A linear Landau damping phenomenon in thick spray models

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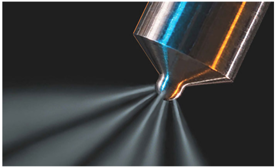
Examples

Thin sprays:



(a) Diesel engine fuel injector

(b) Medical spray

Thick sprays:

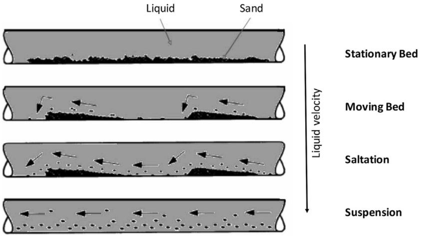
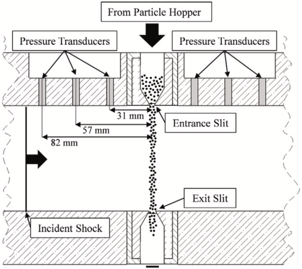


Fig. 1. Sand flow regime in horizontal pipelines.

(c) Leporini at al (2019)



(d) Daniel-Wagner (2022)

Thick spray models

Mixture Fluid-particules: Euler coupled Vlasov

$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = -m_\star \int \boldsymbol{\Gamma} f dv, \\ \partial_t(\alpha\rho e) + \nabla\cdot(\alpha\rho e\mathbf{u}) + p\left(\partial_t\alpha + \nabla\cdot(\alpha\mathbf{u})\right) = D_\star \int |\mathbf{v}-\mathbf{u}|^2 f dv, \\ \partial_t f + \mathbf{v}\cdot\nabla_x f + \nabla_v\cdot(\boldsymbol{\Gamma} f) = 0, \\ \\ m_\star\boldsymbol{\Gamma} = -m_\star\nabla p - D_\star(\mathbf{v}-\mathbf{u}), \\ \alpha = 1 - v_\star \int f dv, \qquad v_\star = \frac{4}{3}\pi r_\star^3, \end{cases}$$

with a perfect gas pressure law $p = (\gamma-1)\rho e$ where the coefficient is $\gamma > 1$. The Vlasov equation is already very simplified with respect to Fox [2023].

The volume fraction of the fluid (in pre-normalized units) is α. The relation

$$\alpha + v_\star \int f dv = 1$$

means that the sum of the partial volumes is the total volume.

Thin spray is the simplification for $\alpha \approx 1$.

Thick sprays $\approx$ all regimes

$$0 < \alpha \leq 1.$$

- Boudin-Desvillettes-Motte [2003], Baranger-Desvillettes [2006], Desvillettes-Mathiaud, ...

References

- Models:
  - in the context of combustion theory Williams [58', 85']
  - Discretization in Dukowicz [80'], advanced methods Fox [23'].
  - Classification of sprays O'Rourke [81']
- Mathematical theory of thin sprays ($\alpha \approx 1$ does not show up):
  - Vlasov-Euler:
    - Local-in-time well posedness for strong solution Baranger-Desvillettes [06'], Mathiaud [10']
    - Global weak solution in 1D with finite energy Cao [22']
  - Vlasov-Navier-Stokes:
    - Large time behavior studied in Choi [2016], Ertzbischoff, Han-Kwan, and Moussa [21'], Han-Kwan, Moussa, and Moyano [20']
- Mathematical theory of thick sprays ($0 < \alpha \leq 1$):
  - Boudin-Desvillettes-Motte [03'],
    numerical work Benjelloun-Desvillettes-Ghidaglia-Nielsen[12']
  - Linear stability studied in Buet-D.-Desvillettes [22']
  - Ongoing PhD V. Fournet
    `https://victorfournet.github.io/publications/`
  - **F. Charles**-L. Desvillettes, From collisional kinetic models to sprays: internal energy exchanges, CMS (2024).

