Mechanical response of metamaterials

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[A metamaterial is] a synthetic composite material engineered to display properties not usually found in natural materials.

Oxford English Dictionary, July 2023. s.v. metamaterial (n.)

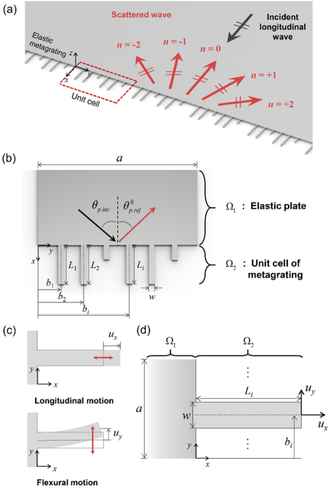

Fig. 1. A schematic illustration of the elastic metagratings constructed with a periodic array of beam-type members. (a) A unit cell composed of multiple beam-type members attached to the elastic continuum plate where the longitudinal waves are scattered, (b) the unit cell of the elastic grating with geometric parameters, (c) illustrations of the longitudinal and flexural motions of a beam-type member, (d) the interface between elastic continuum plate Ω_1 and a beam-type member in Ω_2.

Shin Young Kim, Woorim Lee, Joong Seok Lee, and Yoon Young Kim. Longitudinal wave steering using beam-type elastic metagratings. Mechanical Systems and Signal Processing, 156:107688, 2021

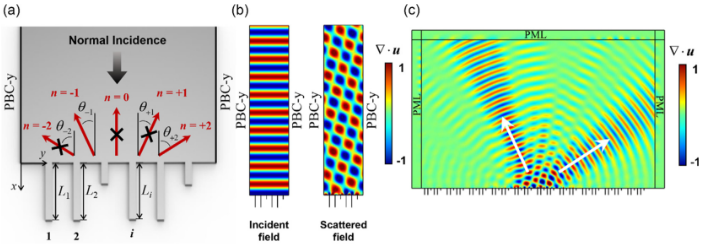

Fig. 5. Beam splitting using the designed metagrating. (a) The problem definition and the designed lengths of the beam-type members ($\theta_{\pm1} = \pm 25^\circ$, $\theta_{\pm2} \approx \pm 57.7^\circ$, $[L_1, \cdots, L_6]$ = [0.06128, 0.06163, 0.01988, 0.06064, 0.06145, 0.02155] m.), (b) the wave fields in the periodic unit cell, (c) the 2D numerical simulation of beam splitting at two different angles.

PML = perfectly matched layer
PBC = periodic boundary conditions

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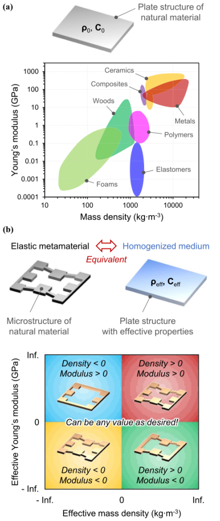
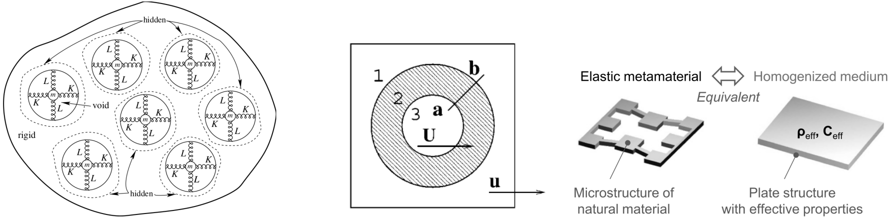


Figure 2. Comparison of the range of (effective) material properties of (a) natural materials and (b) elastic metamaterials.

Jeseung Lee and Yoon Young Kim. Elastic metamaterials for guided waves: from fundamentals to applications. Smart Materials and Structures, 32(12):123001, 2023



(a) Material filled with densely arranged microcavities.

(b) Elementary cell in a three component composite material.

(c) Elementary cell in a microstructured material.

Figure: Schematics of internal arrangement of metamaterials.

Graeme W Milton and John R Willis. On modifications of Newton's second law and linear continuum elastodynamics. Proc. R. Soc. A Math. Phys. Eng. Sci., 463(2079):855–880, 2007

Zhengyou Liu, C. T. Chan, and Ping Sheng. Analytic model of phononic crystals with local resonances. Phys. Rev. B, 71:014103, Jan 2005

Jeseung Lee and Yoon Young Kim. Elastic metamaterials for guided waves: from fundamentals to applications. Smart Materials and Structures, 32(12):123001, 2023

“Negative mass” is influential concept in theory of metamaterials.

PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2007) **463**, 855–880
doi:10.1098/rspa.2006.1795
Published online 9 January 2007

On modifications of Newton’s second law and linear continuum elastodynamics

BY GRAEME W. MILTON[1,*] AND JOHN R. WILLIS[2]

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[2]*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*

In our models, the motion of the rigid material apparently violates Newton’s law owing to the vibration of the internal masses: ‘force equals mass times acceleration’ only applies if mass is replaced by ‘effective mass’, which is a non-local operator in time (equivalently, a function of frequency under any purely harmonic forcing). Furthermore, unless the microstructure is specially constructed so as to be isotropic, the ‘effective mass’ is a second-order tensor, not a scalar.

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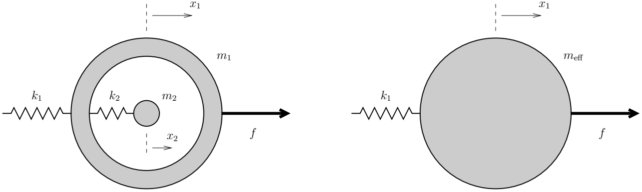
Simple model



(a) Fully resolved system. (b) Effective system.

Figure: Mass-in-mass model.

Governing equations:

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + k_1 x_1 + k_2 (x_1 - x_2) = f$$
$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + k_2 (x_2 - x_1) = 0$$

H.H. Huang, C.T. Sun, and G.L. Huang. On the negative effective mass density in acoustic metamaterials. Int. J. Eng. Sci., 47(4):610–617, 2009

Governing equations:

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + k_1 x_1 + k_2 \left(x_1 - x_2 \right) = f$$
$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + k_2 \left(x_2 - x_1 \right) = 0$$

Harmonic motion:

$$f = \widehat{F} \sin \left(\Omega t \right) \qquad x_i = \widehat{x_i} \sin \left(\Omega t \right)$$

After some algebra:

$$-\underbrace{\left(m_1 + m_2 \frac{\omega_2^2}{\omega_2^2 - \Omega^2} \right)}_{m_{\mathrm{eff}}} \Omega^2 \widehat{x_1} + k_1 \widehat{x_1} = \widehat{F}$$

Fourier image of $m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = f$:

$$-m\Omega^2 \widehat{x} + k\widehat{x} = \widehat{F}$$

Effective mass:

$$m_{\text{eff}} = m_1 + m_2 \frac{\omega_2^2}{\omega_2^2 - \Omega^2}$$

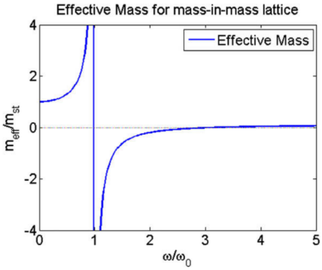


Fig. 5. Dimensionless effective mass m_{eff}/m_{st} as a function of ω/ω_0.

H.H. Huang, C.T. Sun, and G.L. Huang. On the negative effective mass density in acoustic metamaterials. Int. J. Eng. Sci., 47(4):610–617, 2009

Same story for negative effective stiffness.

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On modifications of Newton's second law and linear continuum elastodynamics

By Graeme W. Milton[1,*] and John R. Willis[2]
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In our models, the motion of the rigid material apparently violates Newton's law owing to the vibration of the internal masses: 'force equals mass times acceleration' only applies if mass is replaced by 'effective mass', which is a non-local operator in time (equivalently, a function of frequency under any purely harmonic forcing). Furthermore, unless the microstructure is specially constructed so as to be isotropic, the 'effective mass' is a second-order tensor, not a scalar.

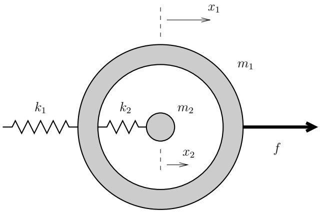
Graeme W Milton and John R Willis. On modifications of Newton's second law and linear continuum elastodynamics. Proc. R. Soc. A Math. Phys. Eng. Sci., 463(2079):855–880, 2007

We like Newton's second law!

