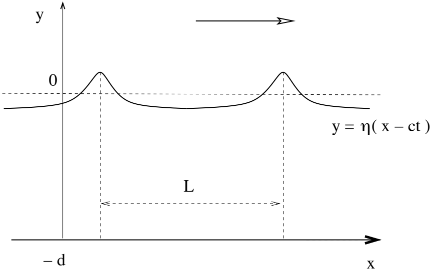
Steady periodic waves

Given $c > 0$, we are looking for two-dimensional periodic steady waves traveling at speed c, that is, the space-time dependence of the free surface, of the pressure, and of the velocity field has the form $(x - ct)$ and is periodic with period $L > 0$. Let the y-axis point vertically upwards, with the origin lying on the mean water level so that the wave profile $y = \eta(x - ct)$ oscillates around the flat surface $y = \int_0^L \eta(x)\,dx = 0$. The flat bed is given by $y = -d$, with $d > 0$ representing the mean water depth. Let (u, v) be the velocity field.



A periodic traveling water wave.

We define the stream function $\psi(x, y)$ by

$$\psi_x = -v, \quad \psi_y = u - c,$$

and let $\omega = u_y - v_x = \Delta\psi$ be the vorticity of the flow.

The free boundary problem for steady periodic water waves can be simplified eliminating time by the change of frame $(x - ct, y) \mapsto (x, y)$. In the new reference frame, in which the origin moves in the direction of propagation of the wave with wave speed c, the wave is stationary and the flow is steady. The mass flux across $x = x_0$ at time t_0 relative to the uniform flow at speed c is $\int_{-d}^{\eta(x_0 - ct_0)} u(x_0 - ct_0, y)\, dy$, and we define the relative mass flux as

$$p_0 = \int_{-d}^{\eta(x)} \Big(u(x, y) - c \Big)\, dy,$$

which is independent of x by the kinematic boundary conditions (tk) and (bk). These boundary conditions also ensure that $\psi(x, y)$ is constant on the free surface $[y = \eta(x)]$ and on the flat bed $[y = 0]$. We normalize ψ by choosing $\psi = 0$ on the free surface. Then, as a consequence of our definition of p_0, the flat bed is the streamline $[\psi = -p_0]$. From the equation of mass conservation (mc) we deduce that $\int_0^L v(\xi, y)\, d\xi$ is independent of y and (bk) shows that this constant is 0, so that ψ is L-periodic in x.

The waves we investigate are such that the propagation speed c of the surface wave is larger than the horizontal velocity u of each individual water particle, that is,

$$u < c \quad \text{throughout the closed fluid domain} \qquad (ns).$$

Field evidence indicates that this assumption holds for wave patterns that are not near the spilling or breaking state, while in laboratory experiments c is an order of magnitude greater than u cf. [Lighthill]. We will show below that the assumption (ns) guarantees the existence of a function γ, called the vorticity function, such that $\omega = \gamma(\psi)$ throughout the fluid. Thus $\Delta\psi = \gamma(\psi)$. Let

$$\Gamma(p) = \int_0^p \gamma(-s)\, ds$$

have maximum value Γ_{max} for $p \in [p_0, 0]$. Notice hat $p_0 < 0$ by (ns) and the definition of p_0.

From the Euler equation (ee) we obtain Bernoulli's law, which states that

$$E = \frac{(c-u)^2 + v^2}{2} + gy + P + \Gamma(-\psi)$$

is constant thoughout the flow. Therefore the dynamic boundary condition (d) is equivalent to stating that

$$\frac{\psi_x^2 + \psi_y^2}{2} + g(y+d) = Q \quad \text{on} \quad y = \eta(x),$$

where $Q = E - P_{atm} + gd$. The hydraulic head E, and hence also Q, has for any flow a constant value that will be regarded as a parameter for the family of solutions that we will construct. In our analysis the mean water depth d is not fixed but will also depend on the particular solution.

Summarizing the above considerations, we can reformulate the free boundary problem as

$$\begin{cases} \Delta\psi = \gamma(\psi) \quad \text{in} \quad -d < y < \eta(x), \\ |\nabla\psi|^2 + 2g(y+d) = Q \quad \text{on} \quad y = \eta(x), \\ \psi = 0 \quad \text{on} \quad y = \eta(x), \\ \psi = -p_0 \quad \text{on} \quad y = \eta(x), \end{cases} \tag{fbp}$$

which is to be solved in the class of functions that are of period $L = 2\pi$ in the x-variable. The main difficulties associated with the problem (fbp) are its nonlinear character and the fact that the free surface $[y = \eta(x)]$ is unknown. The approach we present follows that developed in [Constantin & Strauss], with minor changes — some aspects are simplified and certain inaccuracies are removed (the main one being the fact that the vorticity therein had the opposite sign to the physical vorticity).

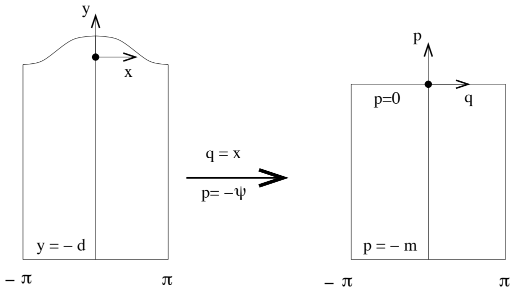
The latter difficulty can be overcome by introducing a coordinate transform devised by Dubreil-Jacotin in 1934. Since ψ is constant both on the free surface and on the flat bed, with $y \mapsto \psi(x, y)$ strictly decreasing in view of (ns), for every fixed x the height

$$h = y + d$$

above the flat bed is a single-valued function of ψ. This leads us to the change of variables

$$q = x, \quad p = -\psi,$$

that transforms the (unknown) fluid domain $D_\eta = \{(x, y) : \ x \in (-\pi, \pi), \ -d < y < \eta(x)\}$ of one wavelength into the (known) rectangular domain $R = (-\pi, \pi) \times (p_0, 0)$. We choose the wave crest on the line $x = 0$.



The hodograph transform .

Since

$$h_q = \frac{v}{u-c}, \qquad h_p = \frac{1}{c-u},$$

we have

$$v = -\frac{h_q}{h_p}, \qquad u = c - \frac{1}{h_p}, \qquad \partial_x = \partial_q - \frac{h_q}{h_p}\,\partial_p, \qquad \partial_y = \frac{1}{h_p}\,\partial_p.$$

Consequently

$$\partial_q\,\omega = \Big(\partial_x + \frac{h_q}{h_p}\,\partial_p\Big)\,\omega = \Big(\partial_x - \frac{v}{c-u}\,\partial_y\Big)\,\omega\,.$$

Taking the curl of the Euler equation (ee) we get $(c-u)\omega_x - v\omega_y = 0$. Consequently $\omega_q = 0$ and ω is a function of p throughout the rectangle, $\omega = \gamma(-p)$, with

$$\gamma(-p) = \gamma(\psi) = \omega = \partial_y u - \partial_x v = \frac{1}{h_p}\,\partial_p\left(c - \frac{1}{h_p}\right) - \left(\partial_q - \frac{h_q}{h_p}\,\partial_p\right)\left(-\frac{h_q}{h_p}\right),$$

and (fbp) becomes

$$\begin{cases} (1+h_q^2)h_{pp} - 2h_q h_p h_{qp} + h_p^2 h_{qq} = \gamma(-p)h_p^3 \quad \text{in} \quad p_0 < p < 0, \\ 1 + h_q^2 + (2gh - Q)h_p^2 = 0 \quad \text{on} \quad p = 0, \\ h = 0 \quad \text{on} \quad p = p_0, \end{cases} \qquad (bp)$$

with h of period 2π in the q-variable and (ns) replaced by

$$h_p > 0 \quad \text{throughout the closed rectangle} \quad \overline{R}. \qquad (hs)$$

The boundary problems (fbp) under the assumption (ns) and (bp) under the assumption (hs) are equivalent. To see this one has to recover ψ, the free surface η and the mean depth d, assuming that $h \in C^k_{per}(\overline{R})$ with $k \geq 2$ solves (bp) and satisfies (hs), where the subscript *per* indicates 2π-periodicity in the q-variable.

