On the problem of a Baer-Nunziato system

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- Weak solutions for a bi-fluid model for a mixture of two compressible non interacting fluids with general boundary data,
  S. Kračmar, Y-S. Kwon, Š. N., A. Novotný
- On existence of weak solutions to a Baer-Nunziato type M. Kalousek, Š. N.
- The existence of a weak solution for a compressible multicomponent fluid structure interaction problem
  M. Kalousek, S. Mitra, Š. N.

One of the acceptable models for description of mixture of several compressible fluids is the so called Two velocity Baer-Nunziato model.

Our goal: to prove the existence of weak solutions for the Baer-Nunziato system with dissipation on an arbitrary large time interval $(0, T)$ in a Lipschitz bounded domain Ω.

Two velocity Baer-Nunziato model

$$\partial_t \alpha_\pm + \vec{v}_I \cdot \nabla_x \alpha_\pm = 0,$$

$$\partial_t(\alpha_\pm \varrho_\pm) + \mathrm{div}(\alpha_\pm \varrho_\pm \vec{u}_\pm) = 0,$$

$$\partial_t(\alpha_\pm \varrho_\pm \vec{u}_\pm) + \mathrm{div}(\alpha_\pm \varrho_\pm \vec{u}_\pm \otimes \vec{u}_\pm) + \nabla_x(\alpha_\pm P_\pm(\varrho_\pm)) - P_I \nabla_x(\alpha_\pm)$$
$$= \alpha_\pm \mu_\pm (\Delta \vec{u}_\pm) + \alpha_\pm(\mu_\pm + \lambda_\pm)\nabla_x \mathrm{div} \vec{u}_\pm$$

$$0 \leqslant \alpha_\pm \leqslant 1, \quad \alpha_+ + \alpha_- = 1.$$

($\alpha_\pm$-concentrations of the $\pm$ species
$\varrho_\pm$- densities of the $\pm$ species
$\vec{u}_\pm$- velocities of the $\pm$ species $\alpha_\pm \varrho_\pm \geqslant 0, \vec{u}_\pm \in R^3$)
$P_\pm$ are two (different) given functions defined on $[0, \infty)$ and P_I, $\vec{v}_I$ are conveniently chosen quantities - they represent pressure and velocity at the interface.
In the multifluid modeling, there are many possibilities how the quantities $\vec{v}_I$, P_I could be chosen, and there is no consensus about this choice.

$$\mu_\pm := \mu,\ \lambda_\pm := \lambda,\ \vec{v}_I = \vec{u}_\pm := \vec{u} \tag{1}$$

$$\alpha P_\pm(s) = \mathcal{P}_\pm(f_\pm(\alpha)s) \text{ for all } \alpha \in (0,1),\ s \in [0,\infty) \tag{2}$$

with some functions $\mathcal{P}_\pm$ defined on $[0,\infty)$ and functions $f_\pm$ defined on $(0,1)$.

- Ishii, Hibiki (2006)
- Bresch et all.(2018)
- Bresch, Mucha, Zatorska: Finite-energy solutions for compressible two-fluid Stokes system (2018), $p = p_- = p_+$
- Li, Zatorska Large time behavior (2020)

One velocity Baer-Nunziato type system

$$\partial_t \alpha + (\vec{u} \cdot \nabla)\alpha = 0, \quad 0 \leqslant \alpha \leqslant 1, \tag{3}$$

$$\partial_t \varrho + \mathrm{div}(\varrho \vec{u}) = 0, \tag{4}$$

$$\partial_t z + \mathrm{div}(z \vec{u}) = 0, \tag{5}$$

$$\partial_t((\rho + z)\vec{u}) + \mathrm{div}((\rho + z)\vec{u} \otimes \vec{u}) + \nabla P(f(\alpha)\varrho, g(\alpha)z) = \mathrm{div}\mathbb{S}(\nabla_x \vec{u}) \tag{6}$$

$P : [0, \infty)^2 \mapsto [0, \infty)$ as well as $f, g : (0, 1) \mapsto [0, \infty)$ are given functions, and

$$\mathbb{S}(\mathbb{Z}) = \mu(\mathbb{Z} + \mathbb{Z}^T) + \lambda \mathrm{Tr}(\mathbb{Z})\mathbb{I}$$

($\mathbb{I}$ is the identity tensor, Tr denotes the trace) is the viscous stress tensor. The constant viscosity coefficients satisfy standard physical assumptions, $\mu > 0$, $\lambda + \frac{2}{3}\mu \geqslant 0$

The system is endowed with initial conditions

$$\alpha|_{t=0} = \alpha_0,\ \varrho|_{t=0} = \varrho_0,\ z|_{t=0} = z_0,\ (\varrho + z)\vec{u}|_{t=0} = (\varrho_0 + z_0)\vec{u}_0, \tag{7}$$

We consider the general inflow-outflow boundary conditions,

$$\vec{u}|_{\partial\Omega} = \vec{u}_B,\ \varrho|_{\Gamma^{\rm in}} = \varrho_B,\ z|_{\Gamma^{\rm in}} = z_B,\ \alpha|_{\Gamma^{\rm in}} = \alpha_B, \tag{8}$$

where

$$\Gamma^{\rm in} = \left\{ x \in \partial\Omega \ \middle|\ \vec{u}_B \cdot \vec{n} < 0 \right\}. \tag{9}$$

$$\Gamma^{\mathrm{out}} = \left\{ x \in \partial\Omega \;\middle|\; \vec{u}_B \cdot \vec{n} > 0 \right\},$$
$$\Gamma^{0} = \left\{ x \in \partial\Omega \;\middle|\; \vec{u}_B \cdot \vec{n} = 0 \text{ or } \vec{n}(x) \text{ does not exist} \right\}. \tag{10}$$

$$P_{\pm}(s) = a_{\pm} s^{\gamma^{\pm}}, \; \gamma_{\pm} > 0; \tag{11}$$

indeed, in this case

$$P(R, Z) = a_+ R^{\gamma^+} + a_- Z^{\gamma^-}, \; f(s) := f_+(s) = s^{\frac{1}{\gamma^+} - 1},$$
$$g(s) := f_-(s) = (1 - s)^{\frac{1}{\gamma^-} - 1}. \tag{12}$$

We shall however be able to treat in system (3–7) more general functions P, f, g than those being given by (12).

Reformulation of system using the change of variables

$$R := f(\alpha)\varrho,\ Z = g(\alpha)z, \tag{13}$$

$$\begin{aligned} \partial_t \rho + \mathrm{div}_x(\rho\vec{u}) &= 0,\\ \partial_t z + \mathrm{div}_x(z\vec{u}) &= 0,\\ \partial_t R + \mathrm{div}_x(R\vec{u}) &= 0,\\ \partial_t Z + \mathrm{div}_x(Z\vec{u}) &= 0,\\ \partial_t\big((\rho+z)\vec{u}\big) + \mathrm{div}_x((\rho+z)\vec{u}\otimes\vec{u}) + \nabla P(R,Z) &= \mathrm{div}\mathbb{S}(\nabla_x\vec{u}) \end{aligned} \tag{14}$$

with boundary and initial conditions

$$\begin{aligned} \vec{u}|_{\partial\Omega} = \vec{u}_B,\ \rho|_{\Gamma^{\rm in}} := \varrho_B,\ z|_{\Gamma^{\rm in}} = z_B := f(\alpha_B)\varrho_B,\\ Z|_{\Gamma^{\rm in}} = Z_B := g(\alpha_B)z_B, R|_{\Gamma^{\rm in}} = \mathbb{R}_B = f(_B)\rho_B, \end{aligned} \tag{15}$$

$$\rho(0,x) = \varrho_0(x),\ R(0,x) = R_0(x) := f(\alpha_0)\varrho_0(x), \tag{16}$$

$$Z(0,x) = Z_0(x) := g(\alpha_0)z_0(x),\ z(0,x) = z_0(x) := (\varrho_0 + z_0)(x),$$

$$(\rho + z)\vec{u}(0,x) := (\varrho_0 + z_0)\vec{u}_0(x)$$

for unknown quintet $(\rho, z, R, Z, \vec{u})$ of functions defined on the space-time cylinder $Q_T = I \times \Omega$. *academic bi-fluid system*.

Existence of multi- component model

- A. Vasseur, H. Wen, C. Yu,(2019) Global weak solution to the viscous two-fluid model with finite energy, $P(R, Z) = R^\gamma + Z^\beta$
- A. Novotny: Weak solutions for a bi-fluid model for a mixture of two compressible non interacting fluids, 2019
- A. Novotný, M. Pokorný. Weak solutions for some compressible multicomponent fluid models (2020)

General boundary conditions- compressible case

- Valli, Zajaczkowski (strong)
- Novo (2005), Girinon (2010), Plotnikov, Sokolowski (2011) - very restrictive assumptions
- A. Novotny, B.J. Jin, T. Chang,H.J. Choe, M. Yang (general inflow-outflow boundary data, on piecewise regular domains)
- E.Feireisl, A. Novotny (2020)-full system
- Young-Sam Kwon, Antonin Novotny (2020)-Construction via a numerical approximation

Definition

A quintet $(\rho, z, R, Z, \vec{v} = \vec{u} - \vec{u}_B)$ *is a bounded energy weak solution to problem (14–16), if the following holds:*

The quintet belongs to the functional spaces $\rho, z, R, Z \geqslant 0$ *a.e. in* $I \times \Omega$, $(\rho, z, R, Z) \in C_{\rm weak}(\bar{I}; L^\gamma(\Omega)) \cap L^\gamma(I; L^\gamma(\partial\Omega; |\vec{u}_B \cdot \vec{n}| {\rm d}S_x))$ *with some* $\gamma > 1$, $\vec{v} \in L^2(I; W_0^{1,2}(\Omega; R^3))$,
$(\rho + z)|\vec{v}|^2 \in L^\infty(I; L^1(\Omega))$, $P(R, Z) \in L^1(I \times \Omega)$,
$(\rho + z)\vec{u} \in C_{\rm weak}(\bar{I}; L^q(\Omega))$ *with some* $q > 1$.

