boldsymbol{v}^\varepsilon = \boldsymbol{v}^\varepsilon(\boldsymbol{x}, t) \in \mathbb{R}^d$ **: friction velocity**

2. Hyperbolic Approximations for the Incompressible NSCH System
Thermodynamical consistency and energy dissipation rate:

$$E^{\varepsilon}[p, \mathbf{u}, c, \mathbf{v}, \mathbf{d}] = \frac{\alpha}{2} p^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{\delta}{2}|\mathbf{v}|^2 + W(c) + \frac{\gamma}{2}|\mathbf{d}|^2$$

Theorem:

Any classical solution of the initial boundary value problem for the friction-type approximation of the NSCH system satisfies for all $t \in (0, T)$ the inequality

$$\frac{d}{dt}\int_D E^{\varepsilon}[p^{\varepsilon}, \mathbf{u}^{\varepsilon}, c^{\varepsilon}, \mathbf{v}^{\varepsilon}, \nabla c^{\varepsilon}](\boldsymbol{x}, t)\, d\boldsymbol{x} = -\int_D \Big(|\mathbf{v}^{\varepsilon}(\boldsymbol{x}, t))|^2 + \nu |\nabla \boldsymbol{u}^{\varepsilon}(\boldsymbol{x}, t)|^2 \Big)\, d\boldsymbol{x} \leq 0.$$

Note:

- We expect convergence to the NSCH model for $|\varepsilon| \to 0$. Friction (or Korteweg) approximations have been studied in e.g. Lattanzio&Tzavaras ’17, R.&von Wolff ’20,...
- Extensions to account for mobility, more general double-well potentials, density variations, more complex boundary conditions...
- The friction approximation still includes a third-order operator.

2. Hyperbolic Approximations for the Incompressible NSCH System

Artificial Compressibility and relaxed friction-type approximation:

Define $\varepsilon = (\alpha, \delta, \beta)$.

$$
\left.
\begin{array}{rcl}
\alpha p_t^\varepsilon + \nabla \cdot \mathbf{u}^\varepsilon &=& 0, \\
\mathbf{u}_t^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon + \nabla p^\varepsilon &=& -c^\varepsilon \big(\nabla (W'(c^\varepsilon) - \beta^{-1}(\omega^\varepsilon - c^\varepsilon))\big) \\
&& + \nu \Delta \mathbf{u}^\varepsilon, \\
c_t^\varepsilon + \nabla \cdot (c^\varepsilon \mathbf{u}^\varepsilon) + \nabla \cdot \mathbf{v}^\varepsilon &=& 0, \\
\delta \mathbf{v}_t^\varepsilon + \nabla \big(W'(c^\varepsilon) - \beta^{-1}(\omega^\varepsilon - c^\varepsilon)\big) &=& -\mathbf{v}^\varepsilon, \\
-\gamma \Delta \omega^\varepsilon + \beta^{-1}(\omega^\varepsilon - c^\varepsilon) &=& 0
\end{array}
\right\} \text{ in } D_T,
$$

$$
\boldsymbol{u}^\varepsilon = \mathbf{0},\ \nabla \omega^\varepsilon \cdot \boldsymbol{n} = 0,\ \boldsymbol{v}^\varepsilon \cdot \boldsymbol{n} = 0 \quad \text{on } \partial D \times (0, T) \quad + \quad \text{IC.}
$$

Unknowns:

$p^\varepsilon \in \mathbb{R}$: pressure, $c^\varepsilon \in [-1, 1]$: phase field, $\omega^\varepsilon \in \mathbb{R}$: relaxation field

$\boldsymbol{u}^\varepsilon \in \mathbb{R}^d$: velocity, $\boldsymbol{v}^\varepsilon \in \mathbb{R}^d$: friction velocity

Artificial Compressibility and relaxed friction-type approximation:

Define $\varepsilon = (\alpha, \delta, \beta)$.

