Existence of nonlinear waves of small amplitude To check that we can apply the Crandall-Rabinowitz theorem in the present setting, we let

$$T = \{(q,p): \ q \in [-\pi,\pi], \ p = 0\}, \qquad B = \{(q,p): \ q \in [-\pi,\pi], \ p = p_0\},$$

be the top, respectively the bottom of the closed rectangle $\overline{R}$, and we define the Banach spaces

$$X = \{w \in C^{3,\alpha}_{per}(\overline{R}): \ w = 0 \quad \text{on} \quad B\}, \qquad Y = C^{1,\alpha}_{per}(\overline{R}) \times C^{2,\alpha}_{per}(T),$$

where the subscript "per" means 2π-periodicity and evenness in the q-variable. If $H(p,\lambda)$ are laminar flows, set

$$h(q,p) = H(p,\lambda) + w(q,p) \quad \text{with} \quad w \in X$$

and write for $\lambda > 2\Gamma_{max}$ the system (bp) in operator form

$$\mathcal{F}(w,\lambda) = 0 \quad \text{with} \quad w \in X,$$

where $\mathcal{F}: X \times (2\Gamma_{max}, \infty) \to Y$ is given by $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ with

$$\mathcal{F}_1(w,\lambda) = (1 + w_q^2)(H_{pp} + w_{pp}) - 2w_q(H_p + w_p)w_{pq} + (H_p + w_p)^2 w_{qq} - \gamma(-p)\,(H_p + w_p)^3, \qquad (F1)$$

$$\mathcal{F}_2(w,\lambda) = \Big(1 + w_q^2 + [2g(H+w) - Q](H_p + w_p)^2\Big)\Big|_T. \qquad (F2)$$

By the equation satisfied by H we have

$$\mathcal{F}(0,\lambda) = 0 \quad \text{for all} \quad \lambda > 2\Gamma_{max}.$$

The linearized operator $\mathcal{F}_w = (\mathcal{F}_{1w}, \mathcal{F}_{2w})$ at $w = 0$ is given by

$$\mathcal{F}_{1w}(0,\lambda) = \partial_p^2 + H_p^2\,\partial_q^2 - 3\gamma(-p)\,H_p^2\,\partial_p \quad \text{in} \quad R, \qquad \mathcal{F}_{2w}(0,\lambda) = 2(\lambda^{-1}g - \lambda^{1/2}\partial_p)\Big|_T. \qquad (Fl)$$

The linear eigenvalue problem (lp) expresses the fact that m belongs to the nullspace of $\mathcal{F}_w(0,\lambda)$.

THE NULL SPACE Assuming (lbc), we know that there is a unique $\lambda^* > 2\Gamma_{max}$ with $\mu(\lambda^*) = -1$ so that the null space $ker\,\{\mathcal{F}_w(0,\lambda^*)\}$ contains at least one element $w(q,p) = M(p)\,\cos(q)$, where $M \in C^{3,\alpha}[p_0,0]$ is the unique eigenfunction of (slps) corresponding to the eigenvalue $\mu(\lambda^*) = -1$. The null space is one-dimensional. Indeed, if $m \in C^{3,\alpha}_{per}(\overline{R})$ belongs to the null space, then we proved before that its Fourier coefficients m_k satisfy (slp) so that $m_1(p)$ is a constant multiple of $M(p)$ while if $m_k \not\equiv 0$ for some $k \geq 2$, we would have

$$\frac{-g\, m_k^2(0) + \displaystyle\int_{p_0}^0 a^3\,(\partial_p m_k)^2\,dp}{\displaystyle\int_{p_0}^0 a\, m_k^2\,dp} = -k^2 < -1\,,$$

contradicting the minimizing property of $\mu(\lambda^*) = -1$. As for m_0, from the differential equation and from the boundary condition at $p = p_0$ in (slp) with $k = 0$ we get

$$m_0(p) = A_0 \int_{p_0}^p a^{-3}(s,\lambda^*)\,ds, \qquad p \in [p_0, 0],$$

for some $A_0 \in \mathbb{R}$, and the boundary condition at $p = 0$ yields $A_0 = 0$ unless $\dfrac{1}{g} = \displaystyle\int_{p_0}^0 a^{-3}(p,\lambda^*)\,dp$, which is impossible. Indeed, the last relation is the defining property of λ_0 as the point where, in the context of laminar flows, the strictly convex function $\lambda \mapsto Q(\lambda)$ attains its minimum. However,

$$\lambda^* < \lambda_0 \qquad (\lambda\lambda),$$

by the monotonicity property of $\mu(\lambda)$ and the fact that $\mu(\lambda^*) = -1$ while $\mu(\lambda_0) = 0$. To see that $\mu(\lambda_0) = 0$, note that $\mathbb{F}\Big(\displaystyle\int_{p_0}^p a^{-3}(s,\lambda_0)\,ds,\ \lambda_0\Big) = 0$ yields $\mu(\lambda_0) \leq 0$. On the other hand, $\mu(\lambda_0) \geq 0$ since $\mathbb{F}(\varphi,\lambda_0) \geq 0$ for any $\varphi \in H^1(p_0,0)$ such that $\varphi(p_0) = 0$, as

$$\begin{aligned} g\,\varphi^2(0) &= g\,\Big(\int_{p_0}^0 \varphi_p(p)\,dp\Big)^2 = \Big(\int_{p_0}^0 a^{3/2}(p,\lambda_0)\,\varphi_p(p)\,a^{-3/2}(p,\lambda_0)\,dp\Big)^2 \\ &\leq g\,\Big(\int_{p_0}^0 a^3(p,\lambda_0)\,\varphi_p^2(p)\,dp\Big)\,\Big(\int_{p_0}^0 a^{-3}(p,\lambda_0)\,dp\Big) = \int_{p_0}^0 a^3(p,\lambda_0)\,\varphi_p^2(p)\,dp\,. \end{aligned}$$