Definition

(Continuation of Definition)

- *Continuity equations*

$$\int_\Omega r(\tau,\cdot)\varphi(\tau,\cdot)\ \mathrm{d}x - \int_\Omega r_0(\cdot)\varphi(0,\cdot)\ \mathrm{d}x + \int_0^\tau \int_{\Gamma^{\rm out}} r\vec{u}_B\cdot\vec{n}\varphi\ \mathrm{d}S_x\mathrm{d}t$$
$$= \int_0^\tau \int_\Omega \big(r\partial_t\varphi + r\vec{u}\cdot\nabla_x\varphi\big)\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\Gamma^{\rm in}} r_B\vec{u}_B\cdot\vec{n}\varphi\ \mathrm{d}S_x\mathrm{d}t \tag{17}$$

are satisfied
for any $\tau\in[0,T]$ and with any $\varphi\in C^1_c([0,T]\times\overline{\Omega})$, where r stands for ρ, z, R, Z.

Definition

(Continuation of Definition)

- *Momentum equation in weak formulation*

- *The energy inequality holds*

Definition

A quartet $(\alpha, \varrho, z, \vec{u})$ *is a bounded energy weak solution to problem (3–7), if the following holds:*

- $\varrho, z \geqslant 0$ *a.e. in* $I \times \Omega$,
  $(\varrho, z) \in C_{\rm weak}(\overline{I}; L^\gamma(\Omega)) \cap L^\gamma(I; L^\gamma(\partial\Omega; |\vec{u}_B \cdot \vec{n}|{\rm d}S_x))$ *with some* $\gamma > 1$,
  $\alpha \in L^\infty(Q_T) \cap C_{\rm weak}(\overline{I}; L^\gamma(\Omega)) \cap L^\infty(I \times \partial\Omega)$, $0 \leqslant \alpha \leqslant 1$,
  $\vec{v} = \vec{u} - \vec{u}_\infty \in L^2(I; W_0^{1,2}(\Omega; R^3))$,
  $(\varrho + z)|\vec{u}|^2 \in L^\infty(I; L^1(\Omega))$, $P(f(\alpha)\varrho, g(\alpha)z) \in L^1(I \times \Omega)$,
  $(\varrho + z)\vec{u} \in C_{\rm weak}(\overline{I}; L^q(\Omega))$ *with some* $q > 1$.

Definition

- *Continuity equations*

$$\int_\Omega r(\tau,\cdot)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega r_0(\cdot)\varphi(0,\cdot)\,\mathrm{d}x + \int_0^\tau \int_{\Gamma^{\mathrm{out}}} r\vec{u}_B\cdot\vec{n}\varphi\,\mathrm{d}S_x\mathrm{d}t$$
$$= \int_0^\tau \int_\Omega \big(r\partial_t\varphi + r\vec{u}\cdot\nabla_x\varphi\big)\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\Gamma^{\mathrm{in}}} r_B\vec{u}_B\cdot\vec{n}\varphi\mathrm{d}S_x\mathrm{d}t \tag{18}$$

are satisfied for all $\tau\in[0,T]$ *and with any* $\varphi\in C_c^1([0,T]\times\overline{\Omega})$*, where* r *stands for* ϱ*,* z*.*

- *Transport equation*

$$\int_\Omega \alpha(\tau,\cdot)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \alpha_0(\cdot)\varphi(0,\cdot)\,\mathrm{d}x + \int_0^\tau \int_{\Gamma_{\mathrm{out}}} \alpha\vec{u}_B\cdot\vec{n}\varphi\mathrm{d}S_x\mathrm{d}t$$
$$= \int_0^T \int_\Omega \big(\alpha\partial_t\varphi + \alpha\vec{u}\cdot\nabla_x\varphi - \varphi\alpha\mathrm{div}\vec{u}\big)\mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \int_{\Gamma_{\mathrm{in}}} \alpha_B\vec{u}_B\cdot\vec{n}\varphi\mathrm{d}S_x\mathrm{d}t \tag{19}$$

holds for all $\tau\in[0,T]$ *with any* $\varphi\in C_c^1([0,T]\times\overline{\Omega})$*.*

Definition

- *weak form of Momentum equation*
- *The energy inequality holds*

Ω is a bounded Lipschitz domain

- *Boundary and initial conditions:*

$$0 < r_B \in C_c(R^3),\ \vec{u}_B \in C_c^1(R^3),\ , \quad r \text{ stands for } \rho,\ R,\ Z,\ \Sigma. \tag{20}$$

$$0 < R_0 \in L^\gamma(\Omega),\ \gamma \geqslant 2,\ Z_0 \in L^\beta(\Omega) \ \text{ if } \beta > \gamma, \quad (\rho_0 + z_0)|\vec{u}_0|^2 \in L^1(\Omega). \tag{21}$$

$$(R_0, Z_0)(x) \in \overline{\mathcal{O}},\ \underline{F} R_0(x) \leqslant \rho_0(x) \leqslant \underline{F} R_0(x), \quad \underline{G} Z_0(x) \leqslant z_0(x) \leqslant \underline{G} Z_0(x), \tag{22}$$

$$(R_B, Z_B)(x) \in \overline{\mathcal{O}},\ \underline{F} R_B(x) \leqslant \rho_B(x) \leqslant \underline{F} R_B(x), \quad \underline{G} Z_B(x) \leqslant z_B(x) \leqslant \underline{G} Z_B(x),$$

In the above $0 < \underline{F} < \overline{F}$, $0 < \underline{G} < \overline{G}$ and

$$\mathcal{O} := (R, Z) \in R^2 \,|\, \underline{a} R < Z < \overline{a} R\} \tag{23}$$

with some $0 \leqslant \underline{a} < \overline{a}$.

- *Domain*

  Ω is a bounded domain with admissible inflow-outflow boundary relative to$\vec{u}_B$.

Regularity and growth of the pressure function P:

$$P \in C^1(\overline{\mathcal{O}}) \cap C^2(\mathcal{O}),\ P(0,0) = 0. \tag{24}$$

$$R^\gamma + Z^\beta - 1 \lesssim P(R,Z) \lesssim R^\gamma + Z^\beta + 1 \text{ in } \mathcal{O}, \tag{25}$$

$$0 \leqslant \partial_Z P(R,Z) \lesssim R^{\underline{\gamma}-1} + R^{\overline{\gamma}-1} \text{ in } \mathcal{O} \text{ with some } \underline{\gamma} \in (0,1],\ 1 \leqslant \overline{\gamma} < \gamma + \gamma_{\mathrm{Bog}} \tag{26}$$

where

$$\gamma \geqslant 2,\ \beta > 0,\ \gamma_{\mathrm{Bog}} = \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\},$$

$$R^{\gamma-1} \lesssim \partial_R P(R,Z) \text{ in } \mathcal{O}. \tag{27}$$

$$H \text{ is convex on } \mathcal{O}. \tag{28}$$

Theorem

Under Hypotheses (20–28), problem (14–16) admits at least one bounded energy weak solution in the sense of Definition 1. *Moreover, for all* $t \in \bar{I}$, $(R(t,x), Z(t,x)) \in \overline{\mathcal{O}}$, $\underline{F}R(t,x) \leqslant \varrho(t,x) \leqslant \overline{F}R(t,x)$ *and* $\underline{G}Z(t,x) \leqslant z(t,x) \leqslant \overline{G}Z(t,x)$ *for a.a.* $x \in \Omega$, *and further for a.a.* $(t,x) \in I \times \partial\Omega$, $(R(t,x), Z(t,x)) \in \overline{\mathcal{O}}$, $\underline{F}R(t,x) \leqslant \varrho(t,x) \leqslant \overline{F}R(t,x)$ *and* $\underline{G}Z(t,x) \leqslant z(t,x) \leqslant \overline{G}Z(t,x)$. *Finally,* $\varrho, z, R, Z \in C(\bar{I}; L^1(\Omega))$, $(\varrho + z)\vec{u} \in C_{weak}([0, T); L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3))$ *and* $P(R,Z) \in L^q(I \times \Omega)$ *for some* $q > 1$ *and* $Z \in C_{weak}([0, T); L^\beta(\Omega)) \cap L^\beta(I, L^\beta(\partial\Omega; |\vec{u}_B \cdot \vec{n}| \mathrm{d}S_x))$ *if* $\beta > \gamma$.