Preliminary remarks

- The mass of each phase is preserved since one has

$$\partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0 \quad \text{and} \quad \partial_t\left(m_\star \int f\,dv\right) + \nabla\cdot\left(m_* \int \mathbf{v}\,f dv\right) = 0.$$

- The total impulse is preserved since one has

$$\partial_t\left(\alpha\rho\mathbf{u} + m_\star\int f\mathbf{v}dv\right) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p + \nabla\cdot\left(m_*\int \mathbf{v}\otimes\mathbf{v}fdv\right) = 0.$$

- The total energy is preserved since one has ($E = e + \frac{1}{2}|\mathbf{u}|^2$)

$$\partial_t\left(\alpha\rho E + m_\star\int f\frac{|\mathbf{v}|^2}{2}\right) + \nabla\cdot\left(\alpha\rho E\mathbf{u} + m_\star\int\frac{|\mathbf{v}|^2}{2}\mathbf{v}fdv + \alpha p\mathbf{u} + pm_\star\int f\mathbf{v}dv\right) = 0.$$

- Thermodynamic correctness is satisfied since one has ($S = \log e/\rho^{\gamma-1}$)

$$\partial_t(\alpha\rho S) + \nabla\cdot(\alpha\rho S\mathbf{u}) = \frac{D_\star}{T}\int|\mathbf{v}-\mathbf{u}|^2 fdv \geq 0.$$

Linearization

Consider the barotropic model

$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = -m_\star \int \boldsymbol{\Gamma} f dv, \\ \partial_t f + \mathbf{v}\cdot\nabla_x f + \nabla_v\cdot(\boldsymbol{\Gamma} f) = 0, \end{cases}$$

where $p = p(\rho)$ with $p'(\rho) > 0$ so that the equation for the energy is not needed.

Lemma (Buet-Desvillettes-D. 2021)

Take a Maxwellian profile for simplicity $F(z) = e^{-z}$.
Then an exact solution is

$$\begin{aligned} &\rho(t,x) = \rho_0, \\ &\mathbf{u}(t,x) = 0, \\ &\alpha(t,x) = \alpha_0 = 1 - m_\star\, n_0, \\ &\boldsymbol{\Gamma} = -d_\star \mathbf{v}, \\ &f(t,\mathbf{x},\mathbf{v}) = e^{3d_\star t} f(0,\mathbf{x},e^{d_\star t}\mathbf{v}) = e^{3d_\star t}\, \frac{n_0}{(K\,T_\star)^{\frac{3}{2}}}\, F\left(\frac{e^{2d_\star t}\,|\mathbf{v}|^2}{2T_\star}\right), \end{aligned} \tag{1}$$

where $d_* = \frac{D_\star}{m_\star} \geq 0$. If the drag/friction coefficient is non zero, then $d_* > 0$.

Linearize around such an exact solution

The first variation of specific volume $\tau = \frac{1}{\rho}$, velocity u and density f are τ_1, u_1 and g_1.
One has

$$\left\{\begin{array}{l} \alpha_0\rho_0\,\partial_t\tau_1 = \alpha_0\,\nabla\cdot\mathbf{u}_1 + m_\star\nabla\cdot\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,\mathbf{v}\,dv, \\ \alpha_0\rho_0\,\partial_t\mathbf{u}_1 = \alpha_0\rho_0^2c_0^2\,\nabla\tau_1 + m_\star d_\star\int\mathbf{v}\sqrt{f_0}\,e^{d_\star t}\,g_1\,dv - m_\star d_\star\mathbf{u}_1\int f_0\,dv, \\ \partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 - \frac{\rho_0^2c_0^2}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\nabla\tau_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) \\ = \frac{d_\star}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\mathbf{u}_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) - d_\star g_1 + \frac{d_\star}{g_1}\nabla_v\cdot\left(\frac{1}{2}\mathbf{v}g_1^2\right). \end{array}\right. \tag{2}$$

This is a linear integro-differential system of equations, with coefficients which are homogeneous in space, but with a dependency in time.
If $d_\star = 0$ (no friction), then the coefficients become constant in space and time.