THE RANGE The fact that the operator $\mathcal{F}_w(0, \lambda^*) : X \longrightarrow Y$ has a closed range of codimension one is ensured if we prove that the pair $(\mathcal{A}, \mathcal{B}) \in Y$ belongs to the range if and only if it satisfies the orthogonality condition

$$\iint_R \mathcal{A}(q,p)\, a^3(p,\lambda^*)\, \varphi^*(q,p)\, dqdp + \frac{1}{2} \int_T \mathcal{B}(q)\, a^2(0,\lambda^*)\, \varphi^*(q,0)\, dq = 0, \qquad (oc)$$

where

$$\varphi^*(q,p) = M(p)\, cos(q) \in X$$

generates the null space $ker\,\{\mathcal{F}_w(0,\lambda^*)\}$. Indeed, assuming the validity of the characterization (oc), the range $\mathcal{R}(\mathcal{F}_w(0,\lambda^*))$ is clearly closed in Y. Notice that $\lambda^* > 0$ as $\lambda^* > 2\Gamma_{max}$ and $\Gamma(0) = 0$ ensures $2\Gamma_{max} \geq 0$. On the other hand, $a(p,\lambda^*) > 0$ for $p \in [p_0, 0]$ and $\mathbb{F}(M,\lambda^*) = -1$ ensure $M(0) \neq 0$. Since $a^2(0,\lambda^*) = \lambda^*$, we deduce from (oc) that

$$(0,\ \cos(q)) \notin \mathcal{R}(\mathcal{F}_w(0,\lambda^*)).$$

Notice that if $(\mathcal{A}_1,\ \mathcal{B}_1),\ (\mathcal{A}_2,\ \mathcal{B}_2) \in Y \setminus \mathcal{R}(\mathcal{F}_w(0,\lambda^*))$ then

$$(\mathcal{A}_1,\ \mathcal{B}_1)\ -\ c\,(\mathcal{A}_2,\ \mathcal{B}_2) \in \mathcal{R}(\mathcal{F}_w(0,\lambda^*))$$

for

$$c = \frac{\displaystyle\iint_R \mathcal{A}_1 a^3 \varphi^*\ dqdp + \frac{1}{2} \int_T \mathcal{B}_1\, a^2\, \varphi^*\ dq}{\displaystyle\iint_R \mathcal{A}_2 a^3 \varphi^*\ dqdp + \frac{1}{2} \int_T \mathcal{B}_2\, a^2\, \varphi^*\ dq},$$

the denominator of which being nonzero as $(\mathcal{A}_2,\ \mathcal{B}_2) \notin \mathcal{R}(\mathcal{F}_w(0,\lambda^*))$. This proves that the Banach space $Y/\mathcal{R}(\mathcal{F}_w(0,\lambda^*))$ is one-dimensional, that is, $\mathcal{R}(\mathcal{F}_w(0,\lambda^*))$ has codimension one.

As for the necessity of (oc), if $\mathcal{F}_w(0,\lambda^*)\varphi = (\mathcal{A},\ \mathcal{B})$, then multiplying the partial differential equation

$$\mathcal{A} = \mathcal{F}_{1w}(0,\lambda^*)\varphi = \varphi_{pp} + H_p^2\, \varphi_{qq} - 3\gamma(-p)\, H_p^2\, \varphi_p$$

by $a^3\varphi^*$, integrating by parts, and using the fact that

$$\mathcal{B} = \mathcal{F}_{2w}(0,\lambda^*)\varphi = 2\left(\frac{g}{\lambda^*}\,\varphi - \varphi_p\,\sqrt{\lambda^*}\right)\Big|_T$$

in combination with $\mathcal{F}(H,\lambda) = 0$, $\mathcal{F}_w(0,\lambda)\varphi^* = 0$ and $H_p = a^{-1}$, one obtains (oc).

The proof of the sufficiency of (oc) is technically more intricate. Let us introduce the closed subspaces

$$X_0 = \{\phi \in X : \int_{-\pi}^{\pi} \phi(q,p)\,dq = 0 \quad \text{for all} \quad p \in [p_0, 0]\} \subset X,$$

$$Y_0 = \{(\mathcal{A},\ \mathcal{B}) \in Y : \int_{-\pi}^{\pi} \mathcal{A}(q,p)\,dqdp = 0 \quad \text{for all} \quad p \in [p_0,0], \quad \int_T \mathcal{B}\,dq = 0\} \subset Y.$$

Given $(\mathcal{A},\ \mathcal{B}) \in Y$ such that (oc) holds, since

$$H_p = a^{-1}, \quad a_p = -\gamma(-p)\,a^{-1}, \quad a(0) = \sqrt{\lambda^*},$$

using (FI), we see that $\mathcal{F}_w(0,\lambda^*)\phi = (\mathcal{A}, \mathcal{B})$ for some $\phi \in X$ if and only if

$$\begin{cases} \left(a^3\,\partial_p\,\phi_0\right)_p = a^3\,\mathcal{A}_0 \quad \text{in} \quad (p_0,0), \\ g\phi_0 - a^3\,\partial_p\phi_0 = \dfrac{1}{2}\,a^2\,\mathcal{B}_0 \quad \text{at} \quad p=0, \\ \phi_0 = 0 \quad \text{at} \quad p = p_0, \end{cases} \tag{oc1}$$

and

$$\begin{cases} (a^3\,\varphi_p)_p + a\,\varphi_{qq} = a^3\,(\mathcal{A} - \mathcal{A}_0) \quad \text{in} \quad R, \\ g\varphi - a^3\,\varphi_p = \dfrac{1}{2}\,a^2\,(\mathcal{B} - \mathcal{B}_0) \quad \text{on} \quad T, \\ \varphi_0 = 0 \quad \text{on} \quad B, \end{cases} \tag{oc2}$$

for

$$\varphi = \phi - \phi_0 \in X_0, \quad (\mathcal{A} - \mathcal{A}_0,\ \mathcal{B} - \mathcal{B}_0) \in Y_0,$$

where $\mathcal{B} \in \mathbb{R}$, $\phi_0 \in C^{3,\alpha}[p_0,0]$, $\mathcal{A}_0 \in C^{1,\alpha}[p_0,0]$ are given by

$$\mathcal{B}_0 = \frac{1}{2\pi}\int_T \mathcal{B}\,dq, \quad \phi_0(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(q,p)\,dqdp, \quad \mathcal{A}_0(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{A}(q,p)\,dqdp, \qquad p \in [p_0,0].$$

Claim 1 *For any* $(\mathcal{A}_0,\ \mathcal{B}_0) \in C^{1,\alpha}[p_0,0] \times \mathbb{R}$ *the problem (oc1) has a unique solution* $\phi_0 \in C^{3,\alpha}[p_0,0]$.

Indeed, from the differential equation in (oc1) we infer that

$$\phi_0'(p) = C + \int_{p_0}^{p} a^3(s,\lambda^*)\,\mathcal{A}_0(s)\,ds, \qquad p \in [p_0, 0],$$

for some constant $C \in \mathbb{R}$, so that the boundary condition at $p = p_0$ yields

$$\phi_0(p) = C\int_{p_0}^{p} a^{-3}(s,\lambda^*)\,ds + \int_{p_0}^{p} a^{-3}(s,\lambda^*)\Big(\int_{p_0}^{s} a^3(\tau,\lambda^*)\,\mathcal{A}_0(\tau)\,d\tau\Big)\,ds, \qquad p \in [p_0, 0].$$

The boundary condition at $p = 0$ becomes

$$\begin{aligned} C\Big\{1 - g\int_{p_0}^{0} a^{-3}(p,\lambda^*)\,dp\Big\} &= g\int_{p_0}^{0} a^{-3}(s,\lambda^*)\Big(\int_{p_0}^{s} a^3(\tau,\lambda^*)\,\mathcal{A}_0(\tau)\,d\tau\Big)\,ds \\ &\quad - \int_{p_0}^{0} a^3(\tau,\lambda^*)\,\mathcal{A}_0(\tau)\,d\tau - \frac{1}{2}\,a^2(0,\lambda^*)\,\mathcal{B}_0. \end{aligned}$$

We already proved (in the analysis of the null space) that

$$\frac{1}{g} = \int_{p_0}^{0} a^{-3}(p,\lambda_0)\,dp < \int_{p_0}^{0} a^{-3}(p,\lambda^*)\,dp.$$

Consequently the constant C is always uniquely determined and the proof of Claim 1 is proved.