The one velocity Baer-Nunziato type system (3–8) and reads:

Theorem

Suppose that $f, g \in C^1(0,1)$ are two strictly monotone and strictly positive functions on interval $(0,1)$ and that the boundary conditions ϱ_B, z_B, $\vec{u}_B$ satisfy conditions (20). Let $\gamma \geqslant 2$, $\beta > 0$ and, in addition,

$$\alpha_B \in C(\overline{\Omega}),\ 0 < \underline{\alpha} \leqslant \alpha_B \leqslant \overline{\alpha} < 1,\ (f(\alpha_B)\varrho_B, g(\alpha_B)z_B)(x) \in \overline{\mathcal{O}},$$

$$\alpha_0 \in L^\infty(\Omega),\ 0 < \underline{\alpha} \leqslant \alpha_0 \leqslant \overline{\alpha} < 1,\ (f(\alpha_0)\varrho_0, g(\alpha_0)z_0)(x) \in \overline{\mathcal{O}}$$

$$0 < \varrho_0 \in L^\gamma(\Omega),\ z_0 \in L^\beta(\Omega)\ \ \textit{if}\ \beta > \gamma,\ (\varrho_0 + z_0)|\vec{u}_0|^2 \in L^1(\Omega). \tag{29}$$

Theorem

(continuation of Theorem)
Suppose that the domain Ω *is a bounded Lipschitz domain with the admissible inflow-outflow boundary with respect to* $\vec{u}_B$*, cf. (22). Finally suppose that the pressure* P *and its Helmholtz function* H *verify hypotheses (24–28). Then the problem (3–8) admits at least one bounded energy weak solution in the sense of Definition 4. Moreover, for all* $t \in \bar{I}$, $(f(\alpha)\varrho(t,x), g(\alpha)z(t,x)) \in \overline{\mathcal{O}}$ *and* $\underline{\alpha} \leqslant \alpha(t,x) \leqslant \overline{\alpha}$ *for a.a.* $x \in \Omega$*, and further for a.a.* $(t,x) \in I \times \partial\Omega$, $(f(\alpha)\varrho(t,x), g(\alpha)z(t,x)) \in \overline{\mathcal{O}}$ *and* $\underline{\alpha} \leqslant \alpha(t,x) \leqslant \overline{\alpha}$*. Finally,* $\alpha, \varrho, z \in C(\bar{I}; L^1(\Omega))$, $z \in C_{weak}(\bar{I}; L^\beta(\Omega)) \cap L^\beta(I, L^\beta(\partial\Omega; |\vec{u}_B \cdot \vec{n}| \mathrm{d}S_x))$ *if* $\beta > \gamma$,
$(\varrho + z)\vec{u} \in C_{weak}([0,T); L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^3))$ *and* $P(f(\alpha)\varrho, g(\alpha)z) \in L^q(I \times \Omega)$ *with some* $q > 1$.

The main steps in our approach are the following:

- - develop the theory of renormalized solutions to families of transport equations with non-homogenous boundary data.
  - Passage (via renormalization) from the (two) continuity equations to a pure transport equation.
  - Passage (via renormalization) from the (two) transport equations and a continuity equation to a continuity equation

  and two consequences of these results
  - *Almost compactness of the ratio of two solutions of continuity equations.*
  - *Almost uniqueness of the solutions to the pure transport equation.*

- *Step 1* Construction of a particular outer neighborhoods of $\Gamma^{\rm in}$, $\Gamma^{\rm out}$ and Γ^0
  we construct convenient outer neighborhoods $\mathcal{U}^+$ of Γ^0 and of each component $\Gamma \subset \Gamma^{\rm out/in}$. In particular, the projection operator$P$ must be sufficiently regular on the outer neighborhoods of components of $\Gamma^{\rm out/in}$.
- *Step 2* Extension of the density beyond the inflow boundary
- *Step 3* Extension of the density beyond the outflow boundary
- *Step 4.*, for Γ^0.
- *Step 5.* Continuity extended
- *Step 6.* Generalization of the DiPerna-Lions
- *Step 7.* Time integrated renormalized continuity equation ($\tau \in L^\infty(\bar{I}, L^\gamma)$ then $\tau \in C_{\rm weak}(\bar{I}, L^\gamma)$.)

- From continuity to pure transport equation
- Almost uniqueness to the pure transport equation
- Almost compactness for the continuity equation

- The main difficulty to pass to the limit in the non-linear pressure term $P(R, Z)$.
- Derivation of the effective viscous flux identity.
- Eliminating oscillations in the sequence of densities by using the theory of renormalized solutions due to DiPerna-Lions which must be modified to accommodate the non homogenous boundary conditions and renormalizing functions of several variables.

Vasseur, Wen Yu (2019)

$$\begin{aligned}&\partial_t Z + \mathrm{div}_x(Z\vec{u}) = 0, Z(0) = Z_0\\&\partial_t R + \mathrm{div}_x(R\vec{u}) = 0, R(0) = R_0\\&\partial_t (R+Z)\vec{u}) + \mathrm{div}_x((R+Z)\vec{u}\otimes\vec{u}) + \nabla P(R,Z) = \mathrm{div}_x \mathbb{S}(\nabla_x \vec{u})\end{aligned} \tag{30}$$

$$P(R,Z) = R^\gamma + Z^\beta$$

.

Almost compactness: Let $(R_n, \vec{u}_n)$, $(Z_n, \vec{u}_n)$ satisfy continuity equation, $0 \leqslant R_n \rightharpoonup_* R$, $0 \leqslant Z_n \rightharpoonup_* Z$ in $L^\infty(I, L^2(\Omega))$, $\vec{u}_n \rightharpoonup_* \vec{u}$ $\in L^2(I, W_0^{1,2}(\Omega))$. Let $0 \leqslant s_n, s \leqslant C$, $s_n R_n = Z_n, sR = Z$ then

$$\int_0^T R_n (s_n - s)^2 \to 0.$$

Application:

$$P(R_n, Z_n) = P(R_n, s_n R_n), s_n = Z_n / R_n$$

$$P(R_n . s_n R_n) = P(R_n, sR_n) + [P(R_n, s_n R_n) - P(R_n s_n, R_n) P(R_n, sR_n)]$$
$$= \Pi(R_n, t, x) + \partial_z P(R_n, s_n) R_n (s_n - s).$$

Passage from transform system to original system- to get pure transport equation- (2020) Novotný, Pokorný (Almost uniqueness)

Lemma

Let $u \in L^2(I, W_0^{1,2}(\Omega, R^3))$, $0 \leqslant s_i \in L^\infty(Q_T), i = 1, 2$ be two weak solution of pure transport equations (up to boundary). Then $s_i \in C(\bar{I}, L^1(\Omega))$. If $s_1(0, .) = s_2(0, .)$ then

$$\text{for all } \tau \in \bar{I}, s_1(\tau, .) = s_2(\tau, .) a.a. x \in \{\rho(\tau, .) > 0\}$$

where ρ is any time integrated weak solution to the continuity equation with the same transport velocity in the class

$$0 \leqslant \rho \in C(\bar{I}, L^1(\Omega) \cap L^2(Q_T) \cap L^\infty(I, L^p(\Omega)), p > 1.$$

$\alpha = f^{-1}(R/\varrho)$ and $\tilde{\alpha} = g^{-1}(Z/z)$ verify transport equation with the same initial condition α_0 and the same boundary condition α_B.

On existence of weak solutions to a Baer–Nunziato type system-full system

M.Kalousek, Š.N,

Problem (P):

$$\begin{aligned}
\partial_t \alpha_\pm + v_I \cdot \nabla \alpha_\pm &= 0, \\
\partial_t(\alpha_\pm \rho_\pm) + \operatorname{div}(\alpha_\pm \rho_\pm u_\pm) &= 0, \\
\partial_t(\alpha_\pm \rho_\pm u_\pm) + \operatorname{div}(\alpha_\pm \rho_\pm u_\pm \otimes u_\pm) + \nabla \left(\alpha_\pm \mathfrak{P}_\pm(\rho_\pm, \vartheta)\right) \\
-P_I \nabla \alpha_\pm - \operatorname{div}\left(\alpha_\pm \mathbb{S}(\vartheta, \nabla u_\pm)\right) &= 0, \\
\partial_t(\alpha_\pm \rho_\pm \mathfrak{e}_\pm(\rho_\pm, \vartheta)) + \operatorname{div}(\alpha_\pm \rho_\pm \mathfrak{e}_\pm(\rho_\pm, \vartheta) u_\pm) + \operatorname{div}(\alpha_\pm q_\pm(\vartheta, \nabla \vartheta)) \\
+\alpha_\pm \mathfrak{P}_\pm(\rho_\pm) \operatorname{div} u_\pm - \alpha_\pm \mathbb{S}(\vartheta, \nabla u_\pm) \cdot \nabla u_\pm - P_I(v_I - u_\pm) \cdot \nabla \alpha_\pm &= 0, \\
\alpha_+ + \alpha_- &= 1.
\end{aligned}$$

Kwon, Y.-S. and Novotný, A. and Arthur Cheng, C.H.: On weak solutions to a dissipative Baer–Nunziato–type system for a mixture of two compressible heat conducting gases, Math. Models Methods Appl. Sci. 30 no. 8, 1517–1553, 2020.

$$
\begin{aligned}
q_\pm(\vartheta, \nabla\vartheta) =& -\kappa_\pm(\vartheta)\nabla\vartheta,\\
\mathbb{S}_\pm(\vartheta, \nabla u) =& \mu_\pm(\vartheta)\left(\nabla u + (\nabla u)^\top - \frac{2}{d}\operatorname{div} u\mathbb{I}\right) + \eta_\pm(\vartheta)\operatorname{div} u\mathbb{I},
\end{aligned}
$$

$$
\begin{aligned}
\partial_t(\alpha_\pm\rho_\pm\mathfrak{s}_\pm(\rho_\pm,\vartheta)) + \operatorname{div}(\alpha_\pm\rho_\pm\mathfrak{s}_\pm(\rho_\pm,\vartheta)u_\pm) + \operatorname{div}\left(\frac{\alpha_\pm}{\vartheta}q_\pm(\vartheta,\nabla\vartheta)\right)\\
-\frac{\alpha_\pm}{\vartheta}\mathbb{S}_\pm(\vartheta,\nabla u_\pm)\cdot\nabla u_\pm - \frac{\alpha_\pm}{\vartheta^2}q_\pm\cdot\nabla\vartheta - \frac{1}{\vartheta}P_I(v_I - u_\pm)\cdot\nabla\alpha_\pm = 0
\end{aligned}
\tag{0.1}
$$

$$\kappa_\pm = \kappa,\ \ \mu_\pm = \mu,\ \ \eta_\pm = \eta,\ \ v_I = u_\pm = u.$$

$$\mathbb{S}(\vartheta, \nabla u) = \mu(\vartheta)\left(\nabla u + (\nabla u)^\top - \frac{2}{d}\operatorname{div} u\mathbb{I}\right) + \eta(\vartheta)\operatorname{div} u\mathbb{I}. \quad (0.2)$$

The pressures $\mathfrak{P}_\pm$:

$$\mathfrak{P}_\pm(r, \vartheta) = \frac{b}{3}\vartheta^4 + P_\pm(r, \vartheta),$$