Lemma (Buet-Desvillettes-D.)

L^2 stability proved by means of a Lyapunov function.

1D Frictionless and dimensionless equations

One gets the integro-differential system

$$\begin{cases} \partial_t \tau = \partial_x u + \partial_x \int_{\mathbb{R}} vg\sqrt{f_0(v)}dv, \\ \partial_t u = \partial_x \tau, \\ \partial_t f + v\partial_x f = \sqrt{f_0(v)}v\partial_t \tau. \end{cases}$$

In what follows we take

$$f_0(v) = e^{-v^2/2}.$$

Landau Fourier-Laplace method

We use the version explained by E. Sonnendrucker.

Goal: Compute the rate of the damping, following the technique of Landau.

Introduce the Fourier-Laplace transform of $\tau \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$:

$$\widetilde{\tau}(\omega, k) = \int_0^{+\infty} \int_{\mathbb{T}} e^{i\omega t} e^{-ikx} \tau(t, x) \, \mathrm{d}x\mathrm{d}t, \quad k \in \mathbb{Z}, \ \Im m(\omega) > 0.$$

Apply Fourier Laplace transform in x-t yields

$$\begin{cases} -i\omega\widetilde{\tau}(\omega, k) = ik\widetilde{u}(\omega, k) + ik \int_{\mathbb{R}} v\sqrt{f_0(v)}\widetilde{f}(\omega, k, v) \, \mathrm{d}v + \widehat{\tau}_{\mathrm{ini}}(k), \quad k \in \mathbb{Z}, \ \mathrm{Im}(\omega) > 0 \\ -i\omega\widetilde{u}(\omega, k) = ik\widetilde{\tau}(\omega, k) + \widehat{u}_{\mathrm{ini}}(k) \\ (-i\omega + ikv)\widetilde{f}(\omega, k, v) - ikv\sqrt{f_0}\widetilde{\tau}(\omega, k) = \widehat{f}_{\mathrm{ini}}(k, v). \end{cases}$$

It yields for $\mathrm{Im}(\omega) > 0$,

$$\widetilde{\tau}(\omega, k) = \frac{\mathcal{N}(\omega, k)}{\mathcal{D}(\omega, k)} \tag{8}$$

with $\mathcal{N}(\omega, k) = \frac{i}{\omega}\left(\widehat{\tau}_{\mathrm{ini}}(k) - \frac{k}{\omega}\widehat{u}_{\mathrm{ini}}(k) + \int_{\mathbb{R}} \frac{v\sqrt{f_0(v)}\widehat{f}_{\mathrm{ini}}(k,v)}{v - \frac{\omega}{k}} \, \mathrm{d}v\right)$ and

$$\mathcal{D}(\omega, k) = 1 - \frac{k^2}{\omega^2} - \int_{\mathbb{R}} \frac{\partial_v f_0(v)}{v - \frac{\omega}{k}} \, \mathrm{d}v.$$

Landau, On the vibration of the electronic plasma. Journal of Physics, 10(1):25–34, 1946.

- To reconstruct the density $\tau(t,x)$, one needs to compute the integral

$$\widehat{\tau(t,k)} = \frac{1}{2i\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{-i\omega t}\widetilde{\tau}(\omega,k)\,\mathrm{d}\omega.$$

- Analytic continuation of $\mathcal{D}_k$ on $\mathbb{C}^*$:

$$\mathcal{D}(\omega,k) = 2 - \frac{k^2}{\omega^2} + \frac{\sqrt{\pi}}{\sqrt{2}}\frac{\omega}{k}e^{-\frac{\omega^2}{2k^2}}\left(i - \frac{2}{\sqrt{\pi}}\int_0^{\frac{\omega}{k\sqrt{2}}} e^{t^2}\,\mathrm{d}t\right).$$