The free surface is easily identified as $\eta(x) = h(x, 0) - d$, where $d = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} h(q, 0)\, dq$. To recover ψ we define the $C^{k-1}_{per}(\overline{R})$ function $F(q, p) = \dfrac{1}{h_p(q, p)}$. For a fixed $x_0 \in \mathbb{R}$ we solve the ordinary differential equation

$$\psi_y(x_0, y) = -F(x_0, -\psi(x_0, y)) \tag{de}$$

with initial data $\psi_y(x_0, \eta(x_0)) = 0$. Since $F \in C^1(\overline{R})$ there is a unique local solution. Moreover, (hs) ensures $F \geq \delta > 0$ throughout $\overline{R}$, for some $\delta > 0$, so that $y \mapsto \psi(x_0, y)$ is strictly increasing as y decreases and the solution can be continued uniquely until it reaches the value $-p_0$ at some $y(x_0) < \eta(x_0)$. This shows that for any $x \in \mathbb{R}$ we can define $\psi(x, y)$ on some interval $[y(x),\ \eta(x)]$ with $y(x) < \eta(x)$. By uniqueness for (de), the fact that F is 2π-periodic in q ensures the 2π-periodicity of ψ in the x-variable. For $x \in \mathbb{R}$ fixed, differentiating the expression with respect to $y \in [y(x),\ \eta(x)]$ and using (de), we see that

$$y + d - h(x, -\psi(x,\ y))$$

is independent of $y \in [y(x),\ \eta(x)]$. Since at $y = \eta(x)$ the expression is zero we infer that

$$y + d - h(x, -\psi(x,\ y)) = 0, \qquad y \in [y(x),\ \eta(x)]. \tag{hp}$$

As $\psi(x,\ y(x)) = -p_0$ yields

$$h\Big(x, -\psi(x,\ y(x))\Big) = 0,$$

we deduce that

$$y(x) = -d, \qquad x \in \mathbb{R}.$$

The fact that ψ is of class C^k follows by the dependence on data for the solutions to (de). To show that the constructed ψ satisfies (fbp), observe that we already know that $\psi = -p_0$ on $y = -d$, and $\psi = 0$ on $y = \eta(x)$. Differentiating (hp) with respect to x, we obtain

$$\psi_x(x, y) = \frac{h_q(x, -\psi(x, y))}{h_p(x, -\psi(x, y))},$$

while (de) and the definition of F yield

$$\psi_y(x, y) = \frac{1}{h_p(x, -\psi(x, y))}.$$

The nonlinear boundary condition in (fbp) follows at once. As for the partial differential equation, differentiating (de) with respect to y, we get

$$\psi_{yy}(x, y) = \psi_y(x, y)\, F_p(x, -\psi(x, y)) = -\, F(x, -\psi(x, y))\, F_p(x, -\psi(x, y)) = \frac{h_{pp}}{h_p^3}(x, -\psi(x, y)),$$

while differentiating the identity $\psi_x(x, y)\, h_p(x, -\psi(x, y)) = h_q(x, -\psi(x, y))$ with respect to x yields

$$\psi_{xx}(x, y) = \Big(\frac{h_{qq}}{h_p} - 2h_{qp}\, \frac{h_q}{h_p} + h_{pp}\, \frac{h_q^2}{h_p^3} \Big)(x, -\psi(x, y)).$$

Now

$$\psi_{xx} + \psi_{yy} = \gamma(\psi)$$

follows from the quasilinear partial differential equation in (bp). Defining $v = -\psi_x$ and $u = \psi_y + c$ we obtain a solution of the original problem, where $c > 0$ specifies the moving frame in which the wave is stationary.

For further considerations we notice that if the solution to (bp) is more regular, $h \in C^{3,\alpha}_{per}(\overline{R})$ for some $\alpha \in (0, 1)$, then $\gamma \in C^{1,\alpha}([0, |p_0|])$ and the physical solution

$$(u, v, \eta) \in C^{2,\alpha}_{per}(\overline{D_\eta}) \times C^{2,\alpha}_{per}(\overline{D_\eta}) \times C^{3,\alpha}_{per}(\mathbb{R}),$$

where $\overline{D_\eta}$ is the closure of the fluid domain

$$D_\eta = \{(x, y) :\ x \in \mathbb{R},\ -d < y < \eta(x)\}.$$

The choice of function spaces

We seek classical solutions so that the function h should be periodic in the q-variable and at least twice continuously differentiable and periodic in both variables. The $C^k_{per}(\overline{R})$ Banach spaces (here $k \geq 0$ is a integer) of all functions which are periodic in q and k times continuously differentiable in $q \in \mathbb{R}$ and $p \in [m, 0]$, with the norm given by

$$\|h\|_{C^k(\overline{R})} = \sum_{0 \leq i+j \leq k} \sup_{(q,p) \in R} |\partial_q^i \partial_p^j f(q,p)|$$

are not appropriate for elliptic PDEs, while the Hölder spaces $C^{k,\alpha}_{per}(\overline{R})$ and the Sobolev spaces $W^{k,s}_{per}(\overline{R})$ are particularly useful.

Hölder spaces Hölder continuity is a quantitative measure of continuity (think of it as fractional differentiability), the Hölder spaces $C^{k,\alpha}_{per}(\overline{R})$ with $0 \leq \alpha < 1$ being designed to fill up the gaps between $C^k_{per}(\overline{R})$ and $C^{k+1}_{per}(\overline{R})$. For $k = 0$ and $0 \leq \alpha \leq 1$ these spaces are defined as the subspace of functions $h \in C^0_{per}(\overline{R})$ which obey the Hölder continuity bound

$$|f(\Theta) - f(\Theta_1)| \leq C\,|\Theta - \Theta_1|^\alpha$$

for some constant $C > 0$ and all Θ, $\Theta_1 \in R$, and endowed with the norm

$$\|f\|_{C^{0,\alpha}(\overline{R})} = \sup_{\Theta \in R} \{|f(\Theta)|\} + \sup_{\Theta,\, \Theta_1 \in R,\ \Theta \neq \Theta_1} \left\{ \frac{|f(\Theta) - f(\Theta_1)|}{|\Theta - \Theta_1|^\alpha} \right\}$$

they are Banach spaces. The restriction to $\alpha \leq 1$ is due to the fact that $\alpha > 1$ would imply that all partial derivatives exist and are zero so that the function would be constant. For $\alpha = 1$ the space $C^{0,1}_{per}(\overline{R})$ is the space of Lipschitz functions - a space slightly larger than $C^1_{per}(\overline{R})$ since e.g. the function

$$\Theta = (q, p) \mapsto |\Theta| = \sqrt{q^2 + p^2}$$

belongs to $C^{0,1}_{per}(\overline{R})$ but is not continuously differentiable, so that the scale $C^{0,\alpha}_{per}(\overline{R})$ with $0 \leq \alpha \leq 1$ provides a near-continuum of spaces between $C^0_{per}(\overline{R})$ and $C^1_{per}(\overline{R})$.

For $k \geq 1$ we define $C^{k,\alpha}_{per}(\overline{R})$ as the subspace of functions $h \in C^k_{per}(\overline{R})$ whose norm

$$\|h\|_{C^{k,\alpha}(\overline{R})} = \|h\|_k + \sum_{i+j=k} \sup_{(q,p)\in R} \|\partial_q^i \partial_p^j f(q,p)\|_{C^{0,\alpha}(\overline{R})}$$

is finite. Equipped with these norms, $C^{k,\alpha}(\overline{R})$ are Banach spaces, and if $f_1 \in C^{k_1,\alpha_1}_{per}(\overline{R})$, $f_2 \in C^{k_2,\alpha_2}_{per}(\overline{R})$, then the product $f_1 f_2 \in C^{k,\alpha}_{per}(\overline{R})$, where $k+\alpha = \min\{k_1+\alpha_1,\ k_2+\alpha_2\}$. Moreover, the Arzela-Ascoli theorem ensures that the inclusion map $C^{k_2,\alpha_2}_{per}(\overline{R}) \hookrightarrow C^{k_1,\alpha_1}_{per}(\overline{R})$ is compact if $k_2+\alpha_2 > k_1+\alpha_1$. A good reference for Hölder spaces is [Gilbarg & Trudinger].