The internal energies $\mathfrak{e}_\pm$

$$\mathfrak{e}_\pm(r, \vartheta) = \frac{b}{r}\vartheta^4 + e_\pm(r, \vartheta),$$

b - the Stefan-Boltzman constant in physics.

$$P_\pm(r,\vartheta) = r^2\partial_r e_\pm(r,\vartheta) + \vartheta\partial_\vartheta P_\pm(r,\vartheta). \tag{0.3}$$

The specific entropies $\mathfrak{s}_\pm$ are defined via

$$\mathfrak{s}_\pm(r,\vartheta) = \frac{4b}{3r}\vartheta^3 + s_\pm(r,\vartheta)$$

and $s_\pm$ satisfy

$$\partial_\vartheta s_\pm = \frac{1}{\vartheta}\partial_\vartheta e_\pm,\ \partial_r s_\pm = -\frac{1}{r^2}\partial_\vartheta P_\pm. \tag{0.4}$$

The relations (0.3) and (0.4) are equivalent to the Gibbs equations

$$\vartheta\partial_r s_\pm = \partial_r e_\pm - \frac{P_\pm}{r^2},\ \vartheta\partial_\vartheta s_\pm = \partial_\vartheta e_\pm \tag{0.5}$$

We introduce new pressure functions $\mathrm{P}_\pm$ via

$$\mathrm{P}_\pm(f_\pm(\alpha_\pm)\alpha_\pm r,\vartheta) = \alpha_\pm P_\pm(r,\vartheta) \tag{0.6}$$

with some functions $f_\pm$ whose properties will be specified later.

$$\alpha = \alpha_+,\ \ \rho = \alpha\rho_+,\ \ z = (1-\alpha)\rho_-$$

$$\partial_t \alpha + u\cdot\nabla\alpha = 0,$$
$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$
$$\partial_t z + \mathrm{div}(zu) = 0,$$
$$\partial_t\left((\rho+z)u\right) + \mathrm{div}\left((\rho+z)u\otimes u\right) +$$
$$+\nabla\left(\tfrac{b}{3}\vartheta^4 + \mathrm{P}_+(f_+(\alpha)\rho,\vartheta) + \mathrm{P}_-(f_-(1-\alpha)z,\vartheta)\right) = \mathrm{div}(\mathbb{S}(\vartheta,\nabla u)),$$
$$\partial_t\left(\rho\mathfrak{s}_+\left(\tfrac{\rho}{\alpha},\vartheta\right) + z\mathfrak{s}_-\left(\tfrac{z}{1-\alpha},\vartheta\right)+\right)$$
$$+\,\mathrm{div}\left(\left(\rho\mathfrak{s}_+\left(\tfrac{\rho}{\alpha},\vartheta\right) + z\mathfrak{s}_-\left(\tfrac{z}{1-\alpha},\vartheta\right)\right)u\right)$$
$$-\,\mathrm{div}\left(\frac{\kappa(\vartheta)}{\vartheta}\nabla\vartheta\right) = \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla u)\cdot\nabla u - \frac{\kappa(\vartheta)}{\vartheta}|\nabla\vartheta|^2\right)$$

The system is endowed with the initial conditions

$$\alpha(0,\cdot) = \alpha_0,\ \rho(0,\cdot) = \rho_0,\ z(0,\cdot) = z_0,\ (\rho+z)u(0,\cdot) = (\rho_0+z_0)u_0,\ \vartheta(0,\cdot) \tag{0.7}$$

and with the boundary conditions

$$\nabla\vartheta \cdot n = 0 \text{ on } (0,T) \times \partial\Omega \tag{0.8}$$

and either

$$u \cdot n = 0 \text{ and } \mathbb{S}(\vartheta, \nabla u)n \times n = 0 \text{ or } u = 0 \text{ on } (0,T) \times \partial\Omega \tag{0.9}$$

for the case of the complete slip, the no slip respectively.

$$\mathsf{e}_\pm(f_\pm(\alpha_\pm)\alpha_\pm r, \vartheta) = \frac{1}{f_\pm(\alpha_\pm)} e_\pm(r, \vartheta), \\ \mathsf{s}_\pm(f_\pm(\alpha_\pm)\alpha_\pm r, \vartheta) = \frac{1}{f_\pm(\alpha_\pm)} s_\pm(r, \vartheta). \tag{0.10}$$

Introducing the changes of variables $\mathfrak{r} = f_+(\alpha)\rho = f_+(\alpha)\alpha\rho_+$, $\mathfrak{z} = f_-(1-\alpha)z = f_-(1-\alpha)(1-\alpha)\rho_-$

$$
\begin{aligned}
\mathsf{e}_+(\mathfrak{r},\vartheta) =& \frac{1}{f_+(\alpha)} e_+\left(\frac{\mathfrak{r}}{f_+(\alpha)\alpha},\vartheta\right),\\
\mathsf{s}_+(\mathfrak{r},\vartheta) =& \frac{1}{f_+(\alpha)} s_+\left(\frac{\mathfrak{r}}{f_+(\alpha)\alpha},\vartheta\right),\\
\mathsf{P}_+(\mathfrak{r},\vartheta) =& \alpha_+ P_+\left(\frac{\mathfrak{r}}{f_+(\alpha)\alpha},\vartheta\right),\\
\mathsf{e}_-(\mathfrak{z},\vartheta) =& \frac{1}{f_-(1-\alpha)} e_-\left(\frac{\mathfrak{z}}{f_-(1-\alpha)(1-\alpha)},\vartheta\right),\\
\mathsf{s}_-(\mathfrak{z},\vartheta) =& \frac{1}{f_-(1-\alpha)} s_-\left(\frac{\mathfrak{z}}{f_-(1-\alpha)(1-\alpha)},\vartheta\right),\\
\mathsf{P}_-(\mathfrak{z},\vartheta) =& (1-\alpha) P_-\left(\frac{\mathfrak{z}}{f_-(1-\alpha)(1-\alpha)},\vartheta\right).
\end{aligned}
\tag{0.11}
$$

$$
\begin{aligned}
\mathfrak{e}_+(\mathfrak{r},\alpha,\vartheta) =& \frac{b}{\mathfrak{r}}\vartheta^4\alpha + \mathsf{e}_+(\mathfrak{r},\vartheta),\\
\mathfrak{e}_-(\mathfrak{z},\alpha,\vartheta) =& \frac{b}{\mathfrak{z}}\vartheta^4(1-\alpha) + \mathsf{e}_-(\mathfrak{z},\vartheta),\\
\mathfrak{s}_+(\mathfrak{r},\alpha,\vartheta) =& \frac{4b}{3\mathfrak{r}}\vartheta^3\alpha + \mathsf{s}_+(\mathfrak{r},\vartheta),\\
\mathfrak{s}_-(\mathfrak{z},\alpha,\vartheta) =& \frac{4b}{3\mathfrak{z}}\vartheta^3(1-\alpha) + \mathsf{s}_-(\mathfrak{z},\vartheta).
\end{aligned}
\tag{0.12}
$$

The Gibbs relations for quantities $\mathsf{s}_\pm$, $\mathsf{e}_\pm$ and $\mathsf{P}_\pm$ have the form

$$
\begin{aligned}
\partial_{\mathfrak{r}}\mathsf{e}_+ = \vartheta\partial_{\mathfrak{r}}\mathsf{s}_+ + \frac{\mathsf{P}_+}{\mathfrak{r}^2},\ \ \partial_\vartheta\mathsf{e}_+ = \vartheta\partial_\vartheta\mathsf{s}_+\\
\partial_{\mathfrak{z}}\mathsf{e}_- = \vartheta\partial_{\mathfrak{z}}\mathsf{s}_- + \frac{\mathsf{P}_-}{\mathfrak{z}^2},\ \ \partial_\vartheta\mathsf{e}_- = \vartheta\partial_\vartheta\mathsf{s}_-.
\end{aligned}
\tag{0.13}
$$

$\Sigma = \rho + z$ and $\mathscr{P}(\mathfrak{r}, \mathfrak{z}, \vartheta) = \frac{b}{3}\vartheta^4 + \mathsf{P}_+(\mathfrak{r}, \vartheta) + \mathsf{P}_-(\mathfrak{z}, \vartheta)$

$$
\begin{aligned}
\partial_t \xi + \operatorname{div}(\xi u) &= 0, \\
\partial_t \mathfrak{r} + \operatorname{div}(\mathfrak{r} u) &= 0, \\
\partial_t \mathfrak{z} + \operatorname{div}(\mathfrak{z} u) &= 0, \\
\partial_t \Sigma + \operatorname{div}(\Sigma u) &= 0, \\
\partial_t (\Sigma u) + \operatorname{div}(\Sigma u \otimes u) + \nabla \mathscr{P}(\mathfrak{r}, \mathfrak{z}, \vartheta) = \operatorname{div} \mathbb{S}(\vartheta, \nabla u) &= 0,
\end{aligned}
$$

$$
\partial_t \left(\left(\mathfrak{r}\mathfrak{s}_+ (\mathfrak{r}, \vartheta) + \mathfrak{z}\mathfrak{s}_- (\mathfrak{z}, \vartheta) \right) u \right) + \operatorname{div} \left(\left(\mathfrak{r}\mathfrak{s}_+ (\mathfrak{r}, \vartheta) + \mathfrak{z}\mathfrak{s}_- (\mathfrak{z}, \vartheta) \right) u \otimes u \right)
$$

$$
+ \operatorname{div} \left(\frac{\kappa(\vartheta)}{\vartheta} \nabla \vartheta \right) - \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla u) \cdot \nabla u + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right) = 0 \tag{0.14}
$$

in Q_T

with initial conditions

$$
\begin{aligned}
\xi(0,\cdot) =& \alpha_0\rho_+(0,\cdot),\\
\mathfrak{r}(0,\cdot) =& f_+(\alpha_0)\alpha_0\rho_+(0,\cdot),\\
\mathfrak{z}(0,\cdot) =& f_-(1-\alpha_0)(1-\alpha_0)\rho_-(0,\cdot),\\
\Sigma u(0,\cdot) =& (\alpha\rho_+ + (1-\alpha)\rho_-)u(0,\cdot),\\
\left(\mathfrak{r}\mathfrak{s}_+(\mathfrak{r},\vartheta) + \mathfrak{z}\mathfrak{s}_-(\mathfrak{z},\vartheta)\right)(0,\cdot) =& \frac{4b}{3}\vartheta_0^3 + \alpha_0\rho_{+,0}\alpha_0 s_+(\rho_{+,0},\vartheta_0) + (1-\alpha_0)\rho_-
\end{aligned}
\tag{0.15}
$$

and boundary conditions

$$
\nabla\vartheta\cdot n = 0,\ \ u = 0 \text{ or } u\cdot n = 0 \text{ and } (\mathbb{S}n)\times n = 0 \text{ on } (0,T)\times\partial\Omega.
$$