The above considerations reduce the proof of the sufficiency of (oc) to showing that if (oc) holds, then (oc2) has a solution $\varphi \in X_0$. Notice that both integrals in (oc) vanish if they are evaluated on $\mathcal{A}_0$, respectively $\mathcal{B}_0$. Consequently we have to prove that if $(\mathcal{A},\, \mathcal{B}) \in Y_0$ are satisfying (oc), then (oc2) has a solution $\varphi \in X_0$.

Claim 2 *With $a = a(p, \lambda^*)$ and $(\mathcal{A},\ \mathcal{B}) \in Y_0$, we claim that for every $\varepsilon \in (0,1)$ the approximate problem*

$$\begin{cases} -\varepsilon\, a^3\, v^{(\varepsilon)} + (1+\varepsilon)\left(a^3\, v_p^{(\varepsilon)}\right)_p + (1+\varepsilon)\, a\, v_{qq}^{(\varepsilon)} = a^3\, \mathcal{A} \quad \textit{in} \quad R, \\ g\, v^{(\varepsilon)} - (1+\varepsilon)\, a^3\, v_p^{(\varepsilon)} = \frac{1}{2}\, a^2\, \mathcal{B} \quad \textit{on} \quad T, \\ v^{(\varepsilon)} = 0 \quad \textit{on} \quad B, \end{cases} \tag{aep}$$

has a unique solution $v^{(\varepsilon)} \in X_0$.

To prove the claim we introduce the space

$$\mathbb{H} = \{\varphi \in H^1_{per}(R):\ \varphi \text{ even in } q,\ \int_{-\pi}^{\pi} \varphi(q,p)\, dq = 0 \text{ a.e. in } [p_0, 0],\quad \varphi = 0 \text{ a.e. on } B\}.$$

Notice that $\varphi \mapsto \int_{-\pi}^{\pi} \varphi(q,p)\, dq$ is by Fubini's theorem a bounded linear map from $H^1_{per}(R) \subset L^1(R)$ to $L^1[p_0, 0]$, and the trace operator is also bounded from $H^1_{per}(R)$ to $L^2(B)$ cf. [Evans] so that $\mathbb{H}$ is a Hilbert space, being a closed subspace of the Hilbert space $H^1_{per}(R)$. A function $\varphi \in \mathbb{H}$ is a weak solution to (aep) if

$$\begin{aligned} (1+\varepsilon) \iint_R a^3 \varphi_p \phi_p\, dqdp + (1+\varepsilon) \iint_R a\varphi_q \phi_q\, dqdp + \varepsilon \iint_R a^3\, \varphi\phi\, dqdp - g \int_T \varphi\phi\, dq \\ = -\frac{1}{2} \int_T \mathcal{B}\, a^2\, \phi\, dq - \iint_R \mathcal{A}\, a^3 \phi\, dqdp \end{aligned} \tag{ws}$$

for all $\phi \in \mathbb{H}$. For $\varphi \in \mathbb{H} \cap C^3_{per}(\overline{R})$ we have

$$\varphi(q,p) = \sum_{k=1}^{\infty} \varphi_k(p)\, \cos(kq) \quad \text{in} \quad C^2_{per}(\overline{R}),$$

with $\varphi_k \in C^3[p_0, 0]$ given by

$$\varphi_k(p) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(q,p)\, \cos(kq)\, dq, \qquad p \in [p_0, 0], \quad k \geq 1.$$

Clearly $\varphi_k(p_0) = 0$ for any $k \geq 1$, and

$$\iint_R a^3 \varphi_p^2 \, dqdp = \pi \sum_{k=1}^{\infty} \int_{p_0}^{0} a^3 \left(\partial_p \varphi_k\right)^2 dp, \quad \iint_R a\varphi_q^2 \, dqdp = \pi \sum_{k=1}^{\infty} k^2 \int_{p_0}^{0} a\, \varphi_k^2 \, dp, \quad \int_T \varphi^2 \, dq = \pi \sum_{k=1}^{\infty} \varphi_k^2(0).$$

Taking into account the mimimization problem (μ), we deduce that

$$\iint_R a^3 \varphi_p^2 \, dqdp + \iint_R a\varphi_q^2 \, dqdp \geq \pi \sum_{k=1}^{\infty} \int_{p_0}^{0} \left\{ a^3 \left(\partial_p \varphi_k\right)^2 + a\, \varphi_k^2 \right\} dp \geq \pi\, g \sum_{k=1}^{\infty} \varphi_k^2(0) = g \int_T \varphi^2 \, dq.$$

Consequently

$$\iint_R a^3 \varphi_p^2 \, dqdp + \iint_R a\varphi_q^2 \, dqdp \geq g \int_T \varphi^2 \, dq \qquad (coe).$$

Moreover, the space $\mathbb{H} \cap C^3_{per}(\overline{R})$ being is dense in $\mathbb{H}$ cf. [Evans & Gariepy], (coe) holds for all $\varphi \in \mathbb{H}$.

Since $\inf_{p\in[p_0,0]} \{a(p, \lambda^*)\} > 0$, using (coe) we see that the left-hand side of (ws) defines a bounded and coercive bilinear form in $\mathbb{H}$, while the right-hand side defines a bounded linear functional on $\mathbb{H}$. An application of the Lax-Milgram theorem (see e.g. [Evans]) yields the existence and uniqueness of a weak solution $v^{(\varepsilon)} \in \mathbb{H}$ to (aep). Standard elliptic regularity theory — see [Brézis] — yields $v^{(\varepsilon)} \in X_0$. Moreover, cf. [Gilbarg & Trudinger, Chapter 8], we have the Schauder estimates

$$\|v^{(\varepsilon)}\|_{C^{1,\alpha}_{per}(\overline{R})} \leq C \left(\|a\|_{C^{1,\alpha}_{per}(\overline{R})} + \|\mathcal{A}\|_{C^{1,\alpha}_{per}(\overline{R})} + \|\mathcal{B}\|_{C^{1,\alpha}_{per}(T)} + \|v^{(\varepsilon)}\|_{L^\infty(R)} \right) \qquad (Sc)$$

with the constant C depending only on $\|a\|_{C^{1,\alpha}_{per}(\overline{R})}$.