Sobolev spaces The spaces $W^{k,s}_{per}(R)$ with $k \geq 0$ integer and $1 < s < \infty$ offer also a suitable setting for elliptic PDEs. They consist of locally integrable functions $h : R \longrightarrow \mathbb{R}$ that are periodic in the q-variable and such that for $i, j \geq 0$ with $i+j \leq k$ the distributional derivative $\partial_q^i \partial_p^j h$ belongs to $L^s(R)$. $W^{k,s}_{per}(R)$ are Banach spaces — respectively Hilbert spaces for $s=2$, in which case they are denoted $H^k(R)$ — if endowed with the norm

$$\|h\|_{W^{k,s}(R)} = \Big(\sum_{0\leq i+j\leq k} \|h\|^s_{L^s(R)}\Big)^{1/s}, \quad \text{where} \quad \|h\|_{L^s(R)} = \Big(\iint_R |h(q,p)|^s \, dqdp\Big)^{1/s}.$$

We denote by $W^{k,s}_0(R)$ the closure in $W^{k,s}_{per}(R)$ of the test functions, that is, of the C^∞ functions of compact support in $(-\pi,\pi)\times(m,0)$ which are extended by periodicity in the q-variable to the strip $\mathbb{R}\times(m,0)$. Since the top and bottom boundary ∂R of R is C^1 (due to periodicity in the q-variable, we may ignore the lateral boundary), functions in $W^{1,s}_{per}(R)$ have a trace on ∂R that belongs to $L^s(\partial R)$, and $h \in W^{1,s}_{per}(R)$ is in $W^{1,s}_0(R)$ if and only if its trace is zero. Notice that ∂R has planar Lebesgue measure zero so that restricting an integrable function in R to the boundary has no direct meaning, the problem being resolved by the notion of the trace operator which associates to $h \in C^0(\overline{R})$ its continuous restriction to ∂R and extends to a bounded linear operator from $W^{1,s}_{per}(R)$ to $L^s(\partial R)$. Good modern sources for Sobolev spaces are [Evans] and [Evans & Gariepy].

SCHAUDER ESTIMATES To explain why it is necessary to introduce Hölder and Sobolev spaces into elliptic theory (as opposed to the more intuitive $C^k(\overline{R})$ spaces), for a given function $f : R \longrightarrow \mathbb{R}$ that is periodic in the q-variable, let us consider the Poisson equation

$$\Delta h = f \quad \text{in} \quad R, \tag{$*$}$$

with homogeneous Dirichlet boundary conditions

$$h = 0 \quad \text{on} \quad \partial R = \{(q, 0) : \; q \in \mathbb{R}\} \, \cup \, \{(q, m) : \; q \in \mathbb{R}\}. \tag{$**$}$$

Then for any $f \in L^s(R)$ with $1 < s < \infty$ there is a unique solution $h \in W_0^{2,s}(R)$ of $(*) - (**)$, and

$$\|h\|_{W_0^{2,s}(R)} \leq \tilde{c}_s \, \|f\|_{L^s(R)} \tag{S_p}$$

where $\tilde{c}_s$ depends only on s. Also, for $f \in C^{0,\alpha}_{per}(\overline{R})$ with $\alpha \in (0, 1)$ there is a unique solution $h \in C^{2,\alpha}_{per}(\overline{R})$ and

$$\|h\|_{C^{2,\alpha}(\overline{R})} \leq c_\alpha \, \|f\|_{C^{0,\alpha}(\overline{R})} \tag{S}$$

where c_α depends only on α. For the proofs of these assertions we refer to [Gilbarg & Trudinger].

The Schauder estimates (S_p) and (S) assert, roughly speaking, that if Δh belongs to $L^s(\overline{R})$ or has $C^{0,\alpha}$ regularity, then all second derivatives of h belong to $L^s(R)$ or have $C^{0,\alpha}$ regularity as well. This remarkable phenomenon — that control of a special linear combination of derivatives of h at some order implies control of all derivatives of h at that order — is known as elliptic regularity. The scales of Hölder and Sobolev spaces are thus suitable to express elliptic regularity, in marked contrast to the $C^k(\overline{R})$ spaces.

The failure of the $C^k(\overline{R})$ spaces to express elliptic regularity can be seen by considering the homogeneous Dirichlet problem for the Poisson equation $(*)$ in the open set $B^* = \{(q, p) \in \mathbb{R}^2 : \; q^2 + p^2 < \frac{1}{4}\}$ with

$$f(q, p) = - \frac{qp}{q^2 + p^2} \left\{ 2 \, f_0^{-1/2} + \frac{1}{4} \, f_0^{-3/2} \right\} \in C^0(\overline{B^*}),$$

where $f_0(q, p) = - \frac{1}{2} \, \ln(q^2 + p^2)$.

Indeed, a direct calculation shows that

$$h^*(q,p) = qp\left(f_0^{1/2} - \sqrt{\ln 2}\right),$$

belongs to $C^1(\overline{B^*})$ with $h^*_{qq},\ h^*_{pp} \in C^0(\overline{B^*})$ and satisfies $\Delta h^* = f$ in B^* with $h^* =$ on ∂B^*, while

$$h^*_{qp} = f_0^{1/2} - \sqrt{\ln 2} - \frac{1}{2}\, f_0^{-1/2} + \frac{q^2 p^2}{(q^2+p^2)^2}\left\{f_0^{-1/2} - \frac{1}{4}\, f_0^{-3/2}\right\} \to \infty \quad \text{for} \quad q, p \to 0.$$

This proves that a Schauder estimate of type (S) on the $C^k(\overline{R})$ scale, asserting that

$$\|h\|_{C^2(\overline{B^*})} \leq c_0\, \|f\|_{C^0(\overline{B^*})},$$

simply does not exist. Moreover, in this case there is no C^2 solution in $\overline{B^*}$: were $h \in C^2(\overline{B^*})$ a solution, the difference $h - h^*$ would be a C^1 harmonic function in $\overline{B^*}$ with zero boundary values on ∂B^* and thus $h - h^* \equiv 0$ throughout $\overline{B^*}$. But then h^* would be C^2 at the origin and this is not the case.

The previous considerations explain why Hölder or Sobolev spaces are proper for elliptic theory. Since we deal with classical solutions and in the degree-theoretic considerations showing the existence of waves of large amplitude as well as in proving the symmetry of the waves we rely upon maximum principles, the most suitable setting for our purposes is that of Hölder spaces — the maximum principles lend themselves well to the suprema that appear in the definition of the norms.

Local bifurcation

When describing the structure of the set of solutions of an equation that depends on a parameter, local bifurcation is the appearance of new solutions when the parameter reaches a critical value. If $\mathbb{X}$, $\mathbb{Y}$ are real or complex Banach spaces — we mainly discuss the case of real Banach spaces but all considerations are equally valid in the case of complex scalars — and $F : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{Y}$ is a map such that

$$F(\lambda, 0) = 0, \qquad \lambda \in \mathbb{R}, \tag{B1}$$

local bifurcation addresses the question: for which λ_0 is there a sequence $(\lambda_n, x_n) \in \mathbb{R} \times \mathbb{X}$ with $x_n \neq 0$ of solutions to $F(\lambda_n, x_n) = 0$, converging to $(\lambda_0, 0) \in \mathbb{R} \times \mathbb{X}$? λ_0 is then called a bifurcation point.