Regularity of initial data

$$\mathfrak{r}_0 \in L^{\gamma_+}(\Omega), \gamma_+ \geq \frac{9}{5}, \int_\Omega \mathfrak{r}_0 > 0, \mathfrak{z}_0 \in L^{\gamma_-}(\Omega) \text{ if } \gamma_- > \gamma_+,$$

$$\int_\Omega \mathfrak{z}_0 > 0,$$

$$(\mathfrak{r}_0, \mathfrak{z}_0)(x) \in \overline{\mathcal{O}_{\underline{a},\overline{a}}},$$

$$(\mathfrak{r}_0 + \mathfrak{z}_0, \Sigma_0)(x) \in \overline{\mathcal{O}_{\underline{b},\overline{b}}}, (\mathfrak{r}_0, \xi_0)(x) \in \overline{\mathcal{O}_{\underline{d},\overline{d}}} \text{ for a.a. } x \in \Omega,$$

$$(\Sigma u)_0 \in L^1(\Omega), \frac{|(\Sigma u)_0|^2}{\Sigma_0} \in L^1(\Omega), \vartheta_0 \in L^4(\Omega), \log \vartheta_0 \in L^2(\Omega),$$

$$\mathfrak{r}_0 \mathfrak{s}_+ (\mathfrak{r}_0, \vartheta_0) + \mathfrak{z}_0 \mathfrak{s}_- (\mathfrak{z}_0, \vartheta_0) \in L^1(\Omega),$$

$$\mathfrak{r}_0 \mathfrak{e}_+ (\mathfrak{r}_0, \vartheta_0) + \mathfrak{z}_0 \mathfrak{e}_- (\mathfrak{z}_0, \vartheta_0) \in L^1(\Omega),$$

$$\mathcal{O}_{\underline{a},\overline{a}} = \{(r, z) \in \mathbb{R}^2 : r \in (0, \infty), \underline{a}r < z < \overline{a}r\} \qquad (0.16)$$

$$0 < \underline{a} < \overline{a} < \infty, 0 < \underline{b} < \overline{b} < \infty, 0 < \underline{d} < \overline{d} < \infty.$$

Transport coefficients $\mu, \eta, \kappa \in C^1([0,\infty))$ satisfy

$$\overline{c}^{-1}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{c}(1+\vartheta), \tag{0.17}$$

$$0 \leq \eta(\vartheta) \leq \overline{c}(1+\vartheta), \tag{0.18}$$

$$\overline{c}^{-1}(1+\vartheta)^{\beta} \leq \kappa(\vartheta) \leq \overline{c}(1+\vartheta)^{\beta}, \ \beta > \frac{4}{3}. \tag{0.19}$$

Structure and regularity of pressure, internal energy and specific entropy
(0.11).

Regularity

$$\mathrm{P}_\pm, \mathrm{e}_\pm \in C([0,\infty)^2) \cap C^2((0,\infty)^2),\ \mathrm{P}_\pm(0,0) = 0,\ \mathrm{e}_\pm(0,0) = 0;$$

Thermodynamic stability conditions

$$\partial_r \mathrm{P}_\pm(r,\vartheta) > 0,\ \partial_\vartheta \mathrm{e}_\pm(r,\vartheta) > 0 \text{ for all } (r,\vartheta) \in (0,\infty)^2; \tag{0.20}$$

Structure of specific entropy

$$\mathrm{s}_\pm(r,\vartheta) = \bar{s}_\pm \log\vartheta - \tilde{s}_\pm \log r + \mathcal{s}_\pm(r,\vartheta) \tag{0.21}$$

for some constants $\bar{s}_\pm, \tilde{s}_\pm \geq 0$;

Structure of the pressure, the precise meaning of (0.6)

There are $f_\pm \in C^1(I_\pm; [0,\infty)^{N_\pm})$, $I_+ = [\underline{\alpha}, \overline{\alpha}]$, $I_- = [1-\overline{\alpha}, 1-\underline{\alpha}]$, $N_\pm \in \mathbb{N}$ and $\mathsf{P}_\pm : [0,\infty)^2 \to [0,\infty)$ such that

$$\alpha_\pm P_\pm(r,\vartheta) = \mathsf{P}_\pm(f_\pm(\alpha_\pm)\alpha_\pm r, \vartheta) \text{ for all } \alpha_\pm \in I_\pm, r \in [0,\infty), \vartheta > 0.$$

Moreover, the function $p : [0,\infty) \times [\underline{a}, \overline{a}] \times (0,\infty) \to \mathbb{R}$ defined as

$$p(r,\zeta,\vartheta) = \mathsf{P}_+(r,\vartheta) + \mathsf{P}_-(r\zeta,\vartheta)$$

is for any $(\zeta,\vartheta) \in [\underline{a}, \overline{a}] \times (0,\infty)$ a nondecreasing function of r on $[0,\infty)$.

Growth conditions for the pressure functions, the internal energies and the specific entropies

Growth conditions for the internal energies $\mathrm{e}_\pm$

$$\overline{c}^{-1}(r^{\gamma_\pm - 1} - 1) \le \mathrm{e}_\pm(r,\vartheta) \le \overline{c}(r^{\gamma_\pm - 1} + d^e_\pm \vartheta^{\omega^e_\pm}) \text{ for all } (r,\vartheta) \in (0,\infty) \tag{0.22}$$

where $\gamma_\pm > 0$ and $\omega^e_\pm > 0$ satisfy

$$\frac{1}{\gamma - 1} + \frac{\omega^e_\pm}{p_\beta} < 1.$$

We denote $\gamma = \max\{\gamma_+, \gamma_-\} \ge \overline{\gamma}_\beta$ where $\overline{\gamma}_\beta$ solves

$$\overline{\gamma}_\beta + \min\left\{\frac{2\overline{\gamma}_\beta}{3} - 1, \overline{\gamma}_\beta\left(\frac{1}{2} - \frac{1}{p_\beta}\right)\right\} = 2 \text{ with } p_\beta = \beta + \frac{8}{3}. \tag{0.23}$$

Let us notice that $\overline{\gamma}_\beta \in \left(\frac{9}{5}, 2\right)$.

Growth conditions for the specific entropies $\mathscr{s}_\pm$

$$|\mathscr{s}_\pm| \leq \overline{c}(r^{\gamma^s_\pm - 1} + d^s_\pm \vartheta^{\omega^s_\pm}) \text{ for all } (r,\vartheta) \in (0,\infty)^2, \qquad (0.24)$$

where $\gamma^s_\pm, \omega^s_\pm > 0$ and $\gamma = \max\{\gamma^s_+, \gamma^s_-\}$,
$\omega^s = \max\{d^s_+\omega^s_+, d^s_-\omega^s_-\}$ obey

$$\frac{\omega^s}{4} + \frac{\gamma^s}{\gamma - 1} < \frac{5}{6}.$$

Growth conditions for the pressures $\mathrm{P}_\pm$

$$\overline{c}^{-1}(r^{\gamma_\pm} - 1) \leq \mathrm{P}_\pm(r,\vartheta) \leq \overline{c}(r^{\gamma_\pm} + d^P_\pm r^{\gamma^P_\pm} \vartheta^{\omega^P_\pm}) \text{ for all } (r,\vartheta) \in (0,\infty)^2 \qquad (0.25)$$

where $\gamma^P_\pm, \omega^P_\pm > 0$ and $\gamma^P = \max\{d^P_+\gamma^P_+, d^P_-\gamma^P_-\}$,
$\omega^P = \max\{d^P_+\omega^P_+, d^P_-\omega^P_-\}$ obey

$$\frac{\gamma^P}{\gamma} + \frac{\omega^P}{p_\beta} < 1.$$

If $\gamma = \overline{\gamma}_\beta$ we assume that

$$p(\rho,\zeta,\vartheta) = \pi(\zeta,\vartheta)\rho^\gamma + \mathscr{p}(\rho,\zeta,\vartheta) \tag{0.26}$$

for some $\pi \in L^\infty((\underline{a},\overline{a}) \times (0,\infty))$ such that

$$\operatorname{ess\,inf}_{(\underline{a},\overline{a})\times(0,\infty)} \pi \geq \underline{\pi} > 0$$

and some $\mathscr{p} \in C([0,\infty) \times [\underline{a},\overline{a}] \times [0,\infty))$ such that the mapping $\rho \mapsto \mathscr{p}(\rho,\zeta,\vartheta)$ is for all (ζ,ϑ) continuous and nondecreasing on $[0,\infty)$.Growth conditions for the entropy and the internal energy variations

$$r|\partial_r \mathscr{s}_\pm|(r,\vartheta) \leq \overline{c}(r^{\underline{\Gamma}^s - 1} + r^{\overline{\Gamma}^s - 1})(1 + \overline{d}^s_\pm \vartheta)^{\overline{\omega}^s}, \tag{0.27}$$

$$r|\partial_\vartheta \mathscr{s}_\pm|(r,\vartheta) \leq \overline{c}(1 + r^{\tilde{\Gamma}^s})(1 + \tilde{d}^s_\pm \vartheta)^{\tilde{\omega}^s}, \tag{0.28}$$

$$r|\partial_r \mathrm{e}_\pm|(r,\vartheta) \leq \overline{c}(r^{\underline{\Gamma}^e - 1} + r^{\overline{\Gamma}^e - 1})(1 + \overline{d}^e_\pm \vartheta)^{\overline{\omega}^e}, \tag{0.29}$$

$$r|\partial_\vartheta \mathrm{e}_\pm|(r,\vartheta) \leq \overline{c}(1 + r^{\tilde{\Gamma}^e})(1 + \tilde{d}^e_\pm \vartheta)^{\tilde{\omega}^e} \tag{0.30}$$

for all $(r,\vartheta) \in (0,\infty)^2$.