Claim 3 *We now claim that if* $(\mathcal{A},\ \mathcal{B}) \in Y_0$ *satisfy (oc), then for any sequence* $\varepsilon_k \downarrow 0$ *the sequence* $\{v^{(\varepsilon_k)}\}_{k \geq 1}$ *is bounded in* $C^{1,\alpha}_{per}(\overline{R})$.

Indeed, the existence of some $\varepsilon_k \downarrow 0$ with $\|v^{(\varepsilon_k)}\|_{C^{1,\alpha}_{per}(\overline{R})} \to \infty$ implies by (Sc) that $\|v^{(\varepsilon_k)}\|_{L^\infty(R)} \to \infty$. But then (Sc) ensures that the normalized functions $v_k = \dfrac{v^{(\varepsilon_k)}}{\|v^{(\varepsilon_k)}\|_{L^\infty(R)}}$ are bounded in $C^{1,\alpha}_{per}(\overline{R})$. The compact embedding $C^{1,\alpha}_{per}(\overline{R}) \subset C^1_{per}(\overline{R})$ enables us to extract a subsequence $\{v_{n_k}\}$ that converges in $C^1_{per}(\overline{R})$ to some v with $\|v\|_{L^\infty(R)} = 1$. Consider now (ws) with $\varepsilon = \varepsilon_{n_k}$ and $\varphi = v^{(\varepsilon_{n_k})}$, divide by $\|v^{(\varepsilon_{n_k})}\|_{L^\infty(R)}$, and pass to the limit $n_k \to \infty$ to obtain

$$\iint_R a^3 v_p \phi_p \, dqdp + \iint_R a v_q \phi_q \, dqdp = g \int_T v\phi \, dq, \qquad \phi \in \mathbb{H}.$$

Thus $v \in C^1_{per}(\overline{R})$ is a weak solution (in $\mathbb{H}$) of the problem

$$\begin{cases} (a^3\, v_p)_p + a\, v_{qq} = 0 \quad \text{in} \quad R, \\ g\, v - a^3\, v_p = 0 \quad \text{on} \quad T, \\ v = 0 \quad \text{on} \quad B. \end{cases} \tag{wl}$$

By elliptic regularity $v \in X_0$. Notice that (wl) is precisely (lp) so that $v \in ker\,\{\mathcal{F}_w(0, \lambda^*)\}$. Since $ker\,\{\mathcal{F}_w(0, \lambda^*)\}$ is one-dimensional, v is a multiple of $\varphi^*(q,p) = M(p)\cos(q)$, where $M \in C^{3,\alpha}[p_0, 0]$ is the eigenfunction of (slps) corresponding to $\mu(\lambda) = -1$, that is,

$$v = \delta\varphi^* \tag{vp}$$

for some $\delta \in \mathbb{R}$. If $(\mathcal{A},\ \mathcal{B}) \in Y_0$ satisfy (oc), we can infer more about this limit v.

Indeed, performing in (ws) with $\varepsilon = \varepsilon_{n_k}$, $\varphi = v^{(\varepsilon_{n_k})}$, $\phi = \varphi^*$, an integration by parts and using the fast that φ^* solves (wl) and (oc) holds true, we get

$$\iint_R a^3\, \varphi^*\, v^{(\varepsilon_{n_k})}\, dqdp \;+\; g\, \int_T a\varphi^*\, v^{(\varepsilon_{n_k})}\, dq = 0.$$

Dividing by $\|v^{(\varepsilon_{n_k})}\|_{L^\infty(R)}$ and passing to the limit $n_k \to \infty$, the previous relation yields

$$\iint_R a^3\, \varphi^*\, v\, dqdp \;+\; g\, \int_T a\varphi^*\, v\, dq = 0.$$

But then (vp) forces $v \equiv 0$, and we had $\|v\|_{L^\infty(R)} = 1$. The obtained contradiction proves Claim 2.

Claim 3 and the compactness of $C^{1,\alpha}_{per}(\overline{R}) \subset C^1_{per}(\overline{R})$ yields the existence of a subsequence $\{v^{(\varepsilon_{n_k})}\}$ converging to some limit w in $C^1_{per}(\overline{R})$. Passing to the limit $n_k \to \infty$ in (ws) with $\varepsilon = \varepsilon_{n_k}$, $\varphi = v^{(\varepsilon_{n_k})}$, we obtain that w is a weak solution (in $\mathbb{H}$), and by elliptic regularity a classical solution $w \in X_0$, of

$$\begin{cases} (a^3\, w_p)_p + \, a\, w_{qq} = a^3\, \mathcal{A} \quad \text{in} \quad R, \\ g\, w - a^3\, w_p = \frac{1}{2}\, a^2\, \mathcal{B} \quad \text{on} \quad T, \\ w = 0 \quad \text{on} \quad B, \end{cases} \tag{w}$$

which is precisely (oc2). This completes the proof of the sufficieny of (oc).

Let us emphasize a delicate technical point. It was essential to split X into X_0 and its topological complement corresponding to zero Fourier mode. Indeed, if try to solve (aep) directly in X for $(\mathcal{A},\, \mathcal{B}) \in Y$, then the function

$$\varphi_0(p) = \int_{p_0}^{p} a^{-3}(s,\, \lambda_0)\, ds, \qquad p \in [p_0\,, 0],$$

would be in corresponding Hilbert space $\mathbb{H}$, in the definition of which we dispense of the condition that the means over $[-\pi,\, \pi]$ should vanish for a.e. $p \in [p_0\,, 0]$. Evaluated on $(\varphi_0\,, \varphi_0)$ the bilinear form in (ws) becomes

$$2\pi(1+\varepsilon)\, \int_{p_0}^{0} a^3(p,\, \lambda^*)\, a^{-6}(p,\, \lambda_0)\, dp + 2\pi\varepsilon\, \int_{p_0}^{0} a^3(p,\, \lambda^*)\, \varphi_0^2(p)\, dp - 2\pi\, g\, \varphi_0^2(0).$$

But $\varphi_0(0) = \dfrac{1}{g}$ by the defining property of λ_0, and $(\lambda\lambda)$ yields $a(p,\, \lambda^*) < a(p,\, \lambda_0)$ on $[p_0\,, 0]$ so that

$$\int_{p_0}^{0} a^3(p,\, \lambda^*)\, a^{-6}(p,\, \lambda_0)\, dp < \int_{p_0}^{0} a^{-3}(p,\, \lambda_0)\, dp = \varphi_0(0) = \frac{1}{g}\,.$$

We see that for $\varepsilon > 0$ small enough coerciveness is lost!