If $F \in C^1(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and if the bounded linear map $\partial_x F(\lambda_0, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is a homeomorphism from $\mathbb{X}$ to $\mathbb{Y}$, by the implicit function theorem all solutions to $F(\lambda, x) = 0$ in a neighborhood of $(\lambda_0, 0) \in \mathbb{R} \times \mathbb{X}$ lie on a unique curve $\{(\lambda, x) : \ x = \varphi(\varepsilon)\}$ with $\varphi : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \longrightarrow \mathbb{X}$ of class C^1, for some $\varepsilon > 0$. From (B1) we conclude that $\varphi(\lambda) = 0$ for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Consequently a necessary condition for λ_0 to be a local bifurcation point is that $\partial_x F(\lambda_0, 0) : \mathbb{X} \longrightarrow \mathbb{Y}$ should not be a homeomorphism. Since $\partial_x F(\lambda_0, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ as F is C^1, in view of the open mapping theorem this is equivalent to $\partial_x F(\lambda_0, 0)$ not being a bijection. This condition is however not sufficient for local bifurcation, as can be seen by regarding $\mathbb{X} = \mathbb{C}$ as a Banach space over $\mathbb{R}$ (we do not identify $\mathbb{X}$ with $\mathbb{R}^2$ since we want to take advantage of multiplication by complex numbers) and consider the compact operator $F \in C^1(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ given by

$$F(\lambda, z) = z - \lambda z - i|z|^2 z, \qquad \lambda \in \mathbb{R}, \quad z \in \mathbb{X}. \tag{B2}$$

Then $\partial_x F(\lambda, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{X})$ is the multiplication operator $(1 - \lambda)$ which fails to be bijective at $\lambda_0 = 1$. But $\lambda_0 = 1$ is not a bifurcation point since a solution $z \neq 0$ of $F(\lambda, z) = 0$ should satisfy $(1 - \lambda)\,|z|^2 = i\,|z|^4$ after multiplication by $\overline{z}$ and this is impossible for $\lambda \in \mathbb{R}$.

Crandall-Rabinowitz theorem *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces and let $F \in C^k(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ with $k \geq 2$ satisfy:*

(i) $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$;

(ii) $L = \partial_x F(\lambda_0, 0) \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is a Fredholm operator of index zero (i.e. its range is closed and has finite codimension equal to the finite dimension of the kernel) with ker(L) one-dimensional;

(iii) the transversality condition $[\partial^2_{\lambda x} F(\lambda_0, 0)]\,(1, \xi_0) \notin$ range(L) holds, where $\xi_0 \in \mathbb{X}$, $\xi_0 \neq 0$, is such that ker$(L) = \{s\xi_0 :\ s \in \mathbb{R}\}$ and $\partial^2_{\lambda x} F(\lambda_0, 0) = \partial_\lambda [\partial_x F(\lambda, 0)]\Big|_{\lambda=\lambda_0} \in \mathcal{L}(\mathbb{R}, \mathcal{L}(\mathbb{X}, \mathbb{Y})) = \mathcal{L}(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$.

Then λ_0 is a bifurcation point: there exists $\varepsilon_0 > 0$ and a branch of solutions

$$\{(\lambda, x) = (\Lambda(s),\ s\,\chi(s)) :\ s \in \mathbb{R},\ |s| < \varepsilon_0\} \subset \mathbb{R} \times \mathbb{X}$$

of $F(\lambda, x) = 0$ with $\Lambda(0) = 0$, $\chi(0) = \xi_0$, and such that $s \mapsto \Lambda(s) \in \mathbb{R}$, $s \mapsto s\chi(s) \in \mathbb{X}$ are of class C^{k-1} on $(-\varepsilon_0, \varepsilon_0)$. Furthermore, there exists an open set $U_0 \subset \mathbb{R} \times \mathbb{X}$ with $(\lambda, 0) \in U_0$ and

$$\{(\lambda, x) \in U_0 :\ F(\lambda, x) = 0,\ x \neq 0\} = \{(\Lambda(s),\ s\,\chi(s)) :\ s \in \mathbb{R},\ 0 < |s| < \varepsilon_0\}.$$

Let us briefly discuss the hypotheses of this theorem — for a proof of the result we refer to [Buffoni & Toland]. Example (B1) shows the importance of hypothesis (ii). Concerning the transversality condition (iii), for

$$F(\lambda, x) = x(\lambda^2 + x^2), \qquad (\lambda, x) \in \mathbb{R} \times \mathbb{R}, \tag{B2}$$

we have $L = \partial_x F(0, 0) = 0 \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ with ker$(L) = \{0\}$ and range$(L) = \{0\}$ of codimension 1 but the transversality condition is not satisfied since $\partial^2_{\lambda x} F(0, 0) = 0 \in \mathcal{L}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. In this case $\lambda_0 = 0$ is not a bifurcation point since $F(\lambda, x) = 0$ is only possible if $x = 0$.

The Crandall-Rabinowitz local bifurcation theorem has wide applicability (and in particular it is well-suited for our purposes) but does not exhaust all possible bifurcations. For example, $\lambda_0 = 0$ is a bifurcation point for

$$F(\lambda, x) = x(\lambda^3 + x^2), \qquad (\lambda, x) \in \mathbb{R} \times \mathbb{R}, \tag{B3}$$

even though the transversality condition fails.

The existence of rotational waves of small amplitude

Given $p_0 < 0$, $\alpha \in (0,1)$ and the vorticity function $\gamma \in C^{1,\alpha}(\overline{R})$, we seek solutions $h \in C^{3,\alpha}(\overline{R})$ to the problem (bp), subject to the condition (hs). Such solutions correspond to solutions

$$\eta \in C^{3,\alpha}_{per}(\overline{D_\eta}), \quad u,\ v \in C^{2,\alpha}_{per}(\overline{D_\eta}), \quad P \in C^{1,\alpha}_{per}(\overline{D_\eta}),$$

of the governing equations, with specified vorticity γ. Notice that a solution h that depends on q corresponds to an undulating free surface, while q-independent solutions h correspond to laminar flows — parallel shear flows with a flat free surface and such that each particle moves horizontally with a speed that depends on the height above the flat bed — representing pure currents.

Local bifurcation theorem *Let* $\gamma_\infty = \|\gamma\|_{C[p_0,0]}$, $p_1 = \max\{p \in [p_0, 0] : \Gamma(p) = \Gamma_{max}\}$ *and assume that*

$$g > \frac{\sqrt{2}}{3}\,\gamma_\infty^{3/2}\,|p_1|^{1/2} + \frac{2\sqrt{2}}{5}\,\gamma_\infty^{1/2}\,|p_1|^{3/2}\,. \tag{lbc}$$

Then there is a C^1*-curve* $\mathcal{C}_{loc}$ *of solutions* $h \in C^{3,\alpha}_{per}(\overline{R})$*. Moreover, the solution curve* $\mathcal{C}_{loc}$ *contains precisely one function that is independent of* q*.*

The proof of this result relies on an application of the Crandall-Rabinowitz theorem. In order to apply it we have to identify the bifurcating parameter. For this notice that the laminar flow solutions to (bp) are given explicitly by

$$H(p,\lambda) = \int_0^p \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}} + \frac{Q-\lambda}{2g} = \int_{p_0}^p \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}}, \qquad p_0 \le p \le 0, \tag{tf}$$

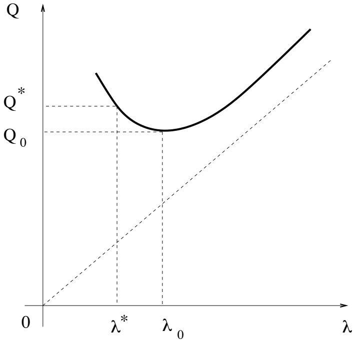
the parameters λ and Q being related by

$$0 < \int_{p_0}^0 \frac{ds}{\sqrt{\lambda - 2\Gamma(s)}} = \frac{Q-\lambda}{2g} \tag{Q}$$

with $0 \le 2\Gamma_{max} < \lambda < Q$.