In the above inequalities we consider $\underline{\Gamma}^a \in [0,1)$, $\overline{\Gamma}^a \in (0,\overline{\gamma})$, $\tilde{\Gamma}^a \in (0,\overline{\gamma})$ such that

$$\frac{\max\{\underline{\Gamma}^a, \overline{\Gamma}^a\}}{\overline{\gamma}} + \frac{\overline{\omega}^a}{p_\beta} < 1, \;\; \frac{\tilde{\Gamma}^a}{\overline{\gamma}} + \frac{\tilde{\omega}^a + 1}{p_\beta} < 1,$$

where the supersript a stand for e and s, $\overline{\omega}^a = \max\{d_+^a \overline{\omega}_+^a, d_-^a \overline{\omega}_-^a\}$,

$$\overline{\gamma} = \gamma + \gamma_{BOG}, \;\; \gamma_{BOG} = \min\left\{\frac{2}{3}\gamma - 1, \left(\frac{1}{2} - \frac{1}{p_\beta}\right)\gamma\right\}. \qquad (0.31)$$

Growth conditions for the pressure variations

$$
\begin{aligned}
|\partial_r \mathsf{P}_\pm|(r,\vartheta) \leq& \overline{c}(r^{\underline{\Gamma}^P - 1} + r^{\overline{\Gamma}^P - 1})(1 + \overline{d}^P_\pm \vartheta)^{\overline{\omega}^P}, \\
|\partial_\vartheta \mathsf{P}_\pm|(r,\vartheta) \leq& \overline{c}(1 + r^{\tilde{\Gamma}^P})(1 + \vartheta)^{\tilde{\omega}^P_\pm}
\end{aligned}
\tag{0.32}
$$

for all $(r,\vartheta) \in (0,\infty)^2$ with some $\underline{\Gamma}^P \in [0,1)$, $\overline{\Gamma}^P \in (0,\gamma)$, $\tilde{\Gamma}^P \in (0,\gamma)$ such that

$$
\frac{\max\{\underline{\Gamma}^P, \overline{\Gamma}^P\}}{\overline{\gamma}} + \frac{\overline{\omega}^P}{p_\beta} < 1, \;\; \frac{\tilde{\Gamma}^P}{\gamma} + \frac{\tilde{\omega}^P + 1}{p_\beta} < 1,
$$

where $\overline{\omega}^P = \max\{d^P_+ \overline{\omega}^P_+, d^P_- \overline{\omega}^P_-\}$, $\tilde{\omega}^P = \max\{d^P_+ \tilde{\omega}^P_+, d^P_- \tilde{\omega}^P_-\}$ and $\overline{\gamma}$ is given by (0.31).

Definition

The quintet $(\alpha, \rho, z, u, \vartheta)$ *is a bounded energy weak solution to original problem if*

$(\alpha, \rho, z, u, \vartheta)$ *possesses the following regularity*

$$
\begin{aligned}
&(\rho, z, \alpha) \in C_w([0,T]; L^\gamma(\Omega)),\\
&(\rho, z)(t,x) \in \overline{\mathcal{O}_{\underline{a},\overline{a}}},\ \underline{\alpha} \le \alpha \le \overline{\alpha} \text{ for all } t \in [0,T] \text{ and a.a. } x \in \Omega,\\
&u \in L^2(0,T; W^{1,2}(\Omega)), (\rho+z)|u|^2 \in L^\infty(0,T; L^1(\Omega)),\\
&(\rho+z)u \in C_w([0,T]; L^q(\Omega)),\\
&\mathrm{P}_+(f_+(\alpha)\rho, \vartheta), \mathrm{P}_-(f_-(1-\alpha)z, \vartheta) \in L^1(Q_T),\\
&\vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),\\
&\log \vartheta \in L^2(0,T; W^{1,2}(\Omega))
\end{aligned}
$$

with some $\gamma, q > 1$*.*

Definition

(continuation)

The continuity equation

$$\int_\Omega r\psi(t) - \int_\Omega r_0\psi(0) = \int_0^t \int_\Omega r(\partial_t\psi + u\cdot\nabla\psi) \qquad (0.33)$$

is fulfilled for any $t \in [0, T]$, $\psi \in C^1([0, T]\times\overline{\Omega})$ *with* r *standing for* ρ *and* z.

The transport equation

$$\int_\Omega \alpha\psi(t) - \int_\Omega \alpha_0\psi(0) = \int_0^t \int_\Omega (\alpha(\partial_t\psi + u\cdot\nabla\psi) - \alpha \operatorname{div} u\psi) \qquad (0.34)$$

is fulfilled for any $t \in [0, T]$, $\psi \in C^1([0, T]\times\overline{\Omega})$.

Definition

The momentum equation

$$\int_\Omega (\rho + z) u\varphi(t) - int_\Omega(\rho_0 + z_0)u_0\varphi(0) =$$
$$\int_0^t \int_\Omega \Bigg(\rho + z) u \cdot \partial_t \varphi + (\rho + z) u \otimes u \cdot \nabla\varphi$$
$$+ \left(\tfrac{b}{3}\vartheta^4 + \mathrm{P}_+(f_+(\alpha)\rho, \vartheta) + \mathrm{P}_-(f_-(1-\alpha)z, \vartheta) \right) \operatorname{div}\varphi -$$
$$\mathbb{S}(\vartheta, \nabla u) \cdot \nabla\varphi \Bigg) \tag{0.35}$$

is fulfilled for any $t \in [0, T]$ *and* $\varphi \in C^1([0, T] \times \overline{\Omega})$, *where either* $\varphi = 0$ *or* $\varphi \cdot n = 0$ *on* $(0, T) \times \partial\Omega$;

Definition

The energy equality

$$\int_\Omega \left(\frac{1}{2}(\rho + z)|u|^2 + \rho \mathfrak{e}_+ \left(\frac{\rho}{\alpha}, \vartheta\right) + z \mathfrak{e}_- \left(\frac{z}{1-\alpha}, \vartheta\right)\right)(t)$$

$$= \int_\Omega \left(\frac{1}{2}(\rho_0 + z_0)|u_0|^2 + \rho_0 \mathfrak{e}_+ \left(\frac{\rho_0}{\alpha_0}, \vartheta_0\right) + z_0 \mathfrak{e}_- \left(\frac{z_0}{1-\alpha_0}, \vartheta_0\right)\right) \tag{0.36}$$

holds for a.a $t \in (0, T)$;

Definition

The balance of entropy holds in the following sense: There is $\sigma \in (C([0,T] \times \overline{\Omega}))^*$ *such that*

$$\langle \sigma, \phi \rangle \geq \int_0^T \int_\Omega \left(\frac{1}{\vartheta} \mathbb{S}(\vartheta, \nabla u) \cdot \nabla u + \frac{\kappa(\vartheta)}{\vartheta^2} |\nabla \vartheta|^2 \right) \phi \quad (0.37)$$

for any $\phi \in C([0,T] \times \overline{\Omega})$, $\phi \geq 0$ *and*

$$\int_\Omega \left(\rho \mathfrak{s}_+ \left(\frac{\rho}{\alpha}, \vartheta \right) + z \mathfrak{s}_- \left(\frac{z}{1-\alpha}, \vartheta \right) \right) \phi(t) -$$
$$\int_\Omega \left(\rho_0 \mathfrak{s}_+ \left(\frac{\rho_0}{\alpha_0}, \vartheta_0 \right) + z_0 \mathfrak{s}_- \left(\frac{z_0}{1-\alpha_0}, \vartheta_0 \right) \right) \phi(0)$$
$$= \int_0^t \int_\Omega \left(\left(\rho \mathfrak{s}_+ \left(\frac{\rho}{\alpha}, \vartheta \right) + z \mathfrak{s}_- \left(\frac{z}{1-\alpha}, \vartheta \right) \right) (\partial_t \phi + u \cdot \nabla \phi) \right.$$
$$\left. - \frac{\kappa(\vartheta)}{\vartheta} \nabla \vartheta \cdot \nabla \phi + \langle \sigma_t, \phi \rangle, \right. \quad (0.38)$$

Definition

where we denoted

$$\langle \sigma_t, \phi \rangle = \int_{[0,t]\times\overline{\Omega}} \phi(\tau, x) \mathrm{d}\mu_\sigma(\tau, x)$$

and μ_σ *is the unique Borel measure associated to* σ *by the Riesz representation theorem.*

