Diffuse Interface Models for Two-Phase Flows of Viscous Incompressible Fluids and Their Sharp Interface Limits

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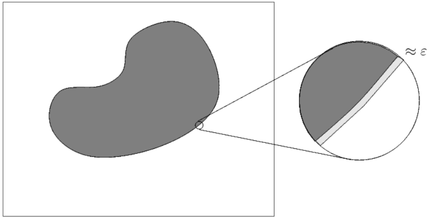
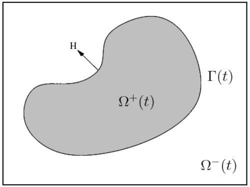
February 6, 2025

Sharp Interface Limits of Diffuse Interface Models

We consider the flow of two macroscopic immiscible, incompressible Newtonian fluids.



$\longrightarrow_{\varepsilon\to 0}$



Diffuse Interface Model (A., Garcke, Grün '12)

We consider

$$\rho\partial_t \boldsymbol{v} + (\rho\boldsymbol{v} + \widetilde{\boldsymbol{J}})\cdot\nabla\boldsymbol{v} - \operatorname{div}(2\nu(c)D\boldsymbol{v}) + \nabla p = \mu\nabla c$$
$$\operatorname{div}\boldsymbol{v} = 0$$
$$\partial_t c + \boldsymbol{v}\cdot\nabla c = m_\varepsilon \Delta\mu$$

in $\Omega\times(0,\infty)$, where $\widetilde{\boldsymbol{J}} = -m_\varepsilon \frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\nabla\mu$ together with

$$\mu = -\varepsilon\Delta c + \frac{1}{\varepsilon}f'(c)$$
$$(\boldsymbol{v}, \partial_{\mathbf{n}} c, \partial_{\mathbf{n}}\mu)|_{\partial\Omega} = 0$$

and initial conditions for $(\boldsymbol{v}, c)$. Here f is a double potential (e.g. $f(c) = (1-c^2)^2$), $\varepsilon > 0$, $m_\varepsilon > 0$

- $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$ – volume averaged velocity.
- $\boldsymbol{v}_j$ – velocity of fluid j.
- c_j – volume fraction of fluid j, $c = c_2 - c_1$.
- $\rho = \rho(c) = \frac{1-c}{2}\tilde{\rho}_1 + \frac{1+c}{2}\tilde{\rho}_2$ and $\tilde{\rho}_j > 0$ are the specific densities.

Alternative models: Lowengrub & Truskinovski '98, Ding, Spelt, Shu '07, ten Eikelder et al. '21, ...

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Conservation of mass:

$$\partial_t \rho + \operatorname{div}\left(\rho\boldsymbol{v} \underbrace{- m_\varepsilon \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \nabla\mu}_{=\widetilde{\mathbf{J}}}\right) = 0$$

Here $-m_\varepsilon \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \nabla\mu$ is a flux relative to $\rho\boldsymbol{v}$ related to diffusion of the particles.

Analytic results: Boyer '03, A. '09 ($\rho \equiv const.$), A., Depner, Garcke '13, Gal, Grasselli, Wu '19, Giorgini '21, A., Garcke, Giorgini '24, A., Garcke, Poiatti '24, ...

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Surface tension: Formally,

$$\mu = 0\cdot\frac{1}{\varepsilon} - \theta_0'(\tfrac{d_{\Gamma_t}}{\varepsilon})\underbrace{\Delta d_{\Gamma_t}}_{=-H_{\Gamma_t}\text{ at }\Gamma_t} + O(\varepsilon)$$

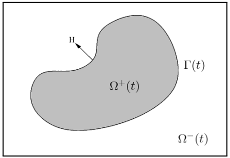
as $\varepsilon\to 0$ provided

$$c(x,t) = \theta_0\left(\frac{d_{\Gamma_t}}{\varepsilon}\right) + O(\varepsilon) \approx 2\chi_{\Omega^+(t)} - 1,$$

where $-\theta_0''(\rho) + f'(\theta_0(\rho)) = 0$ for all $\rho\in\mathbb{R}$ and $d_{\Gamma_t} = \operatorname{sdist}(x,\Gamma_t)$.

Formal Asymptotics $\varepsilon \to 0$ for Navier-Stokes/Cahn-Hilliard system (AGG '12)

Bulk equations: In $\Omega^\pm(t)$ we have



$$\begin{aligned} \rho\partial_t \boldsymbol{v} + \rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \operatorname{div}(2\nu^\pm D\boldsymbol{v}) + \nabla p &= 0 \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}$$

Interface equations:

Case I: $m_\varepsilon = \varepsilon m_0$: On $\Gamma(t)$ we have

$$\begin{aligned} -\left[\boldsymbol{n}_{\Gamma(t)}\cdot(2\nu^\pm D\boldsymbol{v} - p\mathbf{I})\right] &= \sigma H\boldsymbol{n} \\ V_{\Gamma(t)} &= \boldsymbol{n}_{\Gamma(t)}\cdot\boldsymbol{v}|_{\Gamma(t)} \end{aligned}$$

$V_{\Gamma(t)}$ is the normal velocity, $H_{\Gamma(t)}$ is the mean curvature, $\boldsymbol{n}_{\Gamma(t)}$ is a normal.