The bifurcation parameter $\lambda = \dfrac{1}{H_p^2(0,\lambda)}$ is the square of the current velocity at the surface in the moving frame, $\Big(c - u(0,0)\Big)^2$. We point out that λ is not a single-valued function of Q, as the function $\lambda \mapsto Q(\lambda)$ is strictly convex for $\lambda > 0$, its minimum Q_0 on $(0,\infty)$ being attained at the unique point $\lambda_0 > 0$ where

$$\int_{p_0}^{0} \Big(\lambda - 2\Gamma(s)\Big)^{-3/2}\, ds = \frac{1}{g}.$$



For every $Q > Q_0$ there is exactly one $\lambda > \lambda_0$ satisfying (Q), and generally only for certain $Q > Q_0$ there is another solution $\lambda \in (2\Gamma_{max},\ \lambda_0)$. We will see that the bifurcation point λ^* is located to the left of λ_0.

THE LINEARIZATION In order to find waves of small amplitude, we first linearize the problem (bp) about a laminar solution H. We seek solutions $h \in C^{3,\alpha}_{per}(\overline{R})$, even in the q-variable and zero on $p = p_0$, of the form

$$h = H + \varepsilon\, m.$$

The symmetry of the waves is expressed by requiring m to be even. Denoting

$$a(p, \lambda) = \sqrt{\lambda - 2\Gamma(p)},$$

at order ε we obtain for $m \in C^{3,\alpha}_{per}(\overline{R})$ even in the q-variable the boundary problem

$$\begin{cases} (a^3\, m_p)_p + a\, m_{qq} = 0 \quad \text{in} \quad R, \\ a^3\, m_p = gm \quad \text{on} \quad p = 0, \\ m = 0 \quad \text{on} \quad p = p_0\,. \end{cases} \tag{lp}$$

We claim that $m \in C^{3,\alpha}_{per}(\overline{R})$ even in the q-variable has the Fourier series representation

$$m(q, p) = \sum_{k=0}^{\infty} m_k(p)\, \cos(kq) \quad \text{in} \quad C^2_{per}(\overline{R}), \tag{fs}$$

with $C^{3,\alpha}[p_0, 0]$ coefficients

$$m_0(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} m(q, p)\, dq, \qquad m_k(p) = \frac{1}{\pi} \int_{-\pi}^{\pi} m(q, p)\, \cos(kq)\, dq, \qquad k \geq 1.$$

The other cases being similar, it suffices to prove that $\sum_{k=0}^{\infty} m_k''(p)\, \cos(kq)$ converges in $C_{per}(\overline{R})$. Notice that

$$|m_0''(p)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} m_{pp}(q, p)\, dq \right| \leq \|m\|_{C^2_{per}(\overline{R})},$$

$$m_k''(p) = \frac{1}{\pi} \int_{-\pi}^{\pi} m_{pp}(q, p)\, \cos(kq)\, dq = -\frac{1}{k\pi} \int_{-\pi}^{\pi} m_{ppq}(q, p)\, \sin(kq)\, dq, \qquad k \geq 1.$$

Thus for $N \geq n \geq 1$ we have

$$\Big| \sum_{k=n}^{N} m_k''(p)\, \cos(kq)\Big|^2 \leq \Big(\sum_{k=n}^{N} |m_k''(p)|\Big)^2 = \sum_{k=n}^{N} \frac{1}{k\pi}\, \Big| \int_{-\pi}^{\pi} m_{ppq}(q,p)\, \sin(kq)\, dq\Big|\Big)^2$$
$$\leq \Big\{ \sum_{k=n}^{N} \frac{1}{k^2}\Big\} \Big\{ \sum_{k=n}^{N} \Big(\frac{1}{\pi} \int_{-\pi}^{\pi} m_{ppq}(q,p)\, \sin(kq)\, dq\Big)^2 \Big\} \leq \frac{\pi^2}{6} \Big\{ \frac{1}{\pi} \int_{-\pi}^{\pi} m_{ppq}^2(q,p)\, dq\Big\} \leq \frac{\pi}{6}\, \|m\|^2_{C^3_{per}(\overline{R})}.$$

Therefore $\sum_{k=0}^{\infty} m_k''(p)\, \cos(kq)$ converges in $C_{per}(\overline{R})$. Similarly we show that $\sum_{k=0}^{\infty} m_k(p)\, \cos(kq)$ converges in $C_{per}(\overline{R})$ and the convergence of the right-hand side of (fs) to $m(\cdot, p)$ in $L^2[-\pi, \pi]$ for every fixed $p \in [p_0, 0]$ yields $m(q,p) = \sum_{k=0}^{\infty} m_k(p)\, \cos(kq)$ in $C_{per}(\overline{R})$. Then $\sum_{k=0}^{\infty} m_k''(p)\, \cos(kq)$ converges to $m_{pp}(q,p)$ in the sense of distributions and this permits us to identify the limit of $\sum_{k=0}^{\infty} m_k''(p)\, \cos(kq)$ in $C_{per}(\overline{R})$ as being precisely $m_{pp}(q,p)$. Repeating this procedure we establish the validity of (fs).

From (fs) we deduce that m is a solution to (lp) if and only if each m_k solves the Sturm-Liouville problem

$$\begin{cases} (a^3\, M_p)_p = k^2 a\, M \quad \text{in} \quad (p_0, 0), \\ a^3 M_p = gM \quad \text{on} \quad p = 0, \\ M = 0 \quad \text{on} \quad p = p_0, \end{cases} \qquad (slp)$$

with m being q-dependence amounting to $m_k \not\equiv 0$ for some $k \geq 1$. We seek solutions of period 2π so that we investigate (slp) for $k = 1$.

THE VARIATIONAL APPROACH We associate to (slp) with $k = 1$ the minimization problem

$$\mu(\lambda) = \inf_{\varphi \in H^1(p_0,0),\ \varphi(p_0)=0,\ \varphi\not\equiv 0} \{\mathbb{F}(\varphi,\lambda)\} \qquad \text{with} \quad \mathbb{F}(\varphi,\lambda) = \frac{-g\,\varphi^2(0) + \displaystyle\int_{p_0}^0 a^3\varphi_p^2\,dp}{\displaystyle\int_{p_0}^0 a\varphi^2\,dp}\,, \qquad (\mu)$$

suggested by the fact that for a solution M of (slp) we have $\mathbb{F}(M,\lambda) = -k^2$, while the choice of the function space is motivated by the quest for the largest possible Hilbert space for which the $\mathbb{F}$ is well-defined, with the boundary condition on $p = p_0$ captured while that on $p = 0$ is encoded in the form of $\mathbb{F}(\cdot,\lambda)$. For each $\lambda > 2\Gamma_{max}$ the existence of $\mu(\lambda) \in \mathbb{R}$ is ensured since if $\varepsilon(\lambda) = \inf_{p\in[p_0,0]} \{a(p,\lambda)\} > 0$, then

$$\int_{p_0}^0 a^3\varphi_p^2\,dp + \frac{4g^2}{\varepsilon^4(\lambda)}\int_{p_0}^0 a\varphi^2\,dp \geq \varepsilon^3(\lambda)\int_{p_0}^0 \varphi_p^2\,dp + \frac{4g^2}{\varepsilon^3(\lambda)}\int_{p_0}^0 \varphi^2\,dp \geq 4g\int_{p_0}^0 \varphi\varphi_p\,dp = 2g\varphi^2(0) \quad (ivi)$$

whenever $\varphi \in H^1(p_0,0)$ satisfies $\varphi(p_0) = 0$. Thus $\mu(\lambda) > -\dfrac{4g^2}{\varepsilon^4(\lambda)}$. We now claim that the infimum in (μ) is attained by a function $M \in C^{3,\alpha}[p_0,0]$. Choose a minimizing sequence $\varphi_n \in H^1(p_0,0)$ with $\varphi_n(p_0) = 0$ and such that $\mathbb{F}(\varphi_n,\lambda) \to \mu(\lambda)$. Since $\mathbb{F}(\theta\varphi,\lambda) = \mathbb{F}(\varphi,\lambda)$ for any number $\theta \neq 0$, we can normalize the sequence $\{\varphi_n\}_{n\geq 1}$ by setting $\displaystyle\int_{p_0}^0 a\,\varphi_n^2\,dp = 1$ so that, using (ivi), we infer that

$$\mathbb{F}(\varphi_n) = -g\,\varphi_n^2(0) + \int_{p_0}^0 a^3\,(\partial_p\varphi_n)^2\,dp \geq \frac{1}{2}\int_{p_0}^0 a^3\,(\partial_p\varphi_n)^2\,dp - \frac{2g^2}{\varepsilon^4(\lambda)} \geq \frac{\varepsilon^3(\lambda)}{2}\int_{p_0}^0 (\partial_p\varphi_n)^2\,dp - \frac{2g^2}{\varepsilon^4(\lambda)}\,.$$

