Uniformly high-order bound-preserving OEDG schemes for two-phase flows

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Overview

The model: Kapila model

Model

- 2 phases, equation of state $p_i = p_i(\rho_i, e_i)$
- volume fractions $\alpha_1 \in [0,1]$, $\alpha_2 \in [0,1]$, $\alpha_1 + \alpha_2 = 1$
- Density $\rho = \alpha_1\rho_1 + \alpha_2\rho_2$
- Velocity equilibrium: one velocity u
- Energy:$E = e + \frac{1}{2}\rho u^2$, $e = \alpha_1 e_1 + \alpha_2 e_2$
- Pressure equilibrium: $p = p_1(\rho_1, e_1) = p_2(\rho_2, e_2)$

$$
\begin{aligned}
\frac{\partial \alpha_1}{\partial t} + u\frac{\partial \alpha_1}{\partial x} &= K\frac{\partial u}{\partial x} \\
\frac{\partial \alpha_1\rho_1}{\partial t} + \frac{\partial(\alpha_1\rho_1 u)}{\partial x} &= 0 \\
\frac{\partial \alpha_2\rho_2}{\partial t} + \frac{\partial(\alpha_2\rho_2 u)}{\partial x} &= 0 \\
\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[u(E+p)\right] &= 0
\end{aligned}
\qquad
K = \alpha_1\alpha_2\frac{\frac{1}{Z_1} - \frac{1}{Z_2}}{\frac{\alpha_1}{Z_1} + \frac{\alpha_2}{Z_2}}, Z_i = \rho_i a_i^2.
$$

Strictly hyperbolic, $\lambda = u, u, u \pm a$,

$$
\frac{1}{\rho a^2} = \frac{\alpha_1}{\rho_1 a_1^2} + \frac{\alpha_2}{\rho_2 a_2^2}.
$$

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$$\frac{1}{\rho a^2} = \frac{\alpha_1}{\rho_1 a_1^2} + \frac{\alpha_2}{\rho_2 a_2^2}.$$

Justification

Starting from Baer and Nunziato model $i = 1, 2$

$$\frac{\partial \alpha_i}{\partial t} + u_I \frac{\partial \alpha_1}{\partial x} = \lambda(p_i - p_{\bar{i}})$$
$$\frac{\partial \alpha_i \rho_i}{\partial t} + \frac{\partial(\alpha_i \rho_i u^i)}{\partial x} =$$
$$\frac{\partial \alpha_i \rho_i u_i}{\partial t} + \frac{\partial(\alpha_i \rho_i u_i^2 + \alpha_i p_i)}{\partial x} = u_I \frac{\partial \alpha_i}{\partial x} + \mu(u_i - u_{\bar{i}})$$
$$\frac{\partial E_i}{\partial t} + \frac{\partial}{\partial x}\left[u_I(E_i + p_i)\right] = p_I u_I \frac{\partial \alpha_i}{\partial x} + \mu p_i(p_i - p_{\bar{i}}) + \mu u_i(u_i - u_{\bar{i}})$$

$$p_I = \sum_i \alpha_i p_i, u_I = \sum_i \alpha_i u_i / \rho.$$

If λ/μ bounded and $\lambda \to +\infty$ then[1] the BN model "converges" towards Kapila's model.

[1] Angelo Murrone and Hervé Guillard. "A five equation reduced model for compressible two-phase flow problems". In: J. Comput. Phys. 202.2 (2005), pp. 664–698.

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Interpretation

The interpretation of the α equation and the term $K = \alpha_1\alpha_2 \times \ldots$: Take the Lagrangian derivative of

$$p_1(\rho_1, s_1) - p_2(\rho_2, s_2) = 0$$

along isentropes $s_i = cte$.

[1] Angelo Murrone and Hervé Guillard. "A five equation reduced model for compressible two-phase flow problems". In: J. Comput. Phys. 202.2 (2005), pp. 664–698.

Questions

- How to define shocks? In[2], some arguments are given to define shock relations: Hugoniot curve for each phase, and pressure equilibrium.
- Invariant domain:

$$\mathcal{D} = \{\alpha_i \rho_i \geq 0, \alpha_i \in [0,1], e \geq 0\}$$

  How to design numerical scheme that respect that

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dG framework

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Overview

Numerical scheme

Rewrite Kapila's model as:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = \chi \mathbf{z}_1 \frac{\partial u}{\partial x}$$

with

$$U = \begin{pmatrix} \alpha_1 \\ \alpha_1\rho_1 \\ \alpha_2\rho_2 \\ \rho u \\ E \end{pmatrix}, \; F = \begin{pmatrix} \alpha_1 u \\ \alpha_1\rho_1 u \\ \alpha_2\rho_2 u \\ \rho u^2 + p \\ u(E+p) \end{pmatrix} \text{ and } \mathbf{z}_1 = \chi\alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi = \frac{K}{\alpha_1} + 1 = \frac{\frac{1}{Z_1}}{\frac{\alpha_1}{Z_1} + \frac{\alpha_2}{Z_2}}.$$

Numerical scheme: do dG

- $\Omega = \cup I_j$, $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $j = 1, \ldots, N_x$.
- $\mathbb{V}_h^K := \left\{ \mathbf{v} \in L^2(\Omega) : \mathbf{v}|_{I_j} \in \mathbb{P}^K(I_j) \quad \forall I_j \in \Omega \right\}$,
- $\mathbf{U}_h(x,t) = \sum_{k=0}^{K} \mathbf{U}_j^{(k)}(t)\phi_j^{(k)}(x) \quad \forall x \in I_j$, where $\{\phi_j^{(k)}(x)\}_{k=0}^{K}$ is a basis of $\mathbb{P}^K(I_j)$, and $\mathbf{U}_j^{(k)}(t)$ is the corresponding kth degree of freedom.