$$
\begin{aligned}
\alpha p_t^\varepsilon \quad + \quad & \nabla \cdot \mathbf{u}^\varepsilon &=\ & 0,\\
\mathbf{u}_t^\varepsilon \quad + \quad & (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon + \frac{1}{2}(\nabla \cdot \mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon + \nabla p^\varepsilon + c^\varepsilon\big(\nabla(W'(c^\varepsilon) + \beta^{-1}c^\varepsilon)\big) &=\ & \beta^{-1}c^\varepsilon\nabla\omega^\varepsilon\\
& & & +\nu\Delta\mathbf{u}^\varepsilon,\\
c_t^\varepsilon \quad + \quad & \nabla \cdot (c^\varepsilon\mathbf{u}^\varepsilon) + \nabla \cdot \mathbf{v}^\varepsilon &=\ & 0,\\
\delta \mathbf{v}_t^\varepsilon \quad + \quad & \nabla\big(W'(c^\varepsilon) + \beta^{-1}c^\varepsilon\big) &=\ & -\mathbf{v}^\varepsilon + \beta^{-1}\nabla\omega^\varepsilon,\\
& -\gamma\Delta\omega^\varepsilon + \beta^{-1}(\omega^\varepsilon - c^\varepsilon) &=\ & 0,
\end{aligned}
$$

$$
\boldsymbol{u}^\varepsilon = \mathbf{0},\ \nabla\omega^\varepsilon \cdot \boldsymbol{n} = 0,\ \boldsymbol{v}^\varepsilon \cdot \boldsymbol{n} = 0 \quad \text{on } \partial D \times (0, T) \quad + \quad \text{IC.}
$$

Unknowns:

$p^\varepsilon \in \mathbb{R}$: pressure $\quad c^\varepsilon \in [-1, 1]$: phase field $\quad \omega^\varepsilon \in \mathbb{R}$: relaxation field

$\boldsymbol{u}^\varepsilon \in \mathbb{R}^d$: velocity $\quad \boldsymbol{v}^\varepsilon \in \mathbb{R}^d$: friction velocity

2. Hyperbolic Approximations for the Incompressible NSCH System

Thermodynamical consistency and energy dissipation rate:

$$E^{\varepsilon}[p, \mathbf{u}, c, \mathbf{v}, \omega, \mathbf{e}] = \frac{\alpha}{2}p^2 + \frac{1}{2}|\mathbf{u}|^2 + \frac{\delta}{2}|\mathbf{v}|^2 + W(c) + \frac{1}{2\beta}(c-\omega)^2 + \frac{\gamma^2}{2}|\mathbf{e}|^2$$

Theorem:

Any classical solution of the initial boundary value problem for the relaxed friction-type approximation of the NSCH system satisfies for all $t \in (0, T)$ the inequality

$$\begin{aligned} \frac{d}{dt}\int_D E^{\varepsilon}[p^{\varepsilon}, \mathbf{u}^{\varepsilon}, c^{\varepsilon}, \mathbf{v}^{\varepsilon}, \nabla c^{\varepsilon}, \nabla \omega^{\varepsilon}](\boldsymbol{x}, t)\, d\boldsymbol{x} &= -\int_D \Big(|\mathbf{v}^{\varepsilon}(\boldsymbol{x}, t))|^2 + \nu |\nabla \boldsymbol{u}^{\varepsilon}(\boldsymbol{x}, t)|^2 \Big)\, d\boldsymbol{x} \\ &\leq 0. \end{aligned}$$

Note:

We expect convergence to the NSCH model for $|\varepsilon| \to 0$. Relaxation approximations have been studied in e.g. Keim&Munz&R. ’23, Dhaouadi&Dumbser&Gavrilyuk ’24.

2. Hyperbolic Approximations for the Incompressible NSCH System
The relaxed friction-type approximation for $d = 1$: (index ε skipped)

We get for $G'(c) = cW''(c) + \beta^{-1}c$ and the reduced state space

$$\mathcal{Q} = \left\{ \mathbf{Q} = (p, u, c, v)^T \in \mathbb{R}^4 \,\middle|\, c \in [-1, 1] \right\}$$

the system

$$\begin{pmatrix} p \\ u \\ c \\ v \end{pmatrix}_t + \begin{pmatrix} \alpha^{-1} u \\ u^2 + p + G(c) \\ cu + v \\ \delta^{-1}\left(W'(c) + \beta^{-1} c\right) \end{pmatrix}_x = \begin{pmatrix} 0 \\ \beta^{-1} c\omega_x + \nu u_{xx} \\ 0 \\ -\delta^{-1} v + (\delta\beta)^{-1}\omega_x \end{pmatrix} \text{ in } \mathbb{R} \times (0, \infty),$$

which rewrites as

$$\mathbf{Q}_t + \mathbf{f}(\mathbf{Q})_x = \mathbf{S}[\mathbf{Q}, \omega] \text{ in } \mathbb{R} \times (0, \infty).$$