Since $\mathbb{F}(\varphi_n,\lambda) \to \mu(\lambda)$ we deduce that the sequence $\left\{\displaystyle\int_{p_0}^0 (\partial_p\varphi_n)^2\,dp\right\}_{n\geq 1}$ is bounded. As

$$\frac{1}{\varepsilon(\lambda)} = \frac{1}{\varepsilon(\lambda)}\int_{p_0}^0 a\,\varphi_n^2\,dp \geq \int_{p_0}^0 \varphi_n^2\,dp, \qquad n \geq 1,$$

we have that $\{\varphi_n\}_{n\geq 1}$ is bounded in the Hilbert space $H^1(p_0,0)$ and consequently cf. [Evans] has a weakly convergent subsequence $\{\varphi_{n_k}\}$ with limit $M \in H^1(p_0,0)$.

Notice that $\partial_p \varphi_{n_k} \rightharpoonup \varphi_p$ weakly in $L^2(p_0,0)$ and $\varphi_{n_k}(p_0)=0$ yield

$$\varphi_{n_k}(p) = \int_{p_0}^{p} \partial_p \varphi_{n_k}(s)\, ds \to \int_{p_0}^{p} M_p(s)\, ds = M(p) \quad \text{at every} \quad p \in [p_0, 0]\,. \qquad (pl)$$

From $\partial_p \varphi_{n_k} \rightharpoonup \varphi_p$ we can generally not infer the a.e. convergence of some subsequence as wild oscillations are possible, and nonlinear operations with weakly convergent sequences are generally prohibited cf. [Evans].
Fortunately, the sequence $\{\varphi_{n_k}\}$ is minimizing and the functional $\mathbb{F}(\cdot, \lambda)$ has suitable structural properties:

$$\int_{p_0}^{0} a^3\, M_p^2\, dp \le \liminf_{n_k \to \infty} \int_{p_0}^{0} a^3\, (\partial_p \varphi_{n_k})^2\, dp \qquad (il)$$

since

$$\int_{p_0}^{0} a^3\, (\partial_p\varphi_{n_k})^2\, dp - \int_{p_0}^{0} a^3\, M_p^2\, dp = \int_{p_0}^{0} a^3(\partial_p\varphi_{n_k} - M_p)^2\, dp + 2\int_{p_0}^{0} a^3(\partial_p\varphi_{n_k})M_p\, dp - 2\int_{p_0}^{0} a^3\, \varphi_p^2\, dp$$

and $a^3 M_p \in L^2(p_0,0)$ together with $\partial_p\varphi_{n_k} \rightharpoonup M_p$ weakly in $L^2(p_0,0)$ ensure that the last two terms converge towards zero as $n_k \to \infty$. From (pl) and (il) we infer that

$$-g\, M^2(0) + \int_{p_0}^{0} a^3\, M_p^2\, dp \le \liminf_{n_k \to \infty} \Big\{ -g\, \varphi_{n_k}^2(0) + \int_{p_0}^{0} a^3\, (\partial_p\varphi_{n_k})^2\, dp \Big\}$$

and since the sequence $\{\varphi_{n_k}\}$ is minimizing for $\mathbb{F}(\cdot, \lambda)$, the infimum is a minimum attained at $M \in H^1(p_0,0)$.
Actually, $M \in C^{3,\alpha}[p_0,0]$. Indeed, as a minimum M satisfies the Euler-Lagrange equation

$$0 = \frac{d}{d\varepsilon}\, \mathbb{F}(M + \varepsilon\varphi, \lambda)\Big|_{\varepsilon=0} \qquad (el)$$

for every $\varphi \in H^1(p_0,0)$ with $\varphi(p_0)=0$. Observe that $\mathbb{F}(M,\lambda) = \mu(\lambda)$ and $\displaystyle\int_{p_0}^{0} a\, M^2\, dp = 1$ by the dominated convergence theorem in view of (pl), the normalization of $\{\varphi_{n_k}\}$, and the fact that the boundedness of $\{\varphi_{n_k}\}$ in $H^1(p_0,0)$ ensures an $L^\infty[p_0,0]$ uniform bound for $\{\varphi_{n_k}\}$. This allows us to explicitate (el) as

$$-gM(0)\varphi(0) + \int_{p_0}^{0} a^3 \varphi_p M_p\, dp = \mu(\lambda) \int_{p_0}^{0} aM\varphi\, dp \quad \text{for every} \quad \varphi \in H^1(p_0,0) \quad \text{with} \quad \varphi(p_0)=0. \qquad (ele)$$

Choosing φ smooth and compactly supported in $(p_0, 0)$, we infer that

$$(a^3\, M_p)_p = -\mu\, aM \quad \text{in} \quad H^{-1}(p_0, 0)\,. \tag{eli}$$

associated to the variational problem (μ). Since $a \in C^{2,\alpha}[p_0, 0] \subset H^2(p_0, 0)$, we have $aM \in H^1(p_0, 0)$ so that $a^3 M_p \in H^2(p_0, 0)$ and $M_p \in H^2(p_0, 0)$, that is, $M \in H^3(p_0, 0) \subset C^2[p_0, 0]$. Consequently (eli) holds classically and $a \in C^{2,\alpha}[p_0, 0]$ yields $M \in C^{3,\alpha}[p_0, 0]$. Multiplying (eli) by some $\varphi \in H^1(p_0, 0)$ with $\varphi(p_0) = 0$ and integrating yields

$$-a^3(0)M_p(0)\varphi(0) + \int_{p_0}^{0} a^3\varphi_p M_p\, dp = \mu(\lambda)\int_{p_0}^{0} aM\varphi\, dp\,.$$

Choosing above and in (ele) $\varphi(p) = p - p_0$ we obtain the missing boundary condition at $p = 0$ so that the minimizer $M \in C^{3,\alpha}[p_0, 0]$ is a classical solution of the (weighted) Sturm-Liouville problem

$$\begin{cases} (a^3\, M_p)_p = -\mu\, a\, M \quad \text{in} \quad (p_0, 0), \\ a^3 M_p = gM \quad \text{on} \quad p = 0, \\ M = 0 \quad \text{on} \quad p = p_0\,. \end{cases} \tag{slps}$$

For the existence of linear waves it is necessary that $\mu(\lambda) = -1$ for some $\lambda > 2\Gamma_{max}$ and we will see that this also suffices for the existence of nonlinear waves of small amplitude. First we prove that $\mu(\lambda)$ depends smoothly on $\lambda > 2\Gamma_{max}$. Since for $\lambda > g + 2\Gamma_{max}$ we have $a(\lambda, p) = \sqrt{\lambda + 2\Gamma(p)} > \sqrt{g}$ for $p \in [p_0, 0]$ and thus

$$\int_{p_0}^{0}(a^3\varphi_p^2 + a\varphi^2)\, dp > \sqrt{g}\int_{p_0}^{0}(g\varphi_p^2 + \varphi^2)\, dp \geq 2g\int_{p_0}^{0}\varphi\varphi_p\, dp = g\varphi^2(0)$$

whenever $\varphi \in H^1(p_0, 0)$ is such that $\varphi(p_0) = 0$, we deduce from (μ) that $\mu(\lambda) > -1$ for $\lambda > g + 2\Gamma_{max}$. Consequently the existence of some $\lambda > 2\Gamma_{max}$ with $\mu(\lambda) = -1$ is ensured if $\mu(\lambda) \leq -1$ for some $\lambda > 2\Gamma_{max}$. In this context notice that, given $p_0 < 0$, there are constant vorticities $\gamma < 0$ such that $\mu(\lambda) > -1$ for all $\lambda > 2\Gamma_{max}$, which explains the need of some assumption of type (lbc).