Formally:

$$\int_{I_j} \frac{\partial \mathbf{U}_h}{\partial t}\phi_j^{(k)}(x)\mathrm{d}x + \widehat{\mathbf{F}}(\mathbf{U}^-_{j+\frac{1}{2}}, \mathbf{U}^+_{j+\frac{1}{2}})\phi_j^{(k)}(x_{j+\frac{1}{2}}) - \widehat{\mathbf{F}}(\mathbf{U}^-_{j-\frac{1}{2}}, \mathbf{U}^+_{j-\frac{1}{2}})\phi_j^{(k)}(x_{j-\frac{1}{2}})$$

$$- \int_{I_j} \mathbf{F}(\mathbf{U}_h)\frac{\partial \phi_j^{(k)}(x)}{\partial x}\mathrm{d}x = \int_{I_j} \left(\chi)_h(\mathbf{z}_1)\right)_h \frac{\partial u_h}{\partial x}\phi_j^{(k)}(x)\mathrm{d}x, \tag{1}$$

where the superscripts "$-$" and "$+$" represent the left- and right-hand side limits at a cell interface, respectively, and $\widehat{\mathbf{F}}(\mathbf{U}^-, \mathbf{U}^+)$ is the numerical flux.

In this work, we use the HLL flux

Meaning of the non conservative integral

$$\int_{I_j} (\chi)_h(\alpha_1)_h \frac{\partial u_h}{\partial x} \phi_j^{(k)}(x)\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int_{x_{j-\frac{1}{2}}+\varepsilon}^{x_{j+\frac{1}{2}}-\varepsilon} (\chi)_h(\alpha_1)_h \frac{\partial u_h}{\partial x} \phi_j^{(k)}(x)\mathrm{d}x$$

$$+ \lim_{\varepsilon \to 0^+} \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\varepsilon} (\chi)_h(\alpha_1)_h \frac{\partial u_h}{\partial x} \phi_j^{(k)}(x)\mathrm{d}x + \lim_{\varepsilon \to 0^+} \int_{x_{j+\frac{1}{2}}-\varepsilon}^{x_{j+\frac{1}{2}}} (\chi)_h(\alpha_1)_h \frac{\partial u_h}{\partial x} \phi_j^{(k)}(x)\mathrm{d}x. \quad (2)$$

so

$$\lim_{\varepsilon \to 0^+} \int_{x_{j-\frac{1}{2}}+\varepsilon}^{x_{j+\frac{1}{2}}-\varepsilon} (\chi)_h(\alpha_1)_h \frac{\partial u_h}{\partial x} \phi_j^{(k)}(x)\mathrm{d}x \approx \Delta x \sum_{\nu=1}^{Q} \omega_\nu (\chi\alpha_1)_j^{[\nu]} \left(\frac{\partial u}{\partial x}\right)_j^{[\nu]} \phi_j^{(k)}(x_j^{[\nu]}). \quad (3)$$

and:

$$\lim_{\varepsilon \to 0^+} \int_{x_{j+\frac{1}{2}}-\varepsilon}^{x_{j+\frac{1}{2}}+\varepsilon} (\chi)_h(\alpha_1)_h \frac{\partial u_h}{\partial x} \phi_j^{(k)}(x)\mathrm{d}x$$

$$\approx \omega_{j+\frac{1}{2}}^{-} (\chi\alpha_1)_{j+\frac{1}{2}}^{-} [\![u]\!]_{j+\frac{1}{2}} \phi_j^{(k)}(x_{j+\frac{1}{2}}) + \omega_{j+\frac{1}{2}}^{+} (\chi\alpha_1)_{j+\frac{1}{2}}^{+} [\![u]\!]_{j+\frac{1}{2}} \phi_{j+1}^{(k)}(x_{j+\frac{1}{2}}), \quad (4)$$

with:

$$\omega_{j+\frac{1}{2}}^{-} = \frac{-a_{j+\frac{1}{2}}^{-}}{a_{j+\frac{1}{2}}^{+} - a_{j+\frac{1}{2}}^{-}}, \quad \omega_{j+\frac{1}{2}}^{+} = \frac{a_{j+\frac{1}{2}}^{+}}{a_{j+\frac{1}{2}}^{+} - a_{j+\frac{1}{2}}^{-}}.$$

Here, $a_{j+\frac{1}{2}}^{-}$ and $a_{j+\frac{1}{2}}^{+}$ respectively denote the wave speeds estimated in the HLL flux at the interface $x = x_{j+\frac{1}{2}}$.

so . . .

$$
\begin{aligned}
&\int_{I_j} \phi_j^{(k)}(x)\mathrm{d}x \frac{\partial \mathbf{U}_j^{(k)}}{\partial t} \\
=&\Delta x \sum_{\nu=1}^{Q} \omega_\nu \mathbf{F}\big(\mathbf{U}_j^{[\nu]}\big) \frac{\partial \phi_j^{(k)}(x_j^{[\nu]})}{\partial x} + \Delta x \sum_{\nu=1}^{Q} \omega_\nu (\chi\alpha_1)_j^{[\nu]} \left(\frac{\partial u}{\partial x}\right)_j^{[\nu]} \phi_j^{(k)}(x_j^{[\nu]}) \\
& - \frac{a^+_{j+\frac{1}{2}} \mathbf{F}(\mathbf{U}^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}} \mathbf{F}(\mathbf{U}^+_{j+\frac{1}{2}}) + a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}} (\mathbf{U}^+_{j+\frac{1}{2}} - \mathbf{U}^-_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \phi_j^{(k)}(x_{j+\frac{1}{2}}) \\
& + \frac{a^+_{j-\frac{1}{2}} \mathbf{F}(\mathbf{U}^-_{j-\frac{1}{2}}) - a^-_{j-\frac{1}{2}} \mathbf{F}(\mathbf{U}^+_{j-\frac{1}{2}}) + a^+_{j-\frac{1}{2}} a^-_{j-\frac{1}{2}} (\mathbf{U}^+_{j-\frac{1}{2}} - \mathbf{U}^-_{j-\frac{1}{2}})}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}} \phi_j^{(k)}(x_{j-\frac{1}{2}}) \\
& + (\chi\alpha_1)^-_{j+\frac{1}{2}} \frac{-a^-_{j+\frac{1}{2}} [\![u]\!]_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \phi_j^{(k)}(x_{j+\frac{1}{2}}) + (\chi\alpha_1)^+_{j-\frac{1}{2}} \frac{a^+_{j-\frac{1}{2}} [\![u]\!]_{j-\frac{1}{2}}}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}} \phi_j^{(k)}(x_{j-\frac{1}{2}}). \qquad (5)
\end{aligned}
$$