- Following Landau, one formally obtain expansion

$$\widehat{\tau}(t,k) = \sum_{\omega_j} \mathrm{Res}(\widetilde{\tau}(\cdot,k),\omega_j)e^{-i\omega_j t}. \qquad ($$

and the $\omega_j(k)$ are solutions of the dispersion relation

$$\mathcal{D}(\omega,k) = 0.$$

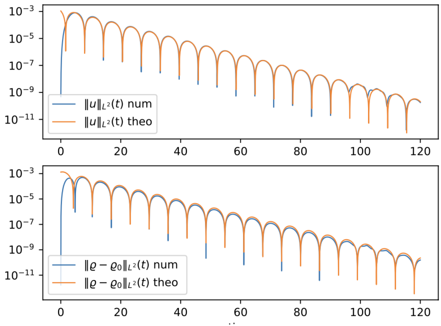
We recover an exponential decay if the ω_j all verify $\mathrm{Im}(\omega_j) < 0$.

Results

Linearize the equations around a solution at rest (nothing moves) with a Gaussian profile $f_0(v) = e^{-v^2/2}$ for the particles.

First pole is $\omega \approx 0.31 - i0.098$.

Initial conditions:
$\varrho_0 = 1, \quad u_0 = 0, \quad f_0(x, v) = (1 + \varepsilon \cos(kx))e^{-v^2/2}, \quad \varepsilon = 10^{-3}$
Orange curve $\propto e^{\Im m(\omega)t} \cos(\Re e(\omega)t)$, $\omega(k)$ solution of $\frac{k^2}{\omega^2} + \int \frac{f_0'(v)}{v - \omega/k} = 1$



Fournet-Buet-D. CMS (204): Analog of Linear Landau Damping in a coupled Vlasov-Euler system for thick sprays

Similarity with Linear Landau Damping in plasma physics

- First predicted by Landau[46'] for the linearized Vlasov-Poisson system

$$\begin{cases}\partial_t f + \mathbf{v}\cdot\nabla_x f - \mathbf{E}\cdot\nabla_v f_0 = 0,\\ \nabla_x\cdot\mathbf{E} = -\int f\,\mathrm{d}v\end{cases}$$

  around maxwellian equilibrium $f_0(v) = e^{-v^2/2}$

- Landau showed the damping of the electric field

$$\|\mathbf{E}(t)\| = \mathcal{O}\left(e^{\mathrm{Im}(\omega)t}\cos(\mathrm{Re}(\omega))\right),$$

  with $\omega(k)\in\mathbb{C}$ verifies a dispersion relation

$$\int_{\mathbb{R}}\frac{\partial_v f_0(v)}{v-\omega/k}\,\mathrm{d}v = k^2.$$

- To show this, take the ansatz $f(t,x,v) = \alpha(v)e^{-i\omega t}e^{ikx}, \quad E(t,x) = \beta e^{-i\omega t}e^{ikx}$.

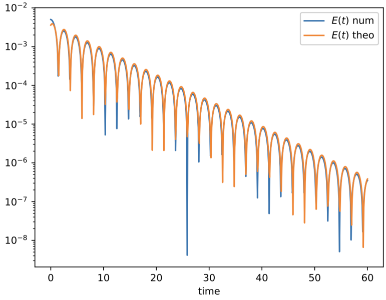


Figure 1: Landau damping for (nonlinear) Vlasov-Poisson

(Linear) Landau Damping yields irreversibility.

Reformulate the fluid part

- Start from

$$\begin{cases} \partial_t(\alpha\varrho) + \partial_x(\alpha\varrho u) = 0 \\ \partial_t(\alpha\varrho u) + \partial_x(\alpha\varrho u^2) + \alpha\partial_x p = 0 \\ \partial_t f + v\partial_x f - \partial_x p \partial_v f = 0 \\ p(\varrho) = \varrho^\gamma, \gamma = 1.4. \end{cases} \tag{3}$$

