Is superfluid helium a mixture?

Michal Pavelka

Mathematical Institute

Joint work with Nadine Cetin, Ondřej Kincl, David Schmoranzer, Marco La Mantia, David Jou, Martin Sýkora, and Miroslav Grmela

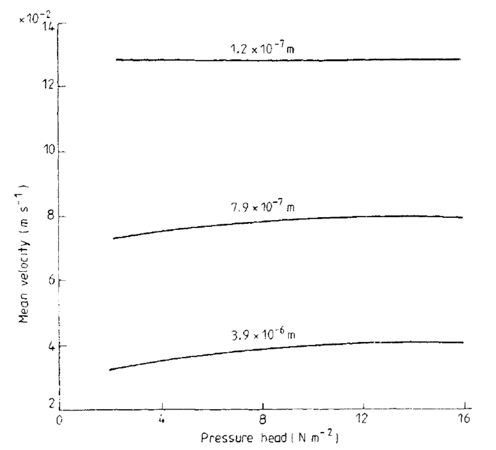
1. Two-component model of superfluid helium-4

2. Geometric mechanics

3. A geometric one-fluid model

Some features of superfluid helium-4

- Below 2.17K flows without resistance (superfluidity)



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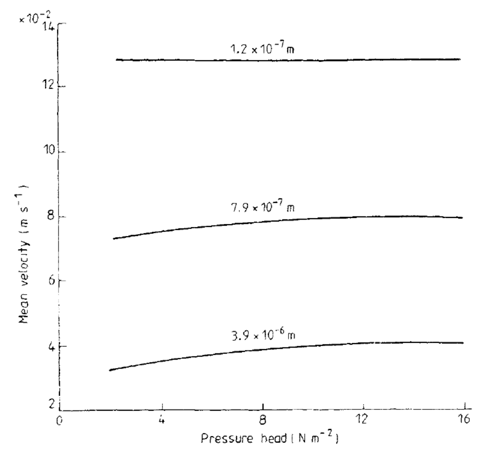

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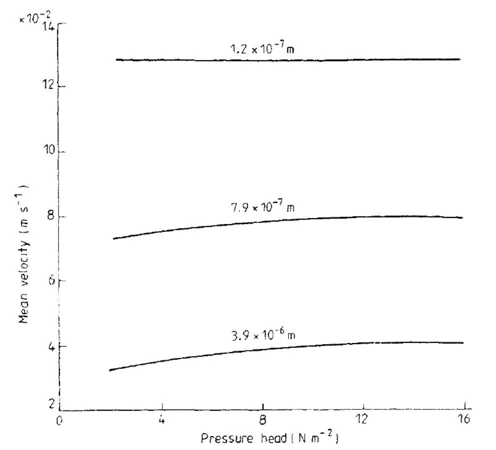
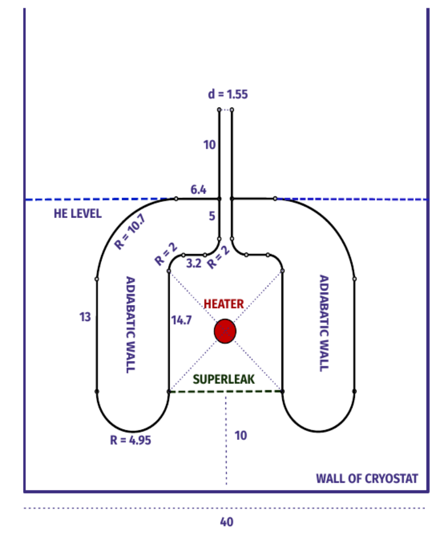


- Temperature gradient $\Rightarrow$ flow (without ∇p)
- Two motions (normal and superfluid)