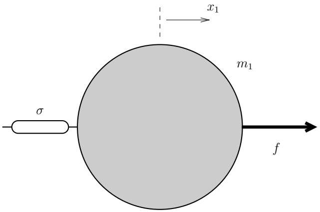
$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = f$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \sigma = f$$

$$\sigma = -kx$$



(a) Fully resolved system.



(b) Effective system.

Figure: Mass-in-mass model.

Original system:

$$
\begin{aligned}
m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + k_1 x_1 + k_2 \left(x_1 - x_2 \right) &= f \\
m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + k_2 \left(x_2 - x_1 \right) &= 0
\end{aligned}
$$

Effective constitutive relation:

$$
\begin{aligned}
& m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} - \sigma = f \\
& \frac{\mathrm{d}^2 \sigma}{\mathrm{d}t^2} + \frac{k_2}{m_2} \sigma = - \left(k_1 + k_2 \right) \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} - \frac{k_1 k_2}{m_2} x_1
\end{aligned}
$$

We are used to this — recall all the spring/dashpot models in theory of viscoelastic materials.

Interesting observation

Conserved quantities:

$$E =_{\mathrm{def}} \frac{1}{2} m_1 \left(\frac{\mathrm{d} x_1}{\mathrm{d} t} \right)^2 + \frac{1}{2} m_2 \left(\frac{\mathrm{d} x_2}{\mathrm{d} t} \right)^2 + k_1 \frac{x_1^2}{2} + k_2 \frac{\left(x_1 - x_2 \right)^2}{2}$$

$$E = \frac{1}{2} m_1 \left(\frac{\mathrm{d} x_1}{\mathrm{d} t} \right)^2 + \frac{1}{2} \frac{m_2}{k_2^2} \left(\frac{\mathrm{d} \sigma}{\mathrm{d} t} + \left(k_1 + k_2 \right) \frac{\mathrm{d} x_1}{\mathrm{d} t} \right)^2 + k_1 \frac{x_1^2}{2} + \frac{\left(\sigma + k_1 x_1 \right)^2}{2 k_2}$$

Lattice model

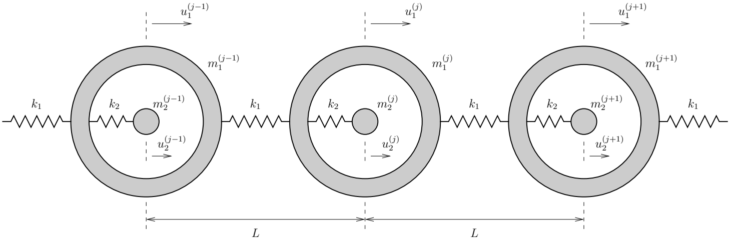


Figure: Infinite mass-in-mass one-dimensional lattice structure.

Governing equations:

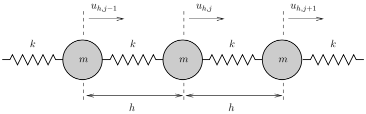
$$m_1^{(j)} \frac{\mathrm{d}^2 u_1^{(j)}}{\mathrm{d}t^2} - k_1 \left(u_1^{(j+1)} - 2u_1^{(j)} + u_1^{(j-1)} \right) + k_2 \left(u_1^{(j)} - u_2^{(j)} \right) = 0$$

$$m_2^{(j)} \frac{\mathrm{d}^2 u_2^{(j)}}{\mathrm{d}t^2} + k_2 \left(u_2^{(j)} - u_1^{(j)} \right) = 0$$

Dispersion relation — wave transmission:

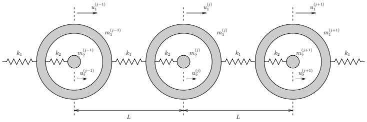
$$u_\gamma^{(j+n)} = B_\gamma \mathrm{e}^{\mathrm{i}(qx+nqL-\omega t)}$$

Inner mass is absent:



$$\omega^2 = \frac{2k_1\left(1-\cos\left(qL\right)\right)}{m_1}$$

Inner mass is present:



$$\omega^2 = \frac{2k_1\left(1-\cos\left(qL\right)\right)}{m_{\mathrm{eff}}}$$

$$m_{\mathrm{eff}} =_{\mathrm{def}} m_{\mathrm{sum}} + \frac{m_2\left(\frac{\omega}{\omega_2}\right)^2}{1-\left(\frac{\omega}{\omega_2}\right)^2}$$

$$m_{\mathrm{sum}} =_{\mathrm{def}} m_1 + m_2$$

Dispersion relation for mass-in-mass lattice structure:

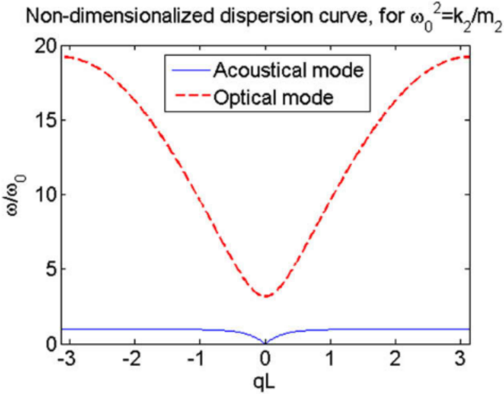


Fig. 3. Nondimensionalized dispersion curve for the mass-in-mass lattice model.

H.H. Huang, C.T. Sun, and G.L. Huang. On the negative effective mass density in acoustic metamaterials. Int. J. Eng. Sci., 47(4):610–617, 2009

Fully resolved system (visible inner mass):

$$m_1^{(j)} \frac{\mathrm{d}^2 u_1^{(j)}}{\mathrm{d}t^2} - k_1 \left(u_1^{(j+1)} - 2u_1^{(j)} + u_1^{(j-1)} \right) + k_2 \left(u_1^{(j)} - u_2^{(j)} \right) = 0$$

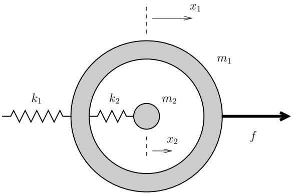
$$m_2^{(j)} \frac{\mathrm{d}^2 u_2^{(j)}}{\mathrm{d}t^2} + k_2 \left(u_2^{(j)} - u_1^{(j)} \right) = 0$$

Equivalent system (effective constitutive relation):

$$m_1^{(j)} \frac{\mathrm{d}^2 u_1^{(j)}}{\mathrm{d}t^2} - \sigma^{(j)} = 0$$

$$\frac{m_2^{(j)}}{k_2} \frac{\mathrm{d}^2 \sigma^{(j)}}{\mathrm{d}t^2} + \left(1 + \frac{m_2^{(j)}}{m_1^{(j)}} \right) \sigma^{(j)} = \frac{m_2^{(j)} k_1}{k_2} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(u_1^{(j+1)} - 2u_1^{(j)} + u_1^{(j-1)} \right) + k_1 \left(u_1^{(j+1)} - 2u_1^{(j)} + u_1^{(j-1)} \right)$$

Mass-in-mass model, nonlinear springs



Generalised energy:

$$W = \left(\frac{1}{2} m_2 \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(x_1 - \frac{\partial g_2}{\partial \sigma_{\mathrm{diff}}} \right) \right)^2 + \psi_1 + \left(-g_2 + \sigma_{\mathrm{diff}} \frac{\partial g_2}{\partial \sigma_{\mathrm{diff}}} \right) \right) \Bigg|_{\sigma_{\mathrm{diff}} = -\left(\frac{\partial \psi_1}{\partial x_1} + \sigma \right)}$$

Effective constitutive relations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(x_1 - \frac{\partial g_2}{\partial \sigma_{\mathrm{diff}}} \right) \right) - \frac{\sigma_{\mathrm{diff}}}{m_2} = 0$$

$$\sigma_{\mathrm{diff}} = -\left(\frac{\partial \psi_1}{\partial x_1} + \sigma \right)$$

Logarithmic corotational derivative:

$$\overset{\circ_{\log}}{\mathbb{H}} = \mathbb{D}$$

Hencky strain $\mathbb{H}$:

$$\mathbb{H} =_{\mathrm{def}} \frac{1}{2} \ln \mathbb{B}$$

Logarithmic corotational derivative:

$$\overset{\circ_{\log}}{\mathbb{A}} =_{\mathrm{def}} \frac{\mathrm{d}\mathbb{A}}{\mathrm{d}t} + \mathbb{A}\Omega^{\log} - \Omega^{\log}\mathbb{A}$$

Logarithmic spin $\Omega^{\log}$ (a skew-symmetric tensor):

$$\Omega^{\log} =_{\mathrm{def}} \mathbb{W} + \sum_{\substack{\sigma,\tau=1 \\ \sigma\neq\tau}}^{3} \left[\left(\frac{1 + \frac{b_\sigma}{b_\tau}}{1 + \frac{b_\sigma}{b_\tau}} + \frac{2}{\ln \frac{b_\sigma}{b_\tau}} \right) \mathbb{P}_\sigma \mathbb{D} \mathbb{P}_\tau \right]$$