- To get rid of the **non conversative product** $\alpha\partial_x p$, we write the fluid part in conservation form

$$\begin{cases} \partial_t(\alpha\varrho) + \partial_x(\alpha\varrho u) = 0 \\ \partial_t\left(\alpha\varrho u + \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} vf \,\mathrm{d}v\right) + \partial_x(\alpha\varrho u^2) + \partial_x p + \partial_x\left(\frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} v^2 f \,\mathrm{d}v\right) = 0 \end{cases} \tag{4}$$

- The fluid part writes as

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, f) = 0.$$

The jacobian of $\mathbf{F}$ w.r.t to $\mathbf{U} = (U_1, U_2)$ is

$$\mathbf{A}(\mathbf{U}, f) := \nabla_{\mathbf{U}}\mathbf{F}(\mathbf{U}, f) = \begin{pmatrix} 0 & 1 \\ \frac{-(U_2 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} fv \,\mathrm{d}v)^2}{U_1^2} + \frac{\gamma U_1^{\gamma-1}}{\alpha^\gamma} & \frac{2(U_2 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}} fv \,\mathrm{d}v)}{U_1} \end{pmatrix}$$

The eigenvalues of the matrix $\mathbf{A}(\mathbf{U}, f)$ are $\lambda_\pm = u \pm \sqrt{\frac{p'(\varrho)}{\alpha}}$.

Numerical methods

Given $(\mathbf{U}^n, f^n)$ at a given time t^n.

- Compute f^* by solving the free transport $\partial_t f + v\partial_x f = 0$ with a semi-lagrangian scheme during a timestep Δt with initial condition f^n.
- Compute f^{n+1} by solving $\partial_t f - \partial_x p \partial_v f = 0$ with a semi-lagrangian scheme during a timestep Δt with initial condition f^*.
- Compute $\mathbf{U}^{n+1}$ by solving $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}, f)$ with a Lax-Wendroff scheme during a timestep Δt with initial condition $(\mathbf{U}^n, f^{n+1})$.

- Given initial data $(\mathbf{U}^n, f^{n+1})$, the quantity $\mathbf{U}^{n+1}$ is computed by the Lax-Wendroff scheme:

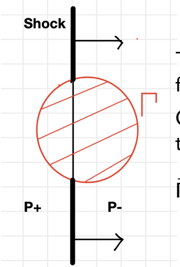
$$
\begin{aligned}
\mathbf{U}^{n+1} = \mathbf{U}^n &- \frac{\Delta t}{2\Delta x}(\mathbf{F}(\mathbf{U}^n_{j+1}, f^{n+1}_{j+1}) - \mathbf{F}(\mathbf{U}^n_{j-1}, f^{n+1}_{j-1})) \\
&+ \frac{\Delta t^2}{2\Delta x^2}\left(\mathbf{A}^n_{j+1/2}(\mathbf{F}(\mathbf{U}^n_{j+1}, f^{n+1}_{j+1}) - \mathbf{F}(\mathbf{U}^n_j, f^{n+1}_j))\right. \\
&\qquad \left. - \mathbf{A}^n_{j-1/2}(\mathbf{F}(\mathbf{U}^n_j, f^{n+1}_j) - \mathbf{F}(\mathbf{U}^n_{j-1}, f^{n+1}_{j-1}))\right).
\end{aligned}
$$

A modification

The barotropic frictionless system writes (all coefficients set to 1)

$$\begin{cases} \partial_t(\alpha\varrho) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \boldsymbol{\nabla}\cdot(\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{\nabla} p \int_{\mathbb{R}^3} \langle f\rangle \,\mathrm{d}v \\ \partial_t f + \boldsymbol{v}\cdot\boldsymbol{\nabla}_x f - \langle \boldsymbol{\nabla}_x p\rangle \cdot \boldsymbol{\nabla}_v f = 0 \\ \alpha = 1 - v_\star \int_{\mathbb{R}^3} \langle f\rangle \,\mathrm{d}v \\ p = p(\varrho) = \varrho^\gamma. \end{cases} \tag{5}$$

A natural convolution principle follows from the reintroduction of the radius of the particles $r_* > 0$.