Simulation of the fountain effect



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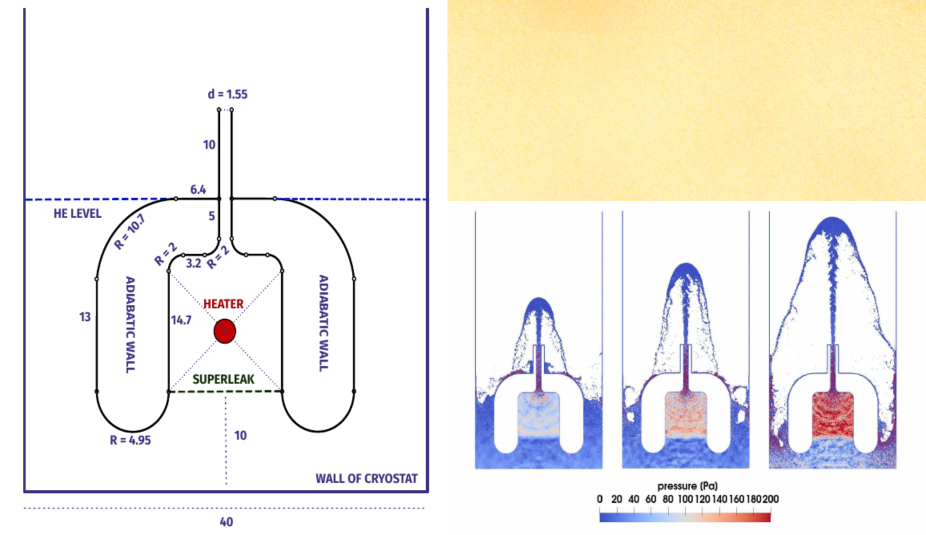

O. Kincl, D. Schmoranzer, M. Pavelka, Simulation of superfluid fountain effect using smoothed particle hydrodynamics, Physics of Fluids 35, 047124 (2023)

Hall-Vinen-Bekarevich-Khalatnikov (HVBK) two-fluid model

$$
\begin{aligned}
\partial_t \mathbf{v}_n + \mathbf{v}_n \cdot \nabla \mathbf{v}_n &= -\nabla p_n + \nu_n \Delta \mathbf{v}_n + \frac{\rho_s}{\rho} \mathbf{F}_{ns} \\
\partial_t \mathbf{v}_s + \mathbf{v}_s \cdot \nabla \mathbf{v}_s &= -\nabla p_s + \mathbf{T} - \frac{\rho_n}{\rho} \mathbf{F}_{ns}
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$$\mathbf{T} = -\nu_s \boldsymbol{\omega} \times (\nabla \times \hat{\boldsymbol{\omega}}), \hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/|\boldsymbol{\omega}|, \quad \boldsymbol{\omega} = \nabla \times \mathbf{v}_s$$

$$\mathbf{F}_{ns} = \frac{1}{2} B \hat{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{c}) + \frac{1}{2} B' \boldsymbol{\omega} \times \mathbf{c}$$

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- Why $\nabla \cdot \mathbf{v}_s = 0 = \nabla \cdot \mathbf{v}_n$?

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}), \rho \partial_t \mathbf{v} \approx -\nabla p \Rightarrow \partial_t \partial_t \rho \approx \Delta p$$

Hamiltonian mechanics from action

$$\delta \int_0^T L(\mathbf{q}, \dot{\mathbf{q}}) \mathrm{d}t = 0 \qquad \Rightarrow \qquad \text{Hamiltonian mechanics for } \mathbf{x} = (\mathbf{q}, \mathbf{p})$$

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General Hamiltonian mechanics

Poisson bracket $\{\mathcal{F}, \mathcal{F}\} \mapsto \mathcal{F}$:

- bilinear: $\{F + G, H\} = \{F, H\} + \{G, H\} \qquad \forall F, G, H \in \mathcal{F}$

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State variables $\mathbf{x}$, Poisson bracket $\{\bullet,\bullet\}$, energy E
$\rightarrow$ Mechanics

Why Poisson brackets?

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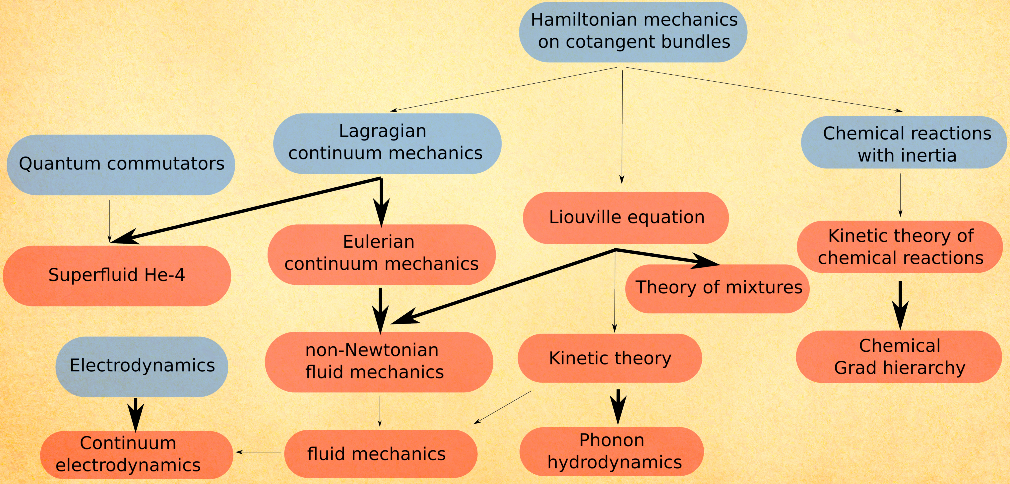
  $=$ Self-consistency of Hamiltonian mechanics