THE TRANSVERSALITY CONDITION To ensure the applicability of the Crandall-Rabinowitz theorem it suffices to check the transversality condition $[\mathcal{F}_{\lambda w}(0,\lambda^*)]\,(1,\varphi^*) \notin \mathcal{R}(\mathcal{F}_w(0,\lambda^*))$. Since $a_p = -\,\gamma(-p)\,a^{-1}$, where $a = a(\cdot,\lambda^*)$, from (FI) we compute

$$\mathcal{F}_{\lambda w}(0,\lambda^*) = -\left(a^{-4}\,\partial_q^2 \,+\, a_p\,a^{-3}\,\partial_p,\ \left\{2g\left(\frac{1}{\lambda^*}\right)^2 \,+\, \frac{1}{\sqrt{\lambda^*}}\,\partial_p\right\}\Big|_T\right).$$

By the previous characterization of the range $\mathcal{R}(\mathcal{F}_w(0,\lambda^*))$, it suffices to check that

$$\iint_R a^3\varphi^*\left\{a^{-4}\,\varphi^*_{qq} \,+\, 3a_p\,a^{-3}\,\varphi^*_p\right\}dqdp \,+\, \int_T\left\{\frac{g}{\lambda^*}\,(\varphi^*)^2 \,+\, \frac{1}{2}\sqrt{\lambda^*}\,\varphi^*\varphi^*_p\right\}dq \neq 0\,. \qquad (neq)$$

Since $\varphi^*(q,p) = M(p)\,\cos(q)$ with $M \not\equiv 0$ solving (slps) with $\mu = -1$, and $a(0) = \sqrt{\lambda^*}$, we have that

$$\begin{cases} a^3\,\varphi^*_{pp} + 3a^2 a_p\,\varphi^*_p - a\,\varphi^* = 0 \quad \text{in} \quad (p_0,0),\\ (\lambda^*)^{3/2}\varphi^*_p = g\,\varphi^* \quad \text{at} \quad p=0,\\ \varphi^* = 0 \quad \text{at} \quad p = p_0. \end{cases} \qquad (\varphi^*)$$

Thus

$$\begin{aligned}\iint_R a_p\,\varphi^*\,\varphi^*_p\,dqdp &= \int_T a\,\varphi^*\,\varphi^*_p\,dq - \iint_R a\,\varphi^*\,\varphi^*_{pp}\,dqdp - \iint_R a\,(\varphi^*_p)^2\,dqdp\\ &= \int_T a\,\varphi^*\,\varphi^*_p\,dq + \iint_R \varphi^*\left(3a_p\,\varphi^*_p - a^{-1}\,\varphi^*\right)dqdp - \iint_R a\,(\varphi^*_p)^2\,dqdp\end{aligned}$$

so that

$$\iint_R a_p\,\varphi^*\,\varphi^*_p\,dqdp = -\frac{1}{2}\int_T a\,\varphi^*\,\varphi^*_p\,dq + \frac{1}{2}\iint_R a^{-1}\,(\varphi^*)^2\,dqdp + \frac{1}{2}\iint_R a\,(\varphi^*_p)^2\,dqdp.$$

Consequently we can express (neq) as

$$\iint_R a^{-1}\,\varphi^*\,\varphi^*_{qq}\,dqdp \quad + \quad \frac{3}{2}\iint_R a^{-1}\,(\varphi^*)^2\,dqdp + \frac{3}{2}\iint_R a\,(\varphi^*_p)^2\,dqdp$$
$$+ \quad \int_T \left\{ \frac{g}{\lambda^*}\,(\varphi^*)^2 \;-\; \sqrt{\lambda^*}\,\varphi^*\,\varphi^*_p \right\} dq \neq 0\,.$$

since $a(0) = \sqrt{\lambda^*}$. But the boundary condition at $p = 0$ in (φ^*) yields the vanishing of the boundary integral, and since $\varphi^*_{qq} = -\varphi^*$ we deduce that (neq) equals

$$\frac{1}{2}\iint_R a^{-1}\,(\varphi^*)^2\,dqdp + \frac{3}{2}\iint_R a\,(\varphi^*_p)^2\,dqdp > 0.$$

This completes the proof of the transversality condition.

Conclusion These considerations show that if we can find $\lambda^* > 2\Gamma_{max}$ for which $\mu(\lambda^*) = -1$, then this ensures the existence of nonlinear waves of small amplitude. The condition (lbc) is sufficient to ensure the existence of λ^*. While the hypotheses of the Crandall-Rabinowitz theorem are generally only sufficient but not necessary for local bifurcation, in our case the condition $\mu(\lambda^*) = -1$ is necessary and sufficient for local bifurcation. Indeed, the proof of sufficiency is also contained in the previous considerations: the existence of a bifurcating curve implies that the linearized problem (lp) has a nontrivial solution and we saw that this is possible only if $\mu(\lambda^*) = -1$ for some $\lambda^* > 2\Gamma_{max}$.

The dispersion relation

Since for the laminar flows

$$H_p^{-1}(0,\lambda) = (c-u)\Big|_{\text{at the flat surface}} = \sqrt{\lambda}$$

we see that waves of small amplitude occur exactly when the velocity of the laminar flows reaches the critical speed $\sqrt{\lambda^*}$ at the flat surface. In some special cases it is possible to compute λ^* explicitly.

IRROTATIONAL FLOW If $\gamma \equiv 0$, then $a(p,\lambda) \equiv \sqrt{\lambda}$ and the problem (slps) with $\mu = -1$ becomes

$$\begin{cases} M_{pp} = \lambda^{-1} M & \text{on} \quad (p_0, 0), \\ M_p = g\,\lambda^{-3} M & \text{at} \quad p = 0, \\ M = 0 & \text{at} \quad p = p_0. \end{cases}$$

The general solution of the differential equation with the boundary condition at $p = p_0$ is

$$M(p) = \delta\, \sinh\left(\frac{p - p_0}{\sqrt{\lambda}}\right), \qquad p \in [p_0, 0],$$

with $\delta \in \mathbb{R}$. For $\delta \neq 0$ the boundary condition at $p = 0$ is equivalent to the implicit equation

$$\lambda - g\, \tanh\left(\frac{|p_0|}{\sqrt{\lambda}}\right) = 0. \qquad (iei)$$

As a function of λ the left-hand side can be easily seen to be a strictly increasing function from $(0,\infty)$ onto $\mathbb{R}$, so that $\lambda^* > 0$ is its unique root. We can compute λ^* explicitely in terms of the depth d of the laminar water flow at which bifurcation occurs. Indeed, by (tf) we have

$$H(p,\lambda^*) = \frac{p - p_0}{\sqrt{\lambda^*}}, \qquad p \in [p_0, 0],$$

so $H_p^{-2}(p,\lambda^*) \equiv \lambda^*$ and in the physical variables we get the uniform current $u(x,y) - c \equiv -\sqrt{\lambda^*}$.