Hyperbolicity of the relaxed friction-type approximation:

Theorem:

Let $\alpha, \delta > 0$, and let β satisfy

$$\beta < -\left(\min_{c\in[-1,1]}\left\{\min\{W''(c), 0\}\right\}\right)^{-1}. \tag{1}$$

Then, the pair $(\eta, q) : \mathcal{Q} \to \mathbb{R}^2$ with

$$\begin{aligned} \eta(\mathbf{Q}) &= \frac{\alpha}{2}p^2 + \frac{1}{2}u^2 + W(c) + \frac{1}{2\beta}c^2 + \frac{\delta}{2}v^2, \\ q(\mathbf{Q}) &= pu + \frac{1}{2}u^3 + c\left(W'(c) + \frac{1}{\beta}c\right)u + \left(W'(c) + \frac{1}{\beta}c\right)v \end{aligned} \tag{2}$$

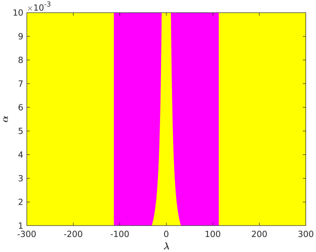
is an entropy/entropy-flux pair for the system of conservation laws

$$\boldsymbol{Q}_t + \mathbf{f}(\mathbf{Q})_x = \mathbf{0} \text{ in } \mathbb{R} \times (0, \infty).$$

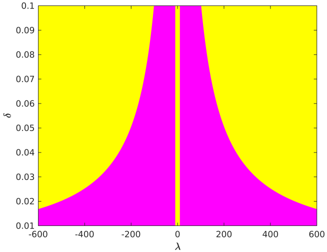
The system is hyperbolic in $\mathcal{Q}$.

2. Hyperbolic Approximations for the Incompressible NSCH System
Characteristic polynomial of the relaxed friction-type approximation:

$$p(\lambda;\mathbf{Q}) = \det(D\mathbf{f}(\mathbf{Q}) - \mathbb{I}\lambda) = \lambda^4 - \frac{5}{2}u\lambda^3 + \left(\frac{3}{2}u^2 - \left(W''(c) + \frac{1}{\beta}\right)\left(\frac{1}{\delta} + c^2\right) - \frac{1}{\alpha}\right)\lambda^2 + \left(\frac{3u}{2\delta}\left(\left(W''(c) + \frac{1}{\beta}\right) + \frac{u}{\alpha}\right)\lambda + \frac{1}{\alpha\delta}\left(W''(c) + \frac{1}{\beta}\right)\right).$$



Color plot of $\mathrm{sgn}(p(.;\mathbf{Q}))$ for fixed $\delta = 0.09$ and $\alpha \in [0.01, 0.001]$.



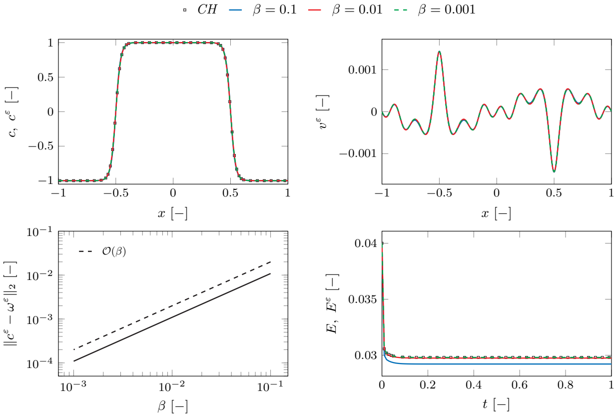
Color plot of $\mathrm{sgn}(p(.;\mathbf{Q}))$ for fixed $\alpha = 0.01$ and $\delta \in [0.01, 0.001]$.