The principle is the same for smooth functions and discontinuous functions.

One extends the function inside the particle.

$$\vec{\Gamma} = \frac{-1}{\pi r_*^2} \oint_\Gamma p\vec{n} d\sigma = \frac{-1}{\pi r_*^2} \int_{|\mathbf{x}|<r_*} \nabla p dx = -\langle \nabla p\rangle .$$

V. Fournet-D. -C. Buet: Local-in-time existence of strong solutions to an averaged thick sprays model KRM 2024.

Formulation of the mathematical result

Claim

Let $\Omega =]0,+\infty[\times\mathbb{R}^3$, $s \in \mathbf{N}$ such that $s > 3/2+1$ and Ω_1, Ω_2 two open sets of Ω such that $\overline{\Omega}_1 \subset \Omega_2$ and Ω_1 and Ω_2 are relatively compact in Ω. Let $(\varrho_0, \varrho_0 \boldsymbol{u}_0) : \mathbb{R}^3 \to \Omega_1$ satisfying $\varrho_0 - 1 \in H^s(\mathbb{R}^3)$ and $\boldsymbol{u}_0 \in H^s(\mathbb{R}^3)$. Let $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ be a function in $C^1_c(\mathbb{R}^3 \times \mathbb{R}^3) \cap H^s(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying

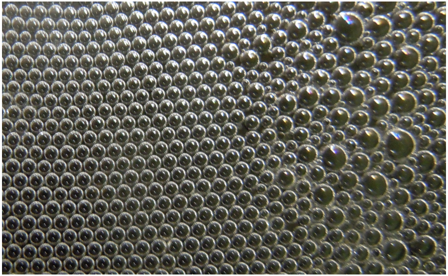
$$\|f_0\|_{L^\infty} < \frac{1}{2^4 \|w\|_{L^1} V_M(0)^3}.$$

Then, one can find $T \in]0,1[$ such that there exists a solution $(\varrho, \varrho\boldsymbol{u}, f)$ of the barotropic frictionless system **with convolution** $r_* > 0$ belonging to $C^1([0,T]\times\mathbb{R}^3, \Omega_2) \times C^1_c([0,T]\times\mathbb{R}^3\times\mathbb{R}^3, \mathbb{R}_+)$. Moreover this solution is unique.

Hint: adapt the iterative scheme proof of Baranger-Desvillettes [2006'] (based on Majda technique).

Illustration

Reintroduce the packing limit (illustrated below with a foam)



$$\alpha = 1 - v_\star \int f dv \geq \alpha_{packing} \qquad (v_\star = \frac{4}{3}\pi r_\star^3).$$

Example: the close packing of spheres in 3D yields $\alpha_{packing} = 1 - \frac{\pi}{3\sqrt{2}} \approx 0.26$.

A typical problem associated to the numerical discretization of thick sprays equations is the **positivity of the fluid volume fraction**, which is a priori not verified by the model.

$$\alpha(t,x) = 1 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v \geq \alpha_{\min}$$

To handle close packing, a possibility (see Fox, Maury) is to introduce a pressure $p_{packing}$=Lagrange multiplier in the Vlasov equation

$$\begin{cases} \partial_t(\alpha\varrho) + \nabla_x \cdot (\alpha\varrho\boldsymbol{u}) = 0 \\ \partial_t(\alpha\varrho\boldsymbol{u}) + \nabla_x \cdot (\alpha\varrho\boldsymbol{u}\otimes\boldsymbol{u}) + \alpha\nabla_x p_{\mathrm{gas}} = D_\star \int_{\mathbb{R}^3} (\boldsymbol{v}-\boldsymbol{u}) f\,\mathrm{d}v \\ \partial_t f + \nabla_x \cdot \left[(\boldsymbol{v} - \nabla_x p_{packing}) f\right] + \nabla_v \cdot (\Gamma f) = 0 \\ \alpha = 1 - \frac{4}{3}\pi r_p^3 \int_{\mathbb{R}^3} f\,\mathrm{d}v \\ m_\star \Gamma = -\frac{4}{3}\pi r_p^3 \nabla_x p_{\mathrm{gas}} - D_\star(\boldsymbol{v}-\boldsymbol{u}) \\ \alpha \geq \alpha_{\min}, \quad (\alpha - \alpha_{\min})\, p_{packing} = 0 \end{cases} \tag{6}$$