Time discretisation: OEDG

- Runge Kutta, for example SSP RK
- To stabilize discontinuities: post processing technique[3]

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\hat{t}} \int_{I_j} \mathbf{U}_\sigma \phi_j^{(k)}(x)\mathrm{d}x + \sum_{m=0}^{K} \beta_j \frac{\sigma_j^m(\mathbf{U}_h^{n,\ell+1})}{\Delta x} \int_{I_j} (\mathbf{U}_\sigma - P^{m-1}\mathbf{U}_\sigma)\phi_j^{(k)}(x)\mathrm{d}x = 0, \\ \mathbf{U}_\sigma(x,0) = \mathbf{U}_h^{n,\ell+1}(x), \end{cases} \tag{6}$$

The damping coefficient $\sigma_j^m(\mathbf{U}_h)$ is defined as

$$\sigma_j^m(\mathbf{U}_h) := \max_{1\le q\le 5} \sigma_j^m(\mathrm{U}_h{}^{(q)})$$

with

$$\sigma_j^m(\mathrm{U}_h{}^{(q)}) = \begin{cases} 0, & \text{if} \quad \mathrm{U}_h{}^{(q)} \equiv \overline{\mathrm{U}}_\Omega^{(q)}, \\ \frac{(2m+1)\Delta x^m}{(2K-1)m!} \frac{\left|[\![\partial_x^m \mathrm{U}_h{}^{(q)}]\!]_{j-\frac{1}{2}}\right| + \left|[\![\partial_x^m \mathrm{U}_h{}^{(q)}]\!]_{j+\frac{1}{2}}\right|}{2\left\|\mathrm{U}_h{}^{(q)} - \overline{\mathrm{U}}_\Omega^{(q)}\right\|_{L^\infty(\Omega)}}, & \text{otherwise,} \end{cases}$$

where $\mathrm{U}_h{}^{(q)}$ is the qth component of $\mathbf{U}_h$, and $\overline{\mathrm{U}}_\Omega^{(q)} = \frac{1}{|\Omega|}\int_\Omega \mathrm{U}_h^{(q)}(x)\mathrm{d}x$ represents the global average of $\mathrm{U}_h^{(q)}$ over the entire computational domain Ω.

[3] M. Peng, Z. Sun, and K. Wu. "OEDG: Oscillation-eliminating discontinuous Galerkin method for hyperbolic conservation laws". In: Mathematics of Computation (Published electronicaly July 30, 2024). in press.

Some properties

Equilibrium condition:

$$\text{if} \quad u^n(x) \equiv u_0, \quad p^n(x) \equiv p_0, \quad \text{then} \quad u^{n+1}(x) \equiv u_0, \quad p^{n+1}(x) \equiv p_0. \tag{7}$$

Result

The fully-discrete OEDG scheme with an SSP RK method maintains the condition (7) condition around an isolated material interface:

$$\text{if} \quad u_\sigma^n(x) \equiv u_0, \quad p_\sigma^n(x) \equiv p_0, \quad \text{then} \quad u_\sigma^{n+1}(x) \equiv u_0, \quad p_\sigma^{n+1}(x) \equiv p_0.$$

Preserving the invariance domain

We use Zhang and Shu's technique[4]:

1. Show that the average values stay in the invariant domain if they are at the previous time step
2. Construct a bound preserving polynomial approximation
3. all this under a CFL like condition

[4] Xiangxiong Zhang and Chi-Wang Shu. "On positivity-preserving high order discontinuous Galerkin schemes for compressible Euler equations on rectangular meshes". In: Journal of Computational Physics 229.23 (2010), pp. 8918–8934.

$$
\begin{aligned}
(\overline{\alpha}_1)_j^{n+1} = & (\overline{\alpha}_1)_j^n + \frac{\tau}{\Delta x}\Bigg(\Delta x \sum_{\nu=1}^{Q} \omega_\nu (\chi\alpha_1)_j^{[\nu]} \left(\frac{\partial u}{\partial x}\right)_j^{[\nu]} \\
& - \frac{a^+_{j+\frac{1}{2}}(\alpha_1 u)^-_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}(\alpha_1 u)^+_{j+\frac{1}{2}} + a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}\left((\alpha_1)^+_{j+\frac{1}{2}} - (\alpha_1)^-_{j+\frac{1}{2}}\right)}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \\
& + \frac{a^+_{j-\frac{1}{2}}(\alpha_1 u)^-_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}(\alpha_1 u)^+_{j-\frac{1}{2}} + a^+_{j-\frac{1}{2}} a^-_{j-\frac{1}{2}}\left((\alpha_1)^+_{j-\frac{1}{2}} - (\alpha_1)^-_{j-\frac{1}{2}}\right)}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}} \\
& + (\chi\alpha_1)^-_{j+\frac{1}{2}} \frac{-a^-_{j+\frac{1}{2}} [\![u]\!]_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + (\chi\alpha_1)^+_{j-\frac{1}{2}} \frac{a^+_{j-\frac{1}{2}} [\![u]\!]_{j-\frac{1}{2}}}{a^+_{j-\frac{1}{2}} - a^-_{j-\frac{1}{2}}}\Bigg), \qquad (8)
\end{aligned}
$$

where $(\overline{\alpha}_1)_j^n := \frac{1}{\Delta x}\int_{I_j} (\alpha_1)_\sigma^n(x)\mathrm{d}x$

We have:

Result

Consider the $\mathbb{P}^K$-based OEDG method for the 1D Kapila five-equation model. If

$$(\widehat{\alpha}_1)_j^{[\alpha]} := (\alpha_1)_\sigma^n(\widehat{x}_j^{[\alpha]}) \in [0,1], \quad (\alpha_1)_j^{[\nu]} := (\alpha_1)_\sigma^n(x_j^{[\nu]}) \in [0,1] \quad \forall j, \alpha, \nu,$$

then the scheme (8) preserves $(\overline{\alpha}_1)_j^{n+1} \in [0,1]$ under the CFL condition

$$\frac{\tau}{\Delta x} \leq \frac{1}{\mathcal{A}_1 + \mathcal{A}_2}, \tag{9}$$

where

$$\begin{aligned} \mathcal{A}_1 &= \Delta x \max\left\{ \left\| \chi \frac{\partial u}{\partial x} \right\|_\infty, \ \left\| \chi_2 \frac{\partial u}{\partial x} \right\|_\infty \right\}, \\ \mathcal{A}_2 &= \frac{1}{\widehat{\omega}_1} \max_j \left\{ \left| a^{\pm}_{j+\frac{1}{2}} \right| + \max\left\{ (\chi)^{\mp}_{j+\frac{1}{2}}, \ (\chi_2)^{\mp}_{j+\frac{1}{2}} \right\} \left| [\![u]\!]_{j+\frac{1}{2}} \right| \right\}. \end{aligned} \tag{10}$$

About the CFL condition

Utilizing the inverse inequalities for polynomials of degree K, we have:

$$\begin{aligned}\left\|\chi_\ell \frac{\partial u_h}{\partial x}\right\|_{L^\infty(I_j)} &\le \|\chi_\ell\|_\infty \left\|\frac{\partial u_h}{\partial x}\right\|_{L^\infty(I_j)} \\ &\le C_1(\Delta x)^{-\frac{1}{2}}\|\chi_\ell\|_\infty \left\|\frac{\partial u_h}{\partial x}\right\|_{L^2(I_j)} \\ &\le C_2(\Delta x)^{-\frac{3}{2}}\|\chi_\ell\|_\infty \|u_h\|_{L^2(I_j)} \\ &\le C_2(\Delta x)^{-1}\|\chi_\ell\|_\infty \|u_h\|_{L^\infty(I_j)}, \quad \ell \in \{1,2\},\end{aligned}$$

where C_1 and C_2 are positive constants depending solely on the polynomial degree K. It follows that

$$\mathscr{A}_1 \le C_2 \|u_h\|_\infty \max\left\{\|\chi\|_\infty, \|\chi_2\|_\infty\right\},$$

which implies the time step constraint (9) is indeed a reasonable CFL-type condition.

Bound preserving

Shu-Wu's technique[5]

$$\mathscr{D} = \underbrace{\{\alpha_1 \in [0,1]\}}_{\mathscr{D}_1} \cap \underbrace{\{\alpha_1\rho_1 \geq 0, u \in \mathbb{R}, \alpha_2\rho_2 \geq o, e \geq 0\}}_{\mathscr{D}_2}$$

$$(\alpha_1\rho_1, \alpha_2\rho_2, u, E) \in \mathscr{D}_1 \iff \forall u_\star, \begin{pmatrix} \alpha_1\rho_1 \\ \alpha_2\rho_2 \\ u, \\ E \end{pmatrix}^T \underbrace{\begin{pmatrix} u_\star^2/2 \\ u_\star^2/2 \\ -u_\star \\ 1 \end{pmatrix}}_{\mathbf{n}_*} \geq 0$$

HLL flux satisfies

Result

For any $\mathbf{U} \in \mathscr{G}_*$ and $q \in \{1,2\}$, the following inequalities hold:

$$\mathbf{F}_q(\mathbf{U}) \cdot \mathbf{n}_\ell > \sigma_q^- \mathbf{U} \cdot \mathbf{n}_\ell, \quad -\mathbf{F}_q(\mathbf{U}) \cdot \mathbf{n}_\ell > -\sigma_q^+ \mathbf{U} \cdot \mathbf{n}_\ell, \qquad \ell = 1, 2,$$
$$\mathbf{F}_q(\mathbf{U}) \cdot \mathbf{n}_* > \sigma_q^- \mathbf{U} \cdot \mathbf{n}_*, \quad -\mathbf{F}_q(\mathbf{U}) \cdot \mathbf{n}_* > -\sigma_q^+ \mathbf{U} \cdot \mathbf{n}_* \qquad \forall \mathbf{u}_* \in \mathbb{R}^2,$$

and

$$\sigma_1^- = \min\{u - c,\ 0\}, \qquad \sigma_1^+ = \max\{u + c,\ 0\},$$
$$\sigma_2^- = \min\{v - c,\ 0\}, \qquad \sigma_2^+ = \max\{v + c,\ 0\}.$$

[5] Kailiang Wu and Chi-Wang Shu. "Geometric quasilinearization framework for analysis and design of bound-preserving schemes". In: SIAM Rev. 65.4 (2023), pp. 1031–1073.

Result

Consider the $\mathbb{P}^K$-based OEDG method for the 1D Kapila five-equation model. If

$$\widehat{\mathbf{U}}_j^{[\alpha]} := \mathbf{U}_\sigma^n(\widehat{x}_j^{[\alpha]}) \in \mathcal{D}, \quad \mathbf{U}_j^{[\nu]} := \mathbf{U}_\sigma^n(x_j^{[\nu]}) \in \mathcal{D} \quad \forall j, \alpha, \nu,$$

then the scheme (8) preserves $\overline{\mathbf{U}}_j^{n+1} \in \mathcal{D}$ under the CFL condition

$$\frac{\Delta t}{\Delta x} \leq \min\left\{ \frac{\widehat{\omega}_1}{\max_j \left\{ a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}} \right\}},\ \frac{1}{\mathcal{A}_1 + \mathcal{A}_2} \right\}, \tag{11}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given in (10).