H. Xiao, O. T. Bruhns, and A. Meyers. Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mech., 124(1-4):89–105, 1997

O. T. Bruhns, A. Meyers, and H. Xiao. On non–corotational rates of Oldroyd's type and relevant issues in rate constitutive formulations. Proc. R. Soc. A: Math. Phys. Eng. Sci., 460(2043):909–928, 2004

Generalised Helmholtz free energy:

$$
\psi\left(\mathbb{H}_1, \mathbb{T}_{\mathrm{th},\,\rho}\right) =_{\mathrm{def}} \frac{1}{2} \overline{\left(\mathbb{H}_1 + \left.\frac{\partial g_2}{\partial \mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}}}\right|_{\mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}} = \mathbb{T}_{\mathrm{th},\,\rho} - \frac{\partial \psi_1}{\partial \mathbb{H}_1}}\right)}^{\circ_{\log}} : \mathcal{C}_{\mathrm{rate}} \overline{\left(\mathbb{H}_1 + \left.\frac{\partial g_2}{\partial \mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}}}\right|_{\mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}} = \mathbb{T}_{\mathrm{th},\,\rho} - \frac{\partial \psi_1}{\partial \mathbb{H}_1}}\right)}^{\circ_{\log}} + \psi_1 + \left.\left(g_2 - \mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}} : \frac{\partial g_2}{\partial \mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}}}\right)\right|_{\mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}} = \mathbb{T}_{\mathrm{th},\,\rho} - \frac{\partial \psi_1}{\partial \mathbb{H}_1}}
$$

Effective constitutive relations:

$$
-\mathcal{C}_{\mathrm{rate}} \overline{\left(\mathbb{H} + \frac{\partial g_2}{\partial \mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}}}\right)}^{\circ_{\log}\circ_{\log}} + \mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}} = \mathbb{O}
$$

$$
\mathbb{T}_{\mathrm{th},\,\rho_2,\,\mathrm{diff}} = \mathbb{T}_{\rho} - \frac{\partial \psi_1}{\partial \mathbb{H}}
$$

Linearisation of governing equations:

$$\rho_{\mathrm{R}} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \operatorname{div} \mathbb{T}$$

$$-\mathcal{C}_{\mathrm{rate}} \rho_{\mathrm{R}} \frac{\partial^2}{\partial t^2} \left(\mathbb{E} - \frac{\rho_{2,\mathrm{R}}}{\rho_{\mathrm{R}}} \mathcal{C}_{\mathrm{diff}}^{-1} \left(\mathbb{T} - \mathcal{C} \mathbb{E} \right) \right) + \left(\mathbb{T} - \mathcal{C} \mathbb{E} \right) = \mathbb{O}$$

Fourier space, after some algebra:

$$-\rho_{\mathrm{R}} \left(1 + \frac{\rho_{\mathrm{R}} c_{\mathrm{rate}} \omega^2}{1 - \rho_{\mathrm{R}} c_{\mathrm{rate}} \left(1 + \frac{\rho_{2,\mathrm{R}}}{\rho_{\mathrm{R}}} \frac{1}{c_{\mathrm{diff}}} \right) \omega^2} \right) \omega^2 \widehat{\mathbf{U}} = - \left[\left(\mu + \lambda \right) \mathbf{k} \otimes \mathbf{k} + \mu \left(\mathbf{k} \cdot \mathbf{k} \right) \mathbb{I} \right] \widehat{\mathbf{U}}$$

Gianluca Rizzi, Marco Valerio d'Agostino, Jendrik Voss, Davide Bernardini, Patrizio Neff, and Angela Madeo. From frequency-dependent models to frequency-independent enriched continua for mechanical metamaterials. European Journal of Mechanics - A/Solids, 106:105269, 2024

Conclusion

- We can avoid negative/motion dependent mass/density concept.
- The alternative to negative mass concept is the effective constitutive relation concept.

David Cichra, Vít Průša, K. R. Rajagopal, Casey Rodriguez, and Martin Vejvoda. The conclusion that metamaterials could have negative mass is a consequence of improper constitutive characterisation, 2024. Accepted in Mathematics and Mechanics of Solids

Vít Průša, K. R. Rajagopal, Casey Rodrigues, Ladislav Trnka, and Martin Vejvoda. Modeling metamaterials by second-order rate-type constitutive relations between only the macroscopic stress and strain. 2025