EXAMPLE: LARGE CONSTANT NEGATIVE VORTICITY If $\gamma < 0$ satisfies

$$|\gamma| > \frac{1}{2} + \frac{g^2}{2\,|p_0|^3}\,\Big(\frac{3}{2} + |p_0|\Big)^2$$

then

$$(|2\gamma|)^{3/2} > \sqrt{|2\gamma|} > \frac{g\,(\frac{3}{2} + |p_0|)}{|p_0|^{3/2}},$$

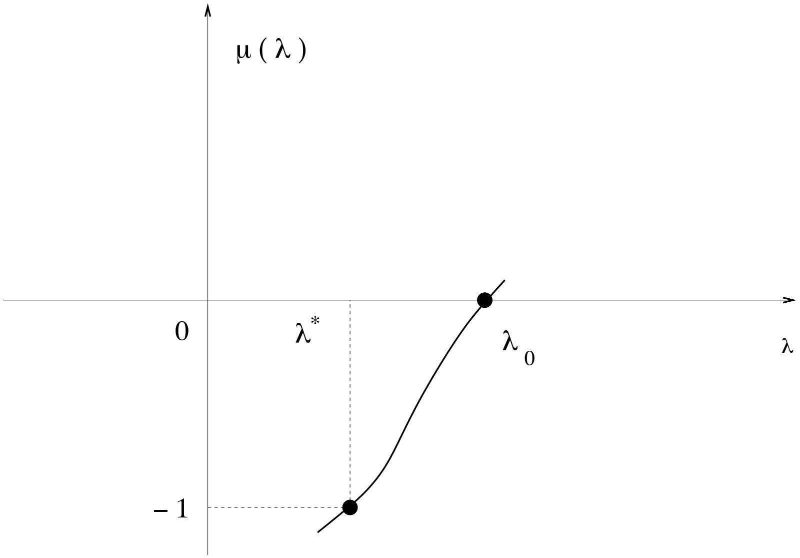
while $\Gamma(p) = \gamma p$ on $[p_0, 0]$ with $\Gamma_{max} = \gamma p_0$, so that

$$a(p,\lambda) = \sqrt{\lambda - 2\Gamma(p)} > \sqrt{2\Gamma(p_0) - 2\Gamma(p)} = \sqrt{2|\gamma|(p-p_0)}, \qquad p \in [p_0, 0], \quad \lambda > 2\Gamma_{max}.$$

Therefore, if $\varphi \in H^1(p_0, 0)$ with $\varphi(p_0) = 0$, $\varphi \not\equiv 0$, then we have

$$\begin{aligned}
\int_{p_0}^0 (a^3\varphi_p^2 + a\varphi^2)\,dp &> (|2\gamma|)^{3/2}\int_{p_0}^0 \varphi_p^2\,(p-p_0)^{3/2}\,dp + \sqrt{|2\gamma|}\int_{p_0}^0 \varphi^2\,\sqrt{p-p_0}\,dp \\
&\geq \frac{g}{|p_0|^{3/2}}\int_{p_0}^0 \varphi_p^2\,(p-p_0)^{3/2}\,dp + \frac{g\,(\frac{3}{2}+|p_0|)}{|p_0|^{3/2}}\int_{p_0}^0 \varphi^2\sqrt{p-p_0}\,dp \\
&= \frac{g}{|p_0|^{3/2}}\Big\{\int_{p_0}^0 \varphi_p^2\,(p-p_0)^{3/2}\,dp + |p_0|\int_{p_0}^0 \varphi^2\,\sqrt{p-p_0}\,dp + \frac{3}{2}\int_{p_0}^0 \varphi^2\sqrt{p-p_0}\,dp\Big\} \\
&\geq \frac{g}{|p_0|^{3/2}}\Big\{\int_{p_0}^0 \varphi_p^2\,(p-p_0)^{3/2}\,dp + \int_{p_0}^0 \varphi^2\,(p-p_0)^{3/2}\,dp + \frac{3}{2}\int_{p_0}^0 \varphi^2\sqrt{p-p_0}\,dp\Big\} \\
&\geq \frac{g}{|p_0|^{3/2}}\Big\{2\int_{p_0}^0 \varphi\varphi_p\,(p-p_0)^{3/2}\,dp + \frac{3}{2}\int_{p_0}^0 \varphi^2\sqrt{p-p_0}\,dp\Big\} = g\,\varphi^2(0)
\end{aligned}$$

and the characterization (μ) yields $\mu(\lambda) > -1$ for $\lambda > 2\Gamma_{max}$.



GROUND STATE DEPENDENCE ON THE PARAMETER We now prove that $\lambda \mapsto \mu(\lambda)$ is real-analytic (in the sense that near any point it may be represented by a convergent power series on some interval of positive length centered at that point) for $\lambda > 2\Gamma_{max}$, with $\mu(\lambda)$ depending monotonically on $\lambda > 2\Gamma_{max}$ whenever $\mu(\lambda) < 0$.

We first prove the smooth dependence of $\mu(\lambda)$ on $\lambda > 2\Gamma_{max}$. Let $M(p, \lambda)$ be the $C^{3,\alpha}[p_0, 0]$-eigenfunction of (slps) corresponding to the eigenvalue $\mu(\lambda)$, normalized by requiring $M(0, \lambda) = 1$; $M(0; \lambda) = 0$ would imply $M \equiv 0$, as can be easily seen by multiplying the differential equation for M by M, and integrating by parts on $[0, p_0]$. Notice that

$$M(p, \lambda) = \varphi(p, \lambda, \mu(\lambda))$$

where $\varphi(p, \lambda, \mu)$ is the unique solution of the linear differential equation

$$(a^3 \varphi_p)_p = -\mu a \varphi \quad \text{in} \quad (p_0, 0) \tag{dem}$$

with initial data

$$\begin{cases} \varphi(0) = 1, \\ \varphi'(0) = \dfrac{g}{a^3(0)}, \end{cases} \tag{idm}$$

depending analytically on (λ, μ) by the dependence of solutions on parameters. The variational approach that provided the existence of $\mu(\lambda)$ yields

$$\mu \int_{p_0}^{0} a M^2 \, dp \,+\, g \,-\, \int_{p_0}^{0} a^3 M_p^2 \, dp = 0, \tag{fem}$$

which we regard as a functional relation between $\mu(\lambda)$ and λ. Differentiating (dem) with respect to μ, multiplying by $= M$ and integrating on $[p_0, 0]$, yields, in view of (idm) and the fact that $M(p_0, \lambda, \mu) = 0$, that

$$2\mu \int_{p_0}^{0} a M M_\mu \, dp \,-\, 2 \int_{p_0}^{0} a^3 M_p M_{p\mu} \, dp = -2 \int_{p_0}^{0} a M^2 \, dp.$$

Therefore the partial derivative with respect to μ of the function of (μ, λ) on the left side of (fem) equals

$$\int_{p_0}^{0} a M^2 \, dp \,+\, 2\mu \int_{p_0}^{0} a M M_\mu \, dp \,-\, 2 \int_{p_0}^{0} a^3 M_p M_{p\mu} \, dp = -\int_{p_0}^{0} a M^2 \, dp < 0.$$

By the implicit function theorem we deduce the real-analytic dependence of $\mu(\lambda)$ on $\lambda > 2\Gamma_{max} > 0$.