Final step: get $\widehat{\mathbf{U}}_j^{[\alpha]} := \mathbf{U}_\sigma^n(\widehat{x}_j^{[\alpha]}) \in \mathscr{D}$

Shu and Zhang[6]

Step 1. First, enforce the volume fraction $0 \le \alpha_1 \le 1$ and partial density $z_\ell \rho_\ell \ge \epsilon_\ell$ with $\epsilon_\ell = \min\left\{(\overline{\alpha_\ell\rho_\ell})_j^n,\ 10^{-13}\right\}$. Modify the OEDG solution $\left(\mathbf{U}_\sigma^n\right)_j(x)$ as

$$\widetilde{\mathbf{U}}_j(x) = \overline{\mathbf{U}}_j^n + \theta_{\min}\left((\mathbf{U}_\sigma^n)_j(x) - \overline{\mathbf{U}}_j^n\right) \quad \text{with} \quad \theta_{\min} = \min\left\{\theta_{\alpha_1}, \theta_{\alpha_1\rho_1}, \theta_{z_2\rho_2}\right\}, \qquad (12)$$

where

$$\theta_z = \min\left\{\frac{1-(\overline{\alpha}_1)_j^n}{\max\limits_{x\in\mathbb{S}_j}\left((\alpha_1)_\sigma^n\right)_j(x) - (\overline{\alpha}_1)_j^n},\ \frac{(\overline{\alpha}_1)_j^n}{(\overline{\alpha}_1)_j^n - \min\limits_{x\in\mathbb{S}_j}\left((\alpha_1)_\sigma^n\right)_j(x)},\ 1\right\},$$

$$\theta_{\alpha_\ell\rho_\ell} = \min\left\{\frac{(\overline{\alpha_\ell\rho_\ell})_j^n - \epsilon_\ell}{(\overline{\alpha_\ell\rho_\ell})_j^n - \min\limits_{x\in\mathbb{S}_j}\left((\alpha_\ell\rho_\ell)_\sigma^n\right)_j(x)},\ 1\right\}, \quad \ell \in \{1,2\}.$$

Step 2. Next, enforce the internal energy $\mathcal{E} \ge \epsilon_3$ with $\epsilon_3 = \min\left\{\mathcal{E}(\overline{\mathbf{U}}_j^n),\ 10^{-13}\right\}$. Modify $\widetilde{\mathbf{U}}_j(x)$ as

$$\mathbf{U}_j(x) = \overline{\mathbf{U}}_j^n + \theta_{\rho e}\left(\widetilde{\mathbf{U}}_j(x) - \overline{\mathbf{U}}_j^n\right) \quad \text{with} \quad \theta_{\rho e} = \min\left\{\frac{\mathcal{E}(\overline{\mathbf{U}}_j^n) - \epsilon_3}{\mathcal{E}(\overline{\mathbf{U}}_j^n) - \min\limits_{x\in\mathbb{S}_j}\mathcal{E}\left(\widetilde{\mathbf{U}}_j(x)\right)},\ 1\right\}. \qquad (13)$$

[6] Xiangxiong Zhang and Chi-Wang Shu. "On positivity-preserving high order discontinuous Galerkin schemes for compressible Euler equations on rectangular meshes". In: Journal of Computational Physics 229.23 (2010), pp. 8918–8934.

Overview

Equation of state

Each fluid follow a gamma model

Accuracy test

$$(\rho_1, \rho_2, u, p, \alpha_1) = \Big(1,\ 1.5 + 0.4\cos(\pi x),\ 1 + 0.4\cos(\pi x),\ 1,\ 0.5 + 0.4\sin(\pi x)\Big).$$

The specific heat ratios $\gamma_1 = 1.4$ and $\gamma_2 = 1.6$. The problem is simulated until $t = 0.05$ in computational domain is $[0, 2]$ with periodic boundary conditions.

Table: Numerical errors in the mixture density ρ at $t = 0.05$ and the corresponding convergence orders for the $\mathbb{P}^2$-BP-OEDG-AVE schemeand our $\mathbb{P}^2$-BP-OEDG scheme.

N_x	l^1 error	order	l^2 error	order	l^∞ error	order
$\mathbb{P}^2$-BP-OEDG-AVE						
40	2.64E-04	–	4.84E-04	–	3.59E-03	–
80	8.56E-06	4.95	1.42E-05	5.09	8.71E-05	5.36
160	9.87E-07	3.12	1.45E-06	3.30	8.32E-06	3.39
320	2.93E-07	1.75	4.04E-07	1.84	2.06E-06	2.01
640	9.07E-08	1.69	1.21E-07	1.73	4.89E-07	2.07
1280	2.53E-08	1.84	3.44E-08	1.82	1.08E-07	2.18
$\mathbb{P}^2$-BP-OEDG						
40	4.54E-04	–	7.60E-04	–	5.16E-03	–
80	1.10E-05	5.37	2.08E-05	5.19	1.65E-04	4.97
160	4.66E-07	4.56	9.52E-07	4.45	7.90E-06	4.38
320	3.42E-08	3.77	8.61E-08	3.47	1.21E-06	2.71
640	3.26E-09	3.39	9.59E-09	3.17	1.83E-07	2.72
1280	3.43E-10	3.25	9.00E-10	3.41	1.58E-08	3.53

BP-OEDG-AVE scheme[7]

[7] Fan Zhang and Jian Cheng. "Analysis on physical-constraint-preserving high-order discontinuous Galerkin method for solving Kapila's five-equation model". In: J. Comput. Phys. 492 (2023), p. 112417.

Table: Numerical errors in the mixture density ρ at $t = 0.05$ and the corresponding convergence orders for the $\mathbb{P}^3$-BP-OEDG-AVE scheme and our $\mathbb{P}^3$-BP-OEDG scheme.

N_x	l^1 error	order	l^2 error	order	l^∞ error	order
$\mathbb{P}^3$-BP-OEDG-AVE						
40	1.99E-04	–	2.81E-04	–	1.34E-03	–
80	2.62E-05	2.93	3.76E-05	2.90	1.49E-04	3.17
160	6.12E-06	2.10	1.06E-05	1.83	5.67E-05	1.39
320	1.60E-06	1.94	3.63E-06	1.55	2.96E-05	0.94
640	4.11E-07	1.96	1.23E-06	1.56	1.33E-05	1.15
1280	1.07E-07	1.95	3.94E-07	1.64	5.36E-06	1.31
$\mathbb{P}^3$-BP-OEDG						
40	2.34E-06	–	3.17E-06	–	1.01E-05	–
80	5.15E-08	5.51	8.18E-08	5.28	3.72E-07	4.76
160	2.20E-09	4.55	4.23E-09	4.27	2.58E-08	3.85
320	9.71E-11	4.50	2.25E-10	4.23	2.14E-09	3.59
640	4.26E-12	4.51	1.10E-11	4.36	1.45E-10	3.88
1280	2.02E-13	4.40	4.80E-13	4.51	5.75E-12	4.66

Strong interface interaction problem

$$(\rho_1, \rho_2, u, p, \alpha_1) = \begin{cases} \left(0.3884,\ 1,\ 27.1123\sqrt{10^5},\ 10^7,\ 10^{-10}\right), & x < 0.2, \\ \left(0.1,\ 1,\ 0,\ 10^5,\ 10^{-10}\right), & 0.2 < x < 0.3, \\ \left(0.1,\ 1,\ 0,\ 10^5,\ 1 - 10^{-10}\right), & x > 0.3. \end{cases}$$

The specific heat ratios $\gamma_1 = \frac{5}{3}$ and $\gamma_2 = 1.4$. The computational domain $[0, 1]$ is divided into 200 uniform cells with outflow boundary conditions.