Definition

The sixtet $(\xi, \mathfrak{r}, \mathfrak{z}, \Sigma, u, \vartheta)$ *is called a bounded energy weak solution to* (0.14) *if*

$$
\begin{aligned}
&\xi, \mathfrak{r}, \mathfrak{z}, \Sigma \geq 0 \text{ a.e. in } Q_T, \ \mathfrak{r}, \mathfrak{z}, \Sigma \in C_w([0,T]; L^\gamma(\Omega)) \\
&\ \text{for some } \gamma > 1, \\
&(\mathfrak{r}, \mathfrak{z})(t,x) \in \overline{\mathcal{O}_{\underline{a},\overline{a}}}, (\mathfrak{r} + \mathfrak{z}, \Sigma)(t,x) \in \overline{\mathcal{O}_{\underline{b},\overline{b}}}, \\
&(\mathfrak{r}, \xi)(t,x) \in \overline{\mathcal{O}_{\underline{d},\overline{d}}} \text{ for all } t \in [0,T] \text{ and a.a. } x \in \Omega, \\
&u \in L^2(0,T; W^{1,2}(\Omega)), \mathscr{P}(\mathfrak{r}, \mathfrak{z}, \vartheta) \in L^1(Q_T), \\
&\Sigma|u|^2 \in L^\infty(0,T; L^1(\Omega)), \Sigma u \in C_w([0,T]; L^q(\Omega)) \text{ for some } q > 1, \\
&\vartheta \in L^\infty(0,T; L^4(\Omega)), \vartheta, \log\vartheta \in L^2(0,T; W^{1,2}(\Omega));
\end{aligned}
\tag{0.39}
$$

Definition

The continuity equation

$$\int_\Omega r\psi(t) - \int_\Omega r_0\psi(0) = \int_0^t \int_\Omega r(\partial_t\psi + u \cdot \nabla\psi) \qquad (0.40)$$

is fulfilled for any $t \in [0, T]$, $\psi \in C^1([0, T]\times\overline{\Omega})$ *with* r *standing for* $\xi, \mathfrak{r}, \mathfrak{z}$ *and* Σ*;*

Definition

The momentum equation

$$\int_\Omega \Sigma u\varphi(t) - \int_\Omega (\Sigma u)_0 \varphi(0) =$$

$$\int_0^t \int_\Omega \left(\Sigma u \cdot \partial_t \varphi + \Sigma u \otimes u \cdot \nabla\varphi + \mathscr{P}(\mathfrak{r}, \mathfrak{z}, \vartheta) \operatorname{div}\varphi - \mathbb{S}(\vartheta, \nabla u) \cdot \nabla\varphi\right) \quad (0.41)$$

is fulfilled for any $t \in [0, T]$ *and* $\varphi \in C^1([0, T] \times \overline{\Omega})$, *where either* $\varphi = 0$ *or* $\varphi \cdot n = 0$ *on* $(0, T) \times \partial\Omega$;

Definition

The energy equality

$$\int_\Omega \left(\frac{1}{2} \Sigma |u|^2 + \mathcal{E}(\mathfrak{r}, \mathfrak{z}, \vartheta) \right)(t) = \int_\Omega \left(\frac{|(\Sigma u)_0|^2}{2\Sigma_0} + \mathcal{E}(\mathfrak{r}_0, \mathfrak{z}_0, \vartheta_0) \right) \tag{0.42}$$

holds for a.a $t \in (0, T)$;

Definition

The balance of entropy holds in the following sense: There is $\sigma \in (C([0,T] \times \overline{\Omega}))^*$ *such that*

$$\langle \sigma, \phi \rangle \geq \int_0^T \int_\Omega \left(\frac{1}{\vartheta} \mathbb{S}(\vartheta, \nabla u) \cdot \nabla u + \frac{\kappa(\vartheta)}{\vartheta^2} |\nabla \vartheta|^2 \right) \phi \qquad (0.43)$$

for any $\phi \in C([0,T] \times \overline{\Omega})$, $\phi \geq 0$ *and*

$$\int_\Omega \mathcal{S}(\mathfrak{r}, \mathfrak{z}, \vartheta)\phi(t) - \int_\Omega \mathcal{S}(\mathfrak{r}_0, \mathfrak{z}_0, \vartheta_0)\phi(0) = \int_0^t \int_\Omega \left(\mathcal{S}(\mathfrak{r}, \mathfrak{z}, \vartheta) \left(\partial_t \phi + u \cdot \nabla \phi \right) - \frac{\kappa(\vartheta)}{\vartheta} \nabla \vartheta \cdot \nabla \phi \right) + \langle \sigma_t, \phi \rangle, \qquad (0.44)$$

Definition

where we denoted

$$\langle \sigma_t, \phi \rangle = \int_{[0,t]\times\overline{\Omega}} \phi(\tau, x) \mathrm{d}\mu_\sigma(\tau, x)$$

and μ_σ is the unique Borel measure associated to σ by the Riesz representation theorem.

Theorem

Let $T > 0$, $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2 and the hypotheses. If $\gamma = \overline{\gamma_\beta}$ we assume $\frac{1}{\gamma+1} + \frac{1}{p_\beta} \leq \frac{1}{2}$ additionaly, where $\overline{\gamma_\beta}$ and p_β are defined in (0.23). Then problem (0.14) admits at least one bounded energy weak solution in the sense of Definition 0.7

Theorem

Let $T > 0$, $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2. Suppose that $f_+, f_- : [0,1] \to [0,1]$ are continuously differentiable, strictly monotone and non-vanishing on $[0,1]$. Let $\gamma_+ \geq \frac{9}{5}$, $\gamma_- > 0$

$$
\begin{aligned}
& 0 < \rho_0(x),\ 0 < \underline{\alpha} \leq \alpha_0(x) \leq \overline{\alpha} < 1,\\
& (f(\alpha_0)\rho_0, f_-(1-\alpha_0)z_0)(x) \in \overline{\mathcal{O}_{\underline{a},\overline{a}}} \text{ for a.a. } x \in \Omega,\\
& \alpha_0 \in L^\infty(\Omega),\ \rho_0 \in L^{\gamma_+}(\Omega),\ \textstyle\int_\Omega \rho_0 > 0,\ z_0 \in L^{\gamma_-}(\Omega) \text{ if } \gamma_- > \gamma_+,\\
& \textstyle\int_\Omega z_0 > 0,\\
& (\rho_0 + z_0)|u_0|^2 \in L^1(\Omega), \vartheta_0 \in L^4(\Omega), \log\vartheta_0 \in L^2(\Omega),\\
& \rho_0 s_+\left(\tfrac{\rho_0}{\alpha_0}, \vartheta_0\right) + z_0 s_-\left(\tfrac{z_0}{1-\alpha_0}, \vartheta_0\right) \in L^1(\Omega),\\
& \rho_0 e_+\left(\tfrac{\rho_0}{\alpha_0}, \vartheta_0\right) + z_0 e_-\left(\tfrac{z_0}{1-\alpha_0}, \vartheta_0\right) \in L^1(\Omega)
\end{aligned}
\tag{0.45}
$$

be given.

Theorem

Let the hypothesis (H) be satisfied. If $\gamma = \gamma_\beta$ we assume $\frac{1}{\gamma+1} + \frac{1}{p_\beta} \leq \frac{1}{2}$ additionaly, where $\overline{\gamma_\beta}$ and p_β are defined in (0.23). Then original problem admits at least one bounded energy weak solution in the sense of Definition 0.1.

Scheme of proof:
Solving the regularized form of (0.14) with two small parameters $\varepsilon > 0$ *characterizing the dissipation in all continuity equations* and $\delta > 0$ characterizing *the artificial pressure* in $(0.14)_5$ and the balance of internal energy is considered instead of the entropy production equation $(0.14)_6$.

(1) *The existence of solution to the approximate problem* is shown via the Faedo – Galerkin scheme in the spirit of monofluid situation.

(a) First, a solution to the Faedo- Galerkin approximation of the regularized momentum equation is shown to exist locally in time by the Schauder fixed point theorem. (b) Next, deriving suitable uniform estimates of the solution allows for extending the solution to the whole given time interval. (c) Then the balance of internal energy is converted to the entropy production equation provided that the approximate tempera- ture is positive. Eventually, the limit passage to infinity with the Faedo-Galerkin parameter is performed.

(2) Performing the passage $\varepsilon \to 0+$, i.e. letting the artificial viscotity in continuity equations tend to zero, (improved uniform integrability of a density approximation sequence, the compensated compactness arguments for showing the compactness of temperature approximations, the almost compactness of the ratio of density approximations and a variant of the effective viscous flux identity.)

(3) Performing the passage $\delta \to 0+$, i.e. letting the artificial pressure term in the momentum equation tend to zero, follows the same procedure as in the previous step with minor differences caused by a lower exponent characterizing the uniform integrability of density approximations.

Multi-component FSI problem

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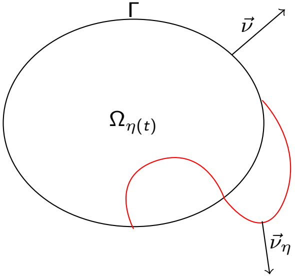
Modelling assumptions

- Flow of a viscous compressible two-fluid model with a pressure law depending on both fluid densities.
- A shell of Koiter type located at the fluid boundary.
- Simplifying assumption: The shell moves only in the normal direction to the reference domain.
- The fluid-solid interface is supplemented with adherence type condition.