But the definition

$$p_0 = \int_{-d}^{0} \Big(u(x,y) - c \Big)\, dy$$

of the relative mass flux yields $d\sqrt{\lambda^*} = |p_0|$ so that (iei) becomes the dispersion relation for 2π-periodic irrotational gravity water waves

$$(c-u)\Big|_{\text{at the flat surface}} = \sqrt{g\ \tanh(d)}\,. \tag{din}$$

To elucidate the meaning of "dispersion", notice that in our analysis we assumed that the wave period is precisely 2π. We have to write the value of the bifurcation parameter for periodic traveling waves with wavelength Ł. For this, given the wavelength Ł > 0, let

$$\kappa = \frac{2\pi}{Ł}$$

be the associated wavenumber, representing the number of cycles of this periodic wave that appear in a spatial window of length 2π in the direction of wave propagation. Consider a solution of period Ł to the governing equations for periodic traveling water waves

$$\begin{cases} (u-c)u_x + vu_y = -P_x \quad \text{in} \quad -d < y < \eta(x-ct), \\ (u-c)v_x + vv_y = -P_y - g \quad \text{in} \quad -d < y < \eta(x-ct), \\ u_x + v_y = 0, \quad \text{in} \quad -d < y < \eta(x-ct), \\ u_y - v_x = \omega, \quad \text{in} \quad -d < y < \eta(x-ct), \\ v = (u-c)\eta' \quad \text{on} \quad y = \eta(x-ct), \\ P = P_{atm} \quad \text{on} \quad y = \eta(x-ct), \\ v = 0 \quad \text{on} \quad y = -d, \end{cases} \tag{Ł}$$

and perform the change of variables (scaling)

$$\tilde{\eta} = \kappa\eta,\ \tilde{c} = c,\ \tilde{u} = u,\ \tilde{v} = v,\ \tilde{P} = P,\ \tilde{x} = \kappa x,\ \tilde{y} = \kappa y,\ \tilde{t} = \kappa t,\ \tilde{g} = \kappa^{-1}g,\ \tilde{\omega} = \kappa^{-1}\omega. \tag{sv}$$

The new variables (dependent and independent) satisfy a 2π-periodic system in the $\tilde{x}$ variable of a form almost identical to (Ł), the only difference being that g should be replaced by $\tilde{g}$ and ω by $\tilde{\omega}$ (corresponding to replacing γ by $\tilde{\gamma} = \kappa^{-1}\gamma$). In view of (din), we obtain the dispersion relation for periodic irrotational gravity water waves

$$(c - u^*) = \sqrt{\frac{g}{\kappa}\ \tanh(\kappa d)}\,, \qquad (di)$$

where $(c - u^*) = (c - u)\Big|_{\text{at the flat surface}}$. Let us now analyze some aspects:

- The function on the right-hand side of (di) being strictly decreasing in κ, we deduce that the speed $(c - u^*)$ exhibits a monotonically increasing dependence on the wavelength $Ł = 2\pi\, \kappa^{-1}$. This is the dispersive effect: within the linear framework (where the superposition principle applies) waves of different lengths travel at different speeds. A group of waves of different wavelengths starting together would spread out, so that after a while the larger waves are at the front.
- Shallow water waves are encountered in the limit $\delta \to 0$, where $\delta = d/Ł$ is the "shallowness parameter". Writing (di) in the form

$$(c - u^*) = \sqrt{gd\ \frac{\tanh(2\pi\delta)}{2\pi\delta}}\,,$$

since $\displaystyle\lim_{\delta\to 0} \frac{\tanh(2\pi\delta)}{2\pi\delta} = 1$, for shallow water waves we have

$$(c - u^*) \approx \sqrt{gd}. \qquad (sww)$$

Numerically $\sqrt{\dfrac{\tanh(s)}{s}}$ lies between 0.97 and 1 for $s < 0.44$ so that (sww) falls short of (di) by at most 3 % provided that $\delta < 0.07$, which for practical purposes is the appropriate range for shallow water waves cf. [Lighthill]. The importance of (sww) lies in that according to it, all waves, if sufficiently long compared with the average water depth, travel at the same speed. This explains why a ride in a speed boat on a relatively calm sea is smoother at higher speeds: if the boat has a speed inferior to the critical speed $\sqrt{gd}$, the waves created by the boat's displacement will overtake it and consequently will create a disturbance in front of the boat, hindering the displacement. To overcome this, the boat must be capable of a sudden burst of power that carries it beyond the critical speed before the wave ahead of it has had the time to form — this is similar to the difficulty in making an aircraft break the sound barrier, that is, travel faster than waves in the air. It is easy to work out the critical speed as $g = 9.8\, m/s^2$. For example, if the depth of the water is $6\, m$, then the critical speed is $7.8\, m/s = 29\, km/h$.

- Another important regime for water waves is that of deep water waves, obtained in the limit $\delta \to \infty$. Writing (di) in the form

$$(c - u^*) = \sqrt{gd\,\frac{\tanh(2\pi\delta)}{2\pi\delta}},$$

since $\lim_{\delta\to\infty}\tanh(2\pi\delta) = 1$, for deep water waves we have

$$(c - u^*) \approx \sqrt{\frac{g\text{Ł}}{2\pi}}. \qquad (dww)$$

Numerically $\sqrt{\tanh(s)}$ lies between 0.97 and 1 for $s > 1.75$ so that (dww) falls short of (di) by at most 3 % provided that $\delta > 0.28$, so $d > 0.28\,\text{Ł}$ characterizes for practical purposes the deep water regime cf. [Lighthill]. The most common cause for waves at sea is due to the wind blowing over the surface of the sea, so that (dww) tells us that waves produced in a storm at sea will travel away from the storm region at speeds proportional to the square root of their wavelengths. In the North Atlantic Ocean, where the average water depth is 3.3 km, under normal conditions waves range from those having lengths of 50 m to those having lengths of 100 m. Formula (dww) tells us that the shorter waves travel at 30 km/h and the longer ones travel faster at 40 km/h to a rough approximation that is, however, not far from reality. In the South Pacific, where the average depth is in excess of 4 km, wavelengths around 300 m can be encountered and such waves travel at speeds around 70 km/h. Notice however that even in the deep sea we can encounter shallow water waves: tsunami waves often have wavelengths in excess of hundreds of km and the deepest point in the ocean is in the Marianas Trench in the Western Pacific Ocean at approximately 11021 m.
- The time period of the wave $\hbar$ (the ratio between wavelength and wave speed) is much easier to measure than wavelength. For waves of small amplitude we obtain from (di) the approximate formula

$$\hbar = \frac{2\pi}{\sqrt{g\kappa\tanh(\kappa d)}},$$

so that, for a fixed undisturbed depth d, the function $\kappa \mapsto \hbar(\kappa)$ is decreasing. Since $\kappa = 2\pi/\text{Ł}$ we infer that the time period of the wave is increasing as a function of wavelength Ł. The numerical value of the period for typical surface gravity waves varies roughly from 0.1 s to 30 s cf. [Lighthill].