Hierarchy of Poisson brackets



Monograph


DE GRUYTER
Michal Pavelka, Václav Klika, Miroslav Grmela
MULTISCALE THERMO-DYNAMICS
INTRODUCTION TO GENERIC

Atoms and phonons

atoms	phonons
$(\mathbf{q}, \mathbf{p})$	$(\mathbf{q}, \mathbf{p})$
$\downarrow$	$\downarrow$
$f^m(\mathbf{q}, \mathbf{p})$	$f^p(\mathbf{q}, \mathbf{p})$
$\downarrow$	$\downarrow$
$\rho = \int m f^m d\mathbf{p}$	$\bar{s} = \int \eta(f^p) d\mathbf{p}$
$\mathbf{u}^m = \int \mathbf{p} f^m d\mathbf{p}$	$\mathbf{u}^p = \int \mathbf{p} f^p d\mathbf{p}$

superfluid helium

$$\rho, \mathbf{m} = \mathbf{u}^m + \mathbf{u}^p, \bar{s}, \mathbf{v}_s = \frac{\mathbf{u}^m}{\rho}$$

Hamiltonian mechanics

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\partial_k(\rho E_{m_k} + E_{v_{sk}}) \\
\frac{\partial m_i}{\partial t} &= -\partial_j(m_i E_{m_j}) - \partial_j(v_{si} E_{v_{sj}}) - \rho \partial_i E_\rho - m_j \partial_i E_{m_j} - \bar{s}\partial_i E_{\bar{s}} \\
&\quad - v_{sk}\partial_i E_{v_{sk}} + \partial_i(E_{v_{sk}} v_{sk}) \\
\frac{\partial \bar{s}}{\partial t} &= -\partial_k\left(\bar{s} E_{m_k}\right) \\
\frac{\partial v_{sk}}{\partial t} &= -\partial_k E_\rho - \partial_k(v_{sj} E_{m_j}) + (\partial_k v_{sj} - \partial_j v_{sk})\left(E_{m_j} + \frac{1}{\rho} E_{v_{sj}}\right)
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$$

M. Sýkora, M. Pavelka, M. La Mantia, D. Jou, and M. Grmela, On the relations between large-scale models of superfluid helium-4, Physics of Fluids 33, 127124 (2021)

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- Effective density ρ_n.

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$\frac{\partial e}{\partial \mathbf{v}_s} = 0$, normal fluids

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- $\mathbf{r} = \mathbf{r}' + \mathbf{V}t$
- $\mathbf{v} = \mathbf{v}' + \mathbf{V}$
- $\mathbf{p} = \mathbf{p}' + M\mathbf{V}$
- $E = \frac{1}{2}M\mathbf{v}^2 = \frac{1}{2}M(\mathbf{v}' + \mathbf{V})^2 = E' + \mathbf{p}\cdot\mathbf{V} + \frac{1}{2}M\mathbf{V}^2$
- $\bar{e} = \frac{(\mathbf{m}-\rho\mathbf{v}_s)^2}{2\rho_n(\rho,\bar{s})} + \bar{\epsilon}(\rho,\bar{s}) + (\mathbf{m} - \rho\mathbf{v}_s)\cdot\mathbf{v}_s + \frac{1}{2}\rho\mathbf{v}_s^2$
- As $\rho_n \to \rho$ at high T

$$\bar{e} \to \frac{(\mathbf{m})^2}{2\rho} + \bar{\epsilon}(\rho,\bar{s})$$

$\frac{\partial e}{\partial \mathbf{v}_s} = 0$, normal fluids

Energy of vorticity

- $\boldsymbol{\omega} = \nabla \times \mathbf{v}_s$
- Energy

$$\bar{e} = \frac{1}{2}\rho \mathbf{v}_s^2 + (\mathbf{m} - \rho \mathbf{v}_s) \cdot \mathbf{v}_s + \frac{(\mathbf{m} - \rho \mathbf{v}_s)^2}{2\rho_n(\rho, \bar{s})} + \bar{\epsilon}(\rho, \bar{s}) + e_\omega(\rho, \bar{s}, |\boldsymbol{\omega}|)$$