The model we consider

Continuity equations: $\begin{cases} \partial_t(\rho) + \operatorname{div}(\rho u) = & 0, \\ \partial_t(Z) + \operatorname{div}(Zu) = & 0 \end{cases}$

Momentum balance: $\left\{ \partial_t \left((\rho + Z)u\right) + \operatorname{div}\left((\rho + Z)u \otimes u\right) = \operatorname{div}\mathbb{S}(\nabla u) - \nabla P(\rho, Z) \right.$

Dynamics of the structure: $\left\{ \partial_t^2 \eta - \zeta \partial_t \Delta \eta + K'(\eta) = F \cdot \nu, \text{ in } (0, T) \times \Gamma \right.$

$$F = (-(\mathbb{S}(\nabla u) - P(\rho, Z)\mathbb{I}_3) \cdot \nu_\eta) \circ \varphi_\eta |\det D\varphi_{\eta(t)}|, \text{ in } (0, T) \times \Gamma.$$

where

$$\mathbb{S}(\nabla u) = 2\mu \left(\frac{\nabla u + \nabla u^T}{2} - \frac{1}{3} \operatorname{div} u \mathbb{I}_3 \right) + \lambda \operatorname{div} u \mathbb{I}_3$$

and

$$\varphi_\eta(x, t) = \varphi(x) + \eta(x, t)\nu(\varphi(x)) \text{ for } x \in \Gamma.$$

Ω is a reference domain, whose boundary is parametrized by a C^4 injective mapping $\varphi : \Gamma \to R^2$, $\eta \in (a_{\partial\Omega}, b_{\partial\Omega})$

Continuity of velocities at the interface: $\left\{ u(t, \varphi_\eta(t, x)) = \partial_t \eta(x, t)\nu(\varphi(x)) \right.$ on $\Gamma \times I$

Supplemented with initial condition:

$$(\rho, Z, (\rho + Z)u, \eta, \partial_t \eta) = (\rho_0, Z_0, (\rho_0 + Z_0)u_0, \eta_0, \eta_1).$$

Some comments on the structure of the pressure and the Koiter energy

Some assumptions on the structure of pressure and initial density:

- For some $0 < \underline{a} < \overline{a} < \infty$ we assume
$$(\rho_0, Z_0) \in \overline{\mathcal{O}_{\underline{a}}} = \{(\rho, Z) \in \mathbb{R}^2 |\ \rho \in [0, \infty),\ \underline{a}\rho \le Z \le \overline{a}\rho\},$$
- $$\text{for all } \ (\rho, Z) \in \overline{\mathcal{O}_{\underline{a}}},\ \ \underline{C}(\rho^\gamma + Z^\beta - 1) \le P(\rho, Z) \le \overline{C}(\rho^\gamma + Z^\beta + 1)$$
with some $\gamma \ge 2$, $\beta > 0$ and positive constants $\underline{C}$, $\overline{C}$ and
$$\text{for all } (\rho, Z) \in \overline{\mathcal{O}_{\underline{a}}},\ \ |\partial_Z P(\rho, Z)| \le C(\rho^{\underline{\kappa}} + \rho^{\overline{\kappa}})$$
for some $\underline{\kappa} \in (0, 1]$ and $\overline{\kappa} \in [1, \tilde{\gamma}), \tilde{\gamma} = \max\{\gamma + \gamma_{bog}, \beta + \beta_{bog}\}$.
- $$P(\rho, \rho s) = \mathcal{P}(\rho, s) - \mathcal{R}(\rho, s),\ \ s = \begin{cases} \frac{Z}{\rho} & \text{if } \rho > 0, \\ 0 & \text{if } \rho = 0. \end{cases}$$
where $[0, \infty) \ni \rho \mapsto \mathcal{P}(\rho, s)$ is non decreasing for any $s \in [0, \overline{a}]$ and
$$\bigcup_{s \in [0, \overline{a}]} \text{supp}\mathcal{R}(\cdot, s) \subset [0, \overline{R}],\ \ \sup_{s \in [0, \overline{a}]} \|\mathcal{R}(\cdot, s)\|_{C^2([0, \overline{R}])} < \infty.$$
- the function $\rho \to P(\rho, Z), Z > 0$ and the function $Z \to \partial_Z P(\rho, Z), \rho > 0$ are Lipschitz on $(Z/\overline{a}, Z/\underline{a}) \cap (r, \infty), (\underline{a}\rho, \overline{a}\rho) \cap (r, \infty)$, respectively.

Example of such a pressure and Koiter energy

A typical example of pressure:

$$P(\rho, Z) = \rho^{\gamma} + Z^{\beta} + \sum_{i=1}^{M} F_i(\rho, Z),$$

where

$$F_i(\rho, Z) = C_i \rho^{r_i} Z^{s_i}, \text{ where } 0 \leqslant r_i < \gamma,\ 0 \leqslant s_i < \beta,\ r_i + s_i < \max\{\gamma, \beta\},\ C_i \geqslant 0.$$

Very roughly the elasticity operator:

$$\langle K'(\eta), b \rangle = \frac{h^3}{24} \int_{\Gamma} \Big(\mathcal{A}(\overline{\gamma}(\eta)\nabla^2\eta) : (\overline{\gamma}'(\eta) b \nabla^2 \eta) + \mathcal{A}(\overline{\gamma}(\eta)\nabla^2\eta) : (\overline{\gamma}(\eta)\nabla^2 b) \Big) + L.O.T.$$

$$\forall\ b \in W^{2,p}(\Gamma),\ p > 2.$$

h : thickness of the structure, $\mathcal{A}$: elasticity tensor

$\overline{\gamma}(\eta)$: a second order polynomial in η.

For details we refer to:
P.G. Ciarlet: Mathematical elasticity, Vol III.

Weak formulations and main result

Comments on weak formulation:

$$\int_0^t \int_{\Omega_\eta(s)} (\rho + Z)u \cdot \partial_t \phi + \int_0^t \int_{\Omega_\eta(s)} ((\rho + Z)u \otimes u) \cdot \nabla \phi - \int_0^t \int_{\Omega_\eta(s)} \mathbb{S}(\nabla u) \cdot \nabla \phi$$
$$+ \int_0^t \int_{\Omega_\eta(s)} P(\rho, Z) \mathrm{div}\, \phi + \int_{(0,t)\times\Gamma} \partial_t \eta \partial_t b - \int_0^t \langle K'(\eta), b\rangle + \zeta \int_0^t \int_\Gamma \partial_t \nabla \eta \nabla b$$
$$= \int_{\Omega_\eta(t)} (\rho + Z)u(t,\cdot)\phi(t,\cdot) - \int_{\Omega_{\eta_0}} M_0 \phi(0,\cdot) + \int_\Gamma \partial_t \eta(t,\cdot) b(t,\cdot) - \int_\Gamma \eta_1 b(0,\cdot)$$

for all $t \in [0, T]$,
$(\phi, b) \in C^\infty([0, T] \times \mathbb{R}^3) \times (L^\infty(0, T; W^{2,2}(\Gamma)) \cap W^{1,\infty}(0, T; L^2(\Gamma))$, with
$tr_{\Sigma_\eta} \phi = b\nu$ and $tr_{\Sigma_\eta} u = \partial_t \eta \nu$.
where the trace tr_{Σ_η} is defined roughly as:

$$tr_{\Sigma_\eta} \phi \approx \phi \circ \eta\nu \text{ on } \Gamma.$$

Please notice that:

- The stress tensor due to the fluid does not appear at the interface in the weak formulation.

Apriori estimate and functional spaces

A formal energy inequality

$$\int_{\Omega_\eta(t)} \left(\frac{1}{2}(\rho+Z)|u|^2 + H_P(\rho,Z)\right)(t,\cdot) + \int_I \int_{\Omega_\eta(t)} \mathbb{S}(\mathbb{D}u)\cdot\nabla u$$
$$+ \left(\int_\Gamma \frac{1}{2}|\partial_t\eta|^2 + K(\eta)(t,\cdot)\right) + \zeta \int_I \int_{\Omega_\eta(t)} |\partial_t\nabla\eta|^2$$
$$\leq \int_{\Omega_{\eta_0}} \left(\frac{|M_0|^2}{2(\rho_0+Z_0)} + H_P(\rho_0,Z_0)\right) + \left(\frac{1}{2}\int_\Gamma |\eta_1|^2 + K(\eta_0)\right)$$

holds for a.a. $t\in I$.
We expect to solve in the frame-work:

$\rho \in C_w([0,T]; L^\gamma(\Omega_\eta(t))),\ Z \in C_w([0,T]; L^\beta(\Omega_\eta(t))),$

$u \in L^2(0,T; W^{1,q}(\Omega_\eta(t))),\ q<2,$ (Korn inequality in uniformly Holder domains)

$\eta \in L^\infty(0,T; W^{2,2}(\Gamma)) \cap L^2(0,T; W^{2,2+\sigma}(\Gamma))$ for $\sigma>0$
(Coercivity of $K(\eta)$ in $W^{2,2}(\Gamma)$),

$\partial_t\eta \in C_w([0,T]; L^2(\Gamma)) \cap L^2([0,T]; W^{\sigma,2}(\Gamma)),$ for $\sigma>0$.

The central result at a glance

Case I: Let $\max\{\gamma, \beta\} > 2$, $0 < \min\{\gamma, \beta\}$, the dissipation parameter $\zeta=0$,

$\rho_0, Z_0 \geqslant 0$, $\rho_0, Z_0 \not\equiv 0$ a.e. in Ω_{η_0}, $\rho_0 \in L^\gamma(\Omega_{\eta_0})$, $Z_0 \in L^\beta(\Omega_{\eta_0})$,
$M_0 = (\rho_0 + Z_0)u_0 \in L^1(\Omega_{\eta_0}), (\rho_0 + Z_0)|u_0|^2 \in L^1(\Omega_{\eta_0})$, $\eta_0 \in W^{2,2}(\Gamma)$, $\eta_1 \in L^2(\Gamma)$,
+compatibity between intial velocities.

Case II: Let $\max\{\gamma, \beta\} \geqslant 2$, $0 < \min\{\gamma, \beta\}$, the dissipation parameter $\zeta > 0$,
Conclusion:
Existence of at least one weak solution.
Moreover, one of the following holds:

- $T = +\infty$,
- $T < \infty$ if
  (a) $\Omega_\eta(s)$ approaches a self–intersection as $s \to T$,
  (b) the Koiter energy degenerates, i.e., $\lim_{s\to T} \bar{\gamma}(\eta(s, y)) = 0$ for some $y \in \Gamma$.

Further in Case II:

$$u \in L^2(0, T; W^{1,2}(\Omega_\eta(t))), \ \eta \in W^{1,2}(0, T; W^{1,2}(\Gamma)).$$

Thank you for your attention