- Let us investigate the dependence of $(c - u^*)$ on the average depth d for fixed frequency $ß = \kappa(c - u^*)$, representing the number of cycles of the wave that pass by any fixed point during a time interval of length 2π. From (di) the get

$$ß^2 = g\kappa\, \tanh(\kappa d) = g\, \frac{ß}{c - u^*}\, \tanh\left(\frac{ßd}{c - u^*}\right). \tag{ß}$$

This shows that as the average depth d varies gradually becoming smaller and smaller (at a fixed frequency ß), the speed $(c - u^*)$ has to decrease. The interest of this consideration lies in the fact that as ocean waves approach the coast, passing through water of gradually less and less depth, the frequency ß is practically constant (so that the number of crests approaching the shore per unit time is equal to the number far from the shore — as a glance at these waves will confirm). Let us consider for example a wave with time period $\hbar = 8\,s$ having wavelength $Ł = 100\,m$ in water of large uniform depth. By (ß) we have

$$c - u^* = \frac{g}{ß}\, \tanh\left(\frac{ßd}{c - u^*}\right) < \frac{g}{ß}$$

which yields $\dfrac{ßd}{c - u^*} > ßd\, \dfrac{ß}{g} = \dfrac{ß^2}{g}\, d$ so that

$$c - u^* = \frac{g}{ß}\, \tanh\left(\frac{ßd}{c - u^*}\right) > \frac{g}{ß}\, \tanh\left(\frac{ß^2}{g}\, d\right).$$

The previously two displayed relations show that for ß fixed we have

$$(c - u^*) \approx \frac{g}{ß} \approx 45\, km/h \quad \text{for large} \quad d.$$

Numerically, since $\tanh(s) > 0.95$ for $s > 1.5$, and $\dfrac{ß^2}{g}\, d \approx \dfrac{d}{16\, m}$, the above estimate has an accuracy within 5 % for $d \geq 24\, m$. Both the wave speed and the wavelength are reduced with passage into gradually shallower water: at a depth of $1\, m$, using (sww), the speed of this wave with time period $\hbar = 8\, s$ (as the frequency ß is preserved, $\hbar = 2\pi/ß$ remains also constant) is

$$(c - u^*) \approx \sqrt{gd} \approx 11\, km/h$$

and, since $Ł = \dfrac{2\pi(c - u^*)}{ß}$, the wavelength is reduced to about $25\, m$ — a fourfold reduction!

Flows with constant vorticity If $\gamma \neq 0$ is a constant, the substitution

$$M(p) = \frac{1}{\sqrt{\lambda - 2\gamma p}}\, M_0\Big(\frac{\sqrt{\lambda - 2\gamma p}}{\gamma}\Big)$$

transforms the differential equation in (slps) with $\mu = -1$ into $M_0'' = M_0$. Since $M_0(p_0) = 0$, we deduce that up to a multiplicative constant

$$M(p) = \frac{1}{\sqrt{\lambda - 2\gamma p}}\, \sinh\Big(\frac{\sqrt{\lambda - 2\gamma p} - \sqrt{\lambda - 2\gamma p_0}}{\gamma}\Big), \quad p \in [p_0, 0].$$

The boundary condition at $p = 0$ in (slps) is then equivalent to $\lambda^* > 0$ being a solution of the equation

$$\tanh\Big(\frac{\sqrt{\lambda} - \sqrt{\lambda - 2\gamma p_0}}{\gamma}\Big) = \frac{\lambda}{\gamma\sqrt{\lambda} - g}. \qquad (dr0)$$

From (tf) we obtain that the bifurcating laminar flow is given by

$$H(p, \lambda^*) = \frac{\sqrt{\lambda^* - 2\gamma p_0} - \sqrt{\lambda^* - 2\gamma p}}{\gamma}, \qquad p \in [p_0, 0].$$

For the corresponding velocity field in physical coordinates we have $v = 0$, $u_y = \gamma$ and $(c - u^*) = (c - u)\Big|_{\text{at the flat surface}} = \dfrac{1}{H_p(0, \lambda^*)}$, so that

$$(u - c, v) = (-\sqrt{\lambda^*} + \gamma y, 0), \qquad -d \leq y \leq 0. \qquad (vf)$$

From the definition

$$p_0 = \int_{-d}^{0} \Big(u(x,y) - c\Big)\, dy$$

of the relative mass flux we get $|p_0| = d\sqrt{\lambda^*} + \frac{\gamma}{2}\, d^2$, thus

$$d = \frac{\sqrt{\lambda^* - 2\gamma p_0} - \sqrt{\lambda^*}}{\gamma} > 0, \tag{d}$$

the other root being negative. The previous formula enables us to express (dr0) as

$$\tanh(d) = \frac{\lambda^*}{g - \gamma\sqrt{\lambda^*}}.$$

Solving for $\sqrt{\lambda^*} = (c - u^*)$, we obtain the dispersion relation for 2π-periodic rotational gravity water waves with constant vorticity

$$(c - u^*) = -\,\frac{\gamma}{2}\, \tanh(d) + \frac{1}{2}\, \sqrt{\gamma^2 \tanh^2(d) + 4g\, \tanh(d)}.$$

As with irrotational waves, replacing g by $\kappa^{-1}g$, d by κd, and γ by $\kappa^{-1}\gamma$, we obtain the dispersion relation for rotational periodic gravity water waves with constant vorticity

$$(c - u^*) = -\,\frac{\gamma}{2\kappa}\, \tanh(\kappa d) + \frac{1}{2\kappa}\, \sqrt{\gamma^2 \tanh^2(\kappa d) + 4g\kappa\, \tanh(\kappa d)}. \tag{drc}$$

For $\gamma = 0$ we see that (drc) particularizes to (di). Since the right-hand side of (drc) is strictly decreasing in γ, a negative vorticity enhances the intrinsic wave speed $(c - u^*)$ with respect to the case of an irrotational flow, whereas a positive vorticity diminishes it. A smaller speed facilitates the appearance of waves and this suggests that a current with $\gamma > 0$ is a favorable current while $\gamma < 0$ corresponds to an adverse current.

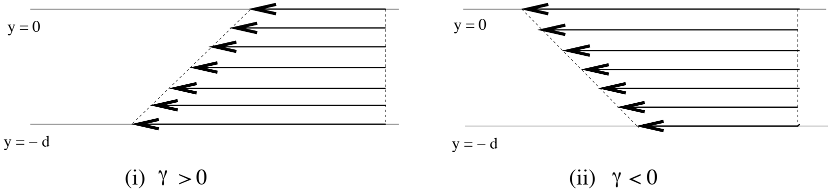
To further clarify this issue, notice that the bifurcating laminar flow with velocity field (vf) is such that

$$(c - u)\Big|_{y=0} = -\sqrt{\lambda^*} < 0$$

along the flat surface and, using (d),

$$(c - u)\Big|_{y=-d} = -\sqrt{\lambda^* - 2\gamma p_0} < 0$$

on the flat bed. This current is positively sheared for $\gamma > 0$ and negatively sheared for $\gamma < 0$.