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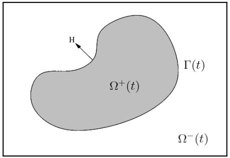

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Case II: $m_\varepsilon = m_0 > 0$: On $\Gamma(t)$ we have

$$\begin{aligned} -\left[\boldsymbol{n}_{\Gamma(t)}\cdot(2\nu^\pm D\boldsymbol{v} - p\mathbf{I})\right] &= \sigma H_{\Gamma(t)} \boldsymbol{n}_{\Gamma(t)} \\ V_{\Gamma(t)} &= \boldsymbol{n}_{\Gamma(t)}\cdot \boldsymbol{v}|_{\Gamma(t)} - \tfrac{m_0}{2}[\boldsymbol{n}_{\Gamma(t)}\cdot\nabla\mu] \\ 2\mu|_{\Gamma(t)} &= \sigma H_{\Gamma(t)} \end{aligned}$$

together with $\Delta\mu = 0$ in $\Omega^\pm(t)$.

Overview of Rigorous Analytic Results

(Navier-)Stokes/Cahn-Hilliard system:

- A. & Röger '09, A. Lengeler '14: Convergence in the case $\varepsilon/m_\varepsilon \to_{\varepsilon\to 0} 0$ for large times in a quite weak "varifold" sense.
- A. & Lengeler '14: Counterexample for convergence if $m_\varepsilon = o(\varepsilon^3)$ for inflow boundary condition.
- A. & Marquardt '20: Convergence for small times with convergence rates in bounded domain in $\mathbb{R}^2$ for Stokes/Cahn-Hilliard system with same viscosities and $m_\varepsilon = m_0$.

Sharp Interface Limit for a Stokes/Cahn-Hilliard System

We consider the sharp interface limit $\varepsilon \to 0$ for

$$
\begin{aligned}
-\Delta \boldsymbol{v}_\varepsilon + \nabla p_\varepsilon &= \mu_\varepsilon \nabla c_\varepsilon && \text{in } \Omega \times [0, T_0], && (1)\\
\operatorname{div} \boldsymbol{v}_\varepsilon &= 0 && \text{in } \Omega \times [0, T_0], && (2)\\
\partial_t c_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla c_\varepsilon &= \Delta \mu_\varepsilon && \text{in } \Omega \times [0, T_0], && (3)\\
\mu_\varepsilon &= -\Delta c_\varepsilon + \tfrac{1}{\varepsilon} f'(c_\varepsilon) && \text{in } \Omega \times [0, T_0], && (4)
\end{aligned}
$$

which formally converges to

$$
\begin{aligned}
-\Delta \boldsymbol{v} + \nabla p &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0], && (5)\\
\operatorname{div} \boldsymbol{v} &= 0 && \text{in } \Omega^\pm(t), t \in [0, T_0], && (6)\\
-\left[\boldsymbol{n}_{\Gamma(t)} \cdot (2D\boldsymbol{v} - pI)\right] &= \sigma H_{\Gamma(t)} \boldsymbol{n}_{\Gamma(t)} && \text{on } \Gamma(t), t \in [0, T_0], && (7)\\
V_{\Gamma(t)} - \boldsymbol{n}_{\Gamma(t)} \cdot \boldsymbol{v}|_{\Gamma(t)} &= \tfrac{m_0}{2} [\boldsymbol{n}_{\Gamma(t)} \cdot \nabla \mu] && \text{on } \Gamma(t), t \in [0, T_0], && (8)\\
2\mu|_{\Gamma(t)} &= \sigma H_{\Gamma(t)} && \text{on } \Gamma(t), t \in [0, T_0] && (9)
\end{aligned}
$$

in a bounded, smooth domain $\Omega \subseteq \mathbb{R}^2$ together with suitable boundary conditions.

Assumption: (5)-(9) possesses a smooth solution for $t \in [0, T_0]$.

Theorem (A. & Marquardt '20)

Let $M \geq 4$. Assume $c_{0,\varepsilon}$, $0 < \varepsilon \leq 1$, are well-prepared initial data, roughly

$$c_{0,\varepsilon}(x) = \theta_0\left(\frac{d_{\Gamma_0}(x)}{\varepsilon}\right) + \dots \quad \textit{close to } \Gamma(t),$$
$$c_{0,\varepsilon}(x) = \pm 1 + \dots \quad \textit{away from } \Gamma(t).$$