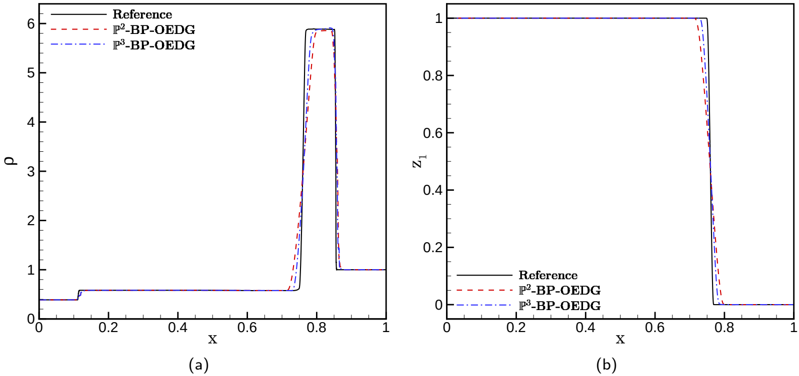


Figure: Numerical results obtained by $\mathbb{P}^2$-BP-OEDG and $\mathbb{P}^3$-BP-OEDG with 200 uniform cells. Left: mixture density ρ; right: volume fraction α_1.

LeBlanc like case

$$(\rho_1, \rho_2, u, e, \alpha) = \begin{cases} (1,\ 0.001,\ 0,\ 10^{-1},\ 1 - 10^{-10}), & x < 3, \\ (1,\ 0.001,\ 0,\ 10^{-7},\ 10^{-10}), & \text{otherwise.} \end{cases}$$

The specific heat ratios $\gamma_1 = \gamma_2 = \frac{5}{3}$. The computational domain $[0, 9]$ is divided into 400 uniform cells with the outflow boundary conditions. Final time$t = 6$.
The reference solution is computed by $\mathbb{P}^2$-BP-OEDG with 10,000 uniform cells.

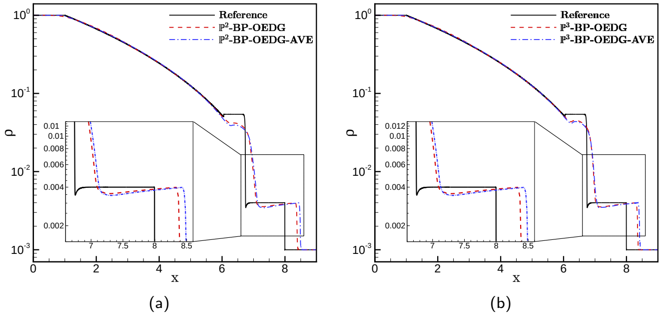


Figure: Numerical results obtained by the BP-OEDG-AVE scheme and our BP-OEDG scheme with 400 uniform cells. Left: $\mathbb{P}^2$; right: $\mathbb{P}^3$.

Lax like problem

$$(\rho_1, \rho_2, u, p, \alpha_1) = \begin{cases} (0.445,\ 0.5,\ 0.698,\ 3.528,\ x+0.5), & x<0, \\ (0.445,\ 0.5,\ 0,\ 0.571,\ x+0.5), & x>0. \end{cases}$$

The specific heat ratios $\gamma_1 = 1.4$ and $\gamma_2 = 1.6$. The computational domain is $[-0.5, 0.5]$ with outflow boundary conditions. Final time: $t = 0.17$.

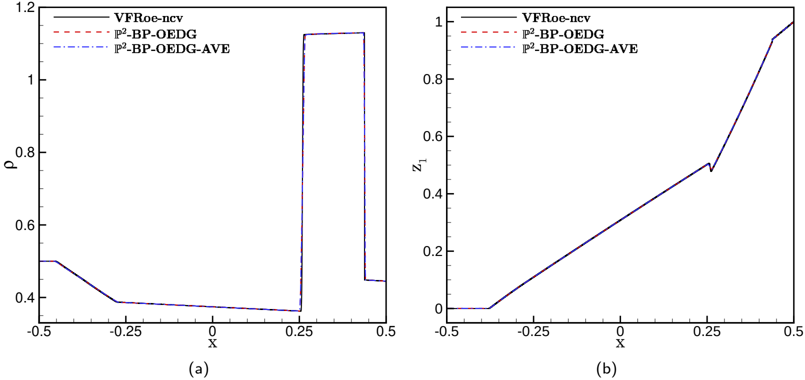


Figure: Numerical results obtained by $\mathbb{P}^2$-BP-OEDG-AVE scheme and our $\mathbb{P}^2$-BP-OEDG scheme with 1000 uniform cells. The reference solution is computed by the VFRoe-ncv scheme[9] with 100,000 uniform cells. Left: mixture density ρ; right: volume fraction α_1.

[8] Angelo Murrone and Hervé Guillard. "A five equation reduced model for compressible two-phase flow problems". In: J. Comput. Phys. 202.2 (2005), pp. 664–698.

[9] Angelo Murrone and Hervé Guillard. "A five equation reduced model for compressible two-phase flow problems". In: J. Comput. Phys. 202.2 (2005), pp. 664–698.