Numerical experiment: stationary droplet ($\alpha = 10^{-4}$, $\delta = 10^{-8}$, $\beta = 10^{-4}$)

Initial data:

$$p(\cdot,0) = u(\cdot,0) = 0,\ c(\cdot,0) = -\tanh\left(10(|\cdot| - 0.5)\right),$$

$$v(\cdot,0) = \left(W'(c(\cdot,0)) + \beta^{-1}(c(\cdot,0) - \omega(\cdot,0))\right),\ -\gamma\Delta\omega(\cdot,0) + \beta^{-1}\omega(\cdot,0) = \beta^{-1}c(\cdot,0).$$



Phase field c^{ε} and friction velocity v^{ε} for $t = 1$.

L^2-difference $\|c^{\varepsilon} - \omega^{\varepsilon}\|_{L^2}$ and energy E^{ε} over time.

Numerical experiment: shock-waves ($\alpha = 0.5$, $\delta = 0.0625$, $\gamma = 10^{-4}$, $\beta = 0.1$)

Discontinuous Riemann initial data:

$$(p, u, c, v)(x, 0) = \begin{cases} (0, +1, 1, 0) : & x \leq 0, \\ (0, -1, 1, 0) : & x > 0, \end{cases} \qquad \text{(compression-like)},$$

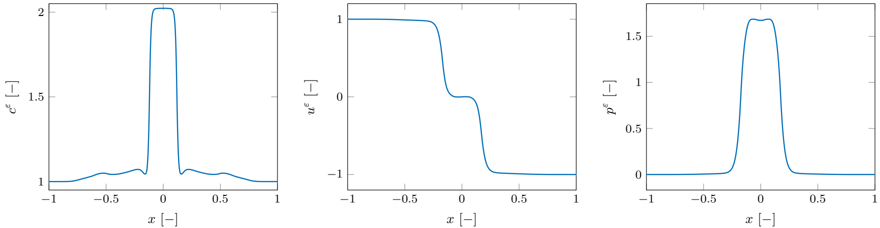
$$(p, u, c, v)(x, 0) = \begin{cases} (0, -1, 2, 0), & x \leq 0, \\ (0, +1, 2, 0), & x > 0. \end{cases} \qquad \text{(expansion-like)}.$$

The double-well potential is chosen according to

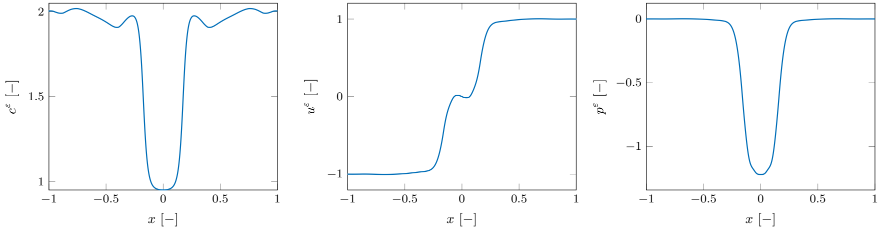
$$W(c) = (c-1)^2(c-2)^2.$$

[2] Alkämper&Magiera&R. '24: ACM Trans. Math. Softw.

Numerical experiment: shock-waves ($\alpha = 0.5$, $\delta = 0.0625$, $\gamma = 10^{-4}$, $\beta = 0.1$)



Phase field c, velocity u and pressure p at $t = 0.15$ for the *compression-like* setup.



Phase field c, velocity u, and pressure p at $t = 0.15$ for the *expansion-like* setup.

3. Conclusions and Outlook

- Generalization of the Artificial-Compressibility approach for the Navier-Stokes-Cahn-Hilliard system
- Development of a (basically hyperbolic) relaxed friction-type approximation
- Usage of asymptotic-preserving IMEX time discretization for the relaxed friction-type approximation ($\rightsquigarrow$ FLEXI)
- Usage of Riemann solvers for consistent solution of the incompressible Navier-Stokes-Cahn-Hilliard system ($\rightsquigarrow$ MIGALE, Bassi et al.)
- Analysis of the asymptotics of the relaxed friction-type approximation (Preliminary result in Huang&R.&Yong&Zhang '24)

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