Then there is some $T \in (0, T_0]$ such that the solutions $(\mathbf{v}_\varepsilon, c_\varepsilon)$ of (5)-(9) satisfy

$$\sup_{0\leq t\leq T} \|c_\varepsilon(t) - c_A(t)\|_{H^{-1}(\Omega)} + \varepsilon^{\frac{1}{2}} \|\nabla_\Gamma(c_\varepsilon - c_A)\|_{L^2((0,T)\times\Omega)} = O(\varepsilon^M),$$

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_A\|_{L^{2-}((0,T)\times\Omega))} = O(\varepsilon^{M-\frac{1}{2}}), \ \textit{where}$$

$$c_A(x,t) = \theta_0\left(\frac{d_{\Gamma(t)}(x) - \varepsilon h_\varepsilon(s,t)}{\varepsilon}\right) + O(\varepsilon) \qquad \textit{in } L^\infty((0,T)\times\Omega),$$
$$\mathbf{v}_A(x,t) = \mathbf{v}(x,t) + O(\varepsilon) \qquad \textit{in } L^\infty((0,T)\times\Omega).$$

Moreover, $c_\varepsilon(x,t) \to_{\varepsilon\to 0} \pm 1$ in $\Omega^\pm(t)$, where $\partial\Omega^\pm(t) = \Gamma(t)$.

Sketch of the Proof (Asymptotic Expansion Method)

We follow the strategy of Alikakos et al. '94 and use ideas of A. & Liu '18.

1. Construction of Approximate Solutions: Using finite pieces of formally matched asymptotics calculations one constructs $(c_A, \mu_A, \boldsymbol{v}_A)$ s.t.

$$-\Delta \boldsymbol{v}_A + \nabla p_A = \mu_A \nabla c_A + O(\varepsilon^M),$$

$$\operatorname{div} \boldsymbol{v}_A = 0 + O(\varepsilon^M),$$

$$\partial_t c_A + \boldsymbol{v}_A \cdot \nabla c_A + (\boldsymbol{v}_\varepsilon - \boldsymbol{v}_A)|_{\Gamma(t)} \cdot \nabla c_A = \Delta \mu_A + O(\varepsilon^M)$$

$$\mu_A = -\varepsilon \Delta c_A + \frac{1}{\varepsilon} f'(c_A) + O(\varepsilon^M),$$

where $\boldsymbol{w}|_{\Gamma(t)}(x,t) = \boldsymbol{w}(P_{\Gamma(t)}(x), t)$.

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\partial_t c_A + \boldsymbol{v}_A \cdot \nabla c_A + (\boldsymbol{v}_\varepsilon - \boldsymbol{v}_A)|_{\Gamma(t)} \cdot \nabla c_A &= \Delta \mu_A + O(\varepsilon^M) \\
\mu_A &= -\varepsilon \Delta c_A + \tfrac{1}{\varepsilon} f'(c_A) + O(\varepsilon^M),
\end{aligned}
$$

where $\boldsymbol{w}|_{\Gamma(t)}(x,t) = \boldsymbol{w}(P_{\Gamma(t)}(x), t)$.

2. Remainder Estimates: Using refined estimates for the linearized Cahn-Hilliard operator due to Chen '94 one proves

$$
\sup_{0 \le t \le T} \|c_\varepsilon(t) - c_A(t)\|_{H^{-1}(\Omega)} + \varepsilon^{\frac{1}{2}} \|\nabla_\Gamma (c_\varepsilon - c_A)\|_{L^2((0,T)\times\Omega)} = O(\varepsilon^M),
$$

$$
\|\boldsymbol{v}_\varepsilon - \boldsymbol{v}_A\|_{L^{2-}((0,T)\times\Omega)} = O(\varepsilon^{M-\frac{1}{2}})
$$

Analytic Results for Navier-Stokes/Allen-Cahn System

If the Cahn-Hilliard system is replaced by an Allen-Cahn equation,

$$\partial_t c_\varepsilon + \boldsymbol{v}_\varepsilon \cdot \nabla c_\varepsilon = m_0 \varepsilon^k (\Delta c_\varepsilon - \tfrac{1}{\varepsilon^2} f'(c_\varepsilon)) :$$

Asymptotic Expansion Method (De Mottoni, Schatzman '89 for Allen-Cahn equation):

- A. & Y. Liu '18, A. & Fei '22: Convergence with same/different viscosities and $k = 0$, $d = 2$.
- A., Fei, Moser '23: Convergence with different viscosities, and $k = \frac{1}{2}$, $d = 2$
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Relative Entropy Method (Fischer, Laux, Simon '20 for Allen-Cahn eq.):

- S. Hensel & Y. Liu '22: Convergence with same viscosities and $k = 0$, $d = 2, 3$.
- A., Fischer, Moser '23: Convergence with same viscosities and $k \in (0, 2)$, $d = 2, 3$.

Here $\Omega \subseteq \mathbb{R}^d$, $\rho \equiv const.$.

Remark: There is a counterexample for convergence if $m_\varepsilon = o(\varepsilon^2)$ with inflow boundary condition. (A. '22 together with A. & Lengeler '14)

Thank you for your attention!

Selected References:

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