About the CFL condition

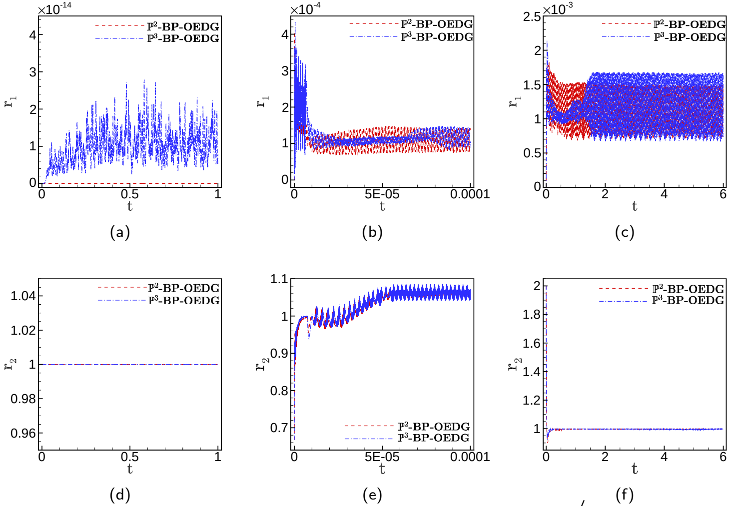


Figure: The time evolution of the ratios $r_1 = \frac{\mathcal{A}_1}{\mathcal{A}_2}$ (top) and $r_2 = \frac{1}{\mathcal{A}_1+\mathcal{A}_2} \Big/ \frac{\widehat{\omega}_1}{\max_j \left\{ a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}} \right\}}$ (bottom). From left to right: Moving interface, Shock interface, and LeBlanc.

Richtmyer–Meshkov Instability

Initial position of the gas-gas interface: $x = 2.9 - 0.1 \sin\big(2\pi(y + 0.25)\big), y \in [0, 1]$.

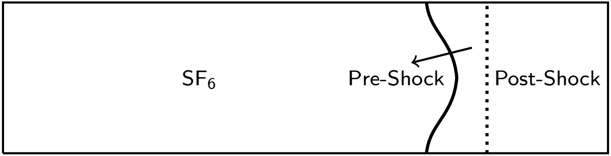


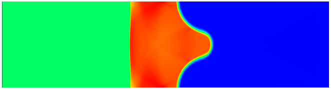
Figure: A schematic of the computational domain for Richtmyer–Meshkov instability problem.

Richtmyer–Meshkov Instability

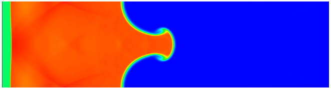
The initial conditions are

$$(\rho_1, \rho_2, u, v, p, \alpha_1) = \begin{cases} \left(1,\ 5.04,\ 0,\ 0,\ 1,\ 10^{-6}\right), & \mathrm{SF}_6, \gamma = 1.093 \\ \left(1,\ 5.04,\ 0,\ 0,\ 1,\ 1-10^{-6}\right), & \text{pre-shock air}, \gamma = 1.4 \\ \left(1.411,\ 5.04,\ -0.39,\ 0, 1.628,\ 1-10^{-6}\right), & \text{post-shock air}, \gamma = 1.4 \end{cases}$$

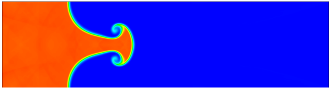
- Initially, a planar shock wave with a Mach number of 1.24 is positioned at $x = 3.2$ in air
- moving from right to left
- Domain: $[0, 4] \times [0, 1]$, mesh 1000×250.
- Boundary conditions:
  - periodic on upper/bottom edges
  - Left is outflow, right: set to post-flow conditions.



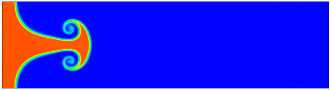
(a) $t = 2.31$



(b) $t = 4.62$



(c) $t = 6.93$



(d) $t = 9.24$

Figure: The mixture density ρ obtained by $\mathbb{P}^3$-BP-OEDG with 1000×250 uniform cells at different time instances.

shock-helium bubble interaction

$d_1 = 50$mm, $d_2 = 25$mm, $d_3 = 100$mm, $d_4 = 300$mm, and $d_5 = 89$mm.

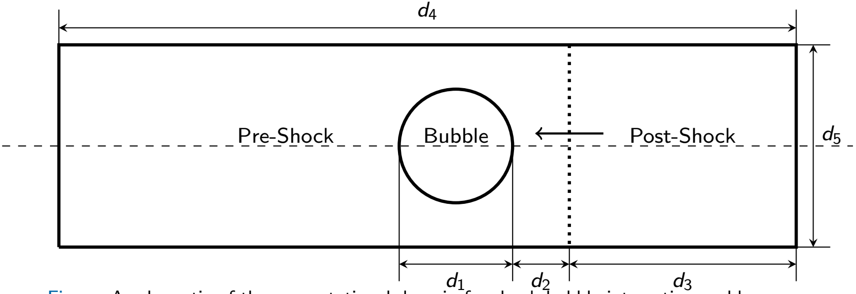
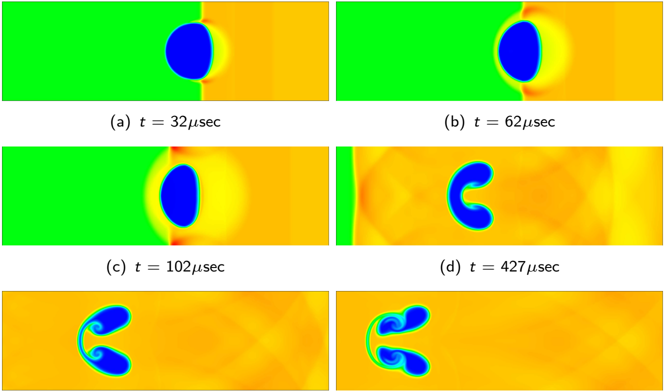


Figure: A schematic of the computational domain for shock-bubble interaction problem.