Denoting $\dot{a} = \dfrac{\partial a}{\partial \lambda}$ and so on, we compute

$$\dot{a} = \frac{1}{2a}, \qquad \dot{a}_p = -\frac{a_p}{2a^2}.$$

From (slps) we obtain that $\dot{M}$ satisfies

$$\begin{cases} (a^3 \dot{M}_p)_p + \dfrac{3}{2}(aM_p)_p = -\dot{\mu}\, aM - \dfrac{1}{2a}\,\mu M - \mu a \dot{M} \quad \text{in} \quad (p_0, 0), \\ \dfrac{3}{2}\, aM_p + a^3 \dot{M}_p = g\dot{M} \quad \text{at} \quad p = 0, \\ \dot{M} = 0 \quad \text{at} \quad p = p_0\,. \end{cases}$$

Multiplying the above differential equation by M and the differential equation in (slps) by $\dot{M}$, integrating on $(p_0, 0)$ and substracting the outcomes we obtain

$$\dot{\mu} \int_{p_0}^{0} aM^2\, dp = -\frac{1}{2}\,\mu \int_{p_0}^{0} a^{-1} M^2\, dp + \frac{3}{2} \int_{p_0}^{0} aM_p^2\, dp\,. \qquad (mu')$$

Consequently $\lambda \mapsto \mu(\lambda)$ is increasing in any interval where it is negative and the solution λ^* to $\mu(\lambda) = -1$, if it exists, is unique. Moreover, a solution exists if and only if $\lim_{\lambda \downarrow 2\Gamma_{max}} \mu(\lambda) < -1$.

Existence of solutions for the linearization Assuming the validity of (lbc) we prove the existence of non-trivial solutions to the linearized problem (lp). The previous developments show that this amounts to proving that for some $\lambda > 2\Gamma_{max}$ we have $\mu(\lambda) \leq -1$. Let $p_1 = \min\{p \in [p_0, 0] : \Gamma(p) = \Gamma_{max}\}$ and define for $k > \frac{1}{2}$ and $n \geq 2$ the function

$$\varphi_n(p) = \begin{cases} 0, & p_0 \leq p \leq p_n, \\ (p - p_n)^k, & p_n \leq p \leq 0, \end{cases}$$

where

$$p_n = \left(1 - \frac{1}{n}\right) p_1 + \frac{1}{n} p_0 < 0.$$

Clearly $\varphi_n \in H^1(p_0, 0)$ is such that $\varphi_n(0) = 0$ and $\varphi_n \not\equiv 0$; the reason why we introduced $p_n < 0$ instead of simply setting $p_n = p_1$ was to prevent $\varphi_n \equiv 0$ in the special case when $p_1 = 0$. We have

$$a(p, 2\Gamma_{max}) = \sqrt{2\Gamma_{max} - 2\Gamma(p)} = \sqrt{2\Gamma(p_1) - 2\Gamma(p)} \leq \sqrt{2\gamma_\infty\, |p_1 - p|}, \qquad p \in [0, p_0],$$

so that

$$\begin{aligned}
&\int_{p_0}^{0} a^3(p, 2\Gamma_{max}) \left(\partial_p \varphi_n(p)\right)^2 dp + \int_{p_0}^{0} a(p, 2\Gamma_{max})\, \varphi_n^2(p)\, dp \\
&\leq (2\gamma_\infty)^{3/2} k^2 \int_{p_n}^{0} |p - p_1|^{3/2} (p - p_n)^{2k-2}\, dp + (2\gamma_\infty)^{1/2} \int_{p_n}^{0} |p - p_1|^{1/2} (p - p_n)^{2k}\, dp \\
&= (2\gamma_\infty)^{3/2} k^2 \left\{ \int_{p_n}^{p_1} |p - p_1|^{3/2} (p - p_n)^{2k-2}\, dp + \int_{p_1}^{0} |p - p_1|^{3/2} (p - p_n)^{2k-2}\, dp \right\} \\
&\quad + (2\gamma_\infty)^{1/2} \left\{ \int_{p_n}^{p_1} |p - p_1|^{1/2} (p - p_n)^{2k}\, dp + \int_{p_1}^{0} |p - p_1|^{1/2} (p - p_n)^{2k}\, dp \right\} \\
&\leq (2\gamma_\infty)^{3/2} k^2 \left\{ |p_1 - p_n|^{3/2} \int_{p_n}^{p_1} (p - p_n)^{2k-2}\, dp + \int_{p_1}^{0} (p - p_n)^{2k-1/2}\, dp \right\} \\
&\quad + (2\gamma_\infty)^{1/2} \left\{ |p_1 - p_n|^{1/2} \int_{p_n}^{p_1} (p - p_n)^{2k}\, dp + \int_{p_1}^{0} (p - p_n)^{2k+1/2}\, dp \right\}.
\end{aligned}$$

The last expression can be computed explicitly as

$$
\begin{aligned}
&(2\gamma_\infty)^{3/2}k^2\left\{\frac{|p_n|^{2k+1/2}}{2k+1/2}+\frac{3(p_1-p_n)^{2k+1/2}}{(2k-1)(4k+1)}\right\}+(2\gamma_\infty)^{1/2}\left\{\frac{|p_n|^{2k+3/2}}{2k+3/2}+\frac{(p_1-p_n)^{2k+3/2}}{(2k+1)(4k+3)}\right\}\\
&=\varphi_n^2(0)\left\{(2\gamma_\infty)^{3/2}k^2\,\frac{|p_n|^{1/2}}{2k+1/2}+(2\gamma_\infty)^{1/2}\,\frac{|p_n|^{3/2}}{2k+3/2}\right\}\\
&\qquad+\varphi_n^2(0)\left\{(2\gamma_\infty)^{3/2}k^2\,\frac{3(p_1-p_n)^{2k+1/2}}{|p_n|^{2k}(2k-1)(4k+1)}+(2\gamma_\infty)^{1/2}\,\frac{(p_1-p_n)^{2k+3/2}}{|p_n|^{2k}(2k+1)(4k+3)}\right\}.
\end{aligned}
$$

since $\varphi_n^2(0)=|p_n|^{2k}$. On the other hand, by construction we have $\dfrac{|p_1-p_n|}{|p_n|}\leq 1$ while $\lim_{n\to\infty}|p_1-p_n|=0$. Using (lbc), we can find $k>\dfrac{1}{2}$ sufficiently small and some integer $N\geq 2$ such that for some $\varepsilon>0$ we have

$$(2\gamma_\infty)^{3/2}\,k^2\,\frac{|p_n|^{1/2}}{2k+1/2}+(2\gamma_\infty)^{1/2}\,\frac{|p_n|^{3/2}}{2k+3/2}<g-\varepsilon\,,$$

whenever $n\geq N$. With this specified value of k we can now choose $n\geq N$ large enough to ensure

$$(2\gamma_\infty)^{3/2}k^2\,\frac{3(p_1-p_n)^{2k+1/2}}{|p_n|^{2k}(2k-1)(4k+1)}+(2\gamma_\infty)^{1/2}\,\frac{(p_1-p_n)^{2k+3/2}}{|p_n|^{2k}(2k+1)(4k+3)}<\varepsilon\,.$$

This provides us with φ_n satisfying

$$\int_{p_0}^{0}a^3(p,2\Gamma_{max})\left(\partial_p\varphi_n(p)\right)^2dp+\int_{p_0}^{0}a(p,2\Gamma_{max})\,\varphi_n^2(p)\,dp<g\,\varphi_n^2(0).$$

Since a depends continuously on λ, the previous inequality ensures that $\mathbb{F}(\varphi_n)<-1$ for some $\lambda>2\Gamma_{max}$. At this specific λ we have $\mu(\lambda)<-1$.

Example: non-negative vorticity If $\gamma\geq 0$ then $\Gamma_{max}=0$ so that $p_1=0$ and the linearized problem has solutions as (lbc) clearly holds true. Earlier we saw that this can not be expected for negative constant vorticities with $|\gamma|$ sufficiently large!