The current in the moving frame .

In the physical variables the waves of small amplitude whose existence is ensured by the local bifurcation result arise as genuine nonlinear solutions representing small perturbations of the laminar flow with a flat free surface

$$(c - \sqrt{\lambda^*} + \gamma y,\ 0), \qquad -d \leq y \leq 0. \tag{blf}$$

Given $p_0 < 0$ and the normalized wavelength $Ł = 2\pi$, we found solutions in the moving frame, specifying thus v, P, the wave profile η and the horizontal velocity $(u - c)$. This raises the issue of how to determine the wave speed — a nontrivial matter even in the context of irrotational flows, where the issue was settled by Stokes in 1847. In the physical variables the velocity field (u, v) beneath the wave can be decomposed into a current of vorticity γ that is steady and aligned in the plane of the wave motion, of the form $\Big(\gamma(y+d), 0\Big)$, and a wave-induced velocity field

$$(U_0(x-ct, y),\ V_0(x-ct, y)) = \Big(u(x-ct, y) - \gamma(y+d),\ v(x-ct, y)\Big)$$

whose horizontal average beneath the trough level is a uniform current of strength $\mathfrak{S} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} u(x, -d)\, dx$. Indeed, for each fixed $y \in [-d, \eta(-\pi)]$ we have

$$\int_{-\pi}^{\pi} v(x, y)\, dx = 0$$

since v is odd in the x-variable as h is even in the q-variable, while

$$\begin{aligned} \int_{-\pi}^{\pi} \Big(u(x,y) - \gamma(y+d)\Big)\, dx - \int_{-\pi}^{\pi} u(x,-d)\, dx \quad &= \quad \int_{-\pi}^{\pi}\int_{-d}^{y} \Big(u_y(x,y) - \gamma\Big)\, dydx \\ &= \quad \int_{-d}^{y}\int_{-\pi}^{\pi} v_x(x,y)\, dxdy = 0 \end{aligned}$$

since v is 2π-periodic in the x-variable. Consequently, at any horizontal level beneath the trough level $y = \eta(-\pi)$ the average of $(U_0,\ V_0)$ over one wavelength is precisely $(\mathfrak{S}, 0)$. We find c by requiring $\mathfrak{S} = 0$: the wave-induced motion is a pure wave motion without an underlying current.

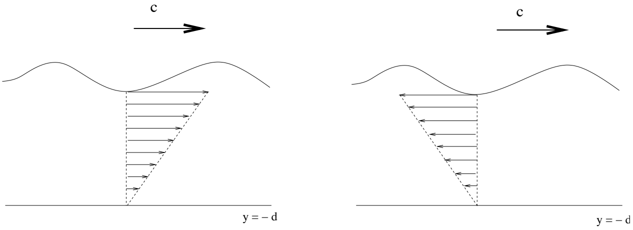
In the case of the laminar flow (blf) we have

$$u(x-ct,y) = c - \sqrt{\lambda^*} + \gamma y, \quad v(x-ct,y) = 0, \qquad -d \le y \le 0,$$

so that

$$c = \sqrt{\lambda^*} + \gamma d = \sqrt{\lambda^* - 2\gamma p_0}$$

in view of (d). This formula is a good approximation of the speed of the waves of small amplitude.



(i) $\gamma > 0$ (ii) $\gamma < 0$

We proved in the context of discussing the possibility $\mu(\lambda) = -1$ that for functions $\gamma \geq 0$ there always exists a solution $\lambda^* > 2\Gamma_{max} = 0$. Notice that a solution $\lambda^* > 2\Gamma_{max}$ of $\mu(\lambda) = -1$, if it exists, is unique since $\lambda \mapsto \mu(\lambda)$ is strictly increasing in wherever $\mu(\lambda) < 0$. We also saw that $\gamma < 0$ constant with $|\gamma|$ sufficiently large might prevent local bifurcation. It is of interest to find the necessary and sufficient condition for local bifurcation in the case of constant vorticity $\gamma < 0$. We discuss the general case in which we do not fix the wavelength $Ł = 2\pi$. Defining the wavenumber $\kappa = 2\pi/Ł$, the necessary and sufficient condition for local bifurcation is the existence of a solution $\lambda > 2\Gamma_{max}$ of (dr0) in which we replace g by $\kappa^{-1}g$, γ by $\kappa^{-1}\gamma$, and p_0 by κp_0: we have

$$\tilde{\psi}(\tilde{x}, \tilde{y}) = \kappa\, \psi(x, y).$$

That is, we seek roots $\lambda > 2\Gamma_{max}$ of

$$f(\lambda) = \tanh\left(\frac{2p_0\kappa}{\sqrt{\lambda} + \sqrt{\lambda - 2\gamma p_0}}\right) - \frac{\lambda\kappa}{\gamma\sqrt{\lambda} - g}. \qquad (dr\kappa)$$

i) For $\gamma = 0$ we have $\Gamma \equiv 0$. The smooth function $f : (0, \infty) \to \mathbb{R}$ is such that $\lim_{\lambda\to\infty} f(\lambda) = \infty$ while $\lim_{\lambda\downarrow 0} f(\lambda) = -1$ so that there exists a root $\lambda^* > 0$.

ii) For constant $\gamma \geq 0$ we have $\Gamma(p) = \gamma p$ so that $\Gamma_{max} = 0$. The smooth function $f : (0, g^2/\gamma^2) \to \mathbb{R}$ has a root since $\lim_{\lambda\uparrow g^2/\gamma^2} f(\lambda) = \infty$ while $\lim_{\lambda\downarrow 0} f(\lambda) = -\tanh\left(\kappa\sqrt{\frac{2|p_0|}{\gamma}}\right) < 0$.

iii) For constant $\gamma < 0$ we have $\Gamma(p) = \gamma p$ so that $\Gamma_{max} = \gamma p_0$ and local bifurcation occurs if and only if there is a solution $\lambda > 2\gamma p_0$ of (drκ). The function $f : (2\gamma p_0, \infty) \to \mathbb{R}$ is smooth and strictly increasing (as a sum of two strictly increasing functions). Clearly $\lim_{\lambda\to\infty} f(\lambda) = \infty$ so that for waves with wavelength $Ł$ in a flow of constant vorticity $\gamma < 0$ the necessary and sufficient condition for local bifurcation is

$$\tanh\left(\kappa\sqrt{\frac{2p_0}{\gamma}}\right) > \frac{2\gamma p_0\kappa}{g - \gamma\sqrt{2\gamma p_0}}, \qquad (nsc)$$

expressing $\lim_{\lambda\downarrow 2\gamma p_0} f(\lambda) < -1$. Passing to the limit $\kappa \to 0$ and $\kappa \to \infty$ we see that for local bifurcation to occur the wavelength $Ł$ must be sufficiently large, while for very short wavelengths there are no genuine waves.