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- Evolution equation for $\mathbf{v}_s$:

$$\begin{aligned}(\partial_t \mathbf{v}_s)_{rev} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = &- \nabla \left(\mu - \frac{1}{2}\mathbf{v}_{ns}^2 \frac{\partial \rho_n}{\partial \rho}\right) \\ &- \nu_s \boldsymbol{\omega} \times (\nabla \times \hat{\boldsymbol{\omega}}) - \frac{\rho_n}{\rho}\boldsymbol{\omega} \times (\mathbf{v}_{ns} - \nu_s \nabla \times \hat{\boldsymbol{\omega}}) \\ &- \nu_s \nabla \left(|\boldsymbol{\omega}| \frac{\partial \rho_s}{\partial \rho}\right) + \frac{\nu_s}{\rho}\boldsymbol{\omega} \times (\hat{\boldsymbol{\omega}} \times \nabla \rho_s)\end{aligned}$$

Dissipative processes

- Gradient dynamics, energy conservation, entropy production:

$$\dot{\mathbf{x}} = -\frac{\delta \mathbb{R}}{\delta E_{\mathbf{x}}}$$

$$\dot{\bar{s}} = \frac{1}{E_{\bar{s}}} \frac{\delta \mathbb{R}}{\delta E_{\mathbf{x}}} \cdot E_{\mathbf{x}} \geq 0$$

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- Rayleigh dissipation potential

$$\mathbb{R} = \int \frac{1}{2} \mu_s (\nabla E_{\mathbf{m}})^2 d\mathbf{r} + \int d\mathbf{r} \frac{\Lambda}{2} \left(\boldsymbol{\omega} \times \frac{\delta E}{\delta \mathbf{v}_s} \right)^2$$

- Irreversible evolution of $\mathbf{v}_s$

$$(\partial_t \mathbf{v}_s)_{irr} = -\frac{\rho_n}{\rho} \frac{B}{2} \hat{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times (\mathbf{v}_{ns} - \nu_s \nabla \times \hat{\boldsymbol{\omega}} - \nu_s \nabla \ln \rho_s \times \hat{\boldsymbol{\omega}}))$$

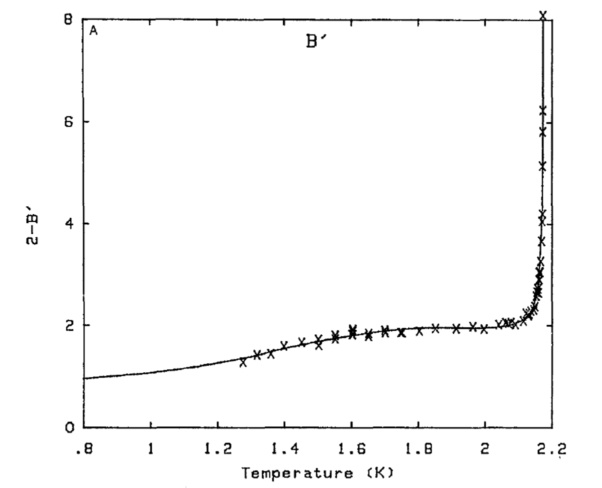
Comparison with HVBK

- Superfluid velocity equation:

$$
\begin{aligned}
\partial_t \mathbf{v}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = &- \nabla \left(\mu - \frac{1}{2}\mathbf{v}_{ns}^2 \frac{\partial \rho_n}{\partial \rho} \right) \\
&- \nu_s \boldsymbol{\omega} \times (\nabla \times \hat{\boldsymbol{\omega}}) - \frac{\rho_n}{\rho} \boldsymbol{\omega} \times (\mathbf{v}_{ns} - \nu_s \nabla \times \hat{\boldsymbol{\omega}}) \\
&- \nu_s \nabla \left(|\boldsymbol{\omega}| \frac{\partial \rho_s}{\partial \rho} \right) + \frac{\nu_s}{\rho} \boldsymbol{\omega} \times (\hat{\boldsymbol{\omega}} \times \nabla \rho_s) \\
&- \frac{\rho_n}{\rho} \frac{B}{2} \hat{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times (\mathbf{v}_{ns} - \nu_s \nabla \times \hat{\boldsymbol{\omega}} - \nu_s \nabla \ln \rho_s \times \hat{\boldsymbol{\omega}}))
\end{aligned}
$$

- $B' = 1$ follows from the energy

$B' = 1?$



Hamiltonian mechanics gives motion of superfluids with vortices

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  - Several fitting parameters

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- Future
  - Vortex line density
  - Numerical schemes