We prefer a random walk

Given a discretized distribution function $f_{i,j}^k$, such that there is a saturated cell i : $\alpha_i^k < \alpha_{\min}$,

1. We start from a cell i where $\alpha_i^k < \alpha_{\min}$.
2. We compute the exceeding mass Δm. We set,

$$f_{i,j}^{k+1} = (1 - \alpha_{\min}) \frac{f_{i,j}^k}{m_\star \Delta v \sum_l f_{i,l}^k}, \text{ for all } j.$$

   Then $\alpha_i^{k+1} = \alpha_{\min}$.
3. Start a random walk (S_m), when the walk meets a cell S_m such that $\alpha_{S_m}^k > \alpha_{\min}$, get rid of as much mass as possible

$$f_{S_m,j}^{k+1} = f_{S_m,j}^k + \frac{\Delta m}{m_\star \Delta v \sum_l f_{i,l}^k} f_{i,j}^k, \text{ for all } j.$$

When all the exceeding mass has been distributed, the volume fraction α_i^{k+1} is admissible.
By construction, this method is conservative

Conjecture: This method approximate the "hard" model with added pressure.

- Particle phase: Each phase has the restriction on the timestep

$$\max_j(|v_j|)\Delta t_1^k \leq \Delta x$$

$$\Delta t_2^k \leq \frac{\Delta v}{\frac{D_\star}{m_\star}\left(\max_j(|v_j|) + \max_i\left(|u_i^k|\right)\right) + \max_i\left(|\partial_x p_i^k|\right)}$$

- For the resolution of the fluid part, the eigenvalues write

$$\lambda_\pm = u \pm \sqrt{\frac{p'(\varrho)}{\alpha}}.$$

For stability reasons, we impose

$$\Delta t_3^k \leq \frac{\Delta x}{\max_i(\lambda_\pm(\mathbf{U}_i^k, f_{i,j}^k))}.$$

- In pratice, we impose

$$\Delta t_{\mathrm{PFC}}^k = \mathrm{CFL}\min(\Delta t_1^k, \Delta t_2^k, \Delta t_3^k)$$

with $\mathrm{CFL} = 0.5$.

The initial condition for the gas is

$$(\varrho_0, u_0)(x) = (1, 0).$$

The initial condition for the particles is

$$f_0(x,v) = \frac{1}{\sqrt{2\pi\sigma_v}} e^{-\frac{(v-6)^2}{2\sigma_v^2}} \frac{1}{\sqrt{2\pi\sigma_x}} e^{\frac{(x-2.6)^2}{2\sigma_x^2}} \mathbf{1}_{v>0}(v) + \frac{1}{\sqrt{2\pi\sigma_v}} e^{-\frac{(v+6)^2}{2\sigma_v^2}} \frac{1}{\sqrt{2\pi\sigma_x}} e^{\frac{(x-3.4)^2}{2\sigma_x^2}} \mathbf{1}_{v<0}(v$$

We set the value for close-packing as $\alpha_{\min} = 0.4$.

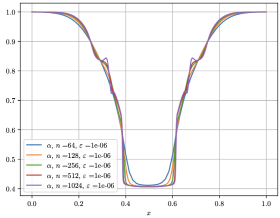


Figure 3: Soft congestion. Video.

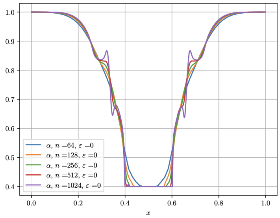


Figure 4: Hard congestion. Video