Initial conditions:

- The bubble is in thermodynamical equilibrium with suriounding air
- A shock wave with a Mach number of 1.22 propagates to the left and hits a helium bubble
- bubble: contaminated with 28% of air.
- Initial conditions are:
  - Helium bubble$\rho_1 = 1.4$, $\rho_2 = 0.25463$, $u = 0$, $v = 0$, $p = 10^5$, $\alpha_1 = 10^{-6}$,
  - Air: pre-shock $\rho_1 = 1.4$, $\rho_2 = 0.25463$, $u = 0$, $v = 0$, $p = 10^5$, $\alpha_1 = 1 - 10^{-6}$, post-shock: $\rho_1 = 1.92691$, $\rho_2 = 0.25463$, $u = -113.5$, $v = 0$, $p = 1.5698 \times 10^5$, $\alpha_1 = 1 - 10^{-6}$, for post-shock air.
  - $\gamma_{air} = 1.4$, $\gamma_{HE} = 1.648$
- solid wall on top/bottom, left is outflow, right is set to post-shock

$\mathbb{P}_2,\ \rho$



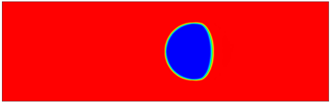
(a) $t = 32\mu$sec (b) $t = 62\mu$sec

(c) $t = 102\mu$sec (d) $t = 427\mu$sec

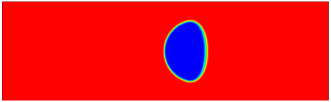
(e) $t = 674\mu$sec (f) $t = 983\mu$sec

Figure: 900×267 uniform cells.

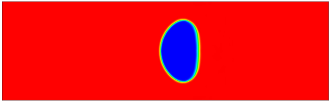
$\mathbb{P}_2$, α



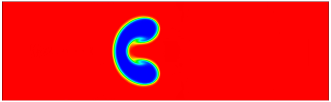
(a) $t = 32\mu$sec



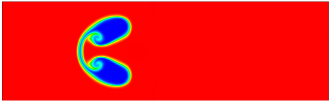
(b) $t = 62\mu$sec



(c) $t = 102\mu$sec



(d) $t = 427\mu$sec

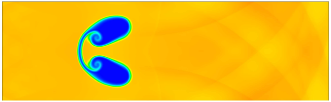


(e) $t = 674\mu$sec

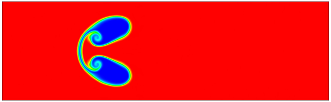
(f) $t = 983\mu$sec

Figure: 900×267 uniform cells.

$\mathbb{P}_3,\ \alpha, \rho$



(a) ρ,$t = 674\mu$sec



(b) α,$t = 674\mu$sec

(c) ρ,$t = 983\mu$sec

(d) α,$t = 983\mu$sec

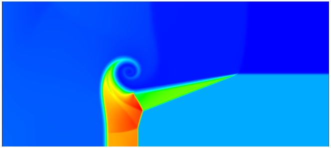
Triple point: 2D Riemann problem

In $[0,7] \times [0,3]$ with 840×360 uniform cells:

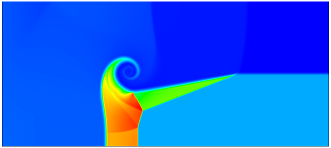
$$(\rho_1, \rho_2, u, v, p, \alpha_1) = \begin{cases} \left(1,\ 1,\ 0,\ 0,\ 1,\ 1 - 10^{-6}\right), & x \leq 1, \\ \left(1,\ 1,\ 0,\ 0,\ 0.1,\ 10^{-6}\right), & x > 1 \text{ and } y < 1.5, \\ \left(0.125,\ 1,\ 0,\ 0,\ 0.1,\ 1 - 10^{-6}\right), & \text{otherwise.} \end{cases}$$

- $\gamma_1 = 1.5$ and $\gamma_2 = 1.4$
- Solid wall on top/bottom
- Ouflow on right and left
- Times: $t = 2.4$ and $t = 5$

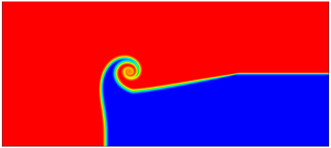
$t = 2.4$, 840×360



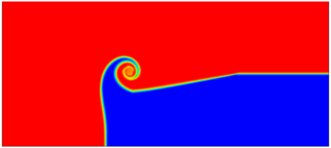
(e) ρ, $\mathbb{P}_2$



(f) $\rho\mathbb{P}_3$

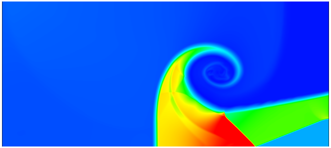


(g) α, $\mathbb{P}_2$

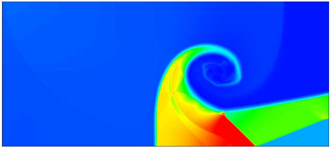


(h) α, $\mathbb{P}_3$

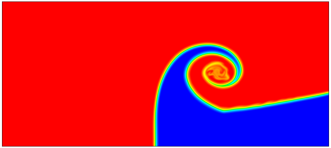
$t = 5,\ 840 \times 360$



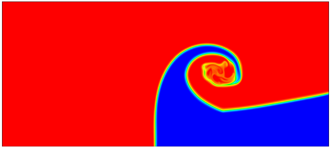
(i) ρ, $\mathbb{P}_2$



(j) $\rho\mathbb{P}_3$



(k) α, $\mathbb{P}_2$



(l) α, $\mathbb{P}_3$

Overview

Conclusions[10]

- A novel high-order bound-preserving oscillation-eliminating discontinuous Galerkin (BP-OEDG) method with the Harten-Lax-van Leer (HLL) flux for the Kapila five-equation model.
- Preserves isolated contact
- Provably: $\alpha \in [0, 1]$, time/space discrete. Carefull discretisation of the α equation
- Bound preserving on $\alpha_i \rho_i$ and internal energy
- Under an almost classical CFL condition.
- Robust

[10] Ruifang Yan, Rémi Abgrall, and Kailiang Wu. "Uniformly high-order bound-preserving OEDG schemes for two-phase flows". In: Mathematical Models & Methods in Applied Sciences 34.13 (2024), pp. 2537–2610.

Conclusions[10]

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What has not been done: respect the Saurel-Gavryliuk shock relations.

[10] Ruifang Yan, Rémi Abgrall, and Kailiang Wu. "Uniformly high-order bound-preserving OEDG schemes for two-phase flows". In: Mathematical Models & Methods in Applied Sciences 34.13 (2024), pp